Summary

We present an example of the solution of a boundary value problem for a two-component porous material with large deformations of the skeleton. This example demonstrates the application of a consistent Lagrangian description of porous materials which has been proposed in paper [1]. Simultaneously we demonstrate the important role of the balance equation of porosity which is an essential part of the thermodynamical model of porous materials proposed in [2,3]. We show as well that a modified set of boundary conditions for permeable boundaries yields a solution of field equations which agrees qualitatively with expectations for the problem of axisymmetric stationary filtration. On the basis of a numerical evaluation of solution we indicate the existence of an instability of the model for very large porosities which could not be explained in this work.

1. Introduction

Diffusion processes in porous materials are connected with numerous additional mathematical problems which do not appear in a usual theory of mixtures of fluids. The two most important of them are connected with the formulation of boundary value problems on permeable boundaries and with changes of microstructure.

The first problem was recognized by K. VON TERZAGHI and O. K. FRÖHLICH [4]. They indicated that the external load on a permeable boundary cannot be *a priori* distributed among components. Such a distribution is time dependent and follows from the solution of field equations. Consequently it was necessary to reconstruct classical boundary conditions for multicomponent systems under the requirement that solely the total external load can be prescribed to a certain combination of partial tractions and partial momentum discontinuities. All other boundary conditions have to have a kinematical character. Such conditions were proposed in [1] and they were investigated numerically for the classical consolidation problem in the work of W. KEMPA [5]. The form of conditions on permeable boundaries is connected with the presence of a boundary layer between the porous material and the external world. An extensive discussion of the microscopical background of such boundary layer problems can be found in the book of M. KAVIANY [6]. This discussion is connected with the problem of boundary conditions for viscous fluid components investigated earlier by Beavers and Joseph (see: [6] for references).

Such layers appear as well in other problems of physics as, for instance, heat conduction problems through thin walls. They indicate the way in which boundary conditions should be constructed. Preliminary results for this problem within the theory of porous materials can be found in the report of B. ALBERS [7].

The second problem is connected with changes of porosity. It was shown [2,3,8] that the model of porous materials based on the balance equation of porosity contains the porosity flux proportional to the diffusion velocity. This balance equation reflects the influence of the microstructure on the behaviour of porous materials.

In this work we illustrate these two features of porous materials by means of a simple stationary boundary value problem. In order to amplify the role of porosity as a field we solve a cylindrical filtration problem for large deformations of the skeleton. Material properties of the skeleton are described by constitutive relations of a Signorini type (see: C. TRUESDELL, W. NOLL [9]). In the case of stationary processes the balance equation for porosity reduces to an algebraic relation for changes of porosity. Such a relation could not appear in any other earlier models of porous materials in which changes of

porosity were driven either by volume changes of the skeleton or by an evolution equation.

The dependence of material parameters on porosity which must be accounted for in the case of large deformations was roughly estimated from the data for propagation of acoustic waves in porous materials (see: T. BOURBIE, O. COUSSY, B. ZINSZNER [10]).

In section 2 we present the field equations of a two-component model for a general three dimensional case. In section 3 modified boundary conditions are discussed. The axial symmetric stationary case is formulated in section 4. In section 5 the method of solution by a regular perturbation is presented. Section 6 is devoted to the numerical evaluation and the discussion of results. In section 7 we indicate an unsolved problem of instability of the model for very large porosities.

2. Governing set of equations for isothermal processes

Two-component porous materials undergoing isothermal processes are described by the following fields

$$(\mathbf{X}, t) \mapsto \left\{\rho^F, \boldsymbol{\chi}^S(\mathbf{X}, t), \mathbf{x}'^F, n\right\} \in \Re^8, \quad \mathbf{X} \in \mathsf{B}, \quad t \in \mathsf{T} \subset \Re^1, \tag{1}$$

where ρ^F is the mass density of the fluid per unit volume of the reference configuration B, $\boldsymbol{\chi}^S$ is the function of motion of skeleton, $\mathbf{x}'^F$ denotes the velocity of the fluid component and n is the porosity.

These quantities satisfy the partial balance equation of mass for the fluid, the partial balance equations of momentum and the balance equation of porosity, whose Lagrangian form [2] is as follows

$$\begin{aligned}
&\frac{\partial \rho^F}{\partial t} + \operatorname{Div}\left(\rho \mathbf{X}'^F\right) = 0, \qquad \mathbf{X}'^F := \mathbf{F}^{S-1}\left(\mathbf{x}'^F - \mathbf{x}'^S\right), \\
&\rho^F\left(\frac{\partial \mathbf{x}'^F}{\partial t} + \mathbf{X}'^F \cdot \operatorname{Grad} \mathbf{x}'^F\right) = \operatorname{Div} \mathbf{P}^F - \hat{\mathbf{p}}, \quad \mathbf{F}^S := \operatorname{Grad} \boldsymbol{\chi}^S, \quad \mathbf{x}'^F := \frac{\partial \boldsymbol{\chi}^S}{\partial t}, \\
&\rho^S \frac{\partial \mathbf{x}'^S}{\partial t} = \operatorname{Div} \mathbf{P}^S + \hat{\mathbf{p}}, \\
&\frac{\partial n}{\partial t} + \operatorname{Div}\left(\Phi_0 \mathbf{X}'^F\right) = \hat{n}.
\end{aligned} \tag{2}$$

In these equations $\mathbf{X}'^F$ is the Lagrangian relative velocity and $\mathbf{F}^S$ is the deformation gradient of skeleton. $\mathbf{P}^F$ and $\mathbf{P}^S$ denote the Piola-Kirchhoff partial stress tensors in the fluid and in the skeleton, respectively. The vector $\hat{\mathbf{p}}$ denotes the source of momentum in the skeleton and $\hat{n}$ is the source of porosity. Φ_0 is the transport coefficient for the porosity flux. ρ^S is a constant mass density of the skeleton in the reference configuration B.

We consider processes near the thermodynamical equilibrium which means that the sources are given by the following relations

$$\hat{\mathbf{p}} = \pi\left(\mathbf{x}'^{F} - \mathbf{x}'^{S}\right),\ \hat{n} = -\frac{\Delta}{\tau},\ \Delta := n - n_E,\ \pi = \pi_0 n,\ \pi_0, \tau - \text{const.}, \tag{3}$$

where π is the permeability coefficient, whose dependence on the porosity, in agreement with simple geometrical considerations, has been assumed to be linear, τ denotes the relaxation time of porosity and n_E is the equilibrium value of porosity.

Constitutive relations which we use in this paper describe an isotropic nonlinear elastic skeleton and an ideal fluid component. For the skeleton we choose the simplest relation proposed by Signorini for one-component compressible materials (see: C. TRUESDELL, W. NOLL [9]). It has the form

$$\begin{aligned} &\mathbf{T}^S = \mathbf{T}_0^S + \left[\lambda^S I + c^S II + \tfrac{1}{2}\left(\lambda^S + \mu^S - \tfrac{1}{2}c^S\right)I^2\right]\mathbf{1} + \\ &+ 2\left[\mu^S - \left(\lambda^S + \mu^S + \tfrac{1}{2}c^S\right)I\right]\mathbf{e}^S + 2c^S\mathbf{e}^{S2} + \beta\,\Delta\,\mathbf{1}, \\ &\mathbf{T}^S := J^{S-1}\mathbf{P}^S\mathbf{F}^{ST}, \quad \mathbf{e}^S := \tfrac{1}{2}\left(\mathbf{1} - \mathbf{F}^{S-T}\mathbf{F}^{S-1}\right), \end{aligned} \tag{4}$$

where $\mathbf{T}^S$ is the partial Cauchy stress tensor in the skeleton and $\mathbf{e}^S$ is the Almansi-Hamel deformation tensor for the skeleton. $\mathbf{T}^S_0$ denotes an initial stress which appears in the reference configuration $\mathbf{e}^S$=0. The principal invariants appearing in (4) are defined as follows

$$\begin{aligned} &I := \operatorname{tr}\mathbf{e}^S, \quad II := \tfrac{1}{2}\left(I^2 - \operatorname{tr}\mathbf{e}^{S2}\right), \quad III := \det\mathbf{e}^S, \\ &J^S := \det\mathbf{F}^S = \left(1 - 2I + 4II - 8III\right)^{-\frac{1}{2}}. \end{aligned} \tag{5}$$

The effective material parameters λ^S, μ^S, c^S and β are independent from the deformation. They may depend solely on the porosity n. Most of available experimental data concern material parameters for small deformations and relatively small porosities ($n \cong 0.2 \div 0.3$). For values of the parameter c^S there are no data at all for porous materials. Numerous papers for one-component nonlinear elastic materials appeared in the 40'ies and early 50'ies under the assumption that this third constant vanishes. This assumption was discussed by C. TRUESDELL [11] who showed that this is consistent with general principles of nonlinear elasticity but it may yield some quantitative discrepancies with expected results. For the purpose of this work we accept the simplifying assumption $c^S \equiv 0$.

The dependence of elastic parameters λ^S, μ^S for small deformations on the porosity can be obtained from data on speeds of propagation of acoustic waves (e.g. see Figure 2.21 in T. BOURBIE, O.COUSSY, B. ZINSZNER [10]). These speeds decay almost linearly with growing porosity. Bearing the approximately linear dependence of the mass density on the porosity in mind we obtain cubic dependence for the material constants. This shall be assumed to hold as well in the case considered in this work. We have then

$$\lambda^S = \lambda_0^S\left(1 - n\right)^3, \quad \mu^S = \mu_0^S\left(1 - n\right)^3, \quad c^S = 0. \tag{6}$$

According to our numerical analysis this assumption seems to be reasonable for not too large values of porosity (smaller than app. 0.7). We return later to this point.

Simultaneously, in the previous works on the model (e.g. K. WILMANSKI [2]) it has been argued that the coupling parameter β is a combination of material parameters independent of porosity.

Now we turn the attention to the partial stress tensor in the fluid. As shown in [2,3,8] for processes near thermodynamic equilibrium the Cauchy stress $\mathbf{T}^F$ in the fluid reduces to the pressure. We have

$$\begin{aligned} &\mathbf{T}^F = -p^F \mathbf{1}, \quad p^F = \wp^F(\rho_t^F, n) + \beta\Delta, \quad \rho_t^F := J^{S-1}\rho^F, \\ &\mathbf{T}^F := J^{S-1}\mathbf{P}^F\mathbf{F}^{ST}. \end{aligned} \tag{7}$$

The relation for the intrinsic pressure $\wp^F$ specifies the material of the fluid component. For the purpose of this work we assume that it is a mixture of waterlike fluid with some amount of bubbles of vapour. In such a case the macroscopic compressibility is not very large and we can choose the linear constitutive relation

$$\wp^F = p_0^F + U^{F2}\left(J^{S-1}\rho^F - \rho_0^F\right), \quad U^F = \text{const.} \tag{8}$$

The constants $p^F{}_0$ and $\rho^F{}_0$ denote the pressure and the mass density of the fluid in the reference configuration $\mathbf{e}^S$=0. The constant U^F is the speed of propagation of the P2-wave. In the above quoted Figure from [10] it can be seen that this speed is almost independent of porosity if the porosity is larger than app. 0.4. We make this assumption for processes considered in this paper.

In order to complete the constitutive relations for the balance equations (2) we have to specify the transport coefficient for porosity Φ_0. It was shown in paper [2] that thermodynamic restrictions yield the following form of this relation

$$\Phi_0 = J^S n_E. \tag{9}$$

This completes the specification of field equations for the fields (1).

3. Boundary conditions

Porous materials with permeable boundaries yield free boundary value problems which require additional conditions on boundaries. This problem was recognized already by K. VON TERZAGHI and O. K. FRÖHLICH [4] who constructed a Gedanken-experiment showing that an a priori division of boundary loading on such a boundary is not possible.

Consequently it was necessary to construct a nondynamical boundary condition connected with a flow through the boundary. Such a proposition was made in paper [1] in which the amount of mass flowing through the boundary, defined by the boundary of the skeleton, was assumed to be determined by the pressure jump in the fluid component. On the one hand side, this condition seems to follow from the existence of boundary layers in a way similar to boundary conditions of the third kind for the heat conduction equation (see: B. ALBERS [7]). On the other hand, numerical simulations for porous materials seem to confirm as well such conditions in purely macroscopic considerations, as the work of W. KEMPA [5] clearly shows.

For the two-component porous materials considered in this work the two vector boundary conditions on the boundary of the skeleton in the current configuration have the form

$$\begin{aligned}
&\mathbf{T}^S\mathbf{n} - p^F\mathbf{n} = \mathbf{t}_{ext}, \quad \rho_t^F\left(\mathbf{v}^F - \mathbf{v}^S\right)\cdot\mathbf{n} = \alpha\left(p^F - np_{ext}\right),\\
&\left(\mathbf{v}^F - \mathbf{v}^F\cdot\mathbf{nn}\right) - \left(\mathbf{v}^S - \mathbf{v}^S\cdot\mathbf{nn}\right) = 0, \qquad \text{on} \quad \partial B_t := \chi^S\left(\partial B, t\right),\\
&\mathbf{v}^F := \mathbf{x}'^F\left(\chi^{S-1}\left(\mathbf{x}, t\right), t\right), \quad \mathbf{v}^S := \mathbf{x}'^S\left(\chi^{S-1}\left(\mathbf{x}, t\right), t\right),
\end{aligned} \tag{10}$$

provided the flow $\rho^F_t(\mathbf{v}^F\text{-}\mathbf{v}^S)\cdot\mathbf{n}$ through the boundary is small enough. Otherwise there is a kinematical contribution to the condition $(10)_1$. In these conditions $\mathbf{t}_{ext}$ denotes the total loading of the boundary and the pressure p_{ext} must be prescribed additionally. This is the extension of boundary conditions caused by the fact that the boundary is not material with respect to the fluid component.

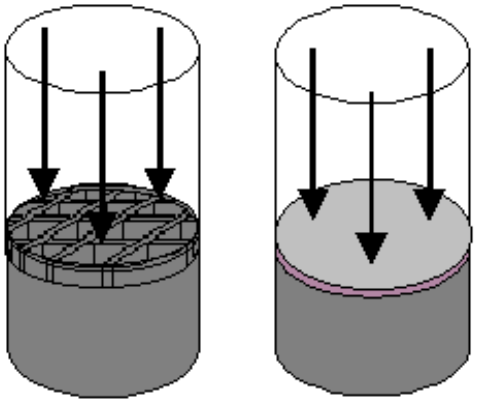

Fig.1: *Two ways of loading a permeable boundary:*

left: the external load is transmitted on both components with the external pressure different from the negative normal component of the loading
right: the external loading is transmitted on both components through a layer of the out-flowing fluid

In some cases the external pressure can be identified with the negative normal component of $\mathbf{t}_{ext}$. This is the case in the example discussed in the present work because the external load is transmitted on the porous material by the fluid appearing outside of the domain B_t (see: Figure 1). Then the conditions (10) can be written in the form

$$\begin{aligned}
&\mathbf{T}^S\mathbf{n} - \frac{1}{\alpha}\rho_t^F\left(\mathbf{v}^F - \mathbf{v}^S\right)\cdot\mathbf{nn} = (1-n)\mathbf{n}\cdot\mathbf{t}_{ext}\mathbf{n} + \left(\mathbf{t}_{ext} - \mathbf{n}\cdot\mathbf{t}_{ext}\mathbf{n}\right),\\
&p^F - \frac{1}{\alpha}\rho_t^F\left(\mathbf{v}^F - \mathbf{v}^S\right)\cdot\mathbf{n} = -n\mathbf{n}\cdot\mathbf{t}_{ext},\\
&\mathbf{v}^F - \mathbf{v}^F\cdot\mathbf{nn} = \mathbf{v}^S - \mathbf{v}^S\cdot\mathbf{nn}.
\end{aligned} \tag{11}$$

In this form it is obvious that the boundary values of the normal component of tractions in the skeleton $\mathbf{n}\cdot\mathbf{T}^S\mathbf{n}$ and the partial pressure p^F on the permeable boundary are not specified by the normal external loading $\mathbf{t}_{ext}\cdot\mathbf{n}$.

The coefficient α characterizes the surface porosity. Not much is known about its values and one of the aims of this work is to check its order of magnitude on simple examples. If it goes to infinity the above conditions reduce to the form characterisitic, for instance, for composites where the external load is divided between components

proportionally to the volume contributions of components. On the other hand, if it is zero then the external load is taken over by both components according to condition $(10)_1$ and, in addition, the second condition is purely kinematical (equal velocities of both components). This is characteristic for impermeable boundaries.

Let us mention that the kinematical condition for the tangential component of the velocity $\mathbf{v}^F$ would have to be modified if the fluid component was viscous.

The Lagrangian form of the conditions $(10)_{1,2}$ which we use in this work is as follows

$$\begin{aligned} &\mathbf{P}^S\mathbf{N} - J^S p^F \mathbf{F}^{S-T}\mathbf{N} = J^S \left(\mathbf{C}^{S-1}\cdot(\mathbf{N}\otimes\mathbf{N})\right)^{\frac{1}{2}} \mathbf{t}_{ext}, \\ &\rho^F \mathbf{X}'^F \cdot \mathbf{N} = \alpha J^S \left(\mathbf{C}^{S-1}\cdot(\mathbf{N}\otimes\mathbf{N})\right)^{\frac{1}{2}} \left(p^F - n p_{ext}\right), \qquad \text{on } \partial B. \end{aligned} \tag{12}$$

Finally let us mention that we do not have to specify a boundary condition for the porosity in the case of stationary processes considered in this work. Namely in such a case the changes of porosity Δ are determined by the algebraic relation following from the porosity equation $(2)_4$

$$\Delta = -\tau \mathrm{Div}\left(\Phi_0 \mathbf{X}'^F\right). \tag{13}$$

Substitution of this relation in the field equations does not increase the order of the operator and, consequently, the two vector boundary conditions (10) yield the solution of the boundary value problem. We illustrate this property further in this paper.

4. Fields and field equations for axisymmetric stationary flow

We consider a cylinder of the radii A and B, A<B loaded by an equal pressure p_0 on the internal and external lateral surfaces. This is the reference configuration. In this configuration the partial mass density of the fluid is $\rho^F{}_0$, the porosity is equal to n_E. The partial stress tensor in the skeleton and the partial pressure in the fluid are given by the relations

$$\mathbf{T}_0^S = -(1-n_E)p_0\mathbf{1}, \quad p_0^F = \hat{p}^F(\rho_0^F, n_E) = n_E p_0. \tag{14}$$

It is assumed that the cylinder is long enough for processes to be independent of the variable z in the axial direction. Then we have plane strains in the skeleton, the flow of the fluid must be radial and, except of a trivial dependence of the position vector of the skeleton on z, all fields depend only on the radius R. We consider a stationary process caused by the following change of the external loading of the cylinder

$$\mathbf{t}_{ext}(r=a) = -p_a\mathbf{n}, \quad \mathbf{t}_{ext}(r=b) = -p_b\mathbf{n}, \quad p_a \equiv p_0, \tag{15}$$

where a is the internal radius and b is the external radius of the cylinder, both after deformation i.e. in the current configuration.

Under the above conditions we seek a time-independent solution for the function of motion of the skeleton χ^S, the velocity of fluid $\mathbf{x}'^F$, the mass density of fluid ρ^F and the porosity n. We choose a natural reference system in which the velocity of the skeleton $\mathbf{x}'^S$ vanishes identically. In the Lagrangian description we have then

$$\mathbf{x} = \Lambda^S(R)R\mathbf{g}_1 + z\mathbf{g}_3, \quad \mathbf{x}'^F = v(R)\mathbf{g}_1, \quad \rho^F = \rho^F(R), \quad n = n(R), \quad A \le R \le B, \tag{16}$$

where Λ^S is the stretch in the radial direction and $\mathbf{g}_1$, $\mathbf{g}_3$ denote the (unit) base vectors of Eulerian cylindrical coordinates in the radial and axial directions, respectively. Consequently, the problem reduces to the four unknown fields: ρ^F, Λ^S, v and Δ as functions of the radius R.

The deformation gradient $\mathbf{F}^S$ in the Lagrangian cylindrical coordinates is as follows

$$\mathbf{F}^S = \frac{d(\Lambda^S R)}{dR}\mathbf{g}_1 \otimes \mathbf{G}^1 + \mathbf{g}_2 \otimes \mathbf{G}^2 + \mathbf{g}_3 \otimes \mathbf{G}^3, \tag{17}$$

where the metric tensors of Lagrangian and Eulerian cylindrical coordinates are as usual

$$(g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\Lambda^S R)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (G_{AB}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$g_{\alpha\beta} := \mathbf{g}_\alpha \cdot \mathbf{g}_\beta, \quad G_{AB} := \mathbf{G}_A \cdot \mathbf{G}_B, \quad \alpha,\beta = 1,2,3, \quad A,B = 1,2,3. \tag{18}$$

The Almansi-Hamel deformation tensor of the skeleton $\mathbf{e}^S$ and its eigenvalues can be easily calculated and we obtain

$$\mathbf{e}^S = \lambda_e^{(1)}\mathbf{g}_1 \otimes \mathbf{g}_1 + \frac{\lambda_e^{(2)}}{\Lambda^{S2}R^2}\mathbf{g}_2 \otimes \mathbf{g}_2,$$
$$\lambda_e^{(1)} = \frac{1}{2}\left(1 - \left(\frac{d\Lambda^S R}{dR}\right)^{-2}\right), \quad \lambda_e^{(2)} = \frac{1}{2}\frac{\Lambda^{S2} - 1}{\Lambda^{S2}}, \quad \lambda_e^{(3)} = 0. \tag{19}$$

Consequently

$$J^S = \Lambda^S \frac{d\Lambda^S R}{dR}. \tag{20}$$

By means of these quantities the constitutive relations and, consequently, the field equations can be written in explicit form. The balance equations (2) reduce to the following set

$$\frac{d}{dR}(R\rho^F W) = 0, \quad W := \mathbf{X}'^F \cdot \mathbf{G}^1 \equiv \left(\frac{d\Lambda^S R}{dR}\right)^{-1} v,$$
$$\rho^F W \frac{dv}{dR} = -\Lambda^S \frac{dp^F}{dR} - \pi v,$$
$$\Lambda^S \frac{dT^{S11}}{dR} + \frac{1}{R}\frac{d\Lambda^S R}{dR}\left(T^{S11} - (\Lambda^S R)^2 T^{S22}\right) + \pi v = 0, \tag{21}$$
$$\Delta = -n_E \tau \frac{1}{R}\frac{d}{dR}\left(\Lambda^S R \frac{d\Lambda^S R}{dR} W\right) \equiv -n_E \tau \frac{1}{R}\frac{d}{dR}(\Lambda^S R v).$$

The pressure p^F and the stress components T^{S11} and T^{S22} are given by the constitutive relations (8) and (4), respectively. The last equation (21) substituted in these constitutive relations yields a dependence of partial stresses on the gradient of velocity. This effect is similar to an effect caused by the bulk viscosity. Consequently, in spite of a leading elastic part of the stress tensor in the skeleton and a pressure similar to this in ideal fluids in the partial stress of the fluid component, the interaction of components through changes of porosity yields a dissipation.

It remains to formulate the boundary conditions for the example considered in this section.

We have

$$\begin{aligned} &T^{S11} - p^F\big|_{R=A} = -p_a, \quad \rho^F W\big|_{R=A} = \alpha\Lambda^S\big|_{R=A}\left(p^F\big|_{R=A} - n\big|_{R=A}\, p_a\right), \\ &T^{S11} - p^F\big|_{R=B} = -p_b, \quad \rho^F W\big|_{R=B} = \alpha\Lambda^S\big|_{R=B}\left(p^F\big|_{R=B} - n\big|_{R=B}\, p_b\right) \end{aligned} \tag{22}$$

The set (21) of four ordinary differential equations is highly nonlinear and cannot be solved analytically. In the next section we present a perturbation method of solution which applies in the case of a small difference between boundary pressures p_a and p_b.

5. Regular perturbation solution

Now we make the assumption that the pressure difference between the lateral surfaces of the cylinder is small. Under this assumption we construct the solution in the form of a power series with respect to the small parameter defined by the pressure difference. Namely

$$\begin{aligned} &\varepsilon := \frac{p_b - p_a}{p_a}, \quad |\varepsilon| << 1 \quad \Rightarrow \\ \Rightarrow \quad &\rho^F = \rho_0^F + \sum_{n=1}^{N} \rho_n(R)\varepsilon^n, \quad W = \sum_{n=1}^{N} W_n(R)\varepsilon^n, \\ &\Delta = \sum_{n=1}^{N} \Delta_n(R)\varepsilon^n, \quad \Lambda^S = 1 + \sum_{n=1}^{N} \Lambda_n(R)\varepsilon^n. \end{aligned} \tag{23}$$

The form of these expansions follows from the fact that the state $p_a=p_b$, i.e. $\varepsilon=0$ is the reference state in which $\rho^F=\rho^F{}_0$, $W=0$, $\Lambda^S=1$ and $\Delta=0$.

We perform calculations up to the second order terms, i.e. N=2. In such a case the boundary value problem formulated in section 4 can be solved analytically. Since the relations which follow for the coefficients in the above series are rather lengthy we present solely some representative examples. Full results will be used in the construction of a numerical example which we present in the next section.

First let us make an important observation. According to equation $(21)_1$ we have in the first order approximation

$$\frac{d}{dR}\left(R\rho_0^F W_1\right) = 0 \quad \Rightarrow \quad W_1 = \frac{A_1}{\rho_0^F R}. \tag{24}$$

Substitution of this result in the first order approximation of equation $(21)_4$ yields immediately

$$\Delta_1 = 0. \tag{25}$$

Consequently, changes of porosity in stationary processes must be much smaller than normalized changes of all other fields. They are made even smaller by the presence of the relaxation time τ in the formula $(21)_4$. This is the case even for large deformations considered in this paper for which this relaxation time is rather large. Solely in nonstationary processes and, in particular, under the presence of mass sources (e.g. due to adsorption or chemical reactions such as combustion of granular materials) we can expect the changes of porosity to be considerable after a long time of duration of the process. This important observation justifies soil engineering calculations based on the assumption of constant porosity.

After simple manipulations the remaining two equations in the first order approximation can be written in the form

$$\begin{aligned}
&\frac{d^2\Lambda_1}{dR^2}+\frac{3}{R}\frac{d\Lambda_1}{dR}=-\pi_0 n_E\frac{A_1}{\rho_0^F\left(\lambda_0^S+2\mu_0^S\right)\left(1-n_E\right)^3}\frac{1}{R^2},\\
&\frac{d\rho_1}{dR}=-\pi_0 n_E\frac{A_1}{\rho_0^F U^{F2}}\left[1+\frac{\rho_0^F U^{F2}}{\left(\lambda_0^S+2\mu_0^S\right)\left(1-n_E\right)^3}\right]\frac{1}{R},
\end{aligned} \tag{26}$$

i.e.

$$\begin{aligned}
&\Lambda_1=B_1+\frac{C_1}{R^2}-\frac{\pi_0 A_1}{2\rho_0^F\left(\lambda_0^S+2\mu_0^S\right)\left(1-n_E\right)^3}\left(\ln R-\frac{1}{2}\right),\\
&\rho_1=D_1-\pi_0 n_E\frac{A_1}{\rho_0^F U^{F2}}\left[1+\frac{\rho_0^F U^{F2}}{\left(\lambda_0^S+2\mu_0^S\right)\left(1-n_E\right)^3}\right]\ln R.
\end{aligned} \tag{27}$$

Constants of integration A_1, B_1, C_1 and D_1 must be found from the boundary conditions. We proceed to formulate them.

Substitution of the constitutive relations for partial stresses (4) and (7) in the relations $(22)_{1,3}$ yields in the first order approximation

$$\begin{aligned}
&\rho_1 U^{F2}-\rho_0^F U^{F2}\left[1+\frac{\left(\lambda_0^S+2\mu_0^S\right)\left(1-n_E\right)^3}{\rho_0^F U^{F2}}\right]\left(R\frac{d\Lambda_1}{dR}+2\Lambda_1\right)+\\
&\qquad+2\mu_0^S\left(1-n_E\right)^3\Lambda_1=\begin{cases}0 & \text{for} \quad R=A,\\ p_a & \text{for} \quad R=B.\end{cases}
\end{aligned} \tag{28}$$

Simultaneously, the kinematical conditions $(22)_{2,4}$ in this order are as follows

$$\rho_0^F W_1 = -\alpha U^{F2}\left[\rho_1 - \rho_0^F\left(R\frac{d\Lambda_1}{dR} + 2\Lambda_1\right)\right] \quad \text{for} \quad R = A,$$
$$\rho_0^F W_1 = \alpha\left\{U^{F2}\left[\rho_1 - \rho_0^F\left(R\frac{d\Lambda_1}{dR} + 2\Lambda_1\right)\right] - n_E p_a\right\} \quad \text{for} \quad R = B. \tag{29}$$

Consequently we obtain four relations for the constants of integrations and the solution in the first order approximation can be fully constructed. We shall not quote this mathematically simple result in this work and proceed to analyse the second order approximation.

It is very easy to construct the mass balance equation in the second order. We obtain

$$\frac{dW_2 R}{dR} = -\frac{1}{\rho_0^F}\frac{d}{dR}\left(R\rho_1 W_1\right) \quad \Rightarrow \quad W_2 = A_2 - \frac{\rho_1}{\rho_0^F}W_1. \tag{30}$$

Certainly, the right hand side is a function of R known form the first order approximation up to the constant A_2.

The porosity equation yields in the second order

$$\frac{\Delta_2}{2\tau n_E} = -\frac{1}{R}\frac{dW_2 R}{dR} - \left\{\frac{d^2\Lambda_1}{dR^2}W_1 R + \frac{d\Lambda_1}{dR}\left(R\frac{dW_1}{dR} + 4W_1\right) + 2\Lambda_1\frac{1}{R}\frac{dW_1 R}{dR}\right\}. \tag{31}$$

Hence, once we know W_2 the changes of porosity in the second approximation can be calculated from this algebraic relation.

We will not quote the momentum balance equations and boundary conditions in the second order approximation. They follow in the same manner as for the first order and they can be solved analytically. However their form is rather involved and it is easier to appreciate their properties on a numerical example which we present in the next section.

6. Numerical example

We illustrate the above presented perturbation solution of a cylindrical filtration problem by a numerical example of a porous rubberlike material with flow of a waterlike fluid. We choose the following material parameters

$$\lambda_0^S = 20\,\text{MPa}, \quad \mu_0^S = 1\,\text{MPa}, \quad c^S = 0, \quad U^F = 750\frac{\text{m}}{\text{s}}. \tag{32}$$

These are the parameters for the solid component as if n_E=0, i.e the material is slightly compressible (Poisson's number ~0.48). The speed of sound U^F in the fluid component corresponds to a mixture of water with a certain amount of vapour bubbles. This mixture has been chosen to expose better effects of large deformations of the skeleton without an influence of support which would appear in the case of an incompressible real fluid component.

The other material parameters are chosen as follows

$$\rho_0^F = 0.5\times 10^3 \frac{kg}{m^3}, \quad n_E = 0.5, \quad \beta = 10^4\,Pa, \quad \tau = 10^{-2}\,s,$$
$$p_a = 0.2MPa, \quad p_b = 0.15MPa. \tag{33}$$

The initial porosity has been chosen to be higher than usual values for rocks or granular materials. However it is still lower than porosities of sponges. Even higher values of porosity seem to be connected with some instabilities of the model which we indicate in the sequel.

The new material parameters β, τ have been chosen on the basis of estimates of the attenuation of acoustic waves. We do not present these estimates in the work as it is solely the order of magnitude of those parameters which bears the hand in the present analysis.

In order to expose the influence of pearmability on the flow through the cylinder we vary the material coefficients π_0 and α. The first one describes, of course, the resistance of the skeleton to the flow of fluid in the interior of the body. On the other hand, the coefficient α describes the surface resistance to the outflow of the fluid from the skeleton. Its physical character is different from π_0 because its appearance is connected with a boundary layer in the transition region between the porous body and the external world.

Let us notice that the value of the parameter ε=-0.25 is not small enough to obtain a quantitative agreement of the approximate solution with an exact solution of the problem. The error is of the order of magnitude of 2-6%. We accept it because qualitative properties of the approximation seem to correspond exactly with those of the exact solution.

Below in the sequence of Figures we present various fields as functions of radius R for two values of pearmability π_0: 10^4 and 10^6 kg/m^{3}s and two values of coefficient α: 10^{-2} and 10^{-4} s/m. They have an order of magnitude one would expect in the case of sponges or some biological tissues. Let us notice that smaller values of π_0 correspond to smaller resistance of the skeleton (see: momentum balance for the fluid) and larger values of α have the similar influence to smaller values of π_0 (see: remarks after the formula (11)).

Figure 2 shows the behaviour of the radial velocity $v \equiv x'^{F1}$ of the fluid as a function of the radius R. Clearly, the maximum value of this velocity (app. 0.28 m/s for the internal radius) corresponds to $\pi_0=10^4$ and $\alpha=10^{-2}$. It is a rather large value due to large deformations of the skeleton. The minimum value of this velocity (app. 0.0018 m/s for the outer radius) corresponds to $\pi_0=10^6$ and $\alpha=10^{-4}$.

In Figure 3 we show changes of the current mass density of the fluid $\rho^F_t=\rho^F J^{S-1}$. In spite of large deformations of the skeleton these changes are very little indeed.

Figure 4 demonstrates the behaviour of the pressure in the fluid. Clearly the changes of pressure depend on the choice of both permeability parameters. However in all cases we see that the values of pressure on the boundaries are smaller from the contribution proportional to the porosity as it would be the case for composites (lack of diffusion!) and as it is sometimes wrongly assumed in papers on porous materials. This observation confirms the results of the Gedankenexperiment of Terzaghi and follows from our choice of boundary conditions.

$\alpha = 10^{-4} \left[\frac{s}{m}\right]$ $\alpha = 10^{-2} \left[\frac{s}{m}\right]$

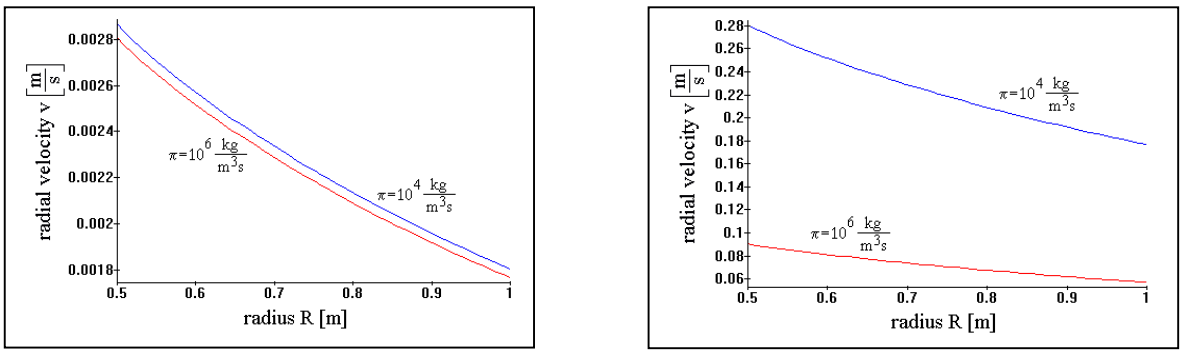

Fig.2: *Radial velocity of the fluid* $v \equiv x'^{F1}$

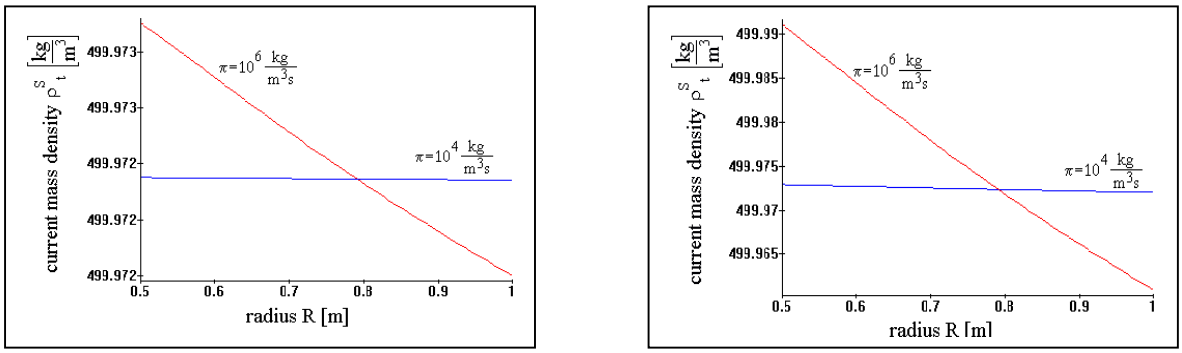

Fig.3: *Current mass density of the fluid* ρ^F_t

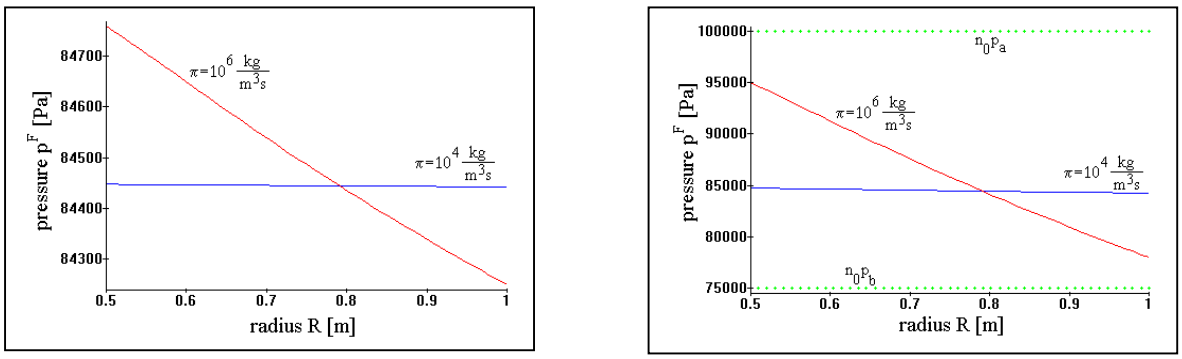

Fig.4: *Pressure* p^F *in the fluid*

$\alpha = 10^{-4} \left[\frac{s}{m}\right]$ $\alpha = 10^{-2} \left[\frac{s}{m}\right]$

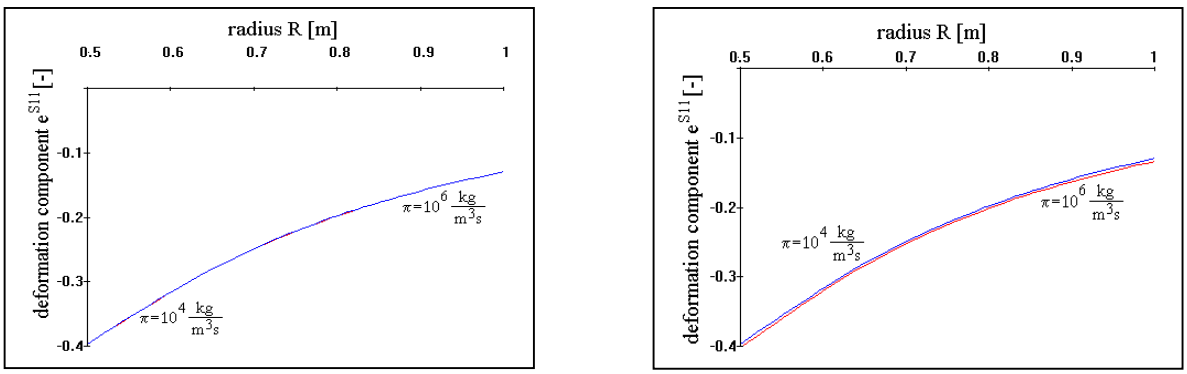

Fig.5: *Radial deformation component* e^{S11} *of the skeleton*

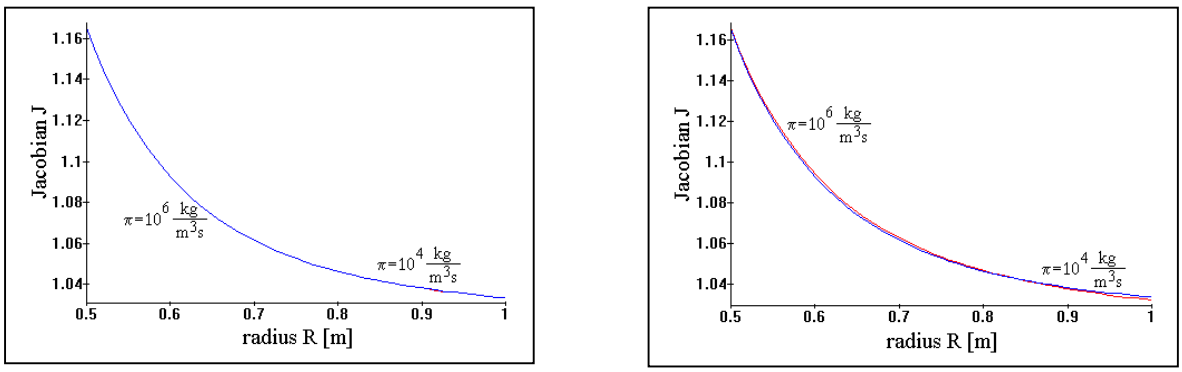

Fig.6: *Volume changes* J^S *of the skeleton*

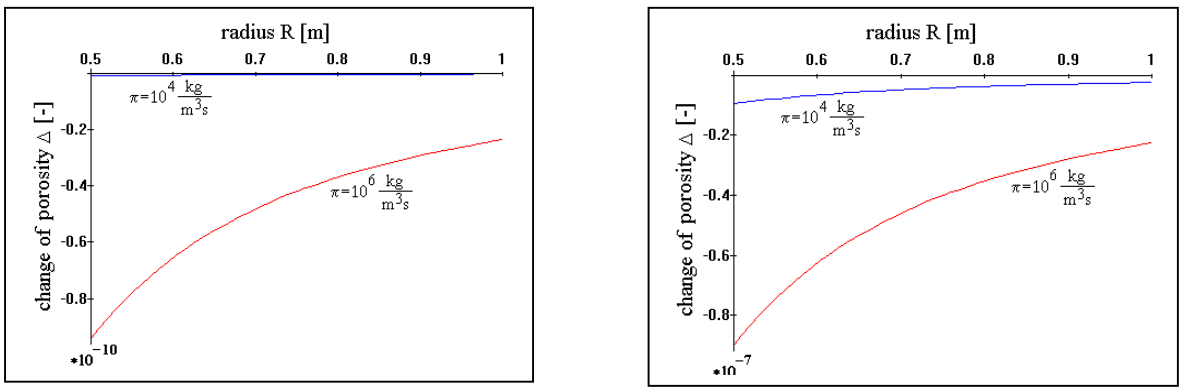

Fig.7: *Changes of the porosity* $\Delta=n-n_E$

In Figure 5 we present the component e^{S11} of the Almansi-Hamel deformation tensor. Even though the dependence on R is qualitatively similar to the deformation obtained in the classical Lamé problem of the linear elasticity theory the values of deformation are much larger than these of the classical elasticity theory and reach the value 40%. The radii of the cylinder grow with the deformation: a>A, b>B and the material is in tension, i.e. $J^S>1$. The last property is demonstrated in Fig.6.

Finally in Figure 7 we present the changes of porosity. It is obvious that, in spite of large deformations of the skeleton, the changes of porosity are very small indeed. They are bigger if the compressibility of the fluid is bigger. This effect could be expected on the account of the physical interpretation of interactions between skeleton and fluid.

7. Final remarks

Analytical results of this paper show that the extension of the model by adding the field of porosity with its own balance equation as well as the new formulation of boundary conditions for two-component porous materials yield qualitatively reasonable results in a rather extreme situation of large deformations. Changes of porosity are small but their influence on the coupling between components is magnified by the static coupling constant β. It means that the porosity itself as a measure of changes of microstructural geometry may be considered to be almost constant but it yields changes in behaviour of other fields.

In some calculations which are not presented in the paper we have found out that higher values of initial porosity may lead to some instabilities of the model which we could not explain. Such a peculiar behaviour seems to appear in the vicinity of the point of inversion of sound velocities. Namely, for sufficiently large values of n_E the speed of the P2-wave, carried primarily by the fluid component becomes larger than the speed of the P1-wave, carried primarily by the skeleton. This is connected with very large deformations of the skeleton. For instance, for n_E=0.6 the maximum value of the component e^{S11} exceeds 100%. One can also observe considerable changes in the behaviour of other fields. Neither the constitutive relations for partial stresses nor the method of regular perturbations applied in this work are suitable for a reliable analysis of this problem. Therefore we are not able to present an explanation of these findings in this work.

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