On Large Deviation Probabilities in Ergodic Theorem for Singularly Perturbed Stochastic Systems

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Abstract

We consider a two scale system of stochastic differential equations. We study asymptotic properties of integral functionals of slow component of this system and establish some large deviation type estimations for these functionals.

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1 Introduction

Let us consider the model describing by the system of singularly perturbed stochastic differential equations:

$$\mathrm{d}x_t^\varepsilon = f(x_t^\varepsilon, y_t^\varepsilon)\mathrm{d}t + g(x_t^\varepsilon)\mathrm{d}w_t, \qquad x_0^\varepsilon = x_0, \tag{1.1}$$

$$\varepsilon \mathrm{d}y_t^\varepsilon = F(y_t^\varepsilon)\mathrm{d}t + \beta\sqrt{\varepsilon}G(y_t^\varepsilon)\mathrm{d}W_t, \qquad y_0^\varepsilon = y_0, \tag{1.2}$$

where $W = (W_t\,,\, t \geq 0)$ and $w = (w_t\,,\, t \geq 0)$ are independent Wiener processes, and ε and β are small parameters.
The processes x^ε and y^ε can be naturally treated as slow and fast components of a stochastic dynamic system. If $\beta = 1$, then the process x^ε obeys the averaging principle, see Freidlin [3], Freidlin, Wentcell [4], Veretennikov [14], which means a convergence of x^ε to a ergodic process arising from (1.1) by substituting in place of y^ε its stationary distribution. The case of a small β i.e. the situation when β tends to zero together with ε, is studied in details in Kabanov and Pergamenshchikov [6]. In this situation the fast component y^ε converges to the root of the equation $F(y) = 0$. In the sequel we shall suppose that the point $y = 0$ is the root of this equation, that is $F(0) = 0$. We establish also some asymptotic expansions with respect to the parameter β for the deviations of x^ε from the limit process u described by the stochastic equation

$$\mathrm{d}u_t = a(u_t)\mathrm{d}t + g(u_t)\mathrm{d}w_t, \qquad u_0 = x_0, \tag{1.3}$$

where $a(x) = f(x, 0)$.
In this paper, we consider a deviation problem for an integral functional

$$\int_{t_0}^{T} \Psi_\varepsilon(x_t^\varepsilon)\,\mathrm{d}t, \tag{1.4}$$

where Ψ_ε is some smooth function.
This study is motivated by the following statistical estimation problem. Similarly to Liptser, Spokoiny [10] we consider the problem of statistical estimating the function $a(x)$ from the observed process x^ε. One may apply usual nonparametric methods, for instance, local polynomial or kernel estimators. If Q is smooth and supported in the interval $[-1,\, 1]$ kernel function, then, given a value $h > 0$ called a bandwidth, the kernel estimate $\widehat{a}_T(x)$ is defined by

$$\widehat{a}_T(x) = \frac{\int_{t_0}^{T} Q\left(\frac{x_t^\varepsilon - x}{h}\right)\,\mathrm{d}x_t^\varepsilon}{\int_{t_0}^{T} Q\left(\frac{x_t^\varepsilon - x}{h}\right)\,\mathrm{d}t}.$$

An asymptotical analysis, as $\varepsilon \to 0$, of such a statistical procedure leads to analyzing integral functionals (1.4) with

$$\Psi_\varepsilon(u) = \frac{1}{h} Q\left(\frac{u - x}{h}\right),$$

where h depends on ε.
The paper is organized as follows. In section 2 we fix assumptions and formulate the main result. Asymptotic properties of the fast and slow components are gathered in Sections 3,4. In section 5 we get upper exponential bound in ergodic theorem for diffusion processes. Proof of the result is given in Section 6.

2 The main results

First we formulate the necessary conditions on the functions f, g, F and G entering in the model equations (1.1) and (1.2), and then state the results.
Below we suppose the following conditions to be fulfilled:

(A_1) the functions F with values in R^n and G with values in the set of $n\times l$-matrices are continuous and locally Lipschitz and satisfy the condition of linear growth;

(A_2) the point $y=0$ is a root of the equation $F(y)=0$, and a solution $\widetilde{y}_s$ of the differential equation

$$\mathrm{d}\widetilde{y}_s = F(\widetilde{y}_s)\mathrm{d}s, \qquad \widetilde{y}_0 = y_0, \tag{2.1}$$

has the limit zero at $s=\infty$,

$$\lim_{s\to\infty} \widetilde{y}_s = 0; \tag{2.2}$$

(A_3) the function F is differentiable with a locally Lipschitz derivative $F'(x)$ which is for every x a $n\times n$-matrix, and all eigenvalues of $F'(0)$ have a strictly negative real part.

Note that the assumption (A_1) ensures existence and uniqueness of the solution of the equation (1.2) (Gikhman, Skorochod [5]).
Furthermore, we suppose that the functions $f(\cdot,\cdot)$, $g(\cdot)$ and $a(\cdot)=f(\cdot,0)$ satisfy the following conditions:

(B_1) the function $f(x,y)$ has a bounded continuous derivatives until second order;

(B_2) the function g is bounded, positive and separated away from zero,

$$g_{\min} \le g(x) \le g_{\max}$$

for some positive constants $g_{\min} < g_{\max}$;

(B_3) the function g is two times differentiable and its second derivative g'' satisfies the Lipschitz condition;

(B_4) the function $a_1(u)$ defined by

$$a_1(u) = a(u)/g(u) - g'(u)/2$$

is differentiable with a strictly negative derivative a_1' i.e. for some $\gamma > 0$

$$a_1'(u) \le -\gamma \qquad\qquad \forall u. \tag{2.3}$$

For example, the condition (B_4) is fulfilled with $a(u) = -u$, $g(u) = \alpha + \pi/4 - \arctan(u^2)/2$, $0 < \alpha < 2 - \pi/4$.
The assumptions (B_1) through (B_3) ensure existence and uniqueness of the solution of equations (1.1), (1.3), see Liptser and Shiryaev [9]. Moreover, (Gikhman, Skorochod [5]) under (B_2), (B_3) and (B_4) the process (1.3) is ergodic with the stationary density

$$q(x) = \frac{\exp\left\{2\int\limits_0^x \frac{a(u)}{g^2(u)}\mathrm{d}u\right\}}{g^2(x)\int\limits_{-\infty}^{\infty} g^{-2}(z)\exp\left\{2\int\limits_0^z \frac{a(u)}{g^2(u)}\mathrm{d}u\right\}\mathrm{d}z}. \tag{2.4}$$

We set

$$m_\varepsilon = \int_{-\infty}^{\infty} \Psi_\varepsilon(x)q(x)\,\mathrm{d}x, \qquad m_\varepsilon^{(1)} = \int_{-\infty}^{\infty} \dot\Psi_\varepsilon(x)q(x)\,\mathrm{d}x,$$

$$|||\Psi_\varepsilon||| = \int_{-\infty}^{\infty} |\Psi_\varepsilon(x)|\,\mathrm{d}x, \qquad \mu_\varepsilon = \sup_{-\infty<x<\infty} |\Psi_\varepsilon(x)|,$$

$$\mu_\varepsilon^{(1)} = \sup_{-\infty<x<\infty} |\dot\Psi_\varepsilon(x)|, \qquad \mu_\varepsilon^{(2)} = \sup_{-\infty<x<\infty} |\ddot\Psi_\varepsilon(x)|.$$

We suppose that the function Ψ_ε satisfies the following conditions:

(C_1) the function $\Psi_\varepsilon(\cdot)$ is twice continuously differentiable;

(C_2)

$$\limsup_{\varepsilon\to 0} |m_\varepsilon| < \infty, \quad \limsup_{\varepsilon\to 0} |||\Psi_\varepsilon||| < \infty;$$

(C_3)

$$\limsup_{\varepsilon\to 0} \frac{|||\dot\Psi_\varepsilon|||}{|m_\varepsilon^{(1)}|} < \infty;$$

(C_4)

$$\lim_{\varepsilon\to 0}(\varepsilon\mu_\varepsilon^{(1)} + \varepsilon^2\mu_\varepsilon^{(2)}) = 0.$$

Let us denote

$$N_\varepsilon = \ln T_\varepsilon/\varepsilon.$$

We also assume that the parameters β, ε, T_ε satisfy the conditions:

(D_1)

$$\lim_{\varepsilon\to 0} \beta N_\varepsilon^2 = 0;$$

(D_2)

$$\lim_{\varepsilon\to 0} \varepsilon/\beta = 0;$$

(D_3)

$$\lim_{\varepsilon\to 0} N_\varepsilon^2/T_\varepsilon = 0;$$

(D_4)

$$\lim_{\varepsilon\to 0} \mu_\varepsilon/T_\varepsilon = 0;$$

(D_5)

$$\lim_{\varepsilon\to 0} \beta^3 N_\varepsilon^4 (m_\varepsilon^{(1)})^2 = 0, \qquad \lim_{\varepsilon\to 0} \beta^3 N_\varepsilon^4 \mu_\varepsilon^{(2)} = 0.$$

Theorem 2.1 *Suppose that conditions $(A_1)-(A_3), (B_1)-(B_4)$ are fulfilled, the function $\Psi_\varepsilon(\cdot)$ satisfies the conditions $(C_1)-(C_4)$, the parameters β, ε, T_ε satisfy the limiting relationships $(D_1)-(D_5)$. Then for any $\lambda > 0$ and $t_0 = o(T_\varepsilon)$*

$$\limsup_{\varepsilon\to 0} \frac{1}{N_\varepsilon^2} \ln P\left(\left| \frac{1}{T_\varepsilon} \int_{t_0}^{T_\varepsilon} \Psi_\varepsilon(x_t^\varepsilon)\, dt - m_\varepsilon \right| > \lambda \right) \le -\kappa,$$

where κ is some positive constant.

Let a function $Q(\cdot)$ be twice continuously differentiable function and supported to the interval $[-1, 1]$ and a function $h = h_\varepsilon$ such that:

$$\lim_{\varepsilon\to 0} h_\varepsilon = 0, \quad \lim_{\varepsilon\to 0} T_\varepsilon h_\varepsilon = \infty, \quad \lim_{\varepsilon\to 0} \varepsilon/h_\varepsilon^2 = 0, \quad \lim_{\varepsilon\to 0} \beta (N_\varepsilon)^{4/3}/h_\varepsilon = 0. \tag{2.5}$$

Then function

$$\Psi_\varepsilon(u) = \frac{1}{h} Q\left(\frac{u-x}{h}\right)$$

satisfies the conditions $(C_1)-(C_4)$ and $(D_4)-(D_5)$.

Theorem 2.2 *Suppose that conditions $(A_1)-(A_3), (B_1)-(B_4)$ are fulfilled, the parameters β, ε, T_ε, h satisfy the limiting relationships $(D_1)-(D_3)$ and (2.5). Then for any $\lambda > 0$ there exists $\kappa > 0$ such that*

$$\limsup_{\varepsilon\to 0} \frac{1}{N_\varepsilon^2} \ln P\left(\left| \frac{1}{T_\varepsilon h} \int_{t_0}^{T_\varepsilon} Q\left(\frac{x_t^\varepsilon - x}{h}\right) dt - Q_0 \right| > \lambda \right) \le -\kappa,$$

where

$$Q_0 = q(x) \int_{-1}^{1} Q(z)\,\mathrm{d}z$$

and $t_0 = o(T_\varepsilon)$.

Theorem 2.2 follows directly from Theorem 2.2.

3 Asymptotical properties of the fast component

As it has been shown in [7] we can represent the solution of (1.2) in the form

$$y_t = v_t^\varepsilon + \beta y_1^\varepsilon(t) + \beta \delta_y(t, \varepsilon), \quad 0 \le t \le T, \tag{3.1}$$

where v_t^ε is a boundary function satisfying equation

$$\varepsilon \frac{dv_t^\varepsilon}{dt} = F(v_t^\varepsilon), \quad v_0 = y_0, \tag{3.2}$$

and the coefficient $y_1^\varepsilon(t), \quad 0 \le t \le T$, is determined by the linear stochastic differential equation

$$\varepsilon dy_1^\varepsilon(t) = F'(v_t^\varepsilon) y_1^\varepsilon(t) dt + \sqrt{\varepsilon} G(v_t^\varepsilon) dW_t \quad y_1(0) = 0. \tag{3.3}$$

Further we need the following lemmas.

Lemma 3.1 *Let $\Phi(t, s)$ be the $l \times l$ fundamental matrix for linear differential equation*

$$\frac{d\Phi(t, s)}{dt} = A_t \Phi(t, s), \quad \Phi(s, s) = I, \quad t \ge s, \tag{3.4}$$

where I is the unit matrix of order l, and $A_t, \quad t \ge 0$ is deterministic function having the following property

$$\lim_{t \to \infty} A_t = A, \tag{3.5}$$

where A is a matrix whose all eigenvalues have negative real parts. Then for the matrix $\Phi(t, s)$ the following exponential bound can be stated:

$$|\Phi(t, s)| \le L \exp\{-\kappa(t - s)\}, \tag{3.6}$$

for some constants $L, \kappa > 0$.

Proof see in [7]. Further, we need to consider linear stochastic differential equation

$$d\xi_t = A_t \xi_t dt + G_t dW_t, \quad \xi_0 = 0, \tag{3.7}$$

supposing the following conditions:
(E_1) A_t is a deterministic function with values in the set of the matrices with the fundamental matrix having exponential bound (3.6).
(E_2) the function G_t is bounded, i.e. $|G_t| \le K, \quad 0 \le t \le T$.

Lemma 3.2 *Let stochastic process ξ_t be the solution of the equation (3.7) in which the coefficients satisfy the conditions $(E_1) - (E_2)$. Then there is a constant $\kappa > 0$ such that for any $T > 0$ and $\lambda > 0$*

$$P\left(\|\xi\|_T > \lambda\right) \le 8T \exp\{-\kappa \lambda^2 / K^2\}, \tag{3.8}$$

where $\|\xi\|_T = \sup_{0 \le t \le T}$.

Proof. Let us consider l - dimensional process η_t

$$d\eta_t = -\eta_t dt + G_t dW_t, \qquad \eta_0 = 0. \tag{3.9}$$

It is well known [6] that for any markovian moment τ with values in $[0, T]$ and any integer $m \ge 1$

$$E|\eta_\tau|^{2m} \le 2m(2m-1)!! \left(K^2/2\right)^m T. \tag{3.10}$$

Further, we define $\Delta_t = \xi_t - \eta_t$. It follows from (3.6) and (3.7) that

$$d\Delta_t = A_t \Delta_t dt + (A_t + I)\eta_t dt, \qquad \Delta_0 = 0. \tag{3.11}$$

This implies

$$\Delta_t = \int_0^t \Phi(t,s)(A_s + I)\eta_s ds, \tag{3.12}$$

where $\Phi(t,s)$ is defined by (3.4). Then using (E_1) we estimate the term (3.12) in the following way

$$|\Delta_t| \le L \int_0^t \exp\{-\kappa(t-s)\} \; |A_s + I| \; |\eta_s| ds \le L \|\eta\|_T \tag{3.13}$$

for some constant $L > 0$. Hence for some constant $L > 0$

$$\|\xi\|_T \le L \|\eta\|_T, \tag{3.14}$$

and therefore

$$P\left(\|\xi\|_T > \lambda\right) \le P\left(\|\eta_\tau\| \ge \lambda / L\right), \tag{3.15}$$

where

$$\tau = \inf\{t \geq 0 : |\eta_t| \geq \lambda/L\} \wedge T. \tag{3.16}$$

Then using the Chebyshev exponential inequality and (3.10) we get

$$P\left(\|\xi\|_T > \lambda\right) \leq \exp\{-\kappa\lambda^2/L^2\} E \exp\{\kappa|\eta_\tau|^2\} =$$

$$= \exp\{-\kappa\lambda^2/L^2\} \sum_{m=0}^{\infty} \frac{\kappa^m}{m!} E|\eta_\tau|^{2m} \leq$$

$$= 2T \exp\{-\kappa\lambda^2/L^2\} \sum_{m=0}^{\infty} \frac{m(2m-1)!!}{m!} \left(\frac{\kappa K^2}{2}\right)^m =$$

$$= 2T \exp\{-\kappa\lambda^2/L^2\} \sum_{m=0}^{\infty} m \left(\kappa K^2\right)^m .$$

By setting here $\kappa = 1/2K^2$ we obtain (3.8). □

We use these lemmas to study asymptotic properties of expansion (3.1).

Proposition 3.1 *Let the conditions $(A_1) - (A_3)$ be fulfilled.Then the boundary function (3.2) satisfies the inequality*

$$|v_t^\varepsilon| \leq L \exp\{-\kappa t/\varepsilon\}, \tag{3.17}$$

for some constants $L > 0$ and $\kappa > 0$.

Proof see in [1].

Proposition 3.2 *Let the conditions $(A_1) - (A_3)$ be fulfilled. Then the process (3.3) satisfies the inequality*

$$P\left(\|y_1^\varepsilon\|_T \geq N_\varepsilon\right) \leq L \exp\{-\kappa N_\varepsilon^2\}, \tag{3.18}$$

for sufficiently small ε and for some constant $\kappa > 0$.

Proof. We make the change of time in the equation (3.3), by setting $r = t/\varepsilon$ and $\widetilde{y}_1(r) = y_1^\varepsilon(r\varepsilon)$. Then

$$d\widetilde{y}_1(t) = F'(\widetilde{y}_t)\widetilde{y}_1(t)dt + G(\widetilde{y}_t)d\widetilde{W}_t, \quad , \ \widetilde{y}_1(0) = 0, \tag{3.19}$$

where $\widetilde{y}_t$ is solution of the equation (1.1), $\widetilde{W}_t = W_{t\varepsilon}/\sqrt{\varepsilon}$. Then

$$P\left(\|y_1^\varepsilon\|_T \geq N_\varepsilon\right) = P\left(\|\widetilde{y}_1\|_{T/\varepsilon} \geq N_\varepsilon\right)$$

and the inequality (3.18) follows from lemmas 3.1-3.2. □

Proposition 3.3 *Let the conditions* $(A_1)-(A_3)$ *and* (D_1) *be fulfilled. Then there exists a constant* L^* *such that*

$$P\left(\|\delta_y\|_T \geq L^\star \beta N_\varepsilon^2\right) \leq L\exp\{-\kappa N_\varepsilon^2\} \tag{3.20}$$

for sufficiently small $\varepsilon > 0$.

Proof. We apply again the change of time now in the expansion (3.1), by letting

$$\widetilde{y}_r^\varepsilon = y_{r\varepsilon}^\varepsilon; \quad \widetilde{y}_r = v_{r\varepsilon}^\varepsilon; \quad \widetilde{y}_1(r) = y_1^\varepsilon(r\varepsilon); \quad \widetilde{\delta}_y(r) = \delta_y(r\varepsilon,\varepsilon),$$

where

$$d\widetilde{y}_r^\varepsilon = F(\widetilde{y}_r^\varepsilon)dr + \beta G(\widetilde{y}_r^\varepsilon)d\widetilde{W}_r, \quad \widetilde{y}_0^\varepsilon = y_0, \tag{3.21}$$

the function $\widetilde{y}_r$ satisfies the equation (2.1) and $\widetilde{y}_1(r)$ is the solution of the equation (3.19). Then

$$d\widetilde{\delta}_y(t) = \beta^{-1}\left(F(\widetilde{y}_t^\varepsilon) - F(\widetilde{y}_t) - \beta F'(\widetilde{y}_t)\widetilde{y}_1(t)\right)dt + (G(\widetilde{y}_t^\varepsilon) - G(\widetilde{y}_t))d\widetilde{W}_t =$$

$$= F'(\widetilde{y}_t)\widetilde{\delta}_y(t)dt + r_t^{(1)}dt + r_t^{(2)}d\widetilde{W}_t, \quad \widetilde{\delta}_y(0) = 0, \tag{3.22}$$

where

$$r_t^{(1)} = \beta^{-1}\left(F(\widetilde{y}_t^\varepsilon) - F(\widetilde{y}_t) - \beta F'(\widetilde{y}_t)\widetilde{y}_1(t)\right) - F'(\widetilde{y}_t)\widetilde{\delta}_y(t),$$

$$r_t^{(2)} = G(\widetilde{y}_t^\varepsilon) - G(\widetilde{y}_t).$$

Define the stopping time τ_0 as

$$\tau_0 = \inf\{t \geq 0 : |\widetilde{y}_1(t)| \geq N_\varepsilon\} \wedge T/\varepsilon, \tag{3.23}$$

Then taking into account inequality (3.18) we derive for sufficiently small $\varepsilon > 0$

$$P(\tau_0 < T/\varepsilon) = P\left(\|\widetilde{y}_1\|_{T/\varepsilon} \geq N_\varepsilon\right) = P(\|y_1\|_T \geq N_\varepsilon) \leq \exp\{-\kappa N_\varepsilon^2\} \tag{3.24}$$

with some constant $\kappa > 0$.
Now we set

$$\tau_\nu = \inf\{t \geq 0 : |\widetilde{\delta}_y(t)| \geq \nu\} \wedge \tau_0, \tag{3.25}$$

where

$$\nu = L^*\beta N_\varepsilon^2. \tag{3.26}$$

The constant L^* will be chosen later. Further we set

$$\widetilde{r}_t^{(i)} = r_{t\wedge\tau_\nu}^{(i)}, \quad i = 1, 2. \tag{3.27}$$

Then by making use of condition (A_3) we obtain the following inequality

$$|\widetilde{r}_t^{(1)}| \le L\left(|\widetilde{y}_1(t\wedge\tau_\nu)|^2 + |\widetilde{\delta}_y(t\wedge\tau_\nu)|^2\right)\beta \le$$

$$\le L\beta(N_\varepsilon^2+\nu^2) \le L\beta N_\varepsilon^2. \tag{3.28}$$

Similarly we get

$$|\widetilde{r}_t^{(2)}| \le L\left(|\widetilde{y}_1(t\wedge\tau_\nu)| + |\widetilde{\delta}_y(t\wedge\tau_\nu)|\right)\beta \le$$

$$\le L\beta(N_\varepsilon+\nu) \le L\beta N_\varepsilon \tag{3.29}$$

for some a constant $L>0$.
Further on the set $\{t\le\tau_\nu\}$ we represent $\widetilde{\delta}_y$ in the following form

$$\widetilde{\delta}_y(t) = \xi_t^{(1)} + \xi_t^{(2)}, \tag{3.30}$$

where

$$d\xi_t^{(1)} = F'(\widetilde{y}_t)\xi_t^{(1)}dt + \widetilde{r}_t^{(1)}dt, \quad \xi_0^{(1)}=0.$$
$$d\xi_t^{(2)} = F'(\widetilde{y}_t)\xi_t^{(2)}dt + \widetilde{r}_t^{(2)}dW_t, \quad \xi_0^{(2)}=0.$$

By the Cauchy formula for linear equations we obtain that

$$\xi_{\tau_\nu}^{(1)} = \int_0^{\tau_\nu}\widetilde{\Phi}(t,s)\widetilde{r}_s^{(1)}ds,$$

where $\widetilde{\Phi}(t,s)$ is the fundamental matrix of the system

$$\frac{d\widetilde{\Phi}(t,s)}{dt} = F'(\widetilde{y}_t)\widetilde{\Phi}(t,s), \quad \widetilde{\Phi}(s,s)=I. \tag{3.31}$$

The conditions $(A_2)-(A_3)$ and Lemma 3.1 imply the exponential bound (3.6) for this matrix. Therefore, it follows from (3.28) that

$$|\xi_{\tau_\nu}^{(1)}| \le L\beta N_\varepsilon^2$$

for some constant $L>0$. Taking into account condition (D_1) we choose L^* in (3.26) such that

$$|\xi_{\tau_\nu}^{(1)}| \le \nu/2. \tag{3.32}$$

Then

$$P\left(\|\delta_y\|_T\ge\nu\right) = P\left(\|\widetilde{\delta}_y\|_{T/\varepsilon}\ge\nu\right) \le P\left(\|\widetilde{\delta}_y\|_{\tau_0}\ge\nu\right) + P\left(\tau_0<T/\varepsilon\right) =$$

$$P\left(|\widetilde{\delta}_y(\tau_\nu)|=\nu\right) + P\left(\tau_0<T/\varepsilon\right) \le P\left(|\xi_{\tau_\nu}^{(1)}| + |\xi_{\tau_\nu}^{(2)}|\ge\nu\right) + P\left(\tau_0<T/\varepsilon\right).$$

Taking into account the inequality (3.32) we get that

$$P\left(\|\delta_y\|_T\ge\nu\right) \le P\left(\|\xi^{(2)}\|_{T/\varepsilon}\ge\nu/2\right) + P\left(\tau_0<T/\varepsilon\right).$$

Now Proposition 3.3 holds by virtue of Lemma 3.2 and inequality (3.29). □

As a corollary of Proposition 3.2 and Proposition 3.3 we obtain

Proposition 3.4 *Under the conditions of Proposition 3.3 the process (3.1) satisfies the inequality*

$$P\left(\|y^\varepsilon - v^\varepsilon\|_T \geq \beta N_\varepsilon\right) \leq L \exp\{-\kappa N_\varepsilon^2\} \tag{3.33}$$

for some a constant $\kappa > 0$ and for sufficiently small ε.

Proposition 3.5 *Let the conditions $(A_1) - (A_3)$ and $(D_1) - (D_2)$ be fulfilled. We suppose also that*

$$\lim_{\varepsilon \to 0} \beta^{-1} \exp\{-\nu t_0/\varepsilon\} = 0 \tag{3.34}$$

for any $\nu > 0$. Then there exists some constant $\kappa > 0$ such that for any fixed $0 < t_0 < T$ and for sufficiently small $\varepsilon > 0$

$$P\left(\|y^\varepsilon\|_{t_0,T} \geq \beta N_\varepsilon\right) \leq L \exp\{-\kappa N_\varepsilon^2\}, \tag{3.35}$$

where $\|y^\varepsilon\|_{t_0,T} = \sup_{t_0 \leq t \leq T} |y_t^\varepsilon|$.

Proof of this proposition follows from Proposition 3.1, the condition (3.34) and Proposition 3.4.

4 Asymptotical properties of the slow component

We set

$$S(x) = \int_0^x \frac{dz}{g(z)} \tag{4.1}$$

It follows from the condition (B_2) that this function has a positive bounded derivative and therefore one can define the function $s(x)$ as the solution of the equation

$$S(s(x)) = x \tag{4.2}$$

for all $x \in (-\infty, \infty)$, and

$$s'(x) = g(s(x)) > 0. \tag{4.3}$$

Next, we set

$$\widehat{x}_t^\varepsilon = S(x_t^\varepsilon). \tag{4.4}$$

Then we obtain from (4.1) and (1.2), using the also Ito's formula that

$$d\widehat{x}_t^\varepsilon = \widehat{f}(\widehat{x}_t^\varepsilon, y_t^\varepsilon)dt + d\omega_t, \quad \widehat{x}_0^\varepsilon = \widehat{x}_0 = S(x_0), \tag{4.5}$$

where

$$\widehat{f}(x,y) = \frac{f(s(x),y)}{g(s(x))} - \frac{g'(s(x))}{2}. \tag{4.6}$$

Following to [12] we represent the solution of the equation (4.5) in the following way:

$$\widehat{x}_t^\varepsilon = \widehat{u}_t^\varepsilon + \beta \widehat{x}_1^\varepsilon(t) + \beta \delta_x^\varepsilon(t), \tag{4.7}$$

where

$$d\widehat{u}_t^\varepsilon = \widehat{f}(\widehat{u}_t^\varepsilon, v_t^\varepsilon)dt + d\omega_t, \quad \widehat{u}_0^\varepsilon = \widehat{x}_0, \tag{4.8}$$

the function v_t^ε is defined by (3.2), the coefficient $\widehat{x}_1^\varepsilon(t)$ satisfies the equation

$$\frac{d\widehat{x}_1^\varepsilon(t)}{dt} = \widehat{f}_x(t,\varepsilon)\widehat{x}_1^\varepsilon(t) + \widehat{f}_y^\star(t,\varepsilon)y_1^\varepsilon(t), \quad \widehat{x}_1^\varepsilon(0) = 0, \tag{4.9}$$

where $\widehat{f}_x(t,\varepsilon) = \widehat{f}_x(\widehat{u}_t^\varepsilon, v_t^\varepsilon), \quad \widehat{f}_y(t,\varepsilon) = \widehat{f}_y(\widehat{u}_t^\varepsilon, v_t^\varepsilon)$, the process $y_1^\varepsilon(t)$ is the solution of the stochastic differential equation (3.3) and $\star$ denote transposition. We need some properties of the asymptotical expansion (4.7). We define

$$d\widehat{u}_t = b(\widehat{u}_t)dt + d\omega_t, \quad , \widehat{u}_0 = \widehat{x}_0, \tag{4.10}$$

where

$$b(u) = \widehat{f}(u, 0). \tag{4.11}$$

Proposition 4.1 *Let the conditions $(A_1) - (A_3)$ and $(B_1) - (B_4)$ be fulfilled. Then the process (4.8) satisfies the following inequality*

$$\sup_{t \ge t_0} |\widehat{u}_t^\varepsilon - \widehat{u}_t| \le L\varepsilon \exp\{-\gamma t_0\} \tag{4.12}$$

for any $t_0 \ge 0$ and for some fixed constants $L > 0$ and $\gamma > 0$.

Proof. At first, we shall show that

$$\sup_{t \ge 0} |\widehat{u}_t^\varepsilon - \widehat{u}_t| \le L\varepsilon \tag{4.13}$$

for some a constant $L > 0$. We set

$$\Delta_t^\varepsilon = \widehat{u}_t^\varepsilon - \widehat{u}_t.$$

In view of (4.8) and (4.10)

$$\frac{d\Delta_t^\varepsilon}{dt} = \kappa_t^\varepsilon \Delta_t^\varepsilon + r_t^\varepsilon, \quad \Delta_0^\varepsilon = 0, \tag{4.14}$$

where $\kappa_t^\varepsilon = (b(\widehat{u}_t^\varepsilon) - b(\widehat{u}_t))/\Delta_t^\varepsilon$, and $r_t^\varepsilon = \widehat{f}(\widehat{u}_t^\varepsilon, v_t^\varepsilon) - \widehat{f}(\widehat{u}_t^\varepsilon, 0)$.
By (2.3) and (4.3)

$$\dot{b}(u) \le -\gamma$$

for all $u \in (-\infty, +\infty)$, and therefore

$$\kappa_t^\varepsilon \le -\gamma. \tag{4.15}$$

Using the Lipschitz condition on the function f and the inequality (3.13) we obtain

$$|r_t^\varepsilon| \le L|v_t^\varepsilon| \le L\exp\{-\alpha t/\varepsilon\}. \tag{4.16}$$

Then solving the equation (4.14) on the interval $[0,t]$ we get

$$\Delta_t^\varepsilon = \int_0^t r_s^\varepsilon \exp\{\int_s^t \kappa_u^\varepsilon\, du\}\, ds.$$

Then (4.13) follows from (4.15) and (4.16). Similarly, we can represent Δ_t^ε on the interval $[t_1, t]$ $(t_1 = t_0/2)$ in the form

$$\Delta_t^\varepsilon = \exp\{\int_{t_1}^t \kappa_s^\varepsilon\, ds\}\Delta_{t_1}^\varepsilon + \int_{t_1}^t r_s^\varepsilon \exp\{\int_s^t \kappa_u^\varepsilon du\}\, ds$$

and taking into account (4.13) we arrive at (4.12). Hence Proposition 4.1. $\square$
Further we need the next auxiliary lemma.

Lemma 4.1 *Under the conditions $(A_1) - (A_3)$ the process (3.3) has the property: for all $t \ge s$*

$$|Ey_1^\varepsilon(t)(y_1^\varepsilon(s))^\star| \le Le^{-\kappa(t-s)/\varepsilon} \tag{4.17}$$

for some fixed constants $L > 0$ and $\gamma > 0$.

Proof. First note that

$$Ey_1^\varepsilon(t)(y_1^\varepsilon(s))^\star = E\widetilde{y}_1(t/\varepsilon)(\widetilde{y}_1(s/\varepsilon))^\star,$$

where $\widetilde{y}_1$ is defined by (3.15). It follows from (3.15) that

$$E\widetilde{y}_1(t/\varepsilon)(\widetilde{y}_1(s/\varepsilon))^\star = \Phi(t/\varepsilon, s/\varepsilon)E\widetilde{y}_1(s/\varepsilon)(\widetilde{y}_1(s/\varepsilon))^\star,$$

where

$$\frac{\Phi(t,s)}{dt} = F'(\widetilde{y}_t)\Phi(t,s), \quad \Phi(s,s) = I.$$

Taking into account the condition (A_2) and Lemma 3.1 we obtain the inequality (4.17). Hence Lemma 4.1. $\square$

Proposition 4.2 *Under the conditions $(A_1) - (A_3)$ and $(B_1) - (B_3)$ the solution of the equation (4.9) for all integer numbers $m \ge 1$ and some positive constant L satisfies the inequality:*

$$\sup_{t\ge 0} E\left\{(\widehat{x}_1^\varepsilon(t))^{2m} | F_T^w\right\} \le (2m-1)!!(L\varepsilon)^m, \tag{4.18}$$

where $F_T^w = \sigma\{w_t, 0 \le t \le T\}$.

Proof. We can represent the solution of (4.9) as

$$\widehat{x}_1^\varepsilon(t) = \int_0^t \widehat{f}_y^*(s,\varepsilon) y_1^\varepsilon(s)\phi_\varepsilon(t,s) \tag{4.19}$$

with

$$\phi_\varepsilon(t,s) = \exp\{\int_s^t \widehat{f}_x(r,\varepsilon)\, dr\}.$$

Since the process (3.3) is Gaussian, and the Wiener processes $(\omega_t, t \geq 0)$, $(W_t, t \geq 0)$ are independent, the process (4.19) is conditionally (with respect to F_T^w) Gaussian with $E\left\{\widehat{x}_1^\varepsilon(t)|F_T^w\right\} = 0$ and

$$E\left\{\left(\widehat{x}_1^\varepsilon(t)\right)^2|F_T^w\right\} = 2\int_0^t \phi_\varepsilon(t,s)\int_s^t \widehat{f}_y^\star(s,\varepsilon) E y_1^\varepsilon(s)(y_1^\varepsilon(\theta))^\star \widehat{f}_y(\theta,\varepsilon)\phi_\varepsilon(t,\theta)\, d\theta\, ds.$$

Note that by (4.6) and conditions $(B_1) - (B_4)$

$$\widehat{f}_x(x,y) \leq \dot{a}_1(s(x))\dot{s}(x) + L|y| \tag{4.20}$$

for some positive constants $\gamma_1 > 0$ and $L > 0$. Therefore, using the inequality (3.13) we get

$$\phi_\varepsilon(t,s) \leq \exp\{-\gamma_1(t-s) + L\int_s^t e^{-\alpha\theta/\varepsilon}\, d\theta\} \leq e^{-\gamma_1(t-s)}. \tag{4.21}$$

Then taking into account (4.17) we obtain

$$E\left\{\left(\widehat{x}_1^\varepsilon(t)\right)^2|F_T^w\right\} \leq L\varepsilon$$

for some constant $L > 0$ and hence (4.18). □
In the sequel we need

Lemma 4.2 *Let η_t be a scalar random process satisfying the linear stochastic differential equation*

$$d\eta_t = \alpha_t\eta_t\, dt + d\omega_t, \quad \eta_0 = 0, \tag{4.22}$$

where ω_t is a standard Wiener process and the coefficient α_t satisfies the inequality

$$\alpha_t \leq -\gamma \tag{4.23}$$

for some constant $\gamma > 0$. Then for any integer $m \geq 1$

$$E\left(\|\eta\|_T\right)^{2m} \leq 1 + 8\gamma^{-1}(\gamma^{-1} + \gamma^{-2})^m m^4 m! T, \tag{4.24}$$

where $\|\eta\|_T = \sup_{0\leq t\leq T}|\eta_t|$, and $T > 0$.

Proof. One can show (see, for example, [6]) that for any stopping time τ with values in the interval $[0,T]$

$$E|\eta_\tau|^{2m} \le m(2m-1)!!T/(2\gamma)^{m-1}. \tag{4.25}$$

We have also

$$E(\|\eta\|_T)^{2m} = 2m\int_0^\infty a^{2m-1}P\{\|\eta\|_T > a\}\,da \le 1 + 2m\int_1^\infty a^{2m-1}P\{|\eta_{\tau_a}| \ge a\}\,da,$$

where

$$\tau_a = \inf\{t \ge 0 : |\eta_t| \ge a\} \wedge T.$$

By letting $\lambda = m^{2m}$ we obtain

$$E(\|\eta\|_T)^{2m} \le 1 + 2m\int_1^\lambda \frac{|\eta_{\tau_a}|^{2m}}{a}\,da + 2m\int_\lambda^\infty \frac{|\eta_{\tau_a}|^{4m}}{a^2}\,da \le$$

$$\le 1 + \frac{2m^2(2m-1)!!T\ln\lambda}{(2\gamma)^{m-1}} + \frac{4m^2(4m-1)!!T}{\lambda(2\gamma)^{2m-1}} \le$$

$$\le 1 + \frac{8m^3 m!T\ln m}{\gamma^{m-1}} + \frac{8m^2(2m)!T}{\gamma^{2m-1}m^{2m}}$$

and hence (4.24). □

We set

$$D_\varepsilon = \{\|y_1^\varepsilon\|_T \le N_\varepsilon, \quad \|\delta_y\|_T \le L^*\beta N_\varepsilon^2\}, \tag{4.26}$$

where δ_y is defined by (3.1), L^* is a constant which fulfills inequality (3.19).

Proposition 4.3 *Under the conditions $(A_1)-(A_3)$, $(B_1)-(B_3)$ and (D_1) the process $\delta_x^\varepsilon(\cdot)$ from (4.7) satisfies the inequality*

$$E\mathbf{1}_{D_\varepsilon}(\delta_x^\varepsilon)^{2m} \le (L\beta N_\varepsilon^2)^m m^4 m!T \tag{4.27}$$

for any integer $m \ge 1$ and some constants $L > 0$ and $T \ge 1$.

Proof. In view of (4.7)-(4.9) the process δ_x^ε obeys the equation

$$d\delta_x^\varepsilon(t) = \kappa_t^\varepsilon\delta_x^\varepsilon(t)\,dt + r_t^\varepsilon\,dt, \quad \delta_x^\varepsilon(0) = 0, \tag{4.28}$$

where

$$\kappa_t^\varepsilon = \frac{\widehat{f}(\widehat{x}_t^\varepsilon, y_t^\varepsilon) - \widehat{f}(\widehat{u}_t^\varepsilon + \beta\widehat{x}_1^\varepsilon(t), y_t^\varepsilon)}{\beta\delta_x^\varepsilon(t)},$$

$$r_t^\varepsilon = \frac{\widehat{f}(\widehat{u}_t^\varepsilon + \beta\widehat{x}_1^\varepsilon(t), y_t^\varepsilon) - \widehat{f}(\widehat{u}_t^\varepsilon, v_t^\varepsilon) - \beta\widehat{f}_x(t,\varepsilon)\widehat{x}_1^\varepsilon(t) - \beta\widehat{f}_y^\star(t,\varepsilon)\widehat{y}_1^\varepsilon(t)}{\beta}$$

Taking into account the asymptotical expansion (3.1) and inequality (4.20) we get on the set D_ε

$$\kappa_t^\varepsilon \le -\gamma_1 + L|v_t^\varepsilon| + L\beta N_\varepsilon + L^*\beta^2 N_\varepsilon^2 \le -\gamma_1/2 + L|v_t^\varepsilon|$$

for sufficiently small $\varepsilon > 0$ and therefore by (3.13) for any $t \ge s$

$$\int_s^t \kappa_u^\varepsilon \, du \le -\gamma_1(t-s)/2 + L\int_s^t e^{-\alpha u/\varepsilon} \, du \le -\gamma_(t-s)/2 + L\varepsilon,$$

that is, for some positive constant $\gamma > 0$

$$\exp\{\int_s^t \kappa_u^\varepsilon \, du\} \le 2e^{-\gamma(t-s)}.$$

Note that by (4.19) and (4.21)

$$\|\widehat{x}_1^\varepsilon\|_T \le L\|\widehat{y}_1^\varepsilon\|_T \tag{4.29}$$

for some constant $L > 0$.
Further, it is easy to get from the definition of the function $\widehat{f}$ in (4.6) that for some constant $L > 0$

$$|\widehat{f}_{xx}(x,y)| \le (1 + |x| + |y|) \tag{4.30}$$

for all x and y; the other second derivatives $\widehat{f}_{xy}$ and $\widehat{f}_{yy}$ are bounded. By applying the finite increments formula we obtain

$$|r_t^\varepsilon| \le L|\delta_y(t,\varepsilon)| + |\widetilde{f}_x(t,\varepsilon) - \widehat{f}_x(t,\varepsilon)||\widehat{x}_1^\varepsilon| + |\widetilde{f}_y(t,\varepsilon) - \widehat{f}_y(t,\varepsilon)||\widehat{y}_1^\varepsilon|,$$

where

$$\widetilde{f}_x(t,\varepsilon) = \widetilde{f}_x(\widehat{u}_t^\varepsilon + \theta\beta\widehat{x}_1^\varepsilon(t), v_t^\varepsilon + \theta\beta y_1^\varepsilon(t)),$$
$$\widetilde{f}_y(t,\varepsilon) = \widetilde{f}_y(\widehat{u}_t^\varepsilon + \theta\beta\widehat{x}_1^\varepsilon(t), v_t^\varepsilon + \theta\beta y_1^\varepsilon(t)), \quad 0 \le \theta \le 1.$$

Similarly, taking into account inequalities (4.12), (4.29) and (4.30) we obtain

$$|r_t^\varepsilon| \le L(|\delta_y(t,\varepsilon)| + \beta(1 + |\widehat{u}_t| + \beta|\widehat{x}_1^\varepsilon(t)| + \beta|\widehat{y}_1^\varepsilon(t)|)(|\widehat{x}_1^\varepsilon(t)|^2 + |\widehat{y}_1^\varepsilon(t)|^2)) \le$$
$$\le L(|\delta_y(t,\varepsilon)| + \beta(1 + \|\widehat{u}\|_T + \beta\|\widehat{y}_1^\varepsilon\|_T)\|\widehat{y}_1^\varepsilon\|_T^2).$$

Therefore on the set D_ε for sufficiently small $\varepsilon > 0$

$$\|r^\varepsilon\|_T \le L(1 + \|\widehat{u}\|_T)\beta N_\varepsilon^2$$

for some constant $L > 0$. Note that the solution of equation (4.28) can be represented in the integral form

$$\delta_x^\varepsilon(t) = \int_0^t r_s^\varepsilon e^{\int_s^t \kappa_u^\varepsilon \, du} \, ds.$$

Then it holds on the set D_ε

$$|\delta_x^\varepsilon(t)| \le 2\int_0^t |r_s^\varepsilon| e^{-\gamma(t-s)}\, ds \le L(1+\|\widehat{u}\|_T)\beta N_\varepsilon^2.$$

Next we study equation (4.10). We rewrite it in the following form

$$d\widehat{u}_t = (b(0) + \alpha_t \widehat{u}_t)\, dt + d\omega_t, \quad \widehat{u}_0 = \widehat{x}_0, \tag{4.31}$$

where $\alpha_t = (b(\widehat{u}_t) - b(0))/\widehat{u}_t$. First, note that condition (B_4) implies the inequality

$$\alpha_t \le -\gamma \tag{4.32}$$

for some constant $\gamma > 0$.
By applying Cauchy formula for linear differential equations we can write the solution of (4.31) in the form

$$\widehat{u}_t = \widehat{x}_0 e^{\int_0^t \alpha_s\, ds} + b(0)\int_0^t e^{\int_s^t \alpha_u\, du}\, ds + \eta_t,$$

where

$$d\eta_t = \alpha_t \eta_t\, dt + d\omega_t, \quad \eta_0 = 0.$$

Inequalities (4.24) and (4.32) imply (4.27). Hence Proposition 4.3. $\square$
In the sequel we need upper exponential bound for the probability of large deviations for $\widehat{x}^\varepsilon$ in the integral metric.

Let ζ_t be a positive F_T^ω - measurable random process. We set

$$\Gamma = \{\int_{t_0}^T \zeta_t\, dt \le K\}, \tag{4.33}$$

where $0 \le t_0 < T, \quad K > 0$.

Proposition 4.4 *Under conditions $(A_1) - (A_3)$, $(B_1) - (B_3)$, and $(D_1) - (D_2)$ the process $\widehat{x}^\varepsilon$ satisfies for some constants $L > 0$ and $\kappa > 0$ and any $\lambda > 0$ and $K > 0$ the inequality*

$$P\left(\int_{t_0}^T |\widehat{x}_t^\varepsilon - \widehat{u}_t^\varepsilon|\zeta_t\, dt > \lambda,\ \Gamma,\ D_\varepsilon\right) \le LT\exp\{-\frac{\kappa\lambda^2}{K^2\beta^3 N_\varepsilon^2}\} \tag{4.34}$$

Proof. It follows from (4.7) that

$$P\left(\int_{t_0}^T |\widehat{x}_t^\varepsilon - \widehat{u}_t^\varepsilon|\zeta_t\, dt > \lambda,\ \Gamma,\ D_\varepsilon\right) \le P(\rho_1^\varepsilon > \lambda/2\beta) + P(\rho_2^\varepsilon > \lambda/2\beta), \tag{4.35}$$

where

$$\rho_1^\varepsilon = \mathbf{1}_\Gamma \int_{t_0}^T |\widehat{x}_1^\varepsilon(t)|\zeta_t\, dt, \quad \rho_2^\varepsilon = \mathbf{1}_{D_\varepsilon}\|\delta_x^\varepsilon\|_T.$$

Now we show that there exists some constant $\kappa > 0$ such that

$$E\exp\{\frac{\kappa(\rho_1^\varepsilon)^2}{K^2\varepsilon}\} \le 2. \tag{4.36}$$

Indeed, by the Hölder inequality and (4.18)

$$E(\rho_1^\varepsilon)^{2m} \le E\mathbf{1}_\Gamma \int_{t_0}^T |\widehat{x}_1^\varepsilon(t)|^{2m}\zeta_t\,dt \left(\int_{t_0}^T \zeta_t\,dt\right)^{2m-1} \le$$

$$\le K^{2m-1}E\mathbf{1}_\Gamma \int_{t_0}^T E\left\{|\widehat{x}_1^\varepsilon(t)|^{2m}|F_T^w\right\}\zeta_t\,dt \le (2m-1)!!(K^2L\varepsilon)^m$$

and therefore

$$E\exp\{\frac{\kappa(\rho_1^\varepsilon)^2}{K^2\varepsilon}\} \le 1 + \sum_{m=1}^\infty \frac{(2m-1)!!}{m!}(\kappa L)^m \le \sum_{m=0}^\infty (2\kappa L)^m.$$

This implies (4.36) for $0 < \kappa < 1/4L$. By making use of the Chebyshev inequality and (4.36) it is easy to get that

$$P(\rho_1^\varepsilon > \lambda/2\beta) \le 2\exp\{-\frac{\kappa\lambda^2}{K^2\beta^2\varepsilon}\} \tag{4.37}$$

for some constant $\kappa > 0$.
Further, taking into account inequality (4.27) one can show that there exists positive constants κ and L, such that

$$E\exp\{\frac{\kappa(\rho_2^\varepsilon)^2}{\beta N_\varepsilon^2}\} \le LT.$$

By applying the Chebyshev inequality we obtain

$$P(\rho_2^\varepsilon > \lambda/2\beta) \le LT\exp\{-\frac{\kappa\lambda^2}{K^2\beta^3 N_\varepsilon^2}\}.$$

Combining this inequality with (4.35), (4.37) and condition (D_2), we obtain (4.34). Hence Proposition 4.4. $\square$

Proposition 4.5 *Under the conditions of Proposition 4.4 the process $\widehat{x}^\varepsilon$ satisfies the inequality*

$$P\left(\int_{t_0}^T |\widehat{x}_t^\varepsilon - \widehat{u}_t^\varepsilon|^2\,dt > \lambda,\ D_\varepsilon\right) \le LT\exp\{-\frac{\kappa\lambda}{\beta^3 N_\varepsilon^2 T}\} \tag{4.38}$$

for any $\lambda > 0$ and some constants $L > 0$ and $\kappa > 0$.

Proof of Proposition 4.5 is similar to the proof of Proposition 4.4.

5 Upper exponential bound for the probability of large deviations in the ergodic theorem for diffusion processes

Let us consider a scalar diffusion process ξ satisfying the stochastic differential equation

$$d\xi_t = b(\xi_t)\,dt + d\omega_t, \quad \xi_0 = const. \tag{5.1}$$

Suppose that the function $b(\cdot)$ is continuously differentiable and

$$\dot{b}(x) \le -\gamma \tag{5.2}$$

for some constant $\gamma > 0$ and all $-\infty < x < \infty$.
It is well known (see, for example, [8]) that in this case the equation (5.1) has an unique strong solution, possessing the stationary distribution with the density:

$$q_1(y) = \frac{\exp\{2\int_0^y b(z)\,dz\}}{\int_{-\infty}^{+\infty} \exp\{2\int_0^u b(z)\,dz\}\,du}. \tag{5.3}$$

Further, for an arbitrary continuous integrable function $\phi(\cdot)$ we define

$$\Delta_T(\phi) = \frac{\int_0^T (\phi(\xi_t) - m(\phi))\,dt}{|||\phi|||\sqrt{T}}, \tag{5.4}$$

where

$$m(\phi) = \int_{-\infty}^{+\infty} \phi(y) q_1(y)\,dy. \tag{5.5}$$

Proposition 5.1 *Let the condition (5.2) for the equation (5.1) be fulfilled. Then there exists an universal constant $\kappa > 0$ such that for any continuous integrable function ϕ and arbitrary $T \ge 1$*

$$E\exp\{\kappa(\Delta_T(\phi))^2\} \le 2. \tag{5.6}$$

Proof. We set $\phi_1(u) = \phi(u) - m(\phi)$. It is obvious that

$$\int_{-\infty}^{+\infty} \phi_1(y)\exp\{2\int_0^y b(z)\,dz\}\,dy = 0. \tag{5.7}$$

Let us define the function

$$V(x) = \int_0^x v(u)\,du, \quad v(u) = -2\int_u^{+\infty} \phi_1(y)\exp\{2\int_u^y b(z)\,dz\}\,dy. \tag{5.8}$$

Now we show that

$$\sup_{-\infty<u<+\infty} |v(u)| \le L|||\phi||| \tag{5.9}$$

for some constant $L > 0$.
Indeed, by applying the finite increments formula and taking into account condition (5.2) for $u > 0$ we get

$$|v(u)| \leq \int_u^{+\infty} (|\phi(y)| + |m(\phi)|) \exp\{-\gamma(y-u)^2 + 2|b(0)|(y-u)\}\, \mathrm{d}y \leq$$

$$\leq L|||\phi||| + L|m(\phi)| \int_0^{+\infty} \exp\{-\gamma z^2 + 2|b(0)|z\}\, dz \leq L|||\phi|||.$$

By (5.7) we obtain that for $u \leq 0$

$$|v(u)| = 2|\int_{-\infty}^u \phi_1(y) \exp\{-2\int_y^u b(z)\, dz\}\, dy| \leq$$

$$\leq 2\int_{-\infty}^u |\phi_1(y)| \exp\{-\gamma(y-u)^2 + 2|b(0)||y-u|\}\, dy \leq L|||\phi|||.$$

These inequalities imply (5.9).
Next note that the function $V(x)$ (5.8) satisfies the differential equation

$$2\dot{V}(x)b(x) + \ddot{V} = 2\phi_1(x).$$

Therefore, by making use of the Itô formula we get

$$\int_0^T \phi_1(\xi_t)\, dt = V(\xi_T) - V(\xi_0) - \int_0^T v(\xi_t)\, d\omega_t.$$

It follows from inequality (5.9) that

$$\left(\int_0^T \phi_1(\xi_t)\, dt\right)^{2m} \leq 3^{2m-1}(|V(\xi_T)|^{2m} + \left|V(\xi_0)|^{2m} + |\int_0^T v(\xi_t)\, d\omega_t\right|^{2m}) \leq$$

$$\leq 3^{2m-1}(L^{2m}|||\phi|||^{2m}|\xi_T|^{2m} + L^{2m}|||\phi|||^{2m}|\xi_0|^{2m} + \left|\int_0^T v(\xi_t)\, d\omega_t\right|^{2m}). \qquad (5.10)$$

Now we show that

$$\sup_{t\geq 0} E|\xi_t|^{2m} \leq (2m-1)!!(L)^m \qquad (5.11)$$

for some constant $L > 0$ and for any integer $m \geq 1$. The function $b(\xi_t)$ can be represented in the form

$$b(\xi_t) = b(0) + \alpha_t \xi_t,$$

with $\alpha_t = (b(\xi_t) - b(0))/\xi_t$. Moreover, we get in view of condition (5.2)

$$\alpha_t \leq -\gamma, \quad t \geq 0.$$

By applying the Cauchy formula for linear differential equations we write the solution of equation (5.1) as

$$\xi_t = \zeta_t + \eta_t,$$

where ξ_t satisfies the ordinary differential equation

$$\frac{d\zeta_t}{dt} = b(0) + \alpha_t \zeta_t, \quad \zeta_0 = \xi_0, \tag{5.12}$$

and η_t satisfies the linear stochastic differential equation

$$d\eta_t = \alpha_t \eta_t \, dt + \, d\omega_t, \quad \eta_0 = 0. \tag{5.13}$$

It is easy to get from (5.12) that

$$\sup_{t \geq 0} |\zeta_t| \leq L$$

for some constant $L > 0$. Next, the process η_t satisfies for any integer $m \geq 1$ the inequality

$$\sup_{t \geq 0} E|\eta_t|^{2m} \leq (2m-1)!!/(2\gamma)^m.$$

(see, [6]) which implies (5.11). Further, the bounds for even moments of stochastic integrals (see, [9]) and inequality (5.9) imply that

$$E(\int_0^T v(\xi_t) \, d\omega_t)^{2m} \leq (2m-1)!!(LT|||\phi|||^2)^m$$

for some constant $L > 0$. This and (5.10) provide for some $L > 0$, $T \geq 1$ and any integer $m \geq 1$

$$E(\Delta_T(\phi))^{2m} \leq (2m-1)!!L^m$$

and hence inequality (5.6). $\square$

Proposition 5.2 *Under the conditions of Proposition 5.1 for any continuous integrable function ϕ and arbitrary $\lambda \geq 0$*

$$P\left(|\frac{1}{T}\int_0^T \phi(\xi_t) \, dt - m(\phi)| \geq \lambda\right) \leq 2 \exp\{-\frac{\kappa \lambda^2 T}{|||\phi|||^2}\} \quad (T \geq 1).$$

This statement follows directly from Proposition 5.1.

6 Proof of Theorem 2.1

To prove Theorem 2.1 we need the following lemmas.

Lemma 6.1 *Let for the process (1.3) the conditions* $(B_2)-(B_4)$ *be fulfilled, the function* Ψ_ε *satisfy the condition* $(C_1)-(C_2)$ *and* T_ε *satisfy the conditions* $(D_3)-(D_4)$. *Then for any* $\lambda > 0$ *there exists* $\kappa > 0$ *such that*

$$\limsup_{\varepsilon\to 0} \frac{1}{N_\varepsilon^2} \ln P\left(\left|\frac{1}{T_\varepsilon}\int_{t_0}^{T_\varepsilon} \Psi_\varepsilon(u_t)\,dt - m_\varepsilon\right| > \lambda\right) \le -\kappa, \tag{6.1}$$

where $t_0 = o(T_\varepsilon)$.

Proof. By change of variables $\widehat{u}_t = S(u_t)$, where $S(\cdot)$ is defined by (4.1), we transform equation (1.3) to equation (4.10) with function $b(u)$ satisfying inequality (5.2). Then

$$P\left(\left|\frac{1}{T_\varepsilon}\int_{t_0}^{T_\varepsilon} \Psi_\varepsilon(u_t)\,dt - m_\varepsilon\right| > \lambda\right) \le P\left(\left|\frac{1}{T_\varepsilon}\int_{0}^{t_0} \varphi_\varepsilon(\widehat{u}_t)\,\mathrm{d}t\right| > \lambda/2\right) +$$

$$+P\left(\left|\frac{1}{T_\varepsilon}\int_{0}^{T_\varepsilon} \varphi_\varepsilon(\widehat{u}_t)\,\mathrm{d}t - m_\varepsilon\right| > \lambda/2\right), \tag{6.2}$$

where

$$\varphi_\varepsilon(u) = \Psi_\varepsilon(s(u)).$$

Next note that

$$|||\varphi||| = \int_{-\infty}^{+\infty} |\Psi_\varepsilon(s(u))|\,\mathrm{d}u = \int_{-\infty}^{+\infty} |\Psi_\varepsilon(u)|\frac{1}{g(u)}\,du \le L|||\Psi_\varepsilon|||.$$

Now we estimate the first term in the right side of inequality (6.2). If $t_0 \le 1$ then

$$\left|\int_0^{t_0} \varphi_\varepsilon(\widehat{u}_t)\,\mathrm{d}t\right| \le \mu_\varepsilon$$

and by condition (D_4)

$$P\left(\frac{1}{T_\varepsilon}\left|\int_0^{t_0} \varphi_\varepsilon(\widehat{u}_t)\,dt\right| > \lambda/2\right) = 0$$

for sufficiently small $\varepsilon > 0$. Now let $t_0 > 1$. Then taking into account condition (C_2) we obtain for sufficiently small $\varepsilon > 0$

$$P\left(\frac{1}{T_\varepsilon}\left|\int_0^{t_0} \varphi_\varepsilon(\widehat{u}_t)\,\mathrm{d}t\right| > \lambda/2\right) \le P\left(\left|\frac{1}{t_0}\int_0^{t_0} \varphi_\varepsilon(\widehat{u}_t)\,\mathrm{d}t - m_\varepsilon\right| > T_\varepsilon\lambda/4t_0\right). \tag{6.3}$$

Therefore, by applying Proposition 5.2 and inequalities (6.2)-(6.3) and taking into account condition (D_3) we come to (6.1). Hence Lemma 6.1. □

Lemma 6.2 *Suppose that conditions* $(A_1)-(A_3),(B_1)-(B_4)$ *are fulfilled, the parameters* $\beta,\ \varepsilon,\ T_\varepsilon$ *satisfy the limiting relationships* $(D_1)-(D_5)$. *Then for any* $\lambda>0$ *there exists* $\kappa>0$ *such that*

$$P\left(\int_{t_0}^{T_\varepsilon}|\Psi_\varepsilon(x_t^\varepsilon)-\Psi_\varepsilon(u_t)|\,\mathrm{d}t>\lambda T_\varepsilon\right)\le e^{-\kappa N_\varepsilon^2} \tag{6.4}$$

for sufficiently small $\varepsilon>0$.

Proof. First, note that Proposition 3.2 and Proposition 3.3 imply the following inequality

$$P(D_\varepsilon^c)\le e^{-\kappa N_\varepsilon^2} \tag{6.5}$$

for some constant $\kappa>0$. We set

$$\Gamma_\varepsilon=\{\int_{t_0}^{T_\varepsilon}|\dot\Psi_\varepsilon(u_t)|\,\mathrm{d}t<T_\varepsilon m_\varepsilon^{(1)}\}. \tag{6.6}$$

It follows from inequality (6.1) that

$$P(\Gamma_\varepsilon^c)\le e^{-\kappa N_\varepsilon^2} \tag{6.7}$$

for some constant $\kappa>0$.
By applying the finite increments formula we obtain

$$|\Psi_\varepsilon(x_t^\varepsilon)-\Psi_\varepsilon(u_t)|\le|x_t^\varepsilon-u_t||\dot\Psi_\varepsilon(u_t)|+\mu_\varepsilon^{(2)}|x_t^\varepsilon-u_t|^2$$

and taking into account inequality (4.13) we get the inequality

$$|\Psi_\varepsilon(x_t^\varepsilon)-\Psi_\varepsilon(u_t)|\le|x_t^\varepsilon-u_t^\varepsilon||\dot\Psi_\varepsilon(u_t)|+\mu_\varepsilon^{(2)}|x_t^\varepsilon-u_t^\varepsilon|^2+\varepsilon\mu_\varepsilon^{(1)}+\varepsilon^2\mu_\varepsilon^{(2)}.$$

It follows from condition (B_2) that the function $s(\cdot)$, defined by (4.2), satisfies the Lipschitz condition. Therefore for some constant $L>0$

$$|\Psi_\varepsilon(x_t^\varepsilon)-\Psi_\varepsilon(u_t)|\le L\left(|\widehat x_t^\varepsilon-\widehat u_t^\varepsilon||\dot\Psi(u_t)|+|\widehat x_t^\varepsilon-\widehat u_t^\varepsilon|^2+\varepsilon\mu_\varepsilon^{(1)}+\varepsilon^2\mu_\varepsilon^{(2)}\right).$$

Then

$$P\left(\int_{t_0}^{T_\varepsilon}|\Psi_\varepsilon(x_t^\varepsilon)-\Psi_\varepsilon(u_t)|\,\mathrm{d}t>\lambda T_\varepsilon\right)\le$$

$$\le P\left(\int_{t_0}^{T_\varepsilon}|\widehat x_t^\varepsilon-\widehat u_t^\varepsilon||\dot\Psi_\varepsilon(u_t)|\,\mathrm{d}t>T_\varepsilon\lambda/4L,\ \Gamma_\varepsilon,\ D_\varepsilon\right)+$$

$$+P\left(\int_{t_0}^{T_\varepsilon}|\widehat x_t^\varepsilon-\widehat u_t^\varepsilon|^2\,\mathrm{d}t>T_\varepsilon\lambda/4L\mu_\varepsilon^{(2)},\ D_\varepsilon\right)+P(\Gamma_\varepsilon^c)+P(D_\varepsilon^c).$$

Combining Propositions 4.4 - 4.5, inequalities (6.5), (6.7) and limiting relationships (D_5) we obtain inequality (6.4). Hence Lemma 6.2. □
Lemma 6.1 and Lemma 6.2 imply the assertion of Theorem 2.1.

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