ORNSTEIN-ZERNIKE BEHAVIOUR AND ANALYTICITY OF SHAPES FOR SELF-AVOIDING WALKS ON $\mathbb{Z}^d$

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ABSTRACT. We derive precise Ornstein-Zernike asymptotics for the decay of the two-point function in any direction of the simple self-avoiding walk on the integer lattice $\mathbb{Z}^d$ in any dimension $d \geq 2$ and for any super-critical value of the parameter $\beta > \beta_c(d)$. The related geometry of the equi-decay level sets is studied as well.

Key words: Ornstein-Zernike behaviour, self-avoiding random, local limit theorems, renewal relations.

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1. INTRODUCTION

Exponential decay of correlations is an essential property of many short range spin lattice systems and sub-critical percolation models. In a variety of cases a typical argument is either based on perturbation type estimates, or on super-multiplicativity properties which, for example, stem from positive association of spins in the underlying field combined with a hard qualitative analysis to assert that the corresponding decay rates are non-trivial. Thus a sharp characterization of the whole of high-temperature region by the exponential decay of connectivities was accomplished in [1] and [2] in the case of the independent percolation and the ferromagnetic spin models with pair interactions respectively.

Many works were also devoted to the study of the, typically polynomial, prefactor near the decay exponent. Results in this direction usually come under the umbrella name of "proving the Ornstein-Zernike behaviour", which is a reference to the original work [12] on classical fluids. Besides providing yet another more precise formula, any such result requires an insight into the fluctuation structure of the corresponding random quantities of interest and is, moreover, naturally related to the intrinsic geometry of the problem.

So far the results were obtained in three main directions (we do not attempt here to provide a complete bibliography and apologize for the missing references):

1. At very high temperatures perturbation techniques such as [13] and, more recently, [11]
2. At any sub-critical temperatures, but for a more restricted class of models and only along specific (for example axis) directions, using coarse graining procedures and analyzing natural renewal structures of the models under considerations [7], [6] and references therein.
3. A completely different robust approach was developed in [3], [4], [5]. It leads to powerful lower bounds on prefactors near the decay exponents for a wide range of models, but fails to pin these prefactors exactly.

Our work here belongs to the second category above and is, in fact, an extension of [7], [6]. In particular we heavily rely on their ideas throughout the article. Our main result asserts complete precise asymptotics of decay in any direction for the two-point function of the simple self-avoiding walk on the integer lattice $\mathbb{Z}^d$. For the sake of convenience we formulate it as two separate theorems: Theorem A below describes the asymptotics itself, whereas Theorem B deals with the properties of related geometric objects.

Below we use $\langle \cdot, \cdot \rangle_d$ to denote the usual scalar product on $\mathbb{R}^d$ and $\| \cdot \|_d$ to denote the corresponding Eucledian norm. Given the value of the parameter $\beta \in (0, \infty)$, the **full** two point function $g_\beta(x, y)$; $x, y \in \mathbb{Z}^d$, of the d-dimensional self-avoiding walk (SAW_d) is defined as

$$g_\beta(x,y) \;=\; \sum_{\omega : x \to y} \mathrm{e}^{-\beta |\omega|}, \tag{1.1}$$

where the above sum is over all lattice self-avoiding paths ω leading from x to y, and $|\omega|$ is the number of steps in ω. We assume that β is chosen above the threshold $\beta > \beta_c(d)$, so that the expression on the right hand side of (1.1) is always finite (see [10] for more details).

Of course, $g_\beta(x, y) = g_\beta(0, x - y) \;\stackrel{\Delta}{=}\; g_\beta(x - y)$, and $g_\beta(\cdot)$ is symmetric in all of its arguments. It is easy to show [10] that $\forall\, \beta > \beta_c$ the "bubble diagram";

$$\mathbf{B}_d(\beta) \;\stackrel{\Delta}{=}\; \sum_x g_\beta(x)^2,$$

is finite. Moreover, $g_\beta/\mathbf{B}_d$ is super-multiplicative;

$$\frac{g_\beta(x+y)}{\mathbf{B}_d(\beta)} \ \geq \ \frac{g_\beta(x)}{\mathbf{B}_d(\beta)}\frac{g_\beta(y)}{\mathbf{B}_d(\beta)}, \tag{1.2}$$

and, consequently, the decay rate

$$\tau_\beta(x) \ = \ -\lim_{n\to\infty}\frac{1}{n}\log g_\beta([nx]) \tag{1.3}$$

is a well defined, sub-additive and homogeneous of order one function on $\mathbb{R}^d$. Moreover, as we shall see in Section 3, τ_β is a norm for each $\beta > \beta_c$. By the super-multiplicativity, $\forall x\ g_\beta(x) \leq \mathbf{B}_d\exp\{-\tau_\beta(x)\}$, and (1.3) can be restated as a logarithmic asymptotic equivalence;

$$g_\beta(x) \ \asymp \ \exp\Big\{ \ -\|x\|_d\tau_\beta\big(\frac{x}{\|x\|_d}\big)\Big\}.$$

In [7] and [6] precise rates of decay of g_β were obtained along lattice directions, that is for $x = (x_1, ..., x_d)$ satisfying $|x_i| \ll |x_1|$; $i > 1$. Our theorem below gives precise decay rates of the two point function g_β in any lattice direction and in any dimension $d \geq 2$:

Theorem A. *In any dimension $d \geq 2$ and for every $\beta > \beta_c(d)$, uniformly in $\|x\|_d$,*

$$g_\beta(x) \ = \ \psi_\beta\big(n(x)\big)\sqrt{\frac{1}{(2\pi\|x\|_d)^{d-1}}}\,e^{-\tau_\beta(x)}\Big(\ 1+o(1) \ \Big), \tag{1.4}$$

where $n(x) = x/\|x\|_d \in \mathbb{S}^{d-1}$ is the unit vector in the direction of x, and the correction coefficient ψ_β is an analytic function; $\psi_\beta : \mathbb{S}^{d-1} \mapsto \mathbb{R}_+$.

Theorem A is a local limit result based on the peculiar renewal structure of connectivities. In Section 2 we prove the corresponding general statement in the context of d-dimensional renewal arrays. The proof of Theorem A is completed in Section 4, where we also obtain an exact expression (4.16) for the correction term ψ_β.

We shall see that the precise assymptotics of the two-point function g_β are closely related to the geometry of the balls $\mathbf{U}^\beta(a)$ in the τ_β-norm;

$$\mathbf{U}^\beta(a) \ \triangleq \ \{x : \tau_\beta(x) \ \leq \ a\ \}.$$

Let us call the level sets of τ_β; $\partial\mathbf{U}^\beta(a)$, **equi-decay profiles**. Since τ_β is homogeneous of order one, equi-decay profiles at different values of a are just dilatations of one another. Thus it would be enough to consider only the unit τ_β-ball. Set $\mathbf{U}^\beta \ \triangleq \ \mathbf{U}^\beta(1)$. The geometry of $\mathbf{U}^\beta$ is studied in Section 3, where we prove:

Theorem B. *In any dimension $d \geq 2$ and for every $\beta > \beta_c(d)$ the boundary of $\partial\mathbf{U}^\beta$ (and hence of any equi-decay profile) is a compact analytic strictly convex surface whose Gaussian curvature is uniformly bounded away from zero.*

2. Ornstein-Zernike Equations

We follow [6] and say that two functions $h : \mathbb{N}\times\mathbb{Z}^d \mapsto \mathbb{R}_+$ and $f : \mathbb{Z}_+\times\mathbb{Z}^d \mapsto \mathbb{R}_+$ satisfy (normalized) Ornstein-Zernike equations, if

$$h(0,k) \ = \ \delta_0(k) \qquad \text{and} \qquad h(n,k) \ = \ \sum_{m=1}^{n}\sum_{l\in\mathbb{Z}^d} f(m,l)h(n-m,k-l). \tag{2.1}$$

Intuitively, $h(n,k)$ represents a connectivity function from the origin $(0,0)$ to the point $(n,k) \in \mathbb{N}\times\mathbb{Z}^d$ inside the $d+1$-dimensional strip $\{0,...n\}\times\mathbb{Z}^d$. Our main objective here is to derive precise large-n local asymptotics of $h(n,k)$. Naturally this task is meaningless in the whole of the generality suggested by (2.1). In view of potential applications we are after as well as for the sake of simplicity we assume that f is strictly positive;

$$f(m,l) \;>\; 0 \qquad \forall m \geq 1 \text{ and } l \in \mathbb{Z}^d. \tag{2.2}$$

More important, we assume that the "mass" of $h(\cdot,\cdot)$;

$$m_{\mathbb{H}}(t) \;\triangleq\; \lim_{n\to\infty} \frac{1}{n} \log \sum_k h(n,k) \mathrm{e}^{(k,t)}, \tag{2.3}$$

is bounded for $t \in \mathbb{R}^d$ in some open neighbourhood of the origin. One of the crucial steps is, then, to verify certain analyticity properties of $m_{\mathbb{H}}$ inside its effective domain. These properties in their turn are intimately related to the renewal structure imposed by (2.1):

Notice that $\forall z \in \mathbb{C}^d$, the functions $h_z(n,k) \;\triangleq\; \mathrm{e}^{\langle z,k\rangle_d} h(n,k)$ and $f_z(n,k) \;\triangleq\; \mathrm{e}^{\langle z,k\rangle_d} f(n,k)$ also obey Ornstein-Zernike equations. In particular, due to the non-negativity of $f(\cdot,\cdot)$ the extended (that is $\overline{\mathbb{R}} \;\triangleq\; \mathbb{R}\cup\{\infty\}$-valued) functions

$$\mathbb{H}_n(t) \;\triangleq\; \sum_k h_t(n,k) \;=\; \sum_k \mathrm{e}^{\langle t,k\rangle_d} h(n,k) \tag{2.4}$$

are super-multiplicative for each real $t \in \mathbb{R}^d$. Consequently $m_{\mathbb{H}}$ in (2.3) is well defined as an extended function. Moreover, $m_{\mathbb{H}}$ is convex. Let us then define the effective domain of $m_{\mathbb{H}}$ as

$$\mathcal{D}_{\mathbb{H}} \;\triangleq\; \left\{\; t \in \mathbb{R}^d \;:\; m_{\mathbb{H}}(t) < \infty \right\}.$$

Then the assumption on the finiteness of the mass we made earlier simply reads as $0 \in \mathrm{int}\big(\mathcal{D}_{\mathbb{H}}\big)$.

In a similar way we define

$$\mathbb{F}_n(t) \;=\; \sum_k \mathrm{e}^{\langle k,t\rangle_d} f(n,k) \;\in\; \bar{\mathbb{R}}_+.$$

In general the existence of the mass of $f(\cdot,\cdot)$ cannot be asserted. We, nonetheless, define:

$$m_{\mathbb{F}}(t) \;\triangleq\; \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{F}_n(t).$$

Clearly, $m_{\mathbb{F}}$ is also convex , and, since,

$$\forall\; (n,k) \in \mathbb{Z}_+ \times \mathbb{Z}^d \qquad 0 \leq f(n,k) \leq h(n,k), \tag{2.5}$$

$m_{\mathbb{F}}$ is finite on $\mathcal{D}_{\mathbb{H}}$.

Because of the non-negativity of the coefficients (2.5) the functions $\mathbb{H}_n$ and $\mathbb{F}_n$ are well defined and analytic in the open strip $\mathcal{S}_{\mathbb{H}} \subset \mathbb{C}^d$,

$$\mathcal{S}_{\mathbb{H}} \;=\; \left\{ z \in \mathbb{C}^d \;:\; \mathrm{Re}(z) \in \mathrm{int}\big(\mathcal{D}_{\mathbb{H}}\big) \right\}.$$

Summing out the $\mathbb{Z}^d$-variable k in the Ornstein-Zernike equations (2.1) for h_z and f_z , we obtain that $\forall z \in \mathcal{S}_{\mathbb{H}}$, the functions $\mathbb{H}_n(z)$ and $\mathbb{F}_n(z)$ obey the usual renewal relation

$$\mathbb{H}_0(z) = 1 \qquad \text{and} \qquad \mathbb{H}_n(z) = \sum_{m=1}^{n} \mathbb{F}_m(z)\mathbb{H}_{n-m}(z). \tag{2.6}$$

These renewal relations have a dramatic impact once a separation of masses type condition is assumed:

Theorem 2.1. *Assume that $t_0 \in int\big(\mathcal{D}_{\mathbb{H}}\big)$ is such that,*

$$m_{\mathbb{H}}(t_0) \;>\; m_{\mathbb{F}}(t_0). \tag{2.7}$$

Then $\exists$ a complex neighbourhood $\mathcal{U}^{\mathbb{C}^d}$ of t_0 in $\mathbb{C}^d$, such that:

(a) *For n large enough,*

$$\mathbb{H}_n(z) \;\neq\; 0 \qquad on\ \mathcal{U}^{\mathbb{C}^d}. \tag{2.8}$$

(b) $m_{\mathbb{H}}$ *can be extended to an analytic function on* $\mathcal{U}^{\mathbb{C}^d}$. *Moreover, there exists* $\alpha > 0$, *such that uniformly in* $z \in \mathcal{U}^{\mathbb{C}^d}$,

$$\frac{1}{n}\log \mathbb{H}_n(z) \;-\; m_{\mathbb{H}}(z) \;=\; \frac{1}{n}\log\mu(z) \;+\; o\big(e^{-n\alpha}\big), \tag{2.9}$$

where,

$$\mu(z) \;=\; \Big(\sum_n n e^{-nm_{\mathbb{H}}(z)}\mathbb{F}_n(z)\Big)^{-1} \;\neq\; 0$$

is analytic on $\mathcal{U}^{\mathbb{C}^d}$.

(c) *If, furthermore, the Hessian* $D^2 m_{\mathbb{H}}(t_0)$ *of* $m_{\mathbb{H}}$ *at* t_0;

$$D^2 m_{\mathbb{H}}(t_0) \text{ is positively definite,} \tag{2.10}$$

then the following asymptotic (in n) description of $h(n,k)$ *for* k/n *close to* $x_0 \;\triangleq\; \nabla m_{\mathbb{H}}(t_0)$ *is valid: Let* $(n,k) \in \mathbb{N}\times\mathbb{Z}^d$ *be such that*

$$\frac{k}{n} \;=\; \nabla m_{\mathbb{H}}(t) \tag{2.11}$$

for some $t \in \mathcal{U}^{\mathbb{C}^d} \cap \mathbb{R}^d$. *Then,*

$$h(n,k) \;=\; \frac{\mu(t)}{\sqrt{(2n\pi)^d det D^2 m_{\mathbb{H}}(t)}}\exp\Big\{-nm_{\mathbb{H}}^*(\frac{k}{n})\Big\}\big(1+o(1)\big), \tag{2.12}$$

where $m_{\mathbb{H}}^*$ *is the Fenchel-Young transform of* $m_{\mathbb{H}}$.

The proof of the first part of the theorem is, to a large extent, built upon the ideas and techniques of [6] and [7]. The link to the local limit behaviour of h comes with the Lee-Yang type condition (2.8) which was apparently overlooked in the later papers.

Notice, first of all, that under the mass-gap condition (2.7) of the theorem, the sequence $\mathrm{e}^{-nm_{\mathbb{H}}(t_0)}\mathbb{F}_n(t_0)$ is a proper probability distribution,

$$\sum_n \mathrm{e}^{-nm_{\mathbb{H}}(t_0)}\mathbb{F}_n(t_0) \;=\; 1. \tag{2.13}$$

Indeed, by the virtue of (2.7), the function

$$\phi(r) \;\triangleq\; \sum_n r^n\mathrm{e}^{-nm_{\mathbb{H}}(t_0)}\mathbb{F}_n(t_0)$$

is well defined and continuous for $r \in [0, \mathrm{e}^{m_{\mathbb{H}}(t_0)-m_{\mathbb{F}}(t_0)}) \supset [0,1]$. If $\phi(1) > 1$, then one can find $r \in (0,1)$, such that $\phi(r) = 1$. Thus $r^n\mathrm{e}^{-nm_{\mathbb{H}}(t_0)}\mathbb{F}_n(t_0)$ becomes a proper probability distribution which generates via the renewal relation the sequence $r^n\mathrm{e}^{-nm_{\mathbb{H}}(t_0)}\mathbb{H}_n(t_0)$. Consequently, by the renewal theorem,

$$\lim_{n\to\infty} r^n\mathrm{e}^{-nm_{\mathbb{H}}(t_0)}\mathbb{H}_n(t_0) \;=\; \Big(\sum_n nr^n\mathrm{e}^{-nm_{\mathbb{H}}(t_0)}\mathbb{F}_n(t_0)\Big)^{-1} \;>\; 0,$$

which implies $m_{\mathbb{H}}(t_0) = m_{\mathbb{H}}(t_0) - \log r$; a contradiction.

If, on the other hand, $\phi(1) < 1$, then, by continuity, $\phi(r) < 1$ for values of r slightly larger than 1 as well. By the renewal theorem this means that for such r-s,

$$\lim_{n\to\infty} r^n\mathrm{e}^{-nm_{\mathbb{H}}(t_0)}\mathbb{H}_n(t_0) \;=\; 0,$$

which again contradicts the definition $m_{\mathbb{H}}(t_0) = \lim_{n\to\infty} 1/n\log\mathbb{H}_n(t_0)$.

Since by convexity both $m_{\mathbb{H}}$ and $m_{\mathbb{F}}$ are continuous on $\mathcal{D}_{\mathbb{H}}$, the mass-gap condition (2.7) persists in some neighbourhood $\mathcal{U}_1^{\mathbb{R}^d}$ of t_0. Applying the above reasoning for each point $t \in \mathcal{U}_1^{\mathbb{R}^d}$, we conclude that,

$$\forall t \in \mathcal{U}_1^{\mathbb{R}^d}(t_0) \qquad m_{\mathbb{H}}(t) > m_{\mathbb{F}}(t) \;\text{ and }\; \sum_n \mathrm{e}^{-nm_{\mathbb{H}}(t)}\mathbb{F}_n(t) = 1. \tag{2.14}$$

We turn now to the proof the first part of the theorem. The crucial statement is the Lee-Yang type assertion (2.8). Define the function Φ via

$$\Phi(\xi, z) \;=\; \sum_n \mathrm{e}^{n\xi}\mathbb{F}_n(z) - 1.$$

Due to the mass-gap condition (2.14) Φ is well defined and analytic in some $\mathbb{C}\times\mathbb{C}^d$ neighbourhood of $(-m_{\mathbb{H}}(t_0), t_0)$. Moreover,

$$\frac{\partial\Phi}{\partial\xi}\big(-m_{\mathbb{H}}(t_0), t_0\big) \;=\; \sum_n n\mathrm{e}^{-nm_{\mathbb{H}}(t_0)}\mathbb{F}_n(t_0) \;\in\; (0,\infty),$$

which again follows by (2.7) Therefore, by the analytic implicit function theorem, one can find a neighbourhood $\mathcal{U}_2^{\mathbb{C}}$ of $-m_{\mathbb{H}}(t_0)$ in $\mathbb{C}$, a $\mathbb{C}^d$-neighbourhood $\mathcal{U}_3^{\mathbb{C}^d} \subset \mathcal{S}_{\mathbb{H}}$ of t_0 and an analytic function $\xi : \mathcal{U}_3^{\mathbb{C}^d} \mapsto \mathcal{U}_2^{\mathbb{C}}$, such that

$$\Phi(\xi(z), z) \;\equiv\; 0 \qquad \text{on } \mathcal{U}_3^{\mathbb{C}^d},$$

and, moreover,

$$\forall (s,z) \in \mathcal{U}_2^{\mathbb{C}} \times \mathcal{U}_3^{\mathbb{C}^d} \qquad \Phi(s,z) = 0 \;\Leftrightarrow\; s = \xi(z). \tag{2.15}$$

Remark 2.2. Since by (2.14) $\Phi(-m_{\mathbb{H}}(t), t) \equiv 0$ on some $\mathbb{R}^d$-neighbourhood $\mathcal{U}_1^{\mathbb{R}^d}$ of t_0, one identifies ξ as the analytic continuation of $-m_{\mathbb{H}}$ on

$$\mathcal{U}_3^{\mathbb{C}^d} \cap \Big\{ z :\ \mathrm{Re}(z) \in \mathcal{U}_1^{\mathbb{R}^d} \Big\}.$$

By itself, however, the analyticity of $m_{\mathbb{H}}$ by no means implies the convergence claim (2.9). Neither it is particularly useful for the derivation of local limit results of the type (2.12).

Let us thus define a new function Ψ;

$$\Psi(z, s) \;=\; \sum_n \mathrm{e}^{-nm_{\mathbb{H}}(z)}\mathbb{F}_n(z)s^n.$$

Conditions (2.2) and (2.15) imply, that there exists a number $\delta > 0$ and a $\mathbb{C}^d$-neighbourhood $\mathcal{U}^{\mathbb{C}^d}$ of t_0, such that $\forall z \in \mathcal{U}^{\mathbb{C}^d}$,

$$\big\{ s \in \mathbb{C} :\ |s| < 1 + \delta \big\} \cap \big\{ s \in \mathbb{C} :\ \Psi(z,s) = 1 \big\} \;=\; \{1\}. \tag{2.16}$$

Define $r_n(z) \;\triangleq\; \mathrm{e}^{-nm_{\mathbb{H}}(z)}\mathbb{F}_n(z)$ and $l_n(z) \;\triangleq\; \mathrm{e}^{-nm_{\mathbb{H}}(z)}\mathbb{H}_n(z))$. Then the above condition reads as

$$\big\{ s \in \mathbb{C} :\ |s| < 1 + \delta \big\} \cap \big\{ s \in \mathbb{C} :\ \mathbb{R}(s) = 1 \big\} \;=\; \{1\}, \tag{2.17}$$

where $\mathbb{R}(s) \;\triangleq\; \sum_n r_n s^n$ is the generating function of the r_n-sequence.

On the other hand, r_n and l_n are in the (complex) renewal relation:

$$l_0(z) = 1 \qquad \text{and} \qquad l_n(z) = \sum_{m=1}^{n} r_m(z) l_{n-m}(z). \tag{2.18}$$

Moreover, possibly after shrinking $\mathcal{U}^{\mathbb{C}^d}$, one easily verifies, that $\forall z \in \mathcal{U}^{\mathbb{C}^d}$,

$$\sum_n n r_n(z) \;\neq\; 0, \tag{2.19}$$

and there exists $\epsilon > 0$, such that

$$\sum_n \mathrm{e}^{n\epsilon}|r_n(z)| \;<\; \infty. \tag{2.20}$$

One can now check that under (2.17)-(2.20) the conclusion of the usual renewal theorem pertains to the complex case as well, that is

$$\lim_{n\to\infty} l_n(z) \;=\; \lim_{n\to\infty} \mathrm{e}^{-nm_{\mathbb{H}}(z)}\mathbb{H}_n(z) \;=\; \Big(\sum_n n r_n(z) \Big)^{-1} \;=\; \mu(z) \;\neq\; 0. \tag{2.21}$$

For example the proof in [15] literally goes through. A Lee-Yang type assertion (2.8) follows.

As soon as (2.8) is established, we easily obtain that

$$m_{\mathbb{H}}(z) \;=\; \lim_{n\to\infty}\frac{1}{n}\log\mathbb{H}_n(z) \;\triangleq\; \lim_{n\to\infty} m_{\mathbb{H},n}(z) \tag{2.22}$$

on $\mathcal{U}^{\mathbb{C}^d}$ in the sense of analytic functions. Indeed, we know that $m_{\mathbb{H}}$ is analytic on $\mathcal{U}^{\mathbb{C}^d}$, and that (2.22) holds for the real values $z \in \mathcal{U}\cap\mathbb{R}^d \subset \mathcal{D}_{\mathbb{H}}$. Consequently, it remains to show that the sequence $\{m_{\mathbb{H},n}\}$ is compact or, equivalently, that it is uniformly bounded on $\mathcal{U}^{\mathbb{C}^d}$. To this end set

$$u_n(z) \;\triangleq\; \mathbb{H}_n(z)^{\frac{1}{n}}.$$

By (2.5), $|u_n(z)| \le u_n(\mathrm{Re}(z))$. Therefore, $\{u_n\}$ is a uniformly bounded sequence of analytic functions on $\mathcal{U}^{\mathbb{C}^d}$. We claim that $\{u_n\}$ is, in addition, uniformly bounded away from zero on $\mathcal{U}^{\mathbb{C}^d}$. Indeed, for real values of z;

$$\forall z \in \mathcal{U}\cap\mathbb{R}^d \subset \mathcal{D}_{\mathbb{H}} \qquad \lim_{n\to\infty} u_n(z) \;=\; \mathrm{e}^{m_{\mathbb{H}}(z)}.$$

Thus, the Vitali's theorem implies that the limit

$$u(z) \;\triangleq\; \lim_{n\to\infty} u_n(z) \tag{2.23}$$

exists and is an analytic function on $\mathcal{U}^{\mathbb{C}^d}$. Since, by (2.21), $u_n \neq 0$ on $\mathcal{U}^{\mathbb{C}^d}$ for all $n \in \mathbb{N}$, and $u(z) = \mathrm{e}^{m_{\mathbb{H}}(z)} \neq 0$ on $\mathcal{U}^{\mathbb{C}^d} \cap \mathbb{R}^d$ as well, it follows from Hurwitz's theorem [17] that $u \neq 0$ on $\mathcal{U}^{\mathbb{C}^d}$ at all. As a result,

$$\lim_{n\to\infty}\inf_{z\in\mathcal{U}^{\mathbb{C}^d}} \left|u_n(z)\right| \;=\; \inf_{z\in\mathcal{U}^{\mathbb{C}^d}}\left|u(z)\right| \;>\; 0.$$

Since $m_{\mathbb{H},n} = \log u_n$ on $\mathcal{U}^{\mathbb{C}^d}$, the assertion (2.22) follows.

In order to check a more refined convergence statement (2.9) we follow [7] and notice that (2.17) and (2.19) imply that for each $z \in \mathcal{U}^{\mathbb{C}^d}$ the function

$$s \;\mapsto\; \frac{1}{1-\Psi(z,s)}$$

has only one pole $s = 1$ on $\{s : |s| < 1+\delta\}$, and, moreover, this pole is simple. Thus, the following representation is valid:

$$\frac{1}{1-\Psi(z,s)} \;=\; \frac{\psi(z,s)}{1-s}, \tag{2.24}$$

where

$$\psi : \mathcal{U}^{\mathbb{C}^d} \times \{s : \; |s| < 1+\delta\} \;\mapsto\; \mathbb{C}$$

is analytic. Writing

$$\psi(z,s) \;=\; \sum_{n=0}^{\infty}\psi_n(z)s^n,$$

we conclude that, after shrinking $\mathcal{U}^{\mathbb{C}^d}$ if necessary, one can find two positive numbers $c, \alpha > 0$, such that the coefficients ψ_n satisfy

$$\left|\psi_n(z)\right| \;\le\; c\mathrm{e}^{-\alpha n} \tag{2.25}$$

uniformly in $z \in \mathcal{U}^{\mathbb{C}^d}$. On the other hand, due to the renewal relation

$$\sum_{n=0}^{\infty}\mathrm{e}^{-nm_{\mathbb{H}}(z)}\mathbb{H}_n(z)s^n \;=\; \frac{1}{1-\Psi(z,s)},$$

(2.21) and the representation formula (2.24) already imply that

$$\mathrm{e}^{-nm_{\mathbb{H}}(z)}\mathbb{H}_n(z) \;=\; \sum_{k=0}^{n}\psi_k(z) \;=\; \mu(z) \;-\; \sum_{k=n+1}^{\infty}\psi_n(z),$$

which , combined with (2.25), leads to (2.9).

The local limit asymptotic (2.12) follows now in a standard way (see [8] for a general but nonetheless lucid exposition) provided a simple decay estimate on characteristic functions, which we prove in Proposition 2.3 below.
We can assume without any loss of generality that

$$\det \mathrm{D}^2 m_{\mathbb{H}} \neq 0 \qquad \forall t \in \mathcal{U}^{\mathbb{R}^d} \triangleq \mathcal{U}^{\mathbb{C}^d} \cap \mathbb{R}^d. \tag{2.26}$$

Let now $t \in \mathcal{U}^{\mathbb{R}^d}$ and a couple $(n,k) \in \mathbb{Z} \times \mathbb{Z}^d$ be as in the assumptions of the theorem, that is $k/n = \nabla m_{\mathbb{H}}(t)$, or, by duality,

$$\langle t, \frac{k}{n}\rangle_d = m_{\mathbb{H}}(t) + m^*_{\mathbb{H}}(\frac{k}{n}). \tag{2.27}$$

Then, using the definition (2.4) of the tilted two-point function h_t;

$$h(n,k) = \mathrm{e}^{-\langle t,k\rangle_d} \mathbb{H}_n(t) \frac{h_t(n,k)}{\mathbb{H}_n(t)}. \tag{2.28}$$

We treat both terms on the right hand side of (2.28) separately: By (2.9) and (2.27),

$$\mathrm{e}^{-\langle t,k\rangle_d} \mathbb{H}_n(t) = \exp\Big\{ -n\big(\langle t, \frac{k}{n}\rangle_d - \frac{1}{n}\log \mathbb{H}_n(t)\big)\Big\} = \mu(t) \mathrm{e}^{-m^*_{\mathbb{H}}(k/n)}\big(1 + o(1)\big). \tag{2.29}$$

As for the second term in (2.28), notice that $h_t(n,\cdot)/\mathbb{H}_n(t)$ is a proper probability distribution. By the Fourier inversion formula,

$$\frac{h_t(n,k)}{\mathbb{H}_n(t)} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\mathbb{H}_n(t+i\tau)}{\mathbb{H}_n(t)} \mathrm{e}^{-i\langle k,\tau\rangle_d} \mathrm{d}\tau. \tag{2.30}$$

We then assert that the integral on the right hand side in (2.30) above is essentially Gaussian with the covariance matrix given by the inverse of $n\mathrm{D}^2 m_{\mathbb{H},n}(t)$. Indeed, for the values of $|\tau|$ sufficiently small the point $t+i\tau$ belongs to $\mathcal{U}^{\mathbb{C}^d}$, and one can, therefore, expand:

$$\frac{1}{n}\log \frac{\mathbb{H}_n(t+i\tau)}{\mathbb{H}_n(t)} = i\langle \nabla m_{\mathbb{H},n}(t), \tau\rangle_d - \frac{1}{2}\langle \mathrm{D}^2 m_{\mathbb{H},n}(t)\tau, \tau\rangle_d + o\big(\|\tau\|_d^2\big),$$

in this region. Due to the non-degeneracy assumption (2.26) and the convergence result (2.9) one can find a positive number $c > 0$, such that

$$\langle \mathrm{D}^2 m_{\mathbb{H},n}(t)\tau, \tau\rangle_d \geq c\|\tau\|_d^2,$$

simultaneously for all $t + i\tau \in \mathcal{U}^{\mathbb{C}^d}$ and n large enough. Moreover, as it follows from (2.9) and the Cauchy formula,

$$n\nabla m_{\mathbb{H},n}(t) - n\nabla m_{\mathbb{H}}(t) = n\nabla m_{\mathbb{H},n}(t) - k = \frac{\nabla \mu(t)}{\mu(t)} + o\big(\mathrm{e}^{-\alpha n}\big).$$

Consequently, there exists $\delta > 0$, such that uniformly in $t \in \mathcal{U}^{\mathbb{R}^d}$ and $\|\tau\|_d < \delta$,

$$\begin{aligned}&\frac{\mathbb{H}_n(t+i\tau)}{\mathbb{H}_n(t)} \mathrm{e}^{-i\langle k,\tau\rangle_d} \\ &\quad = \exp\Big\{ -\frac{n}{2}\langle \mathrm{D}^2 m_{\mathbb{H},n}(t)\tau,\tau\rangle_d + i\langle \nabla \log \mu(t), \tau\rangle_d + no\big(\|\tau\|_d^2\big)\Big\}\Big(1 + o(1) \Big),\end{aligned}$$

and, furthermore,

$$\Big|\frac{\mathbb{H}_n(t+i\tau)}{\mathbb{H}_n(t)}\Big| \leq \exp\{-\frac{cn}{2}\|\tau\|_d^2\}.$$

As a result we obtain that,

$$\int_{|\tau|\leq\delta} \frac{\mathbb{H}_n(t+i\tau)}{\mathbb{H}_n(t)} \mathrm{e}^{-i\langle k,\tau\rangle_d} \mathrm{d}\tau = \sqrt{\frac{(2\pi)^d}{n^d \det \mathrm{D}^2 m_{\mathbb{H}}(t)}}\Big(1 + o(1)\Big),$$

uniformly in $t \in \mathcal{U}^{\mathbb{R}^d}$.

The remaining range of values of τ is controlled by the following proposition

Proposition 2.3. *Let $t_0 \in int\mathcal{D}_{\mathbb{H}}$, and assume that $m_{\mathbb{H}}(t_0) > m_{\mathbb{F}}(t_0)$. Then, there exists a small $\epsilon > 0$, such that for each $\delta > 0$,*

$$\limsup_{n\to\infty} \max_{t\in\mathbb{R}^d:|t-t_0|\leq\epsilon} \max_{|\tau|>\delta} \frac{1}{n} \log \frac{\left|\mathbb{H}_n(t+i\tau)\right|}{\mathbb{H}_n(t)} \ < \ 0. \tag{2.31}$$

Proof. By convexity we can assume that ϵ is chosen so small, that the closed ϵ-ball $\mathcal{U}_\epsilon^{\mathbb{R}^d}$ around t_0 is still in $\mathrm{int}\big(\mathcal{D}_{\mathbb{H}}\big)$, and, moreover,

$$\min_{|t-t_0|\leq\epsilon} \Big(m_{\mathbb{H}}(t) - m_{\mathbb{F}}(t) \Big) \ \stackrel{\Delta}{=} \ 2a \ > \ 0. \tag{2.32}$$

Set

$$\widehat{\mathbb{F}}_n^\delta(t) \ = \ \max_{|\tau|\geq\delta} |\mathbb{F}_n(t+i\tau)| \ < \ \mathbb{F}_n(t).$$

Then, for each $t \in \mathcal{U}_\epsilon^{\mathbb{R}^d}$,

$$\sum_n \mathrm{e}^{-nm_{\mathbb{H}}(t)} \widehat{\mathbb{F}}_n^\delta(t) \ < \ \sum_n \mathrm{e}^{-nm_{\mathbb{H}}(t)} \mathbb{F}_n(t) \ \equiv \ 1,$$

where the second equality above follows by (2.14). Furthermore, the function

$$(\alpha, t) \ \mapsto \ \sum_n \mathrm{e}^{n(\alpha - m_{\mathbb{H}}(t))} \widehat{\mathbb{F}}_n^\delta(t),$$

is well defined and continuous on $(\alpha, t) \in [0, a] \times \mathcal{U}_\epsilon^{\mathbb{R}^d}$. Consequently, one can pick $\bar{\alpha} > 0$ such that

$$\max_{t\in\mathcal{U}_\epsilon^{\mathbb{R}^d}} \sum_n \mathrm{e}^{n(\bar{\alpha} - m_{\mathbb{H}}(t))} \widehat{\mathbb{F}}_n^\delta(t) \ < \ 1. \tag{2.33}$$

Let now $\widehat{\mathbb{H}}_n^\delta(t)$ be the renewal sequence generated by $\widehat{\mathbb{F}}_n^\delta(t)$, that is,

$$\widehat{\mathbb{H}}_0^\delta(t) \ = \ 1 \qquad \text{and} \qquad \widehat{\mathbb{H}}_n^\delta(t) \ = \ \sum_{m+1}^n \widehat{\mathbb{F}}_m^\delta(t) \widehat{\mathbb{H}}_n^\delta(t)$$

Then, for every $n \in \mathbb{N}$ and every $t \in \mathcal{U}_\epsilon^{\mathbb{R}^d}$,

$$\widehat{\mathbb{H}}_n^\delta(t) \ \geq \ \max_{|\tau|\geq\delta} \big|\mathbb{H}_n(t+i\tau)\big|.$$

On the other hand, by the usual renewal theorem and elementary continuity considerations, (2.33) implies that

$$\lim_{n\to\infty} \max_{t\in\mathcal{U}_\epsilon^{\mathbb{R}^d}} \mathrm{e}^{n(\bar{\alpha} - m_{\mathbb{H}}(t))} \widehat{\mathbb{H}}_n^\delta(t) \ = \ 0,$$

and (2.31) follows. □

3. Geometry of Equi-Decay Profiles

For any $\beta > \beta_x$ the decay rate τ_β is indeed an equivalent norm on $\mathbb{R}^d$: a straightforward one-shortest path estimate implies that $g_\beta(x) \geq \exp(-\beta \sum_i |x_i|)$. On the other hand, for each $x \in \mathbb{Z}^d$ the function $\beta \mapsto g_\beta(x)$ is differentiable on $(\beta_c(d), \infty)$. Moreover, for $\beta \in (\beta_c(d), \infty)$,

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \Big\{ -\frac{1}{\|x\|_d} \log g_\beta(x) \Big\} \ = \ \sum_{\omega:0\to x} \frac{|\omega|}{\|x\|_d} \frac{\mathrm{e}^{-\beta|\omega|}}{g_\beta(x)} \ \geq \ 1.$$

Consequently,

$$0 \ < \ \frac{\beta - \beta_c}{2} \ \leq \ \min_{\xi\in\mathbb{S}^{d-1}} \tau_\beta(\xi) \ \leq \ \max_{\xi\in\mathbb{S}^{d-1}} \tau_\beta(\xi) \ \leq \ \beta\sqrt{d} \ < \ \infty, \tag{3.1}$$

There are two compact convex sets with non-empty interiours which are immediately associated with any sub-additive homogeneous of order one function satisfying a two-sided bound like (3.1) above. The first one has been already defined - it is the unit closed ball $\mathbf{U}^\beta$ in the τ_β-norm. The second one is given by

$$\mathbf{K}^\beta \;\triangleq\; \bigcap_{\xi\in\mathbb{S}^{d-1}} \Big\{\, t\in\mathbb{R}^d:\ \langle t,\xi\rangle_d \le \tau_\beta(\xi)\,\Big\}, \tag{3.2}$$

Of course, τ_β is just the support function of $\mathbf{K}^\beta$. In two dimensions $\mathbf{K}^\beta$ has the meaning of its own being the Wulff shape for the scaling model of self-avoiding polygons [9], and we, rather frivolously, shall refer to $\mathbf{K}^\beta$ as to the Wulff shape in the general case of SAW_d, $d\ge 2$. In any dimension, however, $\mathbf{U}^\beta$ and $\mathbf{K}^\beta$ are polar convex sets, that is:

$$\begin{aligned} \mathbf{K}^\beta &= \Big\{\, t:\ \max_{x\in\mathbf{U}^\beta}\langle t,x\rangle_d \;\le\; 1\,\Big\} \\ &\text{or, equivalently,} \\ \mathbf{U}^\beta &= \Big\{\, x:\ \max_{t\in\mathbf{K}^\beta}\langle t,x\rangle_d \;\le\; 1\,\Big\} \end{aligned} \tag{3.3}$$

In particular, the geometry of $\partial\mathbf{U}^\beta$ can be recovered from that of $\partial\mathbf{K}^\beta$ and vice versa. For example, if $\partial\mathbf{K}^\beta$ is smooth and strictly convex, then τ_β is differentiable, and, moreover, the map $\nabla\tau_\beta$ is a bijection from $\partial\mathbf{U}^\beta$ to $\partial\mathbf{K}^\beta$. In such a case for any $x\in\partial\mathbf{U}^\beta$ the point $t\triangleq\nabla\tau_\beta(x)\in\partial\mathbf{K}^\beta$ satisfies:

$$\langle t,x\rangle_d \;=\; 1 \;=\; \max_{s\in\partial\mathbf{K}^\beta}\langle s,x\rangle_d \;=\; \max_{y\in\partial\mathbf{U}^\beta}\langle t,y\rangle_d, \tag{3.4}$$

and we shall call x and t **conjugate** points.

An excellent reference on the theory of convex bodies is [16]. In our case it happens to be more convenient first to derive results on the geometry of the Wulff shape $\mathbf{K}^\beta$. A necessary translation tool to the $\mathbf{U}^\beta$-geometry is given by the following rather standard fact:

Proposition 3.1. *Assume that $\partial\mathbf{K}^\beta$ is an analytic strictly convex surface whose Gaussian curvature is uniformly bounded away from zero. Then the same is also true for $\partial\mathbf{U}^\beta$, that is the conclusions of Theorem A hold. Moreover the Gaussian curvatures of $\partial\mathbf{U}^\beta$ and $\partial\mathbf{K}^\beta$ at any two conjugate points x and t are just reciprocals of one another.*

At this stage it is worthwhile to dwell on the properties of τ_β and $\mathbf{K}^\beta$ in more details. First of all,

$$\begin{aligned} t\in\mathbf{K}^\beta \;&\Leftrightarrow\; \sup_{x\in\mathbb{R}^d}\Big\{\langle x,t\rangle_d \;-\; \tau_\beta(x)\Big\} \;\le\; 0 \\ &\Leftrightarrow\; \sup_{\xi\in\mathbb{S}^{d-1}}\Big\{\langle \xi,t\rangle_d \;-\; \tau_\beta(\xi)\Big\} \;\le\; 0. \end{aligned} \tag{3.5}$$

Similarly,

$$t\in\mathrm{int}\big(\mathbf{K}^\beta\big) \;\Leftrightarrow\; \sup_{\xi\in\mathbb{S}^{d-1}}\Big\{\langle \xi,t\rangle_d \;-\; \tau_\beta(\xi)\Big\} \;<\; 0. \tag{3.6}$$

Moreover, τ_β obviously inherits reflection symmetries from $\mathbb{Z}^d$. In particular,

$$\tau_\beta(x_1,\dots,x_d) \;=\; \tau_\beta(|x_1|,\dots,|x_d|) \;=\; \tau_\beta(x_{\pi(1)},\dots,x_{\pi(d)}), \tag{3.7}$$

for every $x=(x_1,\dots,x_d)\in\mathbb{R}^d$ and any permutation π of the index set $\{1,\dots,d\}$. Consequently, both $\mathbf{K}^\beta$ and $\mathbf{U}^\beta$ are symmetric with respect to all reflections across coordinate hyperplanes. This implies among other things that τ_β is non-decreasing in each $|x_i|$;

$$|x_i|\le|y_i|;\ i=1,\dots,d \;\Rightarrow\; \tau_\beta(x) \;\le\; \tau_\beta(y). \tag{3.8}$$

Indeed, the one-dimensional function $t\mapsto\tau_\beta(x_1,\dots,x_{d-1},t)$ is convex and symmetric around zero for every fixed choice of $x_1,\dots,x_{d-1}$. Consequently, it is non-decreasing on $\mathbb{R}_+$.

As a result, we have the following implication: for any $\tilde{t} \in \mathbb{R}^d$ of the form $\tilde{t} = (t, 0)$; $t \in \mathbb{R}^k$,

$$\sup_{\eta \in \mathbb{S}^{k-1}} \Big\{ \langle \eta, t \rangle_k \; - \; \tau_\beta\big((\eta, 0)\big) \Big\} \; \leq \; 0 \; \Rightarrow \; \tilde{t} \in \mathbf{K}^\beta. \tag{3.9}$$

In order to see this assume on the contrary that $\tilde{t}$ does not belong to $\mathbf{K}^\beta$, which, by (3.5), means that there exists $x = (u, v) \in \mathbb{R}^k \times \mathbb{R}^{d-k}$ such that

$$\langle x, \tilde{t} \rangle_d \; - \; \tau_\beta(x) \; = \; \langle u, t \rangle_k \; - \; \tau_\beta(u, v) \; > \; 0.$$

In view of the lower bound on (3.1) τ_β the u component of x certainly differs from zero, $u \neq 0$. Thus, by (3.8),

$$0 \; < \; \langle x, \tilde{t} \rangle_d \; - \; \tau_\beta(x) \; \leq \; \|u\|_k \Big\{ \langle \frac{u}{\|u\|_k}, t \rangle_k \; - \; \tau_\beta(\frac{u}{\|u\|_k}, 0) \Big\},$$

a contradiction.

We state now our main result on the geometric properties of $\mathbf{K}^\beta$:

Theorem 3.2. *For every $d \geq 2$ and $\beta > \beta_c(d)$, the Wulff shape $\mathbf{K}^\beta$ of SAW_d is a compact strictly convex body with the analytic boundary $\partial\mathbf{K}^\beta$. Moreover the Gaussian curvature of $\partial\mathbf{K}^\beta$ is uniformly bounded below by some positive constant $\alpha = \alpha(d, \beta) > 0$.*

By Proposition 3.1 all the conclusions of Theorem B instantly follow.

The proof of Theorem 3.2 is based on the results on renewal type structures obtained in Section 2 which, in their turn, rely on the techniques and ideas of [7] and [10].
It is convenient to consider from now on self-avoiding walks with values in $\mathbb{Z}^{d+1} = \mathbb{Z} \times \mathbb{Z}^d$, so that $d = 1$ corresponds to SAW_2. The first component ω_1 of ω is singled out. We recall some terminology from [10]. Let $[a, b] = \{j \in \mathbb{N} : a \leq j \leq b\}$ and consider a self-avoiding path ω defined on $[a, b]$. We call ω a **bridge** if

$$\omega_1(a) < \omega_1(j) \leq \omega_1(b) \; , \; a < j \leq b.$$

The initial point of ω is $x = \omega(a)$ and the final point is $y = \omega(b)$; for such a bridge we write $\omega : x \xrightarrow{b} y$. The **span** of a bridge is $\omega_1(b) - \omega_1(a)$. Assume that the span of ω is at least 2; a **break point** of ω is an integer $k \in \mathbb{N}$, $\omega_1(a) < k < \omega_1(b)$, such that there exists $r \in [a, b]$ with the properties

$$\omega_1(j) \leq k \; , \; \forall j \leq r \quad \text{and} \quad \omega_1(j) > k \; , \; \forall j > r.$$

A bridge is called **irreducible** if it has span 1 or if it has no break point. If $x = \omega(a)$ and $y = \omega(b)$, then for such an irreducible bridge we write $\omega : x \xrightarrow{ib} y$.

We define vertical cylinders $\mathcal{C}_n$;

$$\mathcal{C}_n \;\; \stackrel{\Delta}{=} \; \Big\{ \; x = (m, k) \in \mathbb{Z} \times \mathbb{Z}^d : \; 0 \leq m \leq n \; \Big\}.$$

Let $(n, k) \in \mathcal{C}_n$; the **cylindrical** two-point function $h(n, k)$ is defined as:

$$h(n, k) \;\; = \sum_{\omega : (0,0) \xrightarrow{b} (n,k)} \mathrm{e}^{-\beta|\omega|}. \tag{3.10}$$

Obviously, h is super-multiplicative, and the limit

$$\tau_\beta^{\mathbb{H}}(x) \;\; \stackrel{\Delta}{=} \; -\frac{1}{n} \lim_{n \to \infty} \log h(n, [nx]) \tag{3.11}$$

is a well defined and everywhere finite (provided $\beta > \beta_c(d+1)$) convex function on $\mathbb{R}^d$. The relation between full and cylindrical decay rates is as expected:

Proposition 3.3. *Assume that $\beta > \beta_c(d + 1)$. Then,*

$$\tau_\beta^{\mathbb{H}}(\cdot) \;\; \equiv \;\; \tau_\beta(1, \cdot). \tag{3.12}$$

Proof. Since cylindrical connectivities are dominated by the full ones,

$$\tau_\beta(1,x) \;\le\; \tau_\beta^{\mathbb{H}}(x),$$

for every $x \in \mathbb{R}^d$. We need, thereby, to establish the reverse inequality. It is enough to consider only x-s with rational entries; $x \in \mathbb{Q}^d$. Our approach is built upon similar arguments in Section 6 in [14]:

For any $A \subset \mathbb{Z}^{d+1}$ and $x, y \in A$, let us define

$$g_A(x,y) \;\stackrel{\Delta}{=}\; \sum_{\substack{\omega: x\to y\\ \omega\subset A}} \mathrm{e}^{-\beta|\omega|}.$$

Let ∂A to denote the outer boundary of A;

$$\partial A \;=\; \left\{ z \in \mathbb{Z}^{d+1}\setminus A :\; d(z,A) = 1 \right\}.$$

Clearly,

$$\begin{aligned} g_A(x,y) \;&\ge g_\beta(x,y) \;-\; \sum_{z\in\mathbb{Z}^{d+1}\setminus A} g_\beta(x,z) g(y,z) \\ &\ge\; g_\beta(x,y) \;-\; \exp\big(-c_1(\beta) d(\{x,y\},\partial A)\big), \end{aligned} \tag{3.13}$$

where the latter inequality follows from (3.1). We fix next two (large) numbers $l, n \in \mathbb{N}$, such that $lx \in \mathbb{Z}^d$. Iterating (1.2) in the cylindrical region $\mathcal{C}_{nl}$, we obtain,

$$h(nl, nlx) \;\ge\; \Big(\frac{1}{\mathbf{B}_d(\beta)}\Big)^{n-1} \prod_{k=0}^{n-1} g_{\mathcal{C}_{nl}}(u_k, u_{k+1}),$$

where $u_k \;\stackrel{\Delta}{=}\; (kl, klx) \;\in\; \mathcal{C}_{nl}$. Consequently,

$$\begin{aligned} \tau_\beta^{\mathbb{H}}(x) \;&=\; -\lim_{n\to\infty} \frac{1}{nl} h(nl, nlx) \;=\; -\lim_{l\to\infty}\lim_{n\to\infty} \frac{1}{nl} h(nl, nlx) \\ &\le\; \lim_{l\to\infty} \frac{\log \mathbf{B}_d(\beta)}{l} \;-\; \lim_{l\to\infty} \frac{1}{l} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log g_{\mathcal{C}_{nl}}(u_k, u_{k+1}) \\ &=\; -\lim_{l\to\infty} \frac{1}{l} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log g_{\mathcal{C}_{nl}}(u_k, u_{k+1}). \end{aligned}$$

On the other hand, by (3.13),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log g_{\mathcal{C}_{nl}}(u_k, u_{k+1}) \;=\; \log g_\beta\big((l, lx)\big),$$

and (3.12) follows. □

The above proposition asserts that both full and cylindrical two-point functions have the same leading term in the logarithmic asymptotic of decay for every direction $(1,x)$; $x \in \mathbb{R}^d$. Thus, once (3.12) is verified, geometric properties of the SAW_d Wulff shape $\mathbf{K}^\beta$ can be recovered from the analysis of cylindrical two-point functions. On the other hand, there is already a natural renewal mechanism behind the production of cylindrical connectivities $h(\cdot,\cdot)$. Too see this let us follow [7], [10] and to define yet another set of connectivities, the irreducible ones (called direct in [7]). The **irreducible** two-point function f is defined as

$$f(n,k) \;=\; \sum_{\omega:(0,0)\stackrel{ib}{\to}(n,k)} \mathrm{e}^{-\beta|\omega|}. \tag{3.14}$$

Clearly, the functions h and f satisfy the Ornstein-Zernike equations (2.1) of Section 1. We then go on and define all the quantities (that is $m_{\mathbb{H}}$, $\mathcal{D}_{\mathbb{H}}$, $m_{\mathbb{F}}$, $\mathbb{H}_n$, $\mathbb{F}_n$, ...) as it was done there. The following lemma generalizes the separation of decay rates statement in [7]:

Lemma 3.4. *Assume that $\beta > \beta_c(d+1)$, and let the functions h and f be defined by (3.10) and (3.14) respectively. Then, $int(\mathcal{D}_{\mathbb{H}})$ in not empty and $\forall t \in int(\mathcal{D}_{\mathbb{H}})$,*

$$m_{\mathbb{H}}(t) \ > \ m_{\mathbb{F}}(t), \tag{3.15}$$

and, moreover,

$$det\big(D^2 m_{\mathbb{H}}(t)\big) > 0. \tag{3.16}$$

Proof. First of all let us check that $\mathrm{int}(\mathcal{D}_{\mathbb{H}})$ is not empty:

Since h-connectivities are bounded above by the g ones we, in view of the leftmost inequality in (3.1), obtain:

$$\sum_k h(n,k)\mathrm{e}^{\langle t,k\rangle_d} \ \leq \ \sum_k \mathrm{e}^{\langle t,k\rangle_d - \epsilon\sqrt{n^2+\|k\|_d^2}}$$

for some ϵ small enough. Consequently, the above sum is bounded above uniformly in n for $\|t\|_d < \epsilon$, that is

$$\big\{ t \in \mathbb{R}^d : \ \|t\| < \epsilon \ \big\} \ \subseteq \ \mathrm{int}(\mathcal{D}_{\mathbb{H}}).$$

Let $t \in \mathbb{R}^d$ and set $\hat{t} = (t,0)$; we define the susceptibility

$$\chi_{d+1}(t) \ \stackrel{\Delta}{=} \ \sum_{y\in\mathbb{Z}^{d+1}} g(y)\mathrm{e}^{\langle \hat{t},y\rangle_{d+1}}.$$

We claim, that the susceptibility χ_{d+1} is finite,

$$\chi_{d+1}(t) \ < \ \infty, \tag{3.17}$$

whenever $t \in \mathrm{int}(\mathcal{D}_{\mathbb{H}})$. This is the crucial fact: The finitness of the susceptibility was a backbone of the original ingeneous proof [7] of the separation of masses in the particular case $t = 0$. Let us recall the impact of such a condition on the properties of $\mathbb{H}_n$-s and $\mathbb{F}_n$-s. Let:

$$\mathbb{G}_n(t) \ \stackrel{\Delta}{=} \ \sum_k g(n,k)\mathrm{e}^{\langle t,k\rangle_d}. \tag{3.18}$$

Then, exactly as in [7],

$$\mathbb{H}_n(t) \leq \mathbb{G}_n(t) \leq \chi_{d+1}(t)^2 \mathbb{H}_n(t). \tag{3.19}$$

Since $\mathbb{H}_n(t)$ is super-multiplicative and $\mathbb{G}_n(t)$ sub-multiplicative, it follows that

$$\chi_{d+1}(t)^{-2}\mathrm{e}^{nm_{\mathbb{H}}(t)} \leq \mathbb{H}_n(t) \leq \mathrm{e}^{nm_{\mathbb{H}}(t)}. \tag{3.20}$$

Therefore, as it follows from the renewal theory,

$$\sum_n \mathrm{e}^{-nm_{\mathbb{H}}(t)}\mathbb{F}_n(t) \ = \ 1 \quad \text{and} \quad \sum_n n\mathrm{e}^{-nm_{\mathbb{H}}(t)}\mathbb{F}_n(t) < \infty. \tag{3.21}$$

The latter two relations already set up the stage for the renormalization procedure of [7] (or for the more polished version of it in [10]), which extends to our case without problems due to simple cancellations of tilted exponents along self-avoiding paths.

It remains, therefore, to prove (3.17). By the left hand side inequality in (3.1),

$$\lim_{A\to\infty}\lim_{n\to\infty} \log \sum_{\|k\|_d > A} h(n,k) \ = \ -\infty.$$

Thus standard large deviations computations with moment generating functions imply,

$$t \in \mathrm{int}(\mathcal{D}_{\mathbb{H}}) \ \Rightarrow \ m_{\mathbb{H}}(t) \ = \ \sup_{x\in\mathbb{R}^d}\Big\{ \langle t,x\rangle_d \ - \ \tau_\beta^{\mathbb{H}}(x) \Big\}.$$

In view of the Proposition 3.3 the latter supremum can be rewritten as

$$\begin{aligned} &\sup_{x\in\mathbb{R}^d}\big\{ \ \langle t,x\rangle_d \ - \ \tau_\beta^{\mathbb{H}}(x) \ \big\} \\ &\quad = \ \sup_{r>0} \ r \ \sup_{\xi\in\mathbb{S}^{d-1}} \ \big\{ \ \langle t,\xi\rangle_d \ - \ \tau_\beta(\frac{1}{r},\xi) \ \big\} \ < \ \infty. \end{aligned}$$

Consequently,

$$\limsup_{r\to\infty} \sup_{\xi\in\mathbb{S}^{d-1}} \left\{ \langle t,\xi\rangle_d \; - \; \tau_\beta(\frac{1}{r},\xi) \right\} \;\le\; 0,$$

which, by the continuity of τ_β, means that

$$\sup_{\xi\in\mathbb{S}^{d-1}} \left\{ \langle t,\xi\rangle_d \; - \; \tau_\beta(0,\xi) \right\} \;\le\; 0.$$

By (3.9) this implies that the point $\hat{t} = (t,0)$ lies inside the SAW_d Wulff shape $\mathbf{K}^\beta$. Repeating these computations in a small neighbourhood of t in $\mathcal{D}_{\mathbb{H}}$, we conclude that actually,

$$\hat{t} \;\in\; \mathrm{int}\mathbf{K}^\beta.$$

Thus, by (3.6)

$$\sup_{\xi\in\mathbb{S}^d} \Big\{ \langle \hat{t},\xi\rangle_{d+1} \; - \; \tau_\beta(\xi) \Big\} \;<\; -\epsilon,$$

for some small positive ϵ. Combining this with the sub-additive bound (1.2), we obtain that for each $x\in\mathbb{Z}^{d+1}$;

$$\begin{aligned} g_\beta(x) \;&\le\; \mathbf{B}_{d+1}(\beta)\exp\big\{ -\|x\|_{d+1}\tau_\beta(\frac{x}{\|x\|_{d+1}})\big\} \\ &\le\; \mathbf{B}_{d+1}(\beta)\exp\big\{ -\langle \hat{t},x\rangle_{d+1} - \epsilon\|x\|_{d+1}\big\}. \end{aligned} \tag{3.22}$$

It follows that

$$\sum_{x\ne 0} g_\beta(x)\mathrm{e}^{\langle \hat{t},x\rangle_{d+1}} \;\le\; \mathbf{B}_{d+1}(\beta)\sum_{x\ne 0}\mathrm{e}^{-\epsilon\|x\|_{d+1}} \;<\; \infty,$$

which is precisely (3.17) we are after.

Notice, by the way, that the above argument has the following implication: For any point $t\in\mathbb{R}^d$ and the point $\hat{t} \;\triangleq\; (t,0)\in\mathbb{R}^{d+1}$,

$$t\in\mathrm{int}\big(\mathcal{D}_{\mathbb{H}}\big) \;\Longleftrightarrow\; \hat{t}\in\mathrm{int}\big(\mathbf{K}^\beta\big). \tag{3.23}$$

Finally, let us turn to the proof of the non-degeneracy condition (3.16). Fix $t\in\mathrm{int}(\mathcal{D}_{\mathbb{H}})$ and recall, $m_{\mathbb{H},n}(t) \;\triangleq\; 1/n\log\mathbb{H}_n(t)$. By the first part of Theorem 2.1, the sequence $\{m_{\mathbb{H},n}\}$ converges to $m_{\mathbb{H}}$ in a complex neighbourhood of t in the sense of analytic functions. In particular,

$$\mathrm{D}^2 m_{\mathbb{H}}(t) \;=\; \lim_{n\to\infty}\mathrm{D}^2 m_{\mathbb{H},n}(t).$$

Thus, it suffices to show that

$$\liminf_{n\to\infty}\inf_{\lambda\in\mathbb{S}^{d-1}} \big\langle \mathrm{D}^2 m_{\mathbb{H},n}(t)\lambda,\lambda\big\rangle_d \;>\; 0. \tag{3.24}$$

It would be convenient to define some additional notation: Given a bridge ω we use $X(\omega)$ to denote the $\mathbb{Z}^d$ coordinate of its endpoint. A probability measure $\mathbb{P}_n$ on the set of bridges (with the starting point at the origin) of span n, is defined via

$$\mathbb{P}_n(\omega) \;=\; \frac{\mathrm{e}^{-\beta|\omega|+\langle t,X(\omega)\rangle_d}}{\mathbb{H}_n(t)}.$$

Notice that,

$$\big\langle \mathrm{D}^2 m_{\mathbb{H},n}(t)\lambda,\lambda\big\rangle_d \;=\; \mathrm{Var}_{\mathbb{P}_n}\big[\langle\lambda,X(\omega)\rangle_d\big]. \tag{3.25}$$

In a similar way, we define a probability measure $\mathbb{Q}_n$ on the set of irreducible bridges of span n;

$$\mathbb{Q}_n(\omega) \;=\; \frac{\mathrm{e}^{-\beta|\omega|+\langle t,X(\omega)\rangle_d}}{\mathbb{F}_n(t)}.$$

At last we define a probability distribution ν on $\mathbb{Z}_+$;

$$\nu(k) \;=\; \mathrm{e}^{-km_{\mathbb{H}}(t)}\mathbb{F}_k(t).$$

By (3.15) ν has exponentially decaying tails. Let now $\{N_i\}$ be a sequence of i.i.d.-s with common distribution ν, and let $\otimes\nu$ to denote the corresponding product measure. Set

$$\mathcal{R}(n) \triangleq \{\exists k : \sum_1^k N_i = n\},$$

that is $\mathcal{R}(n)$ is the event that n belongs to the range of the $\mathbb{Z}_+$-random walk with steps N_i. Due to the renewal relation (2.6),

$$\otimes\,\nu\big(\mathcal{R}(n)\big) \;=\; \mathrm{e}^{-nm_{\mathbb{H}}(t)}\mathbb{H}_n(t) \;\geq\; \text{const} \;>\; 0. \tag{3.26}$$

For every realization of $\{N_i\}$ in $\mathcal{R}(n)$ we define the hitting time $k(n)$ via

$$\sum_{i=1}^{k(n)} N_i \;=\; n.$$

With this notation,

$$\mathbb{P}_n\big(X(\omega) = x\big) \;=\; \frac{1}{\otimes\nu(\mathcal{R}(n))}\sum_{k=1}^{n}\sum_{n_1+\ldots n_k=n}\prod_1^k \nu(n_i)\bigotimes_1^k \mathbb{Q}_{n_i}\big(\sum_{i=1}^k X(\omega_i) = x\big). \tag{3.27}$$

Consequently, by the conditional variance formula,

$$\mathrm{Var}_{\mathbb{P}_n}\left[\langle\lambda, X(\omega)\rangle_d\right] \;\geq\; \frac{1}{\otimes\nu(\mathcal{R}(n))}\sum_{k=1}^{n}\sum_{n_1+\ldots n_k=n}\prod_1^k \nu(n_i)\Big(\sum_{i=1}^k \mathrm{Var}_{\mathbb{Q}_{n_i}}\left[\langle\lambda, X(\omega)\rangle_d\right]\Big). \tag{3.28}$$

On the other hand, using crude straightforward estimates on the weights of irreducible bridges, one easily verifies that $\forall R \in \mathbb{Z}_+$, there exists a constant $c = c(R) > 0$, such that

$$\min_{m\leq R}\inf_{\lambda\in\mathbb{S}^{d-1}} \mathrm{Var}_{\mathbb{Q}_m}\left[\langle\lambda, X(\omega)\rangle_d\right] \;\geq\; c(R). \tag{3.29}$$

A look at (3.29), (3.28) and (3.25) reveals that the desired bound (3.24) follows as soon as we show that with $\otimes\nu(\cdot|\mathcal{R}(n))$-probability bounded away from zero, a number of irreducible bridges in the decomposition (3.27) of $\mathbb{P}_n$ is proportional to n. This is already a soft task to perform, thanks to the exponential decay of the tails of ν-distribution. Pick, for example, two numbers $\epsilon > 0$ and $T < \infty$, and notice that the right hand side of (3.28) is bounded below by

$$\epsilon n\, c(R) \otimes \nu\Big(\sum_1^{k(n)} \mathbb{I}_{\{N_i\leq R\}} \geq \epsilon n \;\big|\mathcal{R}(n)\Big)$$

$$\geq\; \epsilon n\, c(R)\Big\{ \otimes\,\nu\big(\mathcal{R}(n)\big) \;-\; \otimes\nu\big(k(n) < \frac{n}{T} + 1\big) \;-\; \otimes\nu\Big(\sum_{k=1}^{[n/T]} \mathbb{I}_{\{N_i\leq R\}} < \epsilon n\Big)\Big\}.$$

Because of (3.26) it remains only first to choose T sufficiently large, and then to choose ϵ sufficiently small, and (3.24) follows.

□

Let now $x_0 \in \mathbb{R}^d$ be such that $x_0 = \nabla m_{\mathbb{H}}(t_0)$ for some $t_0 \in \mathrm{int}(\mathcal{D}_{\mathbb{H}})$. By the very definition of τ_β and by (2.12) $\tau_\beta^{\mathbb{H}} = m_{\mathbb{H}}^*$ in some $\mathbb{R}^d$-neighbourhood of x_0. This means by duality that,

$$\langle x_0, t_0\rangle_d \;=\; m_{\mathbb{H}}(t_0) \;+\; \tau_\beta^{\mathbb{H}}(x_0). \tag{3.30}$$

Also, the positivity of $\det\big(D^2 m_{\mathbb{H}}(t_0)\big)$ implies strict convexity of $m_{\mathbb{H}}$ at t_0 which is the dual property to the differentiability of $\tau_\beta^{\mathbb{H}}$ at x_0 Let us see what all this means in $(d+1)$-dimensions:

First of all the point $\tilde{t}_0 \triangleq (t_0, -m_{\mathbb{H}}(t_0))$ lies on the boundary of the Wulff shape $\partial\mathbf{K}^\beta$;

$$\tilde{t}_0 \;\in\; \partial\mathbf{K}^\beta. \tag{3.31}$$

Indeed, by (3.30) and the identification (3.12), the points $\tilde{t}_0$ and $\tilde{x}_0 \triangleq (x_0, 1)$ are conjugate in the sense that

$$0 = \langle \tilde{t}_0, \tilde{x}_0 \rangle_{d+1} - \tau_\beta(\tilde{x}_0) = \max_{x \in \mathbb{R}^d} \Big\{ \langle \tilde{t}_0, \tilde{x} \rangle_{d+1} - \tau_\beta(x, 1) \Big\},$$

where, as in the case of x_0, we have defined $\tilde{x} \triangleq (x, 1)$. (3.31) then instantly follows from (3.5) and (3.6). Furthermore, applying the very same line of reasoning in a small neighbourhood $\mathcal{U} \subset \mathrm{int}(\mathcal{D}_{\mathbb{H}})$ of t_0, we readily obtain that the map

$$\mathcal{U} \ni t \mapsto \big(t, -m_{\mathbb{H}}(t)\big) \tag{3.32}$$

is actually a parametrization of $\partial\mathbf{K}^\beta$ near $\tilde{t}_0$. Consequently, $\partial\mathbf{K}^\beta$ is strictly convex at $\tilde{t}_0$. Analyticity of $\partial\mathbf{K}^\beta$ in a neighbourhood of $\tilde{t}_0$ follows by Theorem 2.1, whereas

$$\rho_{\mathbf{K}}(\tilde{t}_0) \triangleq \frac{\det\big(D^2 m_{\mathbb{H}}(t_0)\big)}{\big(1 + \|\nabla m_{\mathbb{H}}(t_0)\|_d^2\big)^{(d+2)/2}} = \frac{\det\big(D^2 m_{\mathbb{H}}(t_0)\big)}{\big(1 + \|x_0\|_d^2\big)^{(d+2)/2}} \tag{3.33}$$

is identified in this way as the Gaussian curvature of $\partial\mathbf{K}^\beta$ at $\tilde{t}_0$.

Proof of Theorem 3.2. We already know that $\partial\mathbf{K}^\beta$ satisfies the assertion of the theorem at every boundary point $\tilde{t} = (t_1, ..., t_d, t_{d+1}) \in \partial\mathbf{K}^\beta$, as soon as this point is of the form

$$(t_1, ..., t_d) \in \mathrm{int}\big(\mathcal{D}_{\mathbb{H}}\big) \qquad \text{and} \qquad t_{d+1} = -m_{\mathbb{H}}(t_1, ..., t_d). \tag{3.34}$$

It happens that due to the symmetries of $\mathbf{K}^\beta$ (as reflected in (3.7)) this information alone provides all the means to finish the proof of the theorem: Let us denote the "bad" part of $\partial\mathbf{K}^\beta$,

$$\mathrm{Bad}\big(\partial\mathbf{K}^\beta\big) \triangleq \Big\{ t \in \partial\mathbf{K}^\beta : \text{ the assertion of Theorem 3.2 is violated at } t \Big\}.$$

We claim then that $\mathrm{Bad}\big(\partial\mathbf{K}^\beta\big) = \emptyset$. Certainly if $t \in \mathrm{Bad}\big(\partial\mathbf{K}^\beta\big)$, then for every permutation π of the index set $\{1, ..., d+1\}$ all the points

$$\big(\pm |t_{\pi(1)}|, ..., \pm |t_{\pi(d+1)}| \big)$$

belong to $\mathrm{Bad}\big(\partial\mathbf{K}^\beta\big)$ as well.

Let us now define the rank of a point $t \in \mathbb{R}^{d+1}$ as

$$\#(t) \triangleq \#\big\{ i : t_i \neq 0 \big\}.$$

Of course, the only point of rank zero is the origin itself which does not belong to $\partial\mathbf{K}^\beta$ at all. Furthermore, there are exactly $2(d+1)$ points of rank 1 lying on $\partial\mathbf{K}^\beta$. These are just permutations and reflections of $\big(0, 0, ..., 0, -m_{\mathbb{H}}(0)\big)$, that is falling into the framework of (3.34), and thus being "good". The rest of the proof is a rank reduction procedure: contrary to the statement of the theorem assume that there exists a "bad" point t;

$$t = \big(t_1, ..., t_{d+1}\big)$$

As we have just seen the rank $\#(t)$ should be then strictly larger than one. We claim that the assumption $t \in \mathrm{Bad}\big(\partial\mathbf{K}^\beta\big)$ necessarily implies the existence of another "bad" point $s \in \mathrm{Bad}\big(\partial\mathbf{K}^\beta\big)$ satisfying

$$\#(s) = \#(t) - 1. \tag{3.35}$$

Thus $\mathrm{Bad}\big(\partial\mathbf{K}^\beta\big) \neq \emptyset$ would be rendered contradictory in at most d steps.

In order to verify (3.35), notice that we can assume without loss of generality that $t_{d+1} \neq 0$. Writing $t = (t_0, t_{d+1}) \in \mathbb{R}^d \times \mathbb{R}$ we infer from the convexity and symmetries of $\partial\mathbf{K}^\beta$ that the point $s = (t_0, 0)$ also belongs to $\mathbf{K}^\beta$. Furthermore, s actually lies on the boundary $\partial\mathbf{K}^\beta$, for otherwise, by the virtue of (3.23), the $\mathbb{R}^d$-component t_0 should belong to $\mathrm{int}\big(\mathcal{D}_{\mathbb{H}}\big)$ which contradicts the assumption $t \in \mathrm{Bad}\big(\partial\mathbf{K}^\beta\big)$. Thus, again by the symmetry of $\mathbf{K}^\beta$ with respect to $\{t_{d+1} = 0\}$ hyperplane and convexity, the whole linear segment $\big[s, t\big] \subset \partial\mathbf{K}^\beta$. This means, in particular, that $\partial\mathbf{K}^\beta$ fails to be strictly convex at s. Therefore $s \in \mathrm{Bad}\big(\partial\mathbf{K}^\beta\big)$, and since by construction $\#(s) = \#(t) - 1$, we are done ☐

4. Decay of Two-Point Functions

Results on the asymptotic behaviour of the cylindrical two-point function h are stated uniformly over lattice cones

$$\mathcal{K}_a \;\triangleq\; \Big\{x = (n,k) \in \mathbb{N}\times\mathbb{Z}^d :\; |k| \le an\Big\}.$$

Lemma 4.1. *Assume that $\beta > \beta_c(d+1)$. Then, $\forall a \in \mathbb{R}$,*

$$h(n,k) \;=\; \frac{\mu(t)}{\sqrt{(2\pi n)^d det D^2 m_{\mathbb{H}}(t)}} e^{-n\tau_\beta^{\mathbb{H}}(k/n)}\big(1+o(1)\big), \tag{4.1}$$

uniformly in $x \triangleq (n,k) \in \mathcal{K}_a$, where $t = t(n,k)$ is given by $t = \nabla\tau_\beta^{\mathbb{H}}(k/n)$, and, as in the statement of Theorem 2.1, μ is given by

$$\mu(t) \;=\; \Big(\sum_n n e^{-nm_{\mathbb{H}}(t)}\mathbb{F}_n(t)\Big)^{-1}.$$

Proof. By Lemma 3.4 and part c) of Theorem 2.1 local form of the asymptotics (4.1) follows as soon as we show that the point $t = t(n,k)$ is indeed well defined (that is $\tau_\beta^{\mathbb{H}}$ is differentiable at k/n), and, moreover, $t \in \mathrm{int}\big(\mathcal{D}_{\mathbb{H}}\big)$. These are the consequences of the following claim:

$$\bigcup_{t\in\mathrm{int}(\mathcal{D}_{\mathbb{H}})} \nabla m_{\mathbb{H}}(t) \;=\; \mathbb{R}^d. \tag{4.2}$$

In order to verify (4.2) , let us assume to the contrary that there exists a point $x_0 \in \mathbb{R}^d$, such that $x_0 \ne \nabla m_{\mathbb{H}}(t)\ \forall\, t \in \mathrm{int}\big(\mathcal{D}_{\mathbb{H}}\big)$. This means that the supporting hyperplane $\mathcal{H}_{(x_0,1)}$ to $\partial\mathbf{K}^\beta$ in the direction of the vector $(x_0,1)$ does not touch $\partial\mathbf{K}^\beta$ at any point of the form $(t,-m_{\mathbb{H}}(t))$; $t \in \mathrm{int}\big(\mathcal{D}_{\mathbb{H}}\big)$. Due to the symmetry of $\mathbf{K}^\beta$ with respect to $\{t_{d+1}=0\}$, and in view of the strict convexity of $\partial\mathbf{K}^\beta$ established in Theorem 3.2 above, this implies that $\mathcal{H}_{(x_0,1)}$ must then support $\partial\mathbf{K}^\beta$ at some point of the form $(t_0,0)$ with $t_0 \in \partial\mathcal{D}_{\mathbb{H}}$. But at each such point there is already a supporting hyperplane of the form $\mathcal{H}_{(y_0,0)}$ parallel to the t_{d+1}-axis. We thus infer that two distinct hyperplanes support $\partial\mathbf{K}^\beta$ at $(t_0,0)$, which obviously contradicts the analyticity assertion of Theorem 3.2.

It remains only to notice that

$$F_a \;\triangleq\; \Big\{t:\; \nabla m_{\mathbb{H}}(t) \in \mathcal{K}_a\Big\} \;=\; \nabla\tau_{\mathbb{H}}\big(\mathcal{K}_a\big) \tag{4.3}$$

is a compact subset of $\mathrm{int}(\mathcal{D}_{\mathbb{H}})$, which implies that (4.1) is actually a uniform estimate over $(n,k) \in \mathcal{K}_a$. □

We proceed with deriving the local asymptotics for the full two-point function g_β. Due to the lattice symmetries it would be enough to derive the result uniformly over lattice cones $\mathcal{K}_a$ for a sufficiently large.

As before, for a SAW$_{d+1}$ lattice path ω leading from the origin to the lattice hyperplane $\mathcal{P}_n$,

$$\mathcal{P}_n \;\triangleq\; \Big\{(n,k):\; k \in \mathbb{Z}^d\Big\}$$

let $X(\omega)$ to denote the $\mathbb{Z}^d$ coordinate of the end point of ω. Given a path $\omega : 0 \mapsto \mathcal{P}_n$, all break point of ω (if any) belong to the set $\{0,...,n-1\}$. A typical path should have many break points. More precisely, let us say that $\omega : 0 \mapsto \mathcal{P}_n$ is irreducible if it has no break points at all. Set

$$\mathbb{D}_n(t) \;=\; \sum_{\substack{\omega:0\mapsto\mathcal{P}_n\\ \omega \text{ irreducible}}} \mathrm{e}^{\langle t, X(\omega)\rangle_d - \beta|\omega|}.$$

Then, as it was in the case of Lemma 3.4, the finiteness of the susceptibility $\chi_{d+1}(t)$ enables a straightforward generalization of the corresponding arguments in [7], which imply that for any $t \in \mathrm{int}(\mathcal{D}_{\mathbb{H}})$ there exists a neighbourhood $\mathcal{U}^{\mathbb{R}^d}$ of t and constant $\Delta(t) > 0$, such that

$$\mathbb{D}_n(s) \;\le\; e^{-n\Delta(t)}\mathbb{H}_n(t) \;\le\; e^{-n\Delta(t)}\mathbb{G}_n(t). \tag{4.4}$$

for every $s \in \mathcal{U}^{\mathbb{R}^d}$.

It then follows easily that for every number k fixed, the generating function $\mathbb{D}_n^k$ of SAW_{d+1} from the origin to $\mathcal{P}_n$ with exactly k break points satisfies a similar bound:

$$\mathbb{D}_n^k(s) \;\leq\; e^{-n\Delta_k(t)}\mathbb{H}_n(t) \;\leq\; e^{-n\Delta_k(t)}\mathbb{G}_n(t) \tag{4.5}$$

with some $\Delta_k(t) > 0$.

Now any self avoiding path $\omega : 0 \mapsto (n,k)$ contributing to $g_\beta(n,k)$ either has or has not break points. In the former case, let $n_l = n_l(\omega)$ and $n - n_r = n - n_r(\omega)$ to denote respectively the leftmost and the rightmost break points of ω. Accordingly we split ω into three pieces:

$$\omega \;=\; \omega_l \cup \omega_c \cup \omega_r, \tag{4.6}$$

where $\omega_l : 0 \mapsto \mathcal{P}_{n_l}$ and $\omega_r : \mathcal{P}_{n-n_r} \mapsto \mathcal{P}_n$ are irreducible, whereas $\omega_c : \mathcal{P}_{n_l} \mapsto \mathcal{P}_{n-n_r}$ is a bridge. Note, by the way, that ω_l (ω_r) obey cylindrical boundary conditions on the right (respectively on the left):

$$\omega_l \;\subset\; \Big\{(m,k) \in \mathbb{Z}\times\mathbb{Z}^d :\; m \leq n_l\Big\} \qquad \text{and} \qquad \omega_r \;\subset\; \Big\{(m,k) \in \mathbb{Z}\times\mathbb{Z}^d :\; m \geq n - n_r\Big\}. \tag{4.7}$$

The path decomposition (4.6) induces the decomposition of the $\mathbb{Z}^d$-coordinate of the end point $X(\omega)$;

$$k \;=\; X(\omega) \;=\; X(\omega_l) \;+\; X(\omega_c) \;+\; X(\omega_r).$$

On the other hand $X(\omega_l) + X(\omega_r)$ could be equivalently viewed as the $\mathbb{Z}^d$-coordinate of the end point of the path $\tilde{\omega}$ which goes from the origin to the lattice hyperplane $\mathcal{P}_{n_l+n_r}$ and has exactly one break point. Thus, using $d_1(\cdot,\cdot)$ to denote the two point function with exactly one break point;

$$d_1(p,l) \;=\; \sum_{\substack{\tilde{\omega}:0\mapsto(p,l)\\ \text{one break point}}} \mathrm{e}^{-\beta|\omega|},$$

we obtain:

$$g_\beta(n,k) \;=\; d(n,k) \;+\; \sum_{p=1}^{n}\sum_{l\in\mathbb{Z}^d} d_1(p,l)h(n-p,k-l), \tag{4.8}$$

where $d(\cdot,\cdot)$, of course, denotes the irreducible two-point function.

In order to figure out what is the main contribution to the right hand side above, notice, first of all, that due to the exact asymptotics of h-connectivities derived in Lemma 4.1, there is always a lower bound,

$$g_\beta(n,k) \;>\; h(n,k) \;\geq\; \exp\big\{-n\tau_\beta^{\mathbb{H}}(\frac{k}{n}) \;-\; c_a\log n\big\}, \tag{4.9}$$

which holds uniformly in $\mathcal{K}_a$ for some $c_a > 0$ large enough.

We are going to test various terms in (4.8) against this lower bound. The main tool for doing so is the following simple form of the exponential Chebychev inequality adapted to discrete distributions: For any $\mathbb{Z}\times\mathbb{Z}^d$ non-negative array $u(\cdot,\cdot)$ define the moment generation function $\mathbb{U}_n : \mathbb{R}^d \mapsto \bar{\mathbb{R}}_+$; $n = 1,2,\ldots$,

$$\mathbb{U}_n(t) \;=\; \sum_{l\in\mathbb{Z}^d} u(n,l)\mathrm{e}^{\langle t,l\rangle_d}.$$

Then, for each $t \in \mathbb{R}^d$ and every (n,k),

$$u(n,k) \;\leq\; \mathbb{U}_n(t)\mathrm{e}^{-\langle t,k\rangle_d}. \tag{4.10}$$

Set now $t = t(k,n) = \nabla\tau_\beta^{\mathbb{H}}(k/n)$. Thus, for example,

$$d(n,k) \;\leq\; \mathbb{D}_n(t)\mathrm{e}^{-\langle t,k\rangle_d} \;=\; \mathrm{e}^{-n(\tau_\beta^{\mathbb{H}}(k/n)+m_{\mathbb{H}}(t))}\mathbb{D}_n(t).$$

As a result, comparing with the lower bound (4.9), we infer from (4.4) that for some $\epsilon_a > 0$, $d(n,k) \leq \mathrm{e}^{-\epsilon_a n}g_\beta(n,k)$ uniformly in $(n,k) \in \mathcal{K}_a$. In other words the $d(n,k)$-term can be simply dropped down from (4.8).

A slightly more refined analysis shows that for any $\gamma > \delta > 0$,

$$g_\beta(n,k) \;=\; \sum_{p\leq n^\delta}\sum_{\|l\|_d\leq n^\gamma} d_1(p,l)h(n-p,k-l)\Big(1\;+\;o(1)\Big). \tag{4.11}$$

Indeed, in order to rule out $p > n^\delta$ just use (4.10) with the very same $t = t(n,k)$ as above and

$$\begin{aligned} u(n,k) \;&\stackrel{\Delta}{=}\; \sum_{p>n^\delta}\sum_{l\in\mathbb{Z}^d} d_1(p,l)h(n-p,k-l) \;\leq\; \mathrm{e}^{-\langle k,t\rangle_d}\sum_{p>n^\delta}\mathbb{D}_p^1(t)\mathbb{H}_{n-p}(t) \\ &\leq\; \mathrm{e}^{-n\tau_\beta^{\mathbb{H}}(k/n)}\sum_{p>n^\delta}\mathrm{e}^{pm_{\mathbb{H}}(t)}\mathbb{D}_p^1(t). \end{aligned}$$

But by (4.5) the latter quantity is already bounded above by $\exp\big\{-n\tau_\beta^{\mathbb{H}}(k/n) - \Delta_1(t)n^\delta/2\big\}$. Moreover, by (4.3) this translates into a uniform estimate over $\mathcal{K}_a$.

Finally, for every $p < n^\delta$ fixed, redefine $u(\cdot,\cdot)$ as

$$u(n,k) \;\stackrel{\Delta}{=}\; \sum_{\|l\|_d>n^\gamma} d_1(p,l)h(n-p,k-l).$$

Then, using the fact that $t(n,k)\in\mathrm{int}\big(\mathcal{D}_{\mathbb{H}}\big)$, and hence, by (3.23), $(t,0)\in\mathrm{int}\big(\mathbf{K}^\beta\big)$, we infer that there exists $\epsilon > 0$, such that

$$\mathbb{U}_n\big(t(n,k)\big) \;\leq\; \mathrm{e}^{-nm_{\mathbb{H}}(t)-\epsilon n^\gamma}$$

uniformly in $\mathcal{K}_a$. By the lower bound (4.9) and the exponential Chebychev inequality (4.10) the proof of (4.11) is, thereby, concluded.

Choosing δ and γ in (4.11) sufficiently small we notice that by virtue of (4.1),

$$\frac{h(n-p,k-l)}{h(n,k)} \;=\; \exp\big\{-pm_{\mathbb{H}}(t) \;+\; \langle t,l\rangle_d\big\}\big(1+o(1)\big), \tag{4.12}$$

uniformly in $(n,k)\in\mathcal{K}_a$, where, as before, $t = t(k,n)$ satisfies $t = \nabla\tau_\beta^{\mathbb{H}}(k/n)$.

Substituting (4.12) into (4.8) we, obtain:

$$g_\beta(x) \;=\; g_\beta(n,k) \;=\; h(n,k)\sum_p \mathrm{e}^{-pm_{\mathbb{H}}(t)}\mathbb{D}_m^1(t)\big(1+o(1)\big). \tag{4.13}$$

By (4.5) and compactness of F_a in (4.3) the prefactor near $h(n,k)$ above is uniformly bounded over $\mathcal{K}_a$.

In view of the h-asymptotics (4.1) we, thereby, obtain uniformly in $x = (n,k)\in\mathcal{K}_a$;

$$g_\beta(n,k) \;=\; \frac{\mu(t)\sum_p \mathrm{e}^{-pm_{\mathbb{H}}(t)}\mathbb{D}_m^1(t)}{\sqrt{(2\pi n)^d\det D^2 m_{\mathbb{H}}(t)}}\mathrm{e}^{-n\tau_\beta^{\mathbb{H}}(k/n)}\big(1+o(1)\big), \tag{4.14}$$

By Proposition 3.3; $n\tau_\beta^{\mathbb{H}}(k/n) = \tau_\beta(n,k) = \tau_\beta(x)$. Moreover, by (3.33),

$$\begin{aligned} n^d\det D^2 m_{\mathbb{H}}(t) \;&=\; n^d\Big(1+\Big\|\frac{k}{n}\Big\|^2\Big)^{\frac{d+2}{2}}\rho_{\mathbf{K}}(\hat{t}) \\ &=\; \|x\|_{d+1}^d\rho_{\mathbf{K}}(\hat{t})\Big(1+\Big\|\frac{k}{n}\Big\|_d^2\Big), \end{aligned} \tag{4.15}$$

where $\rho_{\mathbf{K}}(\hat{t})$ is the Gaussian curvature of $\partial\mathbf{K}^\beta$ at the boundary point $\hat{t} = (-m_{\mathbb{H}}(t),t)\in\partial\mathbf{K}^\beta$.

Notice that the point $\hat{t}$ is conjugate in the sense of (3.4) to the point $\hat{x}$;

$$\hat{x} \;\stackrel{\Delta}{=}\; \frac{1}{\tau_\beta(n,k)}(n,k)\in\partial\mathbf{U}^\beta.$$

Indeed,

$$\langle\hat{x},\hat{t}\rangle_{d+1} \;=\; \frac{-nm_{\mathbb{H}}(t)+\langle t,k\rangle_d}{\tau_\beta(n,k)} \;=1.$$

Thus, by Proposition 3.1, the right hand side of (4.15), could be rewritten as,

$$\frac{\|x\|_{d+1}^{d}\big(1+\|k/n\|_d^2\big)}{\rho_{\mathbf{U}}(\hat{x})}.$$

Consequently,

$$g_\beta(x) \;=\; \Psi_\beta(x)\sqrt{\frac{1}{(2\pi\|x\|_{d+1})^d}}\mathrm{e}^{-\tau_\beta(x)}\Big(\;1+o(1)\;\Big),$$

whith the prefactor Ψ_β given by,

$$\begin{aligned}\Psi_\beta(x) \;&=\; \frac{\mu(t)\sum_p \mathrm{e}^{-pm_{\mathbb{H}}(t)}\mathbb{D}_p^1(t)}{\sqrt{\rho_{\mathbf{K}}(\hat{t})\big(1+\|k/n\|_d^2\big)}}\\ &=\; \mu(t)\sum_p \mathrm{e}^{-pm_{\mathbb{H}}(t)}\mathbb{D}_p^1(t)\,\sqrt{\frac{\rho_{\mathbf{U}}(\hat{x})}{\big(1+\|k/n\|_d^2\big)}},\end{aligned} \tag{4.16}$$

where, as before, $x=(n,k)$, $\hat{x}=x/\tau_\beta(x)$ and $t=t(x)=t(x/\|x\|_{d+1})$ satisfies $t=\nabla\tau_\beta^{\mathbb{H}}(k/n)$. This already comes very close to the main assertion (1.4) of Theorem A. To conclude the proof: Ψ_β is clearly homogeneous of order zero. Thus we can define

$$\psi_\beta\Big(\frac{x}{\|x\|_{d+1}}\Big) \;=\; \psi_\beta\big(n(x)\big) \;\;\triangleq\;\; \Psi_\beta(x).$$

On the other hand, because of the non-degeneracy (3.16) of the Hessian $\det\mathrm{D}^2 m_{\mathbb{H}}$, $\tau_\beta^{\mathbb{H}}$ is analytic as the Legendre transform of the analytic function [8]. Consequently the tilt $t=t(x)=\nabla\tau_\beta^{\mathbb{H}}(k/n)$ is also analytic, as well as the sum of the exponentially fast convergent series $\sum_p \mathrm{e}^{-pm_{\mathbb{H}}(t)}\mathbb{D}_p^1(t)$. Finally the analyticity of $\mu(t)$ was already asserted in Theorem 2.1.

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