On an operator equation with noise in the operator and the right-hand side with application to an inverse vibration problem

Gottfried Bruckner
Weierstraß-Institut
für
Angewandte Analysis und Stochastik
Mohrenstr. 39
D-10117 Berlin
Germany
bruckner@wias-berlin.de

Masahiro Yamamoto
The University of Tokyo
Department of Mathematical Sciences
3-8-1 Komaba,Meguro
Tokyo 153
Japan
myama@ms.u-tokyo.ac.jp

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Abstract

We consider a linear operator equation with noise in the operator and the right-hand side. As a concept, the ill–posedness of the problem is composed of the ill-posedness with respect to the operator and the ill–posedness with respect to the right–hand side, and in both the cases the ill-posedness can be characterized by an embedding operator. Starting at a numerical procedure for exact data, in the case of noisy data a numerical procedure and error estimates are given. As an example, a Volterra integral equation of the first kind is investigated and finally applied to a point source reconstruction problem for the wave equation.

1 Introduction

We investigate the linear operator equation

$$Au = g, \tag{1.1}$$

where A is an isomorphism of Banach spaces X and Y, i.e. A^{-1} exists and both A and A^{-1} are everywhere defined, linear and continuous mappings. This problem is well–posed, even if g is not exactly given but its deviation from the given g_δ can be measured in the "strong" Y–norm.

Ill–posedness appears if the deviation has to be measured in some weaker norm. A natural description of this situation is by considering an embedding

$$Y \subset U, \tag{1.2}$$

where U is a normed space, and the deviation is described in the norm of U,

$$||g - g_\delta||_U \leq \delta.$$

For an uncertain operator A_ϵ the situation is completely analogous. As will become clear in Section 2 of the paper, ill–posedness comes up only, if the deviation from the exact operator A is measured in a weaker sense (compared to the usual "strong" norm of operators from X to Y). Again this can be naturally described by an embedding

$$Y \subset Z, \tag{1.3}$$

where Z is a normed space, the operators are considered as mappings into Z, and their deviation is measured in this "weak" operator norm,

$$||A - A_\epsilon||_{X \to Z} \leq \epsilon.$$

While the noise levels δ and ϵ give a quantitative description the spaces U and Z indicate the "quality" of the noise. It is clear, that the noise in the operator and the noise in the right–hand side are in nature independent from each other, concerning both, quality and quantity.

On this conceptional basis we are concerned in this paper with the investigation of (1.1) with noisy data. The regularization method consists in a regularization by discretization (also called "selfregularization", c.f. [3] where a similar approach was chosen).

In Section 2 the uncertain operator is studied in an abstract way, by considering the two cases of a "strong" and a "weak" operator norm. Starting at a discretization and a numerical procedure for the exactly given operator A, a numerical procedure and error estimates in the case of an uncertain operator A_ϵ are proved. Independently from this, in Section 3 noisy right–hand sides are treated on the basis of [2]. Here, the discretization, since it is connected to regularization, should be chosen independently too.

Crucial with respect to the ill–posedness is in either case an inverse inequality, reflecting the singular value asymptotics of the embedding (1.2) or (1.3), respectively.

Section 3 concludes with a combination of both kinds of noisy data by giving a numerical procedure and error estimates in the general case.

In Section 4 a concrete example is given. It consists of a linear Volterra integral equation of convolution type, and is applied in Section 5 to a point source reconstruction problem for the wave equation.

2 An abstract operator equation with noisy operator.

Let us consider Banach spaces

$$X, Y, Z,$$

where Y is continuously embedded into Z,

$$Y \subset Z,$$

and isomorphisms (i.e. linear continuous mappings onto)

$$A, A_\epsilon : X \to Y, \quad A^{-1}, A_\epsilon^{-1} : Y \to X.$$

Here $\epsilon > 0$ is a small parameter. The norms in X and Y, also norms of operators from X into Y and from Y into X will be denoted by

$$\|\cdot\|,$$

norms of another space or of operators into another space will carry the name of that other space as an index.

As indicated, in the sequel we are concerned with disturbances A_ϵ of the operator A. We assume properties of the following kind:

$$\|A_\epsilon^{-1}\| \leq \tilde{c}_1, \tag{2.1}$$

$$\|A - A_\epsilon\| \leq \epsilon, \tag{2.2}$$

$$\|A - A_\epsilon\|_Z \leq \epsilon. \tag{2.3}$$

Now, let us prove the following useful

Lemma 2.1 *Let B_1, B_2 be isomorphisms of normed spaces, and B_1^{-1}, B_2^{-1} their inverses. Then*

$$\|B_1^{-1} - B_2^{-1}\| \leq \|B_1^{-1}\|\|B_2^{-1}\|\|B_1 - B_2\|. \tag{2.4}$$

Moreover, if $\|B_1^{-1}\|\|B_1 - B_2\| < 1$, then

$$\|B_2^{-1}\| \leq \|B_1^{-1}\|/(1 - \|B_1^{-1}\|\|B_1 - B_2\|). \tag{2.5}$$

Proof.

$$\begin{aligned}
\|B_1^{-1} - B_2^{-1}\| &= \|B_1^{-1}B_1(B_1^{-1} - B_2^{-1})B_2B_2^{-1}\| \\
&= \|B_1^{-1}(B_2 - B_1)B_2^{-1}\| \\
&\le \|B_1^{-1}\|\|B_2 - B_1\|\|B_2^{-1}\|.
\end{aligned}$$

Additionally,

$$\begin{aligned}
\|B_2^{-1}\| - \|B_1^{-1}\| &\le \|B_1^{-1} - B_2^{-1}\| \\
&\le \|B_1^{-1}\|\|B_2^{-1}\|\|B_1 - B_2\|, \\
\|B_2^{-1}\|(1 - \|B_1^{-1}\|\|B_1 - B_2\|) &\le \|B_1^{-1}\|.
\end{aligned}$$

∎

From Lemma 2.1 we obtain immediately

Proposition 2.1 *If $\epsilon < 1/\|A^{-1}\|$ then (2.2) implies (2.1).*

The objective of this section is to solve the operator equation

$$Au = g, \tag{2.6}$$

where the operator A is not exactly given. To get a discretized version of this task let us introduce finite dimensional subspaces

$$X_1 \subset \ldots \subset X_n \subset \ldots \subset X, \quad \overline{\bigcup_n X_n} = X,$$
$$Y_1 \subset \ldots \subset Y_n \subset \ldots \subset Y, \quad \overline{\bigcup_n Y_n} = Y,$$

and linear mappings

$$A_n, A_{\epsilon,n}: \quad X_n \to Y_n$$

with the stability properties

$$\tilde{c}_0\|A_n v_n\| \ge \|v_n\| \quad \forall v_n \in X_n, \tag{2.7}$$

where $\tilde{c}_0$ does not depend on n,

$$\tilde{c}_1\|A_{\epsilon,n} v_n\| \ge \|v_n\| \quad \forall v_n \in X_n, \tag{2.8}$$

where $\tilde{c}_1$ is independent on n and ϵ. (2.7) implies that A_n^{-1} exists and $\|A_n^{-1}\| \le \tilde{c}_0$ holds, (2.8) implies that $A_{\epsilon,n}^{-1}$ exists and

$$\|A_{\epsilon,n}^{-1}\| \le \tilde{c}_1 \tag{2.9}$$

holds.

Now, let us consider the estimates

$$\|A_n - A_{\epsilon,n}\| \le \epsilon, \tag{2.10}$$

$$\|A_n - A_{\epsilon,n}\|_Z \le \epsilon. \tag{2.11}$$

Remark 2.1 *In favour of a clear presentation we ignore that ϵ here in general may be different from its meaning in (2.2) and (2.3), respectively. Especially, it may depend on n (cf.[3] and Example 2.2). However, in the application considered in this paper (c.f. Section 4), it will be proved that ϵ does not depend on n.*

Let q be a given real number with $0 < q < 1$.

Proposition 2.2 *If $\epsilon < q/\tilde{c}_0$ (2.7) and (2.10) imply (2.8) with*

$$\tilde{c}_1 = \tilde{c}_0/(1-q)$$

and

$$\|A_n^{-1} - A_{\epsilon,n}^{-1}\| \leq c \cdot \epsilon. \tag{2.12}$$

Proof.

$$\begin{aligned} \|A_{\epsilon,n}v_n\| &\geq \|A_n v_n\| - \|A_{\epsilon,n}v_n - A_n v_n\| \\ &\geq (1/\tilde{c}_0 - \epsilon)\|v_n\|. \end{aligned}$$

This means that $A_{\epsilon,n}^{-1}$ exists. From Lemma 2.1 we obtain

$$\begin{aligned} \|A_n^{-1} - A_{\epsilon,n}^{-1}\| &\leq \|A_n^{-1}\|\|A_{\epsilon,n}^{-1}\|\|A_n - A_{\epsilon,n}\| \leq \tilde{c}_0\tilde{c}_1 \cdot \epsilon, \\ \|A_{\epsilon,n}^{-1}\| &\leq \|A_n^{-1}\|/(1 - \|A_n^{-1}\| \cdot \epsilon) \leq \tilde{c}_1. \end{aligned}$$

∎

To treat also estimates of the type (2.11) let us suppose the following inverse inequality:

$$\|\psi\| \leq c \cdot n^\alpha \|\psi\|_Z \quad \forall \psi \in Y_n. \tag{2.13}$$

Here $\alpha \geq 0$ is independent on n.

Proposition 2.3 *The estimates (2.7),(2.9),(2.11) and the inverse inequality (2.13) imply*

$$\|A_n^{-1} - A_{\epsilon,n}^{-1}\| \leq c \cdot n^\alpha \epsilon. \tag{2.14}$$

Proof. Lemma 2.1 gives

$$\|A_n^{-1} - A_{\epsilon,n}^{-1}\| \leq \|A_n^{-1}\|\|A_{\epsilon,n}^{-1}\|\|A_n - A_{\epsilon,n}\|,$$

and (2.13) implies

$$\|A_n - A_{\epsilon,n}\| \leq c \cdot n^\alpha \|A_n - A_{\epsilon,n}\|_Z. \tag{2.15}$$

∎

Example 2.1 *Let*

$$AX_n = Y_n, \quad A_\epsilon X_n = Y_n.$$

Let the restrictions of A, A_ϵ to X_n be denoted by $A_n, A_{\epsilon,n}$, respectively. Then $A_n, A_{\epsilon,n}$ are isomorphisms of the spaces X_n and Y_n with inverses $A_n^{-1}, A_{\epsilon,n}^{-1}$, which are the restrictions of A^{-1}, A_ϵ^{-1} to Y_n, respectively. It is easy to deduce $\|A_n^{-1}\| \leq \|A^{-1}\|$ and (2.9),(2.10),(2.11) from (2.1),(2.2),(2.3),respectively.

Example 2.2 *Let* $AX_n = A_\epsilon X_n = Y_n'$ *and* $Q_n : Y \to Y_n$, $\|Q_n\| \le c$, *be linear operators. Denote the restrictions of* $Q_n A$, $Q_n A_\epsilon$ *to* X_n *by* A_n, $A_{\epsilon,n}$, *respectively. Then*

$$\|A_n - A_{\epsilon,n}\| \le c \cdot \|A - A_\epsilon\|, \tag{2.16}$$

and, provided (2.13) holds for Y_n',

$$\|A_n - A_{\epsilon,n}\|_Z \le c \cdot n^\alpha \|A - A_\epsilon\|_Z. \tag{2.17}$$

Here, the first estimate (2.16) is immediate, while the second one (2.17) follows from

$$\sup_{x \in X_n, \|x\| \le 1} \|(A_n - A_{\epsilon,n})x\|_Z \le \sup_{x \in X_n, \|x\| \le 1} \|(A - A_\epsilon)x\|_Z \cdot \sup_{y \in Y_n', \|y\|_Z \le 1} \|Q_n y\|_Z,$$

and

$$\sup_{y \in Y_n', \|y\|_Z \le 1} \|Q_n y\|_Z \le c \cdot n^\alpha \sup_{y \in Y_n', \|y\| \le 1} \|Q_n y\|_Z,$$

where the last inequality follows from (2.13), giving $\|y\|_Z \le 1/cn^\alpha \Rightarrow \|y\| \le 1$ *and the trivial equality*

$$\sup_{\|y\| \le a} \|Qy\| = a \cdot \sup_{\|y\| \le 1} \|Qy\|.$$

For solving (2.6) let us consider the approximate operator equations

$$A_n u_n = g_n \tag{2.18}$$

and

$$A_{\epsilon,n} u_{\epsilon,n} = g_n \tag{2.19}$$

where $g_n = Q_n g \in Y_n$ is an approximation of g,

$$Q_n : Y \to Y_n, \quad \|Q_n\| \le c,$$

and $u_n, u_{\epsilon,n} \in X_n$. The stability properties (2.7) and (2.8) imply that $u_n, u_{\epsilon,n}$ are unique. This way, we are given operators

$$\begin{aligned} T(n, A) : Y &\to X_n,\ T(n, A) = A_n^{-1} Q_n,\ T(n, A)g = u_n \\ T(n, A_\epsilon) : Y &\to X_n,\ T(n, A_\epsilon) = A_{\epsilon,n}^{-1} Q_n,\ T(n, A_\epsilon)g = u_{\epsilon,n}. \end{aligned}$$

Next, we need a convergence assumption to the solution of (2.6) with unperturbed operator. To give it in a rather general form (applicable, e.g., in the theory of boundary integral equations) let us introduce scales of Banach spaces $X = X^{\lambda_0} \supset \ldots \supset X^\lambda \supset \ldots$ and $Y = Y^{\lambda_0} \supset \ldots \supset Y^\lambda \supset \ldots$ and let A be an isomorphism of X^λ to Y^λ, $\lambda_0 \le \lambda$. Then we have, if $g \in Y^\mu, \mu \ge \lambda_0$,

$$\|T(n, A)g - A^{-1}g\| \le c \cdot n^{\lambda_0 - \mu} \|g\|_{Y^\mu}. \tag{2.20}$$

Remark 2.2 *From (2.20) the operators T(n,A) are uniformly bounded in n.*

Theorem 2.1 *Let be $g \in Y^\mu, \mu \geq \lambda_0$, and let the assumptions (2.7) and (2.20) hold. Moreover, let us assume (2.10) and $\epsilon < q/\tilde{c}_0$. Then*

$$\begin{aligned} \|T(n, A_\epsilon)g - A^{-1}g\| &\leq c \cdot (n^{\lambda_0 - \mu}\|g\|_{Y^\mu} + \epsilon\|g\|) \\ &\leq C \cdot (n^{-\kappa} + \epsilon) \end{aligned}$$

where $\kappa = \mu - \lambda_0 (\geq 0)$ and C depends on g. For $n \sim \epsilon^{-\frac{1}{\kappa}}$ we have

$$\|T(n, A_\epsilon)g - A^{-1}g\| = O(\epsilon).$$

Proof.

$$\begin{aligned} \|T(n, A_\epsilon)g - A^{-1}g\| &\leq \|T(n, A)g - A^{-1}g\| + \|T(n, A_\epsilon) - T(n, A)\|\|g\| \\ \|T(n, A_\epsilon) - T(n, A)\| &= \|(A_{\epsilon,n}^{-1} - A_n^{-1})Q_n\| \\ &\leq \|A_{\epsilon,n}^{-1} - A_n^{-1}\|\|Q_n\|. \end{aligned}$$

Then, using (2.20) and Proposition 2.2 we get the assertions. ■

Theorem 2.2 *Let be $g \in Y^\mu, \mu \geq \lambda_0$, and let the assumptions (2.7) and (2.20) hold. Moreover, let us assume (2.9), (2.11), (2.13). Then*

$$\begin{aligned} \|T(n, A_\epsilon)g - A^{-1}g\| &\leq c \cdot (n^{\lambda_0 - \mu}\|g\|_{Y^\mu} + n^\alpha\epsilon\|g\|) \\ &\leq C \cdot (n^{-\kappa} + n^\alpha\epsilon) \end{aligned}$$

where $\kappa = \mu - \lambda_0 (\geq 0)$ and C depends on g. For $n \sim \epsilon^{-\frac{1}{\kappa+\alpha}}$ we have

$$\|T(n, A_\epsilon)g - A^{-1}g\| = O(\epsilon^{\frac{\kappa}{\kappa+\alpha}}).$$

Proof.

$$\begin{aligned} \|T(n, A_\epsilon)g - A^{-1}g\| &\leq \|T(n, A)g - A^{-1}g\| + \|T(n, A_\epsilon) - T(n, A)\|\|g\| \\ \|T(n, A_\epsilon) - T(n, A)\| &= \|(A_{\epsilon,n}^{-1} - A_n^{-1})Q_n\| \\ &\leq \|A_{\epsilon,n}^{-1} - A_n^{-1}\|\|Q_n\|. \end{aligned}$$

Then, using (2.20) and Proposition 2.3 we get the assertions. ■

3 Combination with noisy right–hand side.

The objective of this section is to solve the operator equation

$$Au = g, \tag{3.1}$$

where the operator A and the right–hand side g are not exactly given.
As in Section 2 let us consider Banach spaces X, Y, Z, U, with norms $\|\cdot\|$ in both X and Y, $\|\cdot\|_Z$ in Z, $\|\cdot\|_U$ in U. Let Y be continuously embedded into Z and additionally Y be continuously embedded into U,

$$Y \subset U. \tag{3.2}$$

Moreover, as in (2.20), let us consider Banach scales $X = X^{\lambda_0} \supset \ldots \supset X^\lambda \supset \ldots$ and $Y = Y^{\lambda_0} \supset \ldots \supset Y^\lambda \supset \ldots$ and let A be an isomorphism of X^λ to $Y^\lambda, \lambda_0 \leq \lambda$.

3.1 Regularization of the embedding.

The regularization of the embedding (3.2) is an important step in solving (3.1). In [1] for the regularization of an embedding (3.2) a truncated singular value decomposition method was considered, while in [2] an approximation problem was solved.
In this subsection we summarize the results of [2] concerning the regularization of (3.2).
Let $\delta > 0$ be a small real parameter and $g_\delta \in U$ a noisy right–hand side with

$$\|g - g_\delta\|_U \leq \delta. \tag{3.3}$$

Let ℓ be a regularization parameter. The task is to find

$$P(\ell, \delta) \in Y$$

such that

$$\|g - P(\ell, \delta)\| \to 0$$

if $\ell = \ell(\delta)$ and $\delta \to 0$, where the rate of convergence should be as high as possible.
Now, let us consider a more concrete situation. Let be $\ell \in \mathbb{N}$ and let us suppose that Y, Z are spaces of complex valued functions over a bounded domain or manifold Ω and let $Y \subset C(\bar{\Omega})$. Moreover, let a finite mesh

$$G_\ell = \{t_1, \ldots, t_\ell\}$$

be imposed on $\bar{\Omega}$, and let (inaccurate) measurements g_δ^j of g be given at the mesh points $t_j \in \bar{\Omega}$ with

$$|g(t_j) - g_\delta^j| \leq \delta, \quad j = 1, \ldots, \ell. \tag{3.4}$$

Let us introduce finite dimensional subspaces

$$\hat{Y}_1 \subset \ldots \subset \hat{Y}_\ell \subset \ldots \subset Y, \quad \overline{\bigcup_\ell \hat{Y}_\ell} = Y.$$

The choice of $\hat{Y}_\ell$ here is completely independent on the choice of Y_ℓ in Section 2.
Additionally, for arbitrary $\underline{g} = (g_1, \ldots, g_\ell), g_j \in \mathbb{C}, j = 1, \ldots, \ell$, let there exist a uniquely determined interpolation polynomial

$$S_\ell \underline{g} \in \hat{Y}_\ell$$

with the property

$$(S_\ell \underline{g})(t_j) = g_j, \; j = 1, \ldots, \ell.$$

For $g \in C(\Omega), \underline{g} = (g(t_1), \ldots, g(t_\ell))$, let us define

$$S_\ell g = S_\ell \underline{g}.$$

The just defined operator S_ℓ is a projector from Y onto $\hat{Y}_\ell$.
For the operators S_ℓ and the subspaces $\hat{Y}_\ell$ we consider the following properties:

Approximation property. If $g \in Y^\mu, \mu \geq \lambda_0$, we have

$$\|g - S_\ell g\| \leq c \cdot \ell^{\lambda_0 - \mu} \|g\|_{Y^\mu}. \tag{3.5}$$

Inverse property. There exists a real $\beta \geq 0$ such that we have

$$\|\hat{\psi}\| \leq c \cdot \ell^{\beta} \|\hat{\psi}\|_U \quad \forall \hat{\psi} \in \hat{Y}_\ell. \tag{3.6}$$

Finite property. For all $\hat{\psi} \in \hat{Y}_\ell$ we have

$$\|\hat{\psi}\|_U \leq c \cdot \max_{1 \leq j \leq \ell} |\hat{\psi}(t_j)|. \tag{3.7}$$

Remark 3.1 *1) Examples, where (3.5),(3.6),(3.7) hold, are given in [2].*
2) There are examples, where other functions of n (such as $\log n, \exp n$) appear as rates or factors in (3.5), (3.6),(3.7) (cf. [2]). In those cases one has to proceed analogously.
3) The rates in the error estimates (2.20) and (3.5) are chosen identically. This is intended as it seems to be natural in our situation.
4) The real number β in the inverse inequality represents an ill–posedness measure of the embedding operator (3.2). It is connected to the behavior of its singular values. Therefore, it seems to be natural to suppose it independent on the choice of the scale of finite–dimensional subspaces. Notice that (2.13) concerns another embedding.

Now, take

$$P(\ell, \delta) = S_\ell \underline{g}^\delta, \tag{3.8}$$

where

$$\underline{g}^\delta = (g_\delta^1, \ldots, g_\delta^\ell)$$

is the vector of measurements with the property (3.4). We obtain the following

Proposition 3.1 *Let the operators S_ℓ and the spaces $\hat{Y}_\ell$ fulfil the properties (3.5), (3.6), (3.7). Then for $g \in Y^\mu$ we get*

$$\|g - P(\ell, \delta)\| \leq c \cdot (\ell^{\lambda_0 - \mu} \|g\|_{Y^\mu} + \ell^\beta \delta). \tag{3.9}$$

Let be $\kappa = \mu - \lambda_0$. If

$$\ell \sim \delta^{-\frac{1}{\kappa + \beta}}, \tag{3.10}$$

then we have

$$\|g - P(\ell, \delta)\| = O(\delta^{\frac{\kappa}{\kappa + \beta}}). \tag{3.11}$$

The proof of Proposition 3.1 and further remarks can be found in [2].

3.2 Noisy operator and noisy right–hand side combined.

Before turning to the general case of both, noisy operator and noisy right–hand side, let us first consider the case of an exact operator and a noisy right– hand side.

Let us suppose that we are in the situation of (2.20) and Proposition 3.1, in particular we are given operators $T(n, A)$ and elements $P(\ell, \delta)$ with the properties (2.20) and (3.9), respectively.

Proposition 3.2 *Let $g \in Y^\mu$ and the relations (2.20), (3.5), (3.6), (3.7) hold. Let again $\kappa = \mu - \lambda_0$. Then*

$$\begin{aligned}\|A^{-1}g - T(n,A)P(\ell,\delta)\| &\leq c \cdot (n^{-\kappa}\|g\|_{Y^\mu} + \ell^{-\kappa}\|g\|_{Y^\mu} + \ell^\beta \delta) \\ &\leq C \cdot (n^{-\kappa} + \ell^{-\kappa} + \ell^\beta \delta),\end{aligned}$$

where C depends on g. If $\ell \sim \delta^{-\frac{1}{\kappa+\beta}}$ we have

$$\|A^{-1}g - T(n,A)P(\ell,\delta)\| \leq C \cdot (n^{-\kappa} + \delta^{\frac{\kappa}{\kappa+\beta}}).$$

It is natural to choose $n \sim \ell$. In that case

$$\|A^{-1}g - T(n,A)P(\ell,\delta)\| = O(\delta^{\frac{\kappa}{\kappa+\beta}}).$$

Proof. Using Remark 2.2 we obtain

$$\begin{aligned}\|A^{-1}g - T(n,A)P(\ell,\delta)\| &= \|A^{-1}g - T(n,A)g + T(n,A)(g - P(\ell,\delta))\| \\ &\leq \|A^{-1}g - T(n,A)g\| + c \cdot \|g - P(\ell,\delta)\|.\end{aligned}$$

Then, (2.20) and Proposition 3.1 give the result. ■

Now, let us combine Proposition 3.2 with uncertain operators, i.e. with the situation of Section 2. Besides of $P(\ell,\delta)$ we are given operators $T(n,A_\epsilon) = A_{\epsilon,n}^{-1}Q_n$. According to Theorems 2.1 and 2.2 let us consider two cases.

Theorem 3.1 *Let the assumptions of Theorem 2.1 and of Proposition 3.1 hold. Let $\kappa = \mu - \lambda_0$. Then*

$$\begin{aligned}\|A^{-1}g - T(n,A_\epsilon)P(\ell,\delta)\| &\leq c \cdot (n^{-\kappa}\|g\|_{Y^\mu} + \epsilon\|g\| + \ell^{-\kappa}\|g\|_{Y^\mu} + \ell^\beta \delta) \\ &\leq C \cdot (n^{-\kappa} + \epsilon + \ell^{-\kappa} + \ell^\beta \delta),\end{aligned}$$

where C depends on g. If

$$n \sim \epsilon^{-\frac{1}{\kappa}}$$

and

$$\ell \sim \delta^{-\frac{1}{\kappa+\beta}}$$

then we obtain

$$\|A^{-1}g - T(n,A_\epsilon)P(\ell,\delta)\| = O(\epsilon + \delta^{\frac{\kappa}{\kappa+\beta}}),$$

for $\epsilon, \delta \to 0$ independently.

Proof. Using (2.9) (that follows from Proposition 2.2) we get

$$\begin{aligned}\|A^{-1}g - T(n,A_\epsilon)P(\ell,\delta)\| &= \|A^{-1}g - T(n,A_\epsilon)g + A_{\epsilon,n}^{-1}Q_n(g - P(\ell,\delta))\| \\ &\leq \|A^{-1}g - T(n,A_\epsilon)g\| + c \cdot \|g - P(\ell,\delta)\|.\end{aligned}$$

Then Theorem 2.1 and Proposition 3.1 give the first assertion. The other assertions are easy consequences. ■

Theorem 3.2 *Let the assumptions of Theorem 2.2 and of Proposition 3.1 hold. Let $\kappa = \mu - \lambda_0$. Then*

$$\begin{aligned}\|A^{-1}g - T(n, A_\epsilon)P(\ell,\delta)\| &\le c\cdot(n^{-\kappa}\|g\|_{Y^\mu} + n^\alpha\epsilon\|g\| + \ell^{-\kappa}\|g\|_{Y^\mu} + \ell^\beta\delta)\\ &\le C\cdot(n^{-\kappa} + n^\alpha\epsilon + \ell^{-\kappa} + \ell^\beta\delta),\end{aligned}$$

where C depends on g. If

$$n \sim \epsilon^{-\frac{1}{\kappa+\alpha}}$$

and

$$\ell \sim \delta^{-\frac{1}{\kappa+\beta}}$$

then we obtain

$$\|A^{-1}g - T(n, A_\epsilon)P(\ell,\delta)\| = O(\epsilon^{\frac{\kappa}{\kappa+\alpha}} + \delta^{\frac{\kappa}{\kappa+\beta}}),$$

for $\epsilon, \delta \to 0$ independently.

Proof. Using (2.9) we get

$$\begin{aligned}\|A^{-1}g - T(n, A_\epsilon)P(\ell,\delta)\| &= \|A^{-1}g - T(n, A_\epsilon)g + A_{\epsilon,n}^{-1}Q_n(g - P(\ell,\delta))\|\\ &\le \|A^{-1}g - T(n, A_\epsilon)g\| + c\cdot\|g - P(\ell,\delta)\|.\end{aligned}$$

Then Theorem 2.2 and Proposition 3.1 give the first assertion. The other assertions are easy consequences. ∎

4 Application to a model operator equation.

Here we shall apply the abstract theory of the former sections to a model integral equation of the first kind that will be important in the parameter determination problem of Section 5.

Let us consider the (Hilbert) spaces

$$L^2 = L^2(0,1), \quad {}_0H^1 = \{w \in H^1(0,1), w(0) = 0\},$$

where norm and scalar product in ${}_0H^1$ are induced by $H^1 = H^1(0,1)$, and the (compact) embedding

$${}_0H^1 \subset L^2,$$

whose singular values decrease like n^{-1} (cf. the Appendix, where the singular value decomposition is given). Again, let us denote by

$$\|\cdot\|$$

the norm in both L^2 and ${}_0H^1$, also the norm of operators in or between those spaces, if no confusion is possible.

Now, consider a function $f \in C^1[0,1]$ with $f(0) \neq 0$ and define

$$A : L^2 \to {}_0H^1$$

as

$$(Au)(t) = \int_0^t f(t-s)u(s)ds.$$

That A is an isomorphism can be seen easily from

$$A = D^{-1}B,$$

where

$$D = d/dt, \quad D^{-1} = \int_0^t \cdot\, (s)ds,$$

are isomorphisms (the well-known differential and integral operators) between L^2 and ${}_0H^1$ with norms $\|D\| \leq 1$, $\|D^{-1}\| \leq \sqrt{3/2}$, and B,

$$(Bu)(t) = f(0)u(t) + \int_0^t f'(t-s)u(s)ds,$$

is an isomorphism of L^2 to itself.

To solve numerically the operator equation

$$Au = g$$

with exact data, let us consider for $n \in \mathbb{N}$ the equidistant discretization

$$t_i = i/n, \quad i = 0, 1, \cdots, n,$$

and let us define

$$X_n = span\{e_i,\ i = 1, \cdots, n\}, \quad Y_n = span\{d_i,\ i = 1, \cdots, n\},$$

where $e_i(t) = 1$ for $t_{i-1} \leq t \leq t_i$, $e_i(t) = 0$ else, $i = 1, \cdots, n$, and d_i is linear and continuous with $d_i(t_j) = 1$ for $j = i$ and $= 0$ for $j \neq i$, $i = 1, \cdots, n$. (I.e. d_i is the hat–function with top at t_i.)

It should be noted, that D is an isomorphism of Y_n onto X_n.

Now, to an arbitrary function $u \in L^2$ let us consider the mean values

$$M_i(u) = n \int_{t_{i-1}}^{t_i} u(s)ds, \ i = 1, \cdots, n,$$

and define the projector $P_n : L^2 \to X_n$,

$$P_n u = \sum_{i=1}^{n} M_i(u)e_i.$$

Denoting the scalar product in L^2 by $(\cdot,\cdot)$ we have that P_n is the orthoprojector to X_n,

$$P_n u = \sum_{i=1}^{n} \frac{(e_i, u)}{(e_i, e_i)} e_i.$$

Lemma 4.1 *Let be $u \in L^2$. Then*

$$\|P_n u\| \le \|u\|. \tag{4.1}$$

Let be $u \in H^\nu, \nu \ge 0$. Then

$$\|P_n u - u\| \le c \cdot n^{-\nu} \|u\|_{H^\nu}. \tag{4.2}$$

P r o o f. Lemma 4.1 is well known from approximation theory. ■

Let us note that the operator

$$Q_n = D^{-1} P_n D$$

is a projector from ${}_0H^1$ to Y_n with the following properties:

Lemma 4.2

$$\|Q_n\| \le \sqrt{3/2}. \tag{4.3}$$

Let ${}_0H^{1+\nu} = {}_0H^1 \cap H^{1+\nu}, \nu \ge 0$. If $w \in {}_0H^{1+\nu}$ then

$$\|Q_n w - w\| \le c \cdot n^{-\nu} \|w\|_{H^{1+\nu}}. \tag{4.4}$$

P r o o f. This is immediately clear from Lemma 4.1, taking into account that $w = D^{-1}u \in H^{1+\nu}$ if and only if $u \in H^\nu$. ■

Let us define the operators

$$A_n = Q_n A, \quad B_n = P_n B,$$

mapping X_n to Y_n and X_n to X_n, respectively, let be $g_n \in Y_n,\ h_n \in X_n$, and consider the finite dimensional problems

$$A_n v = g_n \tag{4.5}$$

and

$$B_n v = h_n. \tag{4.6}$$

Since $B_n = DA_n$, the solution sets of (4.5) and (4.6) coincide, if $h_n = Dg_n$.

Our next step is to give a matrix representation for B_n and to show the stability property (2.7). Then the stability holds also for A_n.

Let $v = \sum_{j=1}^n x_j e_j$ and consider the Ritz-Galerkin method

$$n(B_n v, e_i) = n(h_n, e_i), \quad i = 1, \ldots, n,$$

giving rise to the linear system

$$\sum_{j=1}^n b_{ji} x_j = y_i, \quad i = 1, \cdots, n,$$

where

$$b_{ji} = n(B_n e_j, e_i), \quad y_i = n(h_n, e_i), \quad i, j = 1, \cdots, n,$$

hold and the multiplication by $n = 1/(e_i, e_i)$ is motivated. We have

$$P_n B e_j = \sum_{k=1}^{n} M_k(Be_j) e_k$$

implying

$$b_{ji} = M_i(Be_j) = f(0)\, n \int_{t_{i-1}}^{t_i} e_j(t)dt + M_i(F_j),$$

where $F_j(t) = \int_0^t f'(t-s)e_j(s)ds$. Since $\int_{t_{i-1}}^{t_i} e_j(t)dt = \delta_{ij}/n$,

$$F_j(t) = \begin{cases} 0, & 0 \le t \le t_{j-1} \\ f(t-t_{j-1}) - f(0), & t_{j-1} \le t \le t_j \\ f(t-t_{j-1}) - f(t-t_j), & t_j \le t \le 1 \end{cases},$$

$$M_i(F_j) = \begin{cases} 0, & i < j \\ \mu_1 - f(0), & i = j \\ \mu_{i-j+1} - \mu_{i-j}, & i > j \end{cases},$$

where $\mu_k = M_k(f),\ k = 1, \cdots, n$, we finally get

$$b_{ji} = \begin{cases} 0, & i < j \\ a_{i-j}, & i \ge j \end{cases},$$

where $a_0 = \mu_1, a_k = \mu_{k+1} - \mu_k, k = 1 \cdots, n-1$. From this it is clear, that $\mathbf{B}_n = (b_{ji})$ is invertible, and $b_{ii} = \mu_1 \to f(0), b_{ji} \to 0, j \ne i$, if $n \to \infty$.

The representation $\mathbf{B}_n = \sum_{k=0}^{n-1} a_k \mathbf{J}^k$, where $\mathbf{J}$ is the $n \times n$-matrix $(\eta_{ji}), \eta_{ji} = 1$ if $i = j+1$; $\eta_{ji} = 0$ else, $j = 1, \cdots, n-1$, implies

$$\|\mathbf{B}_n\| \le \sum_{k=0}^{n-1} |a_k|,$$

and the mean-value theorem gives

$$|a_k| \le c_0/n, \quad c_0 = 2|f'|_C, \quad k = 1, \cdots, n-1. \tag{4.7}$$

Indeed, $\forall k \exists \xi_k, t_{k-1} \le \xi_k \le t_k$ such that $M_k(f) = f(\xi_k)$ and we have

$$|a_k| = |M_{k+1}(f) - M_k(f)| = |f(\xi_{k+1}) - f(\xi_k)| \le |f'|_C |\xi_{k+1} - \xi_k|,$$

proving (4.7).

The inverse $\mathbf{B}_n^{-1} = (b'_{ji})$ has the same structure

$$b'_{ji} = \begin{cases} 0, & i < j \\ a'_{i-j}, & i \ge j \end{cases},$$

where a'_k solves the recursion system

$$a_0 a'_0 = 1,\ \sum_{\nu=0}^{k} a_\nu a'_{k-\nu} = 0,\ k = 1, \cdots, n-1, \tag{4.8}$$

and can be calculated in a simple way, providing an easily implementable numerical procedure for the solution of the problem (4.5) with exact data. To prove the stability (2.7) or equivalently $\|\mathbf{B}_n^{-1}\| \le c$, we need the

Lemma 4.3
$$\sum_{k=0}^{n-1} |a'_k| \le |a'_0| \left(1 + \frac{c_0|a'_0|}{n}\right)^{n-1}. \tag{4.9}$$

P r o o f. From (4.8) we obtain immediately

$$|a'_0| = 1/|a_0|, \quad a'_k = -\sum_{\nu=1}^{k} a'_0\, a_\nu\, a'_{k-\nu} \ , \quad k = 1, \cdots, n-1,$$

and, using (4.7),

$$|a'_k| \le \frac{c_0|a'_0|}{n} \sum_{\nu=0}^{k-1} |a'_\nu|, \ k = 1, \cdots, n-1. \tag{4.10}$$

Let us set $c_n = c_0|a'_0|/n$, and let us show by induction over k that

$$|a'_k| \le c_n|a'_0|(1+c_n)^{k-1} \ , \quad k = 1, \cdots, n-1, \tag{4.11}$$

holds. Indeed, from (4.10) and (4.11) we get

$$\begin{aligned} |a'_1| &\le c_n|a'_0|, \\ |a'_{k+1}| &\le c_n \sum_{\nu=0}^{k} |a'_\nu| \le c_n|a'_0| + c_n^2|a'_0| \sum_{\nu=0}^{k-1}(1+c_n)^\nu \\ &= c_n|a'_0| + c_n^2|a'_0| \cdot \frac{1}{c_n}\Big((1+c_n)^k - 1\Big) = c_n|a'_0|(1+c_n)^k, \end{aligned}$$

proving (4.11) for $k+1$. The Lemma is proved by summing up (4.11) over k. ■

Let $|a'_0| \le c'_0$, where c'_0 does not depend on n. Since $|a'_0| \to 1/|f(0)|$ if $n \to \infty$ such a bound must exist. In the case when $|f(t)|$ is not decreasing in a small intervall $[0,\tau)$

$$|a'_0| \le 1/|f(0)|$$

holds for $n > N(\tau)$. We have the

Proposition 4.1
$$\|\mathbf{B}_n^{-1}\| \le c'_0 \cdot e^{c_0 c'_0} \tag{4.12}$$

holds, where

$$c_0 = 2|f'|_C,$$

and, if $|f(t)|$ is not decreasing in $[0,\tau)$ and $n > N(\tau)$,

$$c'_0 = 1/|f(0)|.$$

P r o o f. Lemma 4.3 gives

$$\|\mathbf{B}_n^{-1}\| \le c'_0(1+\hat{c}/n)^n,$$

where $\hat{c} = c_0 c'_0$. The elementary estimate $1 + \hat{c}/n \le e^{\hat{c}/n}$ then implies the assertion. ■

The convergence of the procedure

$$T(n,A)g = A_n^{-1}Q_n g$$

to the solution $A^{-1}g$ can be described as follows. Consider the scale of Banach spaces

$$Y^0 = {}_0H^1 \supset \cdots \supset Y^\nu = {}_0H^{1+\nu} \supset \cdots, \quad \nu \ge 0.$$

Realize (by interpolation theory) that A is bijective from H^ν to ${}_0H^{1+\nu}$ for every $\nu \ge 0$. We have the

Proposition 4.2 *If $g \in {}_0H^{1+\nu}, \nu \geq 0$, then*

$$\|T(n,A)g - A^{-1}g\| \leq c \cdot n^{-\nu}\|g\|_{H^{1+\nu}}. \tag{4.13}$$

P r o o f. Proposition 4.2 follows from the stability (2.7) and Lemmas 4.1 and 4.2. Indeed, from page 26 (1.23.Proposition (c)) of [10] we obtain in our notation

$$\|A^{-1}g - T(n,A)g\| \leq \inf_{v \in X_n} (\|A^{-1}g - v\| + \|A_n^{-1}\|\|A_n v - Q_n g\|).$$

The assertion follows by putting $v = P_n A^{-1} g$. ∎

Moreover, since the singular values of the embedding ${}_0H^1 \subset L^2$ decrease like n^{-1}, (as is proved in the Appendix) we expect an inverse inequality of the following kind: For each $\psi \in Y_n$ we have

$$\|\psi\| \leq c \cdot n\|\psi\|_{L^2}. \tag{4.14}$$

Now, bringing noisy data into the discussion, let us assume, that instead of the exact function f we are given a function f_ϵ with $f_\epsilon \in C^1$ and $f_\epsilon(0) \neq 0$. For small $\epsilon \geq 0$ and c positive we consider the following properties:

$$\|f_\epsilon - f\|_{H^1} \leq \epsilon, \tag{4.15}$$

$$\|f_\epsilon - f\|_{L^2} \leq \epsilon, \tag{4.16}$$

$$|f_\epsilon(0) - f(0)| \leq \epsilon, \tag{4.17}$$

$$|f_\epsilon(0)| \geq c > 0, \tag{4.18}$$

$$|f_\epsilon'|_C \leq c. \tag{4.19}$$

Let us now consider uncertain operators

$$(A_\epsilon u)(t) = \int_0^t f_\epsilon(t-s)u(s)ds, \quad (B_\epsilon u)(t) = f_\epsilon(0)u(t) + \int_0^t f_\epsilon'(t-s)u(s)ds,$$

being isomorphisms from L^2 onto ${}_0H^1$ and onto itself, respectively. Concerning the deviation of A_ϵ from the exact operator A we have the following estimates:

Proposition 4.3

$$\|A_\epsilon - A\|_{L^2}^2 \leq \|f_\epsilon - f\|_{L^2}^2, \tag{4.20}$$

$$\|A_\epsilon - A\|_{H^1}^2 \leq 2(|f_\epsilon(0) - f(0)|^2 + \|f_\epsilon - f\|_{H^1}^2). \tag{4.21}$$

P r o o f. For the L^2-functions h and w, using the Hölder inequality and a simple substitution we have

$$|\int_0^t h(t-s)w(s)ds| \leq \|h\|_{L^2}\|w\|_{L^2}.$$

Then

$$\|(A_\epsilon - A)w\|_{L^2}^2 = \int_0^1 \left|\int_0^t (f_\epsilon(t-s) - f(t-s))w(s)ds\right|^2 dt \leq \|f_\epsilon - f\|_{L^2}\|w\|_{L^2},$$

proving (4.20), and $\|(A_\epsilon - A)w\|_{H^1}^2 = \|(A_\epsilon - A)w\|_{L^2}^2 + \|D(A_\epsilon - A)w\|_{L^2}^2$, where

$$\begin{aligned}\|D(A_\epsilon - A)w\|_{L^2}^2 &= \int_0^1 \left|(f_\epsilon(0) - f(0))w(t) + \int_0^t (f_\epsilon'(t-s) - f'(t-s))w(s)ds\right|^2 dt \\ &\leq 2\left(\int_0^1 (f_\epsilon(0) - f(0))^2 |w(t)|^2 dt + \|f_\epsilon' - f'\|_{L^2}^2 \|w\|_{L^2}^2\right),\end{aligned}$$

proving (4.21). ∎

Defining

$$A_{\epsilon,n} = Q_n A_\epsilon,\ B_{\epsilon,n} = P_n B_\epsilon,$$

it is clear now, how to get error estimates of the kind (2.10) and (2.11) from given estimates (4.15),(4.16),(4.17) using (4.3), Proposition 4.3 and Example 2.2. We have

$$\|A_n - A_{\epsilon,n}\| \leq \sqrt{3}\cdot(|f_\epsilon(0) - f(0)| + \|f_\epsilon - f\|_{H^1}), \tag{4.22}$$

$$\|A_n - A_{\epsilon,n}\|_{L^2} \leq c\cdot n\|f_\epsilon - f\|_{L^2}. \tag{4.23}$$

Remark 4.1 *The estimate (4.23) is not sharp. We have*

$$\begin{aligned}\|(A_{\epsilon,n} - A_n)u\|_{L^2} &= \|Q_n(A_\epsilon - A)u\|_{L^2} \\ &\leq \|(A_\epsilon - A)u\|_{L^2} + \|Q_n(A_\epsilon - A)u - (A_\epsilon - A)u\|_{L^2} \\ &\leq \|f_\epsilon - f\|_{L^2}\|u\|_{L^2} + G(n,\epsilon,u),\end{aligned}$$

where $G(n,\epsilon,u) \to 0$ if $n \to \infty$ and ϵ, u are fixed. More exactly, for $u \in X_n$ we obtain

$$\|Q_n(A_\epsilon - A)u - (A_\epsilon - A)u\|_{L^2} \leq c\cdot n^{-1}\|u\|_{L^2}, \tag{4.24}$$

such that we have

$$\|A_n - A_{\epsilon,n}\|_{L^2} \leq c\cdot(\|f_\epsilon - f\|_{L^2} + n^{-1}).$$

Let us prove (4.24). Using well–known results of approximation theory (cf. [10] or [11]) one can show that for $w \in H^{1+\lambda}, 1 < \lambda < 1/2$,

$$\|Q_n w - w\|_{L^2} \leq c\cdot n^{-(1+\lambda)}\|w\|_{H^{1+\lambda}} \tag{4.25}$$

holds. Taking $u \in X_n$ we have $u \in H^\lambda, 0 < \lambda < 1/2$, and since A, A_ϵ are isomorphisms of H^λ to ${}_0H^{1+\lambda}$, we have

$$Au, A_\epsilon u \in H^{1+\lambda},$$

and $\|(A - A_\epsilon)u\|_{H^{1+\lambda}} \leq c\cdot\|u\|_{H^\lambda}$. Now, setting $w = (A_\epsilon - A)u$ in (4.25) and using an inverse inequality of the kind

$$\|u\|_{H^\lambda} \leq c\cdot n^\lambda\|u\|_{L^2},$$

we get (4.24).

Moreover, in our concrete situation we can even prove the following

Proposition 4.4

$$\|A_n - A_{\epsilon,n}\|_{L^2} \le \|f - f_\epsilon\|_{L^2},$$

where A_n and $A_{\epsilon,n}$ are considered as operators mapping X_n to L^2.

P r o o f. Since the operator $A_n = A_n[f]$ depends linearly on f and $A_{\epsilon,n} = A_n[f_\epsilon]$ it suffices to prove that

$$\|A_n u\|_{L^2} \le \|f\|_{L^2}\|u\|_{L^2} \tag{4.26}$$

holds for each $u \in X_n$. Moreover, instead of (4.26) it is enough to prove

$$\|A_n e_j\|_{L^2} \le \|f\|_{L^2}/n\,,\ j = 1, \cdots, n. \tag{4.27}$$

Indeed, for $u = \sum_{i=1}^n x_i e_i$ we have $\|u\|^2 = \sum_{i=1}^n |x_i|^2/n$. In addition, $\|A_n u\|^2 =$

$$\|\sum x_i A_n e_i\|^2 \le (\sum |x_i|\|A_n e_i\|)^2 \le \sum |x_i|^2 \sum \|A_n e_i\|^2 \le \sum |x_i|^2 \cdot \|f\|^2/n = \|f\|^2\|u\|^2,$$

where we have used triangle inequality, Hölder inequality and (4.27).

Now, let us prove (4.27). We have

$$\begin{aligned} A_n e_j &= D^{-1}P_n B e_j = D^{-1}\sum_{k=1}^n b_{jk}e_k = \sum_{k=1}^n b_{jk}\int_0^t e_k(s)ds, \\ \|A_n e_j\|^2 &= \int_0^1 \left|\sum_{k=1}^n b_{jk}\int_0^t e_k(s)ds\right|^2 dt = \int_0^1 \left|\sum_{k=j}^n a_{k-j}\int_0^t e_k(s)ds\right|^2 dt = \int_0^1 |\phi_j(t)|^2 dt. \end{aligned}$$

Using $a_0 = \mu_1, a_\nu = \mu_{\nu+1} - \mu_\nu, \nu = 1, \cdots, n-1$ and

$$\int_0^t e_k(s)ds = \begin{cases} 0 & \text{if } 0 \le t \le t_{k-1} \\ t - t_{k-1} & \text{if } t_{k-1} \le t \le t_k \\ 1/n & \text{if } t_k \le t \le 1 \end{cases},$$

we obtain

$$\phi_j(t) = \begin{cases} 0 & \text{if } 0 \le t \le t_{j-1} \\ \mu_1(t - t_{j-1}) & \text{if } t_{j-1} \le t \le t_j \\ \mu_r/n + (\mu_{r+1} - \mu_r)(t - t_{j+r-1}) & \text{if } t_{j+r-1} \le t \le t_{j+r},\ r = 1, \cdots, n-j. \end{cases}$$

A straightforward calculation gives

$$\int_0^1 |\phi_j(t)|^2 dt = \sum_{r=0}^{n-j}\int_{t_{j+r-1}}^{t_{j+r}} |\phi_j(t)|^2 dt = \frac{1}{3n^3}\left(\mu_1^2 + \sum_{r=1}^{n-j}(\mu_r\mu_{r+1} + \mu_r^2 + \mu_{r+1}^2)\right) \le \frac{1}{n^3}\sum_{r=1}^n \mu_r^2.$$

Using

$$\mu_r^2 = n^2\left(\int_{t_{r-1}}^{t_r} f(s)ds\right)^2 \le n\int_{t_{r-1}}^{t_r} |f(s)|^2 ds,$$

(4.27) follows immediately. ∎

Another important assumption in the general part (Section 2) is the stability estimate (2.9). Proposition 4.1 gives

Proposition 4.5 *(4.18) and (4.19) imply (2.9).*

Finally, after these preparations, it is not difficult to define a numerical procedure in the case of an uncertain operator A_ϵ and noisy right-hand sides g_δ and to give error estimates.

In favour of a clear presentation let us begin with the just considered situation of an uncertain operator and an exactly given right-hand side, using Theorems 2.1 and 2.2.

Theorem 4.1 *Let be $g \in {}_0H^{1+\nu}, \nu \geq 0$. Moreover, let f_ϵ and $\epsilon > 0$ be such, that*

$$\sqrt{3} \cdot (|f_\epsilon(0) - f(0)| + \|f_\epsilon - f\|_{H^1}) \leq \epsilon$$

and

$$\epsilon < q/\tilde{c}_0, \; \tilde{c}_0 = e^{2|f'|_C/|f(0)|}/|f(0)|,$$

q fixed with $0 < q < 1$, hold. Let $T(n, A_\epsilon) = A_{\epsilon,n}^{-1}Q_n$. Then

$$\begin{aligned} \|T(n, A_\epsilon)g - A^{-1}g\| &\leq c \cdot (n^{-\nu}\|g\|_{{}_0H^{1+\nu}} + \epsilon\|g\|) \\ &\leq C \cdot (n^{-\nu} + \epsilon) \end{aligned}$$

where C depends on g. For $n \sim \epsilon^{-\frac{1}{\nu}}$ we have

$$\|T(n, A_\epsilon)g - A^{-1}g\| = O(\epsilon).$$

P r o o f. Using Propositions 4.1, 4.2 and (4.22) we can show that the assumptions of Theorem 2.1 are fulfilled. ∎

Theorem 4.2 *Let be $g \in {}_0H^{1+\nu}, \nu \geq 0$. Moreover, let*

$$\|f_\epsilon - f\|_{L^2} \leq \epsilon$$

hold, let (4.18) and (4.19) be true and $T(n, A_\epsilon) = A_{\epsilon,n}^{-1}Q_n$. Then

$$\begin{aligned} \|T(n, A_\epsilon)g - A^{-1}g\| &\leq c \cdot (n^{-\nu}\|g\|_{{}_0H^{1+\nu}} + n\epsilon\|g\|) \\ &\leq C \cdot (n^{-\nu} + n\epsilon) \end{aligned}$$

where C depends on g. For $n \sim \epsilon^{-\frac{1}{\nu+1}}$ we have

$$\|T(n, A_\epsilon)g - A^{-1}g\| = O(\epsilon^{\frac{\nu}{\nu+1}}).$$

P r o o f. Using Propositions 4.1, 4.2, 4.4, 4.5 and (4.14) we see that the assumptions of Theorem 2.2 are fulfilled. ∎

To incorporate a noisy right-hand side into the consideration, in what follows we apply the abstract theory of Section 3. We have to take $\Omega = [0, 1]$ and consider the case, where

$$U = L^2$$

and

$$\hat{Y}_n = Y_n, \; S_n = Q_n, \; n = 1, \cdots, \; t_i = i/n, \; i = 1, \cdots, n$$

hold. It is easy to see, that $Q_n w$ is indeed the linear interpolation polynomial, since we have by direct computation

$$(D^{-1}P_n u)(t_i) = (D^{-1}u)(t_i), \; i = 1, \cdots, n$$

for an arbitrary $u \in L^2$ using that $D^{-1}e_j$ is piecewise linear with the property

$$(D^{-1}e_j)(t_i) = \begin{cases} 0 & \text{if } i < j \\ 1/n & \text{if } i \geq j \end{cases}, \; i, j = 1, \cdots, n.$$

Let us suppose that we are given measurements g_δ^j at the points $t_j = j/\ell \in [0,1], \; j = 1, \cdots, \ell$ with the property (3.4), where $\delta > 0$ is the noise level of the measurements.

Theorem 4.3 *Let be $g \in {}_0H^{1+\nu}, \nu \geq 0$. Moreover, let be $\epsilon > 0, \delta > 0$, and let ϵ be such that*

$$\sqrt{3} \cdot (|f_\epsilon(0) - f(0)| + \|f_\epsilon - f\|_{H^1}) \leq \epsilon$$

and

$$\epsilon < q/\tilde{c}_0, \; \tilde{c}_0 = e^{2|f'|_C/|f(0)|}/|f(0)|,$$

q fixed with $0 < q < 1$, hold. Let $T(n, A_\epsilon) = A_{\epsilon,n}^{-1}Q_n$ and let $P(\ell, \delta)$ be the linear interpolation polynomial of the measurements. Then

$$\begin{aligned} \|T(n, A_\epsilon)P(\ell, \delta) - A^{-1}g\| &\leq c \cdot ((n^{-\nu} + \ell^{-\nu})\|g\|_{{}_0H^{1+\nu}} + \epsilon\|g\| + \ell\delta) \\ &\leq C \cdot (n^{-\nu} + \ell^{-\nu} + \epsilon + \ell\delta) \end{aligned}$$

where C depends on g. For $n \sim \epsilon^{-\frac{1}{\nu}}, \; \ell \sim \delta^{-\frac{1}{\nu+1}}$ we have

$$\|T(n, A_\epsilon)P(\ell, \delta) - A^{-1}g\| = O(\epsilon + \delta^{\frac{\nu}{\nu+1}}).$$

P r o o f. Using Propositions 4.1, 4.2 and (4.22) we can show that the assumptions of Theorem 2.1 are fulfilled. Moreover, (4.4), (4.14) imply the approximation property and the inverse property. The property (3.7) is clear taking into account that $\|\psi\|_{L^2} \leq c \cdot \|\psi\|_C$ holds for $\psi \in Y_n$. This means, that the assumptions of Proposition 3.1 are also fulfilled. The assertions then follow from Theorem 3.1. ■

Theorem 4.4 *Let be $g \in {}_0H^{1+\nu}, \nu \geq 0$. Moreover, let be $\epsilon > 0, \delta > 0$, and let ϵ be such that*

$$\|f_\epsilon - f\|_{L^2} \leq \epsilon$$

holds, let (4.18) and (4.19) be true and $T(n, A_\epsilon) = A_{\epsilon,n}^{-1}Q_n$. Let $P(\ell, \delta)$ be the linear interpolation polynomial of the measurements. Then

$$\begin{aligned} \|T(n, A_\epsilon)P(\ell, \delta) - A^{-1}g\| &\leq c \cdot ((n^{-\nu} + \ell^{-\nu})\|g\|_{{}_0H^{1+\nu}} + n\epsilon\|g\| + \ell\delta) \\ &\leq C \cdot (n^{-\nu} + \ell^{-\nu} + n\epsilon + \ell\delta) \end{aligned}$$

where C depends on g. For $n \sim \epsilon^{-\frac{1}{\nu+1}}, \; \ell \sim \delta^{-\frac{1}{\nu+1}}$ we have

$$\|T(n, A_\epsilon)P(\ell, \delta) - A^{-1}g\| = O(\epsilon^{\frac{\nu}{\nu+1}} + \delta^{\frac{\nu}{\nu+1}}).$$

P r o o f. Using Propositions 4.1, 4.2, 4.4, 4.5 and (4.14) we see that the assumptions of Theorem 2.2 are fulfilled. As in the proof of Theorem 4.3 the assumptions of Proposition 3.1 are also fulfilled. Then the assertions follow from Theorem 3.2. ■

5 Stable point source reconstruction in the wave equation.

This section is based on a joint work of the authors (cf. [4, 5]).

We consider the stable reconstruction of point sources in the 1D–wave equation from observations at a single inner point. Here we are in a position, where the investigations of Section 4 can be applied.

As the direct problem let us consider the following initial–boundary value problem:

$$\begin{aligned} w_{tt}(x,t) &= w_{xx}(x,t) + f(t)\sum_{j=1}^{m}\alpha_j\delta(x-x_j), \quad 0<x<1, 0<t<T \\ w(x,0) &= w_t(x,0) = 0, \qquad 0<x<1 \\ w(0,t) &= w(1,t) = \ 0, \qquad 0<t<T, \end{aligned}$$

where α_j are real numbers, $x_j \in (0,1), 1 \le j \le m,\ f \in C^1[0,1], f(0) \neq 0$, are given and $\delta(\cdot - z)$ is Dirac's distribution with $\int_0^1 \delta(x-z)\varphi(x)dx = \varphi(z), \varphi \in C_0^\infty$. The direct problem is solved by the following

Proposition 5.1 *For a given parameter set*

$$P = \{m, \alpha_1, \dots, \alpha_m, x_1, \dots, x_m\}$$

there exists a unique weak solution

$$w \in C^1([0,T], L^2(0,1)) \cap C^0([0,1], {}_0H^1), \quad {}_0H^1 = \{\tilde{w} \in H^1(0,T), \tilde{w}(0) = 0\},$$

Moreover, there holds the series representation

$$w(x,t) = \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\Big(\sum_{j=1}^{m}\alpha_j \sin k\pi x_j\Big)\sin k\pi x\Big(f(t) - f(0)\cos k\pi t - \int_0^t f'(t-s)\cos k\pi s\, ds\Big).$$

The proof follows from [6], [7], [8]. We have $w(y_0,\cdot) \in {}_0H^1$ for $y_0 \in [0,1]$.

Let the inverse problem consist in the determination of the parameter set

$$P = \{m, \alpha_1, \dots, \alpha_m, x_1, \dots, x_m\}$$

from the domain observation $w(y_0,t), 0<t<T$, for an arbitrary fixed $y_0 \in (0,1)$. (As to boundary observations cf.[4], and in [9] other kinds of sources are considered.)

Denote $L^2 = L^2(0,T)$, and let be

$$T = 1.$$

We have $w(x,t) = (Au(x,\cdot))(t)$, where

$$\begin{aligned} u(x,t) &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\Big(\sum_{j=1}^{m}\alpha_j \sin k\pi x_j\Big)\sin k\pi x\ \sin k\pi t, \\ (Av)(t) &= \int_0^t f(t-s)v(s)ds, \quad A: L^2 \longrightarrow {}_0H^1. \end{aligned}$$

Besides, let $\phi_k(t) = \sin k\pi t,\ k = 1, 2, \ldots$ and $u_0 = u(y_0, \cdot)$, such that for the data $w_0 = w(y_0, \cdot)$

$$w_0 = Au_0 \tag{5.1}$$

holds. Then we obtain the

Proposition 5.2 *For every $k = 1, 2, \ldots$ the following is true: If $\sin k\pi y_0 \neq 0$, then*

$$\sum_{j=1}^{m} \alpha_j \sin k\pi x_j = \frac{k\pi}{\sin k\pi y_0}(u_0, \phi_k),$$

where $(\cdot,\cdot)$ is the scalar product in L^2.

P r o o f. We have

$$u_0(t) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \Big(\sum_{j=1}^{m} \alpha_j \sin k\pi x_j \Big) \sin k\pi y_0 \ \sin k\pi t.$$

Since $\sqrt{2}\phi_k,\ k = 1, \cdots,$ is an orthonormal basis in L^2, we get the assertion by scalar multiplication in L^2 by ϕ_k. ■

In a special case the reconstruction can be given explicitely from Proposition 5.2:

Corollary 5.1 *Let be a–priori known that $m = 2,\ \alpha_1 = \alpha_2 = 1,\ 0 < x_1 < x_2 \leq 1/2$, and let y_0 be such that $\sin \pi y_0 \neq 0, \sin 3\pi y_0 \neq 0$. Then*

$$x_1 = \arcsin \theta_1, \quad x_2 = \arcsin \theta_2,$$

where θ_1, θ_2 are the roots of the quadratic equation

$$\theta^2 - a\theta + \frac{b + 4a^3 - 3a}{12a} = 0,\ a = \frac{\pi}{\sin \pi y_0}(u_0, \phi_1),\ b = \frac{3\pi}{\sin 3\pi y_0}(u_0, \phi_3).$$

P r o o f. Proposition 5.2 implies in our case

$$\begin{aligned} \sin \pi x_1 + \sin \pi x_2 &= a \\ \sin 3\pi x_1 + \sin 3\pi x_2 &= b. \end{aligned}$$

Using the identity $\sin 3\rho = 3 \sin \rho - 4 \sin^3 \rho$, we see that both, $\theta_i = \sin \pi x_i,\ i = 1, 2$, satisfy the quadratic equation. ■

The reconstruction depends continuously on u_0. But u_0 is unknown, it has to be found from (5.1). We are given the observation w_0 and the operator A, both perturbed by noise. Therefore, in the case of uncertain data, a regularization is in order. This regularization can be done on the whole nonlinear problem. We prefer however to regularize the linear operator equation (5.1), where the ill–posedness arises from.

Here we are in a position to apply the considerations of the previous section. For a stable reconstruction in the case of noisy data (inexact measurements g^δ at the point y_0 and uncertain function f_ϵ) let us propose the following performance.

In accordance with the kind of noise and the magnitude of noise levels ϵ and δ we apply Theorems 4.3 or 4.4, where $g = w_0$ has to be taken. By a well–defined procedure we obtain elements

$$T(n, A_\epsilon)P(\ell, \delta) = u_{\epsilon,\delta}$$

converging to u_0 in a stable way if $\epsilon, \delta \to 0$. Then, as the nonlinear reconstruction of P from u_0 is well–posed, $u_{\epsilon,\delta}$ (instead of u_0) in Proposition 5.2 or Corollary 5.1 takes us into a neighborhood of P.

6 Appendix. Singular value decomposition.

Here, a singular value decomposition for the embedding operator ${}_0H^1 \subset L^2$ is given being relevant to the regularization approach in Section 4. We again consider the general case $T > 0$.

Proposition 6.1 *A singular value decomposition of the embedding*

$$E : {}_0H^1 \to L^2$$

is given by $\{G_k, g_k, \sigma_k\}_1^\infty$, *where*

$$\begin{aligned} g_k(t) &= \sqrt{\frac{2}{T}} \cos \frac{(k-1/2)\pi(t-T)}{T}, \\ G_k &= \sigma_k g_k, \\ \sigma_k &= \Big(1 + \frac{(k-1/2)^2\pi^2}{T^2}\Big)^{-1/2}. \end{aligned}$$

P r o o f. Let us define

$$\hat{H} = \{u \in H_0^1(0, 2T);\ u(t) = u(2T - t)\,,\ 0 < t < T\}\,.$$

We equip ${}_0H^1$ and $\hat{H}$ with the scalar products and norms of $H^1(0,T)$ and $H_0^1(0,2T)$, respectively. For instance,

$$\begin{aligned} (u,v)_{{}_0H^1} &= (u,v) + \left(\frac{du}{dt}, \frac{dv}{dt}\right), \quad u, v \in {}_0H^1, \\ (U,V)_{\hat{H}} &= (U,V)_{L^2(0,2T)} + \left(\frac{dU}{dt}, \frac{dV}{dt}\right)_{L^2(0,2T)}, \quad U, V \in \hat{H}. \end{aligned}$$

Further, let us define an extension operator γ from ${}_0H^1 \to \hat{H}$ by

$$(\gamma u)(t) = \begin{cases} u(t)\,, & 0 \le t \le T \\ u(2T-t)\,, & T < t < 2T \end{cases}.$$

By direct calculations, we see that

$$\frac{d\gamma u}{dt}(t) = \begin{cases} \frac{du}{dt}(t)\,, & 0 < t < T \\ -\frac{du}{dt}(2T-t)\,, & T < t < 2T \end{cases}$$

in the sense of $\mathcal{D}'(0,2T)$ (the distributions in $(0,2T)$). Therefore $\gamma u \in \hat{H}$ and moreover we obtain

$$(\gamma u, \gamma v)_{\hat{H}} = 2(u,v)_{{}_0H^1}, \quad u, v \in {}_0H^1 .$$

Consequently by γ, the Hilbert spaces ${}_0H^1$ and $\hat{H}$ are isomorphic.

Let us set

$$L = \gamma^{-1},$$

i.e., L is the restriction operator of functions on $(0,2T)$ to $(0,T)$. Then we have $L\hat{H} = {}_0H^1$.

Moreover,

$$g_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cos \frac{(k-1/2)\pi(t-T)}{T}, \quad k \in \mathbb{N},$$

is an orthonormal basis in $L^2(0,T)$. In fact, the orthonormality is straightforward. To prove the completeness, let us consider the eigenvalue problem

$$\begin{aligned} -\frac{d^2\phi}{dt^2}(t) &= \lambda^2\phi(t), \quad 0 < t < T \\ \phi(0) &= \frac{d\phi}{dt}(T) = 0 . \end{aligned}$$

Then, as is easily checked,

$$\lambda_k^2 = \frac{(k-1/2)^2\pi^2}{T^2}, \quad k \in \mathbb{N},$$

is the set of eigenvalues and g_k, $k \in \mathbb{N}$, is an eigenfunction for λ_k^2. Therefore by a well–known result on the Sturm–Liouville problem, we see that g_k, $k \in \mathbb{N}$ is complete in $L^2(0,T)$.

Let

$$\sigma_k = \left(1 + \frac{(k-1/2)^2\pi^2}{T^2}\right)^{-1/2}.$$

Easily we can verify that

$$\Gamma_k(t) = \frac{1}{\sqrt{2}}\, \sigma_k\, \gamma g_k(t), \quad k \in \mathbb{N},$$

is an orthonormal basis of $\hat{H}$.

In fact, the orthonormality in $\hat{H}$ is straightforward. For the completeness, we can proceed as follows. Let $v \in \hat{H}$ satisfy $(v, \Gamma_k)_{\hat{H}} = 0$, $k \in \mathbb{N}$. Then since v is symmetric with respect to $t = T$, we have by integration by parts,

$$0 = (v, \Gamma_k)_{\hat{H}} = \sqrt{2}\sigma_k^{-1}(Lv, g_k), \ k \in \mathbb{N},$$

namely $(Lv, g_k) = 0$, $k \in \mathbb{N}$. By the completeness of g_k, $k \in \mathbb{N}$, in $L^2(0,T)$, we can conclude that $v = 0$. Thus, Γ_k, $k \in \mathbb{N}$, is an orthonormal basis in $\hat{H}$. We obtain that

$$G_k(t) = \sqrt{2}\, L\Gamma_k = \sigma_k\, g_k(t)$$

is an orthonormal basis in ${}_0H^1$.
Then, a singular value decomposition is

$$\{G_k, g_k, \sigma_k\}\ ,\ k \in \mathbb{N}\,.$$

In fact, $EG_k = \sigma_k g_k$, $k \geq 1$. In view of

$$(G_j, E^* g_k)_{{}_0H^1} = (EG_j, g_k) = \sigma_j \delta_{jk} = \sigma_k \delta_{jk} = \sigma_k (G_j, G_k)_{{}_0H^1} = (G_j, \sigma_k G_k)_{{}_0H^1},\ k \geq 1,$$

we obtain $E^* g_k = \sigma_k G_k$. ∎

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