One-Particle Stochastic Lagrangian Model for Turbulent Dispersion in Horizontally Homogeneous Turbulence

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The paper was partially written while the authors were visiting WIAS, Berlin

Abstracts. A one-particle stochastic Lagrangian model in 2D and 3D dimensions is constructed for transport of particles in horizontaly homogeneous turbulent flows with arbitrary one-point probability density function. It is shown that in the case of anisotropic turbulence with gaussian pdf, this model essentially differs from the known Thomson's model. The results of calculations according to our model in the case of neutrally stratified atmospheric surface layer agree satisfactorily with the measurements known from the literature.
Mathematics Subject Classification. 65C05, 76F99, 65C20,
Key words and phrases. Stochastic models of turbulence, Generalized Langevin stochastic differential equation, Eulerian and Lagrangian one- and two-particle stochastic models of turbulent transport, particle dispersion in the surface layer of the turbulent atmosphere.

1 Introduction

This paper deals with one-particle stochastic Lagrangian models for 2D and 3D turbulent transport. Here we treat the fully developed turbulence (i.e., a flow with very high Reynolds number) as a random velocity field (u, v, w) which is assumed to be incompressible. Therefore, the trajectories of particles in such flows are stochastic processes. To simulate these stochastic processes, two different approaches are known in the literature. The first one is based on the numerical solution of the system of random equations

$$\begin{aligned} \frac{\partial X}{\partial t} &= u(X, Y, Z, t), \\ \frac{\partial Y}{\partial t} &= v(X, Y, Z, t), \\ \frac{\partial Z}{\partial t} &= w(X, Y, Z, t). \end{aligned} \tag{1.1}$$

Here $X(t), Y(t), Z(t)$ are the coordinates of the Lagrangian trajectory at the time t. The random fields u, v, w are simulated by Monte Carlo methods (e.g., see [2], [4], [5], [8], [9], [13]), and the random trajectories are then obtained by numerical solution of (1.1) with the relevant initial data.

In the second approach the true trajectory $X(t), Y(t), Z(t)$ is assumed to be approximated by a model trajectory $\hat{X}(t), \hat{Y}(t), \hat{Z}(t)$, a solution to a stochastic differential equation of Ito type (e.g., see [10], [12] and the list of references in these papers):

$$\begin{aligned} d\hat{X} &= \hat{U}dt, \quad d\hat{Y} = \hat{V}dt, \quad d\hat{Z} = \hat{W}dt, \\ d\hat{U} &= a_u(t, \hat{X}, \hat{Y}, \hat{Z}, \hat{U}, \hat{V}, \hat{W})dt + b_u(t, \hat{X}, \hat{Y}, \hat{Z}, \hat{U}, \hat{V}, \hat{W})\, dB_u(t), \\ d\hat{V} &= a_v(t, \hat{X}, \hat{Y}, \hat{Z}, \hat{U}, \hat{V}, \hat{W})dt + b_v(t, \hat{X}, \hat{Y}, \hat{Z}, \hat{U}, \hat{V}, \hat{W})\, dB_v(t), \\ d\hat{W} &= a_w(t, \hat{X}, \hat{Y}, \hat{Z}, \hat{U}, \hat{V}, \hat{W})dt + b_w(t, \hat{X}, \hat{Y}, \hat{Z}, \hat{U}, \hat{V}, \hat{W})\, dB_w(t). \end{aligned} \tag{1.2}$$

Here $\hat{U}, \hat{V}, \hat{W}$ are the components of the model Lagrangian velocity, $B_u(t), B_v(t), B_w(t)$ are three standard independent Wiener processes.

Ideally, one would have an approximation such that the true and the model Lagrangian velocities coincide:

$$\begin{aligned} \hat{U}(t) &= u(\hat{X}(t), \hat{Y}(t), \hat{Z}(t), t), \\ \hat{V}(t) &= v(\hat{X}(t), \hat{Y}(t), \hat{Z}(t), t), \\ \hat{W}(t) &= w(\hat{X}(t), \hat{Y}(t), \hat{Z}(t), t), \end{aligned} \tag{1.3}$$

which would assure that the true and the model trajectories are the same. However it is unrealistic to satisfy (1.3), therefore one uses different consistency principles. Namely, the general consistency principle says that the statistics of the model process $\hat{X}(t), \hat{Y}(t), \hat{Z}(t),\ \hat{U}(t), \hat{V}(t), \hat{W}(t)$ satisfies the same relations which are satisfied by the true process $X(t), Y(t), Z(t),\ \ U(t),\ V(t),\ W(t),$ where $U(t) = u(X(t), Y(t), Z(t), t)$, $V(t) = v(X(t), Y(t), Z(t), t)$, $W(t) = w(X(t), Y(t), Z(t), t)$ are the components of the true Lagrangian velocity.

Two consistency criteria used in the literature are:

(A) Consistency with the Kolmogorov similarity theory,
(B) The Novikov integral relation.

Here (A) reads

$$\langle (dU)^2 \rangle = \langle (dV)^2 \rangle = \langle (dW)^2 \rangle = C_0 \varepsilon dt,$$

and

$$\langle dU\, dV \rangle = \langle dU\, dW \rangle = \langle dW\, dV \rangle = 0,$$

where dU, dV, dW are the components of the increments of the Lagrangian velocity, ε is the mean rate of the dissipation of turbulence energy, C_0 is the universal constant (e.g., see [6], [10], [12]); here and in what follows, the angle brackets stand for the ensemble average over the samples of the random velocity field.

Note that (A) implies (e.g., see [12]) that in (1.2), all the terms b_u, b_v, b_w are equal to $\sqrt{C_0 \varepsilon}$:

$$b_u = b_v = b_w = \sqrt{C_0 \varepsilon}. \tag{1.4}$$

Novikov's integral relation has the form [7]

$$p_E(u, v, w; x, y, z, t) = \int\limits_{R^3} p_L(x, y, z, u, v, w; x_0, y_0, z_0, t) dx_0 dy_0 dz_0. \tag{1.5}$$

Here p_E is the probability density function (pdf) of the Eulerian velocity u, v, w, in the fixed point x, y, z, at the time t, and p_L is the joint pdf of the true Lagrangian phase point X, Y, Z, U, V, W defined by the trajecory started at x_0, y_0, z_0.

Thus the consistency with the Novikov relation (1.5) means that the pdf of the model phase point governed by (1.2), say $\hat{p}_L$, satisfies

$$p_E(u, v, w; x, y, z, t) = \int\limits_{R^3} \hat{p}_L(x, y, z, u, v, w; x_0, y_0, z_0, t) dx_0 dy_0 dz_0. \tag{1.6}$$

Note that (1.6), the Focker-Planck-Kolmogorov equation for $\hat{p}_L$ and (1.4) lead to the well-mixed condition due to D. Thomson [12]:

$$\begin{aligned} \frac{\partial p_E}{\partial t} &+ u\frac{\partial p_E}{\partial x} + v\frac{\partial p_E}{\partial y} + w\frac{\partial p_E}{\partial z} + \frac{\partial}{\partial u}(a_u p_E) + \frac{\partial}{\partial v}(a_v p_E) + \frac{\partial}{\partial w}(a_w p_E) \\ &= \frac{C_0 \varepsilon}{2} \left\{ \frac{\partial^2 p_E}{\partial u^2} + \frac{\partial^2 p_E}{\partial v^2} + \frac{\partial^2 p_E}{\partial w^2} \right\}. \end{aligned} \tag{1.7}$$

In this paper we study a horizontally homogeneous turbulent flow which implies that p_E does not depend on x, y. Therefore, in the left-hand side of (1.7) the second and third terms vanish. Here the main problem is that (1.7) does not define the coefficients a_u, a_v and a_w of the model (1.2) uniquely. Indeed, even for the homogeneous turbulence, in [11] two different choices of a_u, a_v, a_w are presented both satisfying the well-mixed condition (1.7) but whose statistical characteristics are different. For the gaussian form of p_E, one of appropriate technique of getting the coefficients a_u, a_v, a_w is given in [12]. In the nongaussian 3D case, to the authors knowledge, there is no appropriate choice of these coefficients. In 2D, the nongaussian case was treated by Flesch and Wilson in [3]. These authors mentioned that the two different models do not lead to essentially different results in the case of gaussian p_E. As reported in [3], the same is true for two models considered in [11].

In this paper we suggest a proper choice of the coefficients a_u, a_v, a_w in a general case of the pdf p_E. Our derivation is based on some assumptions which ensure a unique choice of the model. It should be stressed that even in the gaussian case our model essentially differs, as shown below (Sect.4), from the model given by Thomson [12]. This confirms our opinion that it is necessary, along theoretical studies, to extract additional information from experiments.

2 Choice of the coefficients in (1.2)

Let us formulate the main assumptions about the Lagrangian model of the type (1.2). We consider a horizontally homogeneous incompressible high-Reynolds number turbulent flow in the space R^3. Thus the mean velocity has no vertical component. In addition we assume that the mean velocity is directed along the x-axis. Thus the mean velocity vector is $(\bar{u}(x,y,z,t),0,0)$, while p_E and $\bar{u}$ do not depend on x,y. We will write the pdf p_E in the form

$$p_E(u,v,w;z,t) = p'_E(u',v',w';z,t)$$

where $u' = u - \bar{u}(z,t)$, $v' = v$ and $w' = w$.

By (1.4), the equation (1.2) in these variables has the form:

$$\begin{aligned} d\hat{X} &= (\hat{U}' + \bar{u}(\hat{Z},t))dt,\ d\hat{Y} = \hat{V}'dt,\ d\hat{Z} = \hat{W}'dt, \\ d\hat{U}' &= a'_u(t,\hat{Z},\hat{U}',\hat{V}',\hat{W}')dt + \sqrt{C_0\varepsilon}\, dB_u(t), \\ d\hat{V}' &= a'_v(t,\hat{Z},\hat{U}',\hat{V}',\hat{W}')dt + \sqrt{C_0\varepsilon}\, dB_v(t), \\ d\hat{W}' &= a'_w(t,\hat{Z},\hat{U}',\hat{V}',\hat{W}')dt + \sqrt{C_0\varepsilon}\, dB_w(t). \end{aligned} \tag{2.1}$$

The well mixed condition in new variables is

$$\begin{aligned} &\frac{\partial p'_E}{\partial t} + w'\frac{\partial p'_E}{\partial z} + \frac{\partial}{\partial u'}(a'_u p'_E) + \frac{\partial}{\partial v'}(a'_v p'_E) + \frac{\partial}{\partial w'}(a'_w p'_E) \\ &= \frac{C_0\varepsilon}{2}\left\{\frac{\partial^2 p'_E}{\partial (u')^2} + \frac{\partial^2 p'_E}{\partial (v')^2} + \frac{\partial^2 p'_E}{\partial (w')^2}\right\}. \end{aligned} \tag{2.2}$$

Assumption. *We assume in addition that a'_u does not depend on v' while a'_w does not depend on u',v': $a'_u = a'_u(t,z,u',w')$, $a'_w = a'_w(t,z,w')$.*

Then the model (2.1) reads

$$
\begin{aligned}
d\hat{X} &= (\hat{U}' + \bar{u}(\hat{Z},t))dt,\ d\hat{Y} = \hat{V}'dt,\ d\hat{Z} = \hat{W}'dt,\\
d\hat{U}' &= a'_u(t,\hat{Z},\hat{U}',\hat{W}')dt + \sqrt{C_0\varepsilon}\, dB_u(t),\\
d\hat{V}' &= a'_v(t,\hat{Z},\hat{U}',\hat{V}',\hat{W}')dt + \sqrt{C_0\varepsilon}\, dB_v(t),\\
d\hat{W}' &= a'_w(t,\hat{Z},\hat{W}')dt + \sqrt{C_0\varepsilon}\, dB_w(t).
\end{aligned}
\tag{2.3}
$$

Integrating (2.2) over u' and v' yields

$$
\frac{\partial p'_{1E}}{\partial t} + w'\frac{\partial p'_{1E}}{\partial z} + \frac{\partial}{\partial w'}(a'_w(t,z,w')p'_{1E}) = \frac{C_0\varepsilon}{2}\frac{\partial^2 p'_{1E}}{\partial (w')^2}\,, \tag{2.4}
$$

where

$$
p'_{1E} = p'_{1E}(w';z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p'_E(u',v',w';z,t)\, du'dv'. \tag{2.5}
$$

Here we have assumed that

$$
a'_u p'_E,\ a'_v p'_E,\ \frac{\partial p'_E}{\partial u'},\ \frac{\partial p'_E}{\partial v'} \quad \text{all tend to zero as} \quad (u')^2 + (v')^2 \to \infty.
$$

Similarly, the integration of (2.2) over v' leads to

$$
\begin{aligned}
\frac{\partial p'_{2E}}{\partial t} &+ w'\frac{\partial p'_{2E}}{\partial z} + \frac{\partial}{\partial u'}(a'_u(t,z,u',w')p'_{2E}) + \frac{\partial}{\partial w'}(a'_w(t,z,w')p'_{2E})\\
&= \frac{C_0\varepsilon}{2}\left(\frac{\partial^2 p'_{2E}}{\partial (u')^2} + \frac{\partial^2 p'_{2E}}{\partial (w')^2}\right),
\end{aligned}
\tag{2.6}
$$

where

$$
p'_{2E} = p'_{2E}(u',w';z,t) = \int_{-\infty}^{\infty} p'_E(u',v',w';z,t)\, dv'. \tag{2.7}
$$

Now, under the assumption about the behaviour in the infinity, it is possible to define uniquely the coefficients a'_u, a'_v and a'_w. Indeed, from (2.4) one gets a'_w, then from (2.6) one finds a'_u, and from (2.2) one obtains a'_v. This yields

$$
a'_w(t,z,w) = \frac{1}{p'_{1E}(w;z,t)}\left\{\frac{C_0\varepsilon}{2}\frac{\partial p'_{1E}}{\partial w} - \left(\frac{\partial f_{1E}}{\partial t} + \frac{\partial F_{1E}}{\partial z}\right)\right\}, \tag{2.8}
$$

where

$$
f_{1E}(w;z,t) = \int_{-\infty}^{w} p'_{1E}(w';z,t)\, dw',
$$

$$
F_{1E}(w;z,t) = \int_{-\infty}^{w} w'p'_{1E}(w';z,t)\, dw',
$$

and

$$
a'_u(t,z,u,w) = \frac{1}{p'_{2E}}\left\{\frac{C_0\varepsilon}{2}\left(\frac{\partial p'_{2E}}{\partial u} + \frac{\partial^2 f_{2E}}{\partial w^2}\right) - \left(\frac{\partial f_{2E}}{\partial t} + w\frac{\partial f_{2E}}{\partial z}\right) - \frac{\partial}{\partial w}\Big(a'_w\, f_{2E}\Big)\right\}, \tag{2.9}
$$

where

$$f_{2E}(u,w;z,t)=\int\limits_{-\infty}^{u}p'_{2E}(u',w;z,t)\,du'.$$

Finally,

$$\begin{aligned}a'_v(t,z,u,w)=\frac{1}{p'_E}&\left\{\frac{C_0\varepsilon}{2}\left(\frac{\partial^2 f_E}{\partial u^2}+\frac{\partial p'_E}{\partial v}+\frac{\partial^2 f_E}{\partial w^2}\right)-\left(\frac{\partial f_E}{\partial t}+w\,\frac{\partial f_E}{\partial z}\right)\right.\\&\left.-\ \frac{\partial}{\partial u}\left(a'_u f_E\right)-\frac{\partial}{\partial w}\left(a'_w\, f_E\right)\right\},\end{aligned}\tag{2.10}$$

where

$$f_E(u,v,w;z,t)=\int\limits_{-\infty}^{v}p'_E(u,v',w;z,t)\,dv'.$$

Thus (2.3), with the coefficients (2.8)-(2.10) define a unique stochastic model (2.1) through the pdf p'_E.

Remark 2.1. This model is a natural extension of the one-dimensional (in z direction) Thomson's model [12] in the sense that the vertical coordinates (z,w) are governed in our model by a SDE which coincides with Thomson's model. Note that in Thomson's 3D model (with gaussian p'_E) this is not the case: the statistics of (z,w) in his 3D model essentially differs from that of (z,w) in his one-dimensional model (see Sect.4).

3 2D stochastic model with gaussian pdf

In this section we present concrete expressions for the coefficients in the case of gaussian pdf. We extract the 2D model from 3D model as

$$\begin{aligned}d\hat{X}\ &=\ (\hat{U}'+\bar{u}(\hat{Z},t))dt,\ d\hat{Z}=\hat{W}'dt,\\d\hat{U}'\ &=\ a'_u(t,\hat{Z},\hat{U}',\hat{W}')dt+\sqrt{C_0\varepsilon}\,dB_u(t),\\d\hat{W}'\ &=\ a'_w(t,\hat{Z},\hat{W}')dt+\sqrt{C_0\varepsilon}\,dB_w(t).\end{aligned}\tag{3.1}$$

In the gaussian case,

$$p'_{2E}(u,w;z,t)=\frac{1}{2\pi\sigma_{u/w}\sigma_w}\exp\left\{-\frac{1}{2\sigma^2_{u/w}}(u-\mu)^2-\frac{w^2}{2\sigma^2_w}\right\}\tag{3.2}$$

where

$$\sigma_{u/w}=\frac{\Delta^{1/2}}{\sigma_w},\quad \mu=\frac{\overline{uw}}{\sigma^2_w}\,w,\quad \Delta=\sigma^2_u\sigma^2_w-(\overline{uw})^2,$$

and σ^2_u, σ^2_w are the variances of the x- and z- velocity components, respectively. From (3.2),

$$p'_{1E}(w;z,t)=\frac{1}{\sqrt{2\pi}\sigma_w}\exp\left\{-\frac{w^2}{2\sigma^2_w}\right\},\tag{3.3}$$

then,

$$f_{1E}(w;z,t) = \int_{-\infty}^{\frac{w}{\sigma_w}} \frac{1}{\sqrt{2\pi}} \exp\left(-t^2/2\right) dt = \Phi\left(\frac{w}{\sigma_w}\right),$$

$$F_{1E}(w;z,t) = -\sigma_w^2 p'_{1E}(w;z,t).$$

Note that

$$\frac{1}{p'_{1E}}\frac{\partial p'_{1E}}{\partial w} = -\frac{w}{\sigma_w^2}, \quad -\frac{1}{p'_{1E}}\frac{\partial F_{1E}}{\partial z} = \frac{1}{2}(w^2+1)\frac{\partial \sigma_w^2}{\partial z},$$

and

$$\frac{\partial f_{1E}}{\partial t} = -\frac{w}{\sigma_w^2}\frac{\partial \sigma_w}{\partial t}\dot{\Phi}(w/\sigma_w),$$

where $\dot{\Phi}(\tau) = \frac{d\Phi}{d\tau}$, and

$$\Phi(\tau) = \int_{-\infty}^{\tau} \frac{1}{\sqrt{2\pi}} \exp\left(-t^2/2\right) dt.$$

From (2.8) we find

$$a'_w(t,z,w) = -\left(\frac{C_0\varepsilon}{2\sigma_w^2} - \frac{1}{\sigma_w}\frac{\partial \sigma_w}{\partial t}\right) w + \frac{1}{2}\frac{\partial \sigma_w^2}{\partial z}\left(\frac{w^2}{\sigma_w^2}+1\right). \tag{3.4}$$

Note that this coincides with Thomson's relevant expression in his 1D model [12].

By the definition we have

$$f_{2E}(u,w,z,t) = p'_{1E}(w;z,t)\Phi\left(\frac{u-\mu}{\sigma_{u/w}}\right).$$

To find a'_u from (2.9) we need the expressions for

$$\frac{\partial f_{2E}}{\partial t}, \quad \frac{\partial f_{2E}}{\partial z}, \quad \frac{\partial f_{2E}}{\partial w}, \quad \frac{\partial p'_{2E}}{\partial u}, \quad \frac{\partial^2 f_{2E}}{\partial w^2}.$$

By definition we get

$$\begin{aligned}
\frac{\partial p'_{2E}}{\partial u} &= -\frac{(u-\mu)}{\sigma_{u/w}^2}p'_{2E} \quad \frac{\partial f_{2E}}{\partial t} = f_{2E}\left\{\frac{1}{2\sigma_w^2}\frac{\partial \sigma_w^2}{\partial t}\left(\frac{w^2}{\sigma_w^2}-1\right) + \Psi(\xi)\frac{\partial \xi}{\partial t}\right\}, \\
\frac{\partial f_{2E}}{\partial z} &= f_{2E}\left\{\frac{1}{2\sigma_w^2}\frac{\partial \sigma_w^2}{\partial z}\left(\frac{w^2}{\sigma_w^2}-1\right) + \Psi(\xi)\frac{\partial \xi}{\partial z}\right\}, \quad \frac{\partial f_{2E}}{\partial w} = f_{2E}\left\{-\frac{w}{\sigma_w^2} - \Psi(\xi)\eta\right\}, \\
\frac{\partial^2 f_{2E}}{\partial w^2} &= f_{2E}\left\{\left[-\frac{w}{\sigma_w^2} - \Psi(\xi)\eta\right]^2 - \frac{1}{\sigma_w^2} + \eta^2\dot{\Psi}(\xi)\right\},
\end{aligned} \tag{3.5}$$

where

$$\Psi(\tau) = \frac{d}{d\tau}\ln\Phi(\tau), \;\; \dot{\Psi}(\tau) = \frac{d\Psi(\tau)}{d\tau}, \;\; \xi = \frac{u-\mu}{\sigma_{u/w}}, \;\; \eta = \frac{\overline{uw}}{\sigma_{u/w}\sigma_w^2}.$$

Substituting (3.5) in (2.9) yields

$$
\begin{aligned}
a'_u \;=\; & -\frac{C_0\varepsilon}{2\sigma^2_{u/w}}(u-\mu) + \frac{1}{p'_{2E}}\left\{-\frac{\partial f_{2E}}{\partial t} - w\frac{\partial f_{2E}}{\partial z} - f_{2E}\frac{\partial a'_w}{\partial w} - a'_w\frac{\partial f_{2E}}{\partial w} + \frac{C_0\varepsilon}{2}\frac{\partial^2 f_{2E}}{\partial w^2}\right\} \\
=\; & -\frac{C_0\varepsilon}{2\sigma^2_{u/w}}(u-\mu) + \frac{f_{2E}}{p'_{2E}}\left\{-\frac{1}{2\sigma^2_w}\frac{\partial \sigma^2_w}{\partial t}\left(\frac{w^2}{\sigma^2_w}-1\right) - \Psi(\xi)\frac{\partial \xi}{\partial t}\right. \\
& - w\left[\frac{1}{2\sigma^2_w}\frac{\partial\sigma^2_w}{\partial z}\left(\frac{w^2}{\sigma^2_w}-1\right) + \Psi(\xi)\frac{\partial\xi}{\partial z}\right] - \frac{\partial a'_w}{\partial w} - a'_w\left[-\frac{w}{\sigma^2_w} - \Psi(\xi)\eta\right] \\
& \left. + \frac{C_0\varepsilon}{2}\left[\left(-\frac{w}{\sigma^2_w} - \Psi(\xi)\eta\right)^2 - \frac{1}{\sigma^2_w} + \eta^2\dot{\Psi}(\xi)\right]\right\}.
\end{aligned}
\tag{3.6}
$$

Since

$$
\frac{f_{2E}\Psi}{p'_{2E}} = \sigma_{u/w}, \quad \dot{\Psi}(\xi) = -\Psi(\xi)(\xi + \Psi(\xi)),
$$

we find from (3.6)

$$
\begin{aligned}
a'_u(t,z,u,w) \;=\; & -\frac{C_0\varepsilon(1+\rho^2)}{2\sigma^2_{u/w}}(u-\rho w) + \frac{\rho}{2\sigma^2_w}\left(C_0\varepsilon + \frac{\partial \sigma^2_w}{\partial t}\right)w \\
& + \frac{\rho}{2}\frac{\partial\sigma^2_w}{\partial z}\left(\frac{w^2}{\sigma^2_w}+1\right) - \sigma_{u/w}\left(\frac{\partial\xi}{\partial t} + w\frac{\partial \xi}{\partial z}\right).
\end{aligned}
\tag{3.7}
$$

Here

$$
\rho = \frac{\overline{uw}}{\sigma^2_w}, \quad \xi = \frac{u-\rho w}{\sigma_{u/w}}.
$$

Note that in the stationary case these expressions can be simplified to

$$
\begin{aligned}
a'_w(t,z,w) \;=\; & -\frac{C_0\varepsilon}{2\sigma^2_w}w + \frac{1}{2}\frac{\partial\sigma^2_w}{\partial z}\left(\frac{w^2}{\sigma^2_w}+1\right), \\
a'_u(t,z,u,w) \;=\; & -\frac{C_0\varepsilon(1+\rho^2)}{2\sigma^2_{u/w}}(u-\rho w) + \frac{\rho\, C_0\varepsilon}{2\sigma^2_w}w + \frac{\rho}{2}\frac{\partial \sigma^2_w}{\partial z}\left(\frac{w^2}{\sigma^2_w}+1\right) - \sigma_{u/w}\frac{\partial\xi}{\partial z}w.
\end{aligned}
\tag{3.8}
$$

In the next section we present some numerical experiments which show that the model presented essentially differs from Thomson's model [12].

4 Numerical experiments

In this section we compare our model against Thomson's model in the case of a 2D stationary turbulence. First we consider the case of a 2D homogeneous turbulence specified by

$$
\sigma_u = b_u u_*, \quad \sigma_w = b_w u_*, \tag{4.1}
$$

where u_* is defined by $u_*^2 = -\overline{uw}$; b_u and b_w are some dimensionless constants. The mean velocity field is zero.

Thomson's model in the stationary homogeneous case reads [12] [3]

$$
\begin{aligned}
d\hat{X} \;=\; & \hat{U}'dt, \; d\hat{Z} = \hat{W}'dt, \\
d\hat{U}' \;=\; & a'_u(\hat{U}', \hat{W}')dt + \sqrt{C_0\varepsilon}\, dB_u(t), \\
d\hat{W}' \;=\; & a'_w(\hat{U}', \hat{W}')dt + \sqrt{C_0\varepsilon}\, dB_w(t),
\end{aligned}
\tag{4.2}
$$

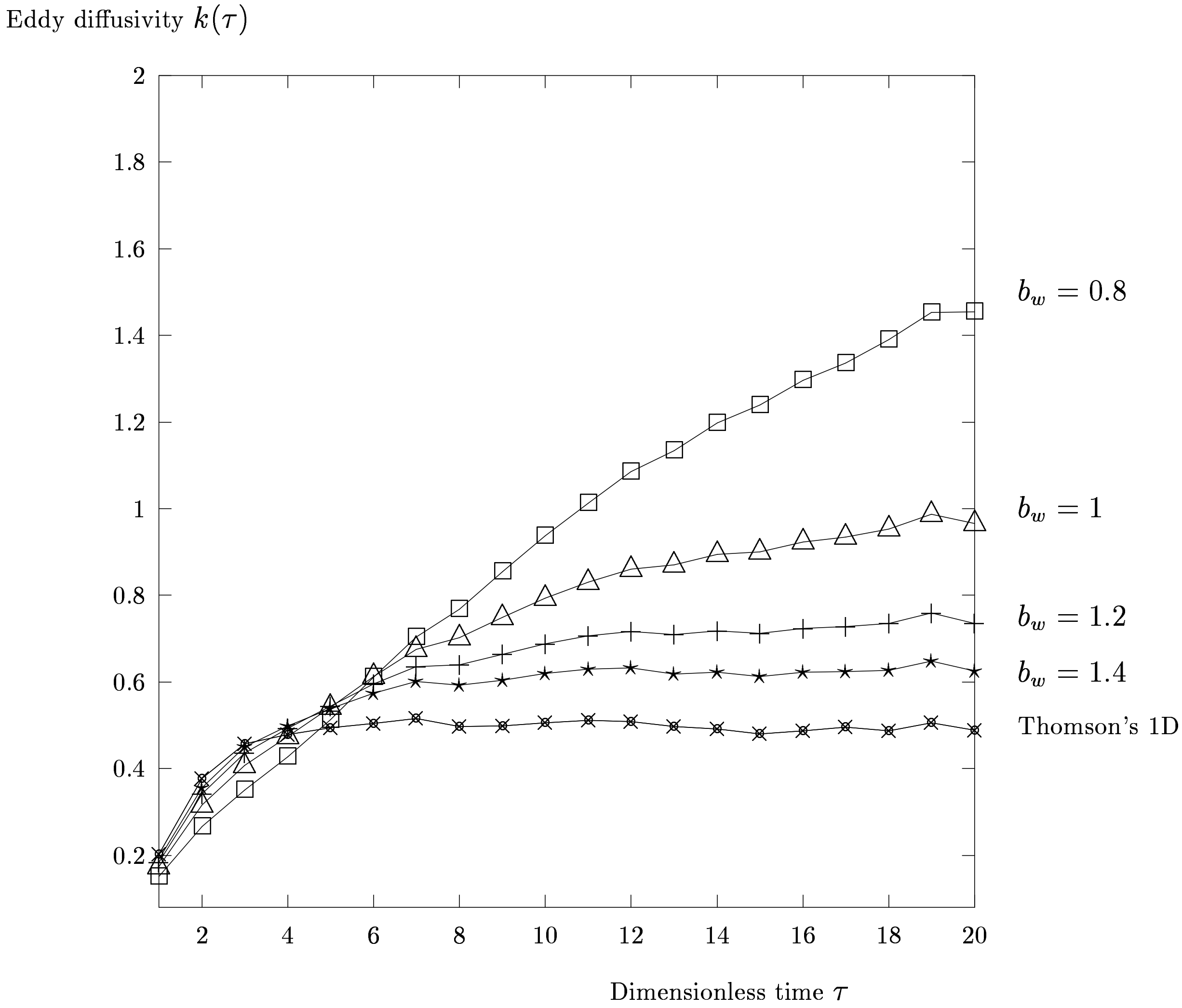

Fig.4.1 Dimensionless vertical eddy diffusivity coefficient as a function of τ, for different values of b_w. The lowest curve corresponds to Thomson's 1D model.

with

$$a'_u(u,w) = -\frac{C_0\varepsilon}{2\Delta}\Big(\sigma_w^2 u + u_*^2 w\Big), \quad a'_w(u,w) = -\frac{C_0\varepsilon}{2\Delta}\Big(\sigma_u^2 w + u_*^2 u\Big), \tag{4.3}$$

where $\Delta = \sigma_u^2\sigma_w^2 - u_*^4$.

We calculated the dimensionless vertical eddy diffusivity

$$k(\tau) = \frac{\varepsilon\langle \hat{Z}\hat{W}\rangle}{u_*^4}, \tag{4.4}$$

where $\tau = t/T_L$. Here $T_L = \frac{2\sigma_w^2}{C_0\varepsilon}$ is the Lagrangian time scale in z-direction. Since this characteristic depends only on z, w, it is sufficient to take in our model (3.1) only the equation governing z, w:

$$\begin{aligned} d\hat{Z} &= \hat{W}' dt, \\ d\hat{W}' &= -\frac{C_0\varepsilon}{2\sigma_w^2}\hat{W}' dt + \sqrt{C_0\varepsilon}\, dB_w(t). \end{aligned} \tag{4.5}$$

We note again (see Remark 2.1) that (4.5) is exactly Thomson's 1D model [12]. In our numerical experiments we have fixed $b_u = 2.3$, and have made calculations for $b_w = 0.8, 1., 1.2$ and 1.4. These values characterize the anisotropy in the neutrally stratified surface layer of the atmosphere [1]. The values of C_0 and ε were taken as 4 and $1m^2/sec^3$, respectively, while $u_* = 0.4\ m/sec$.

In Fig.4.1 we show the function $k(\tau)$ (see (4.4)) obtained by Thomson's 2D model (4.2) for different values of b_w and by our model (4.5) (which coincides with Thomson's 1D model [12]).

It is clearly seen that the diffusivity coefficients of these two models essentially differ, e.g., for $b_w = 0.8$ this difference is about a factor of 3 at the steady-state values of $k(\tau)$. With decreasing of anisotropy (i.e., when the ratio b_u/b_w decreases), this difference becomes smaller.

Thus we conclude that even in the homogeneous turbulence (which is however anisotropic) the two studied models may give essentially different results. To choose a proper case, one would need relevant measurements. We have no such experimental results in homogeneous case, while in the neutrally stratified surface layer (NSSL) the measurements are at hand (e.g., see [1]), therefore, it is interesting to compare Thomson's 2D model with our model defined by (3.1). (3.8).

Thomson's 2D model of one-particle dispersion in horizontally homogeneous stationary turbulent flow reads [12], [3]

$$\begin{aligned} d\hat{X} &= (\hat{U}' + \bar{u}(\hat{Z}, t))dt, \ d\hat{Z} = \hat{W}'dt, \\ d\hat{U}' &= a'_u(\hat{Z}, \hat{U}', \hat{W}')dt + \sqrt{C_0\varepsilon}\, dB_u(t), \\ d\hat{W}' &= a'_w(\hat{Z}, \hat{U}', \hat{W}')dt + \sqrt{C_0\varepsilon}\, dB_w(t), \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} a'_u(z,u,w) &= -\frac{C_0\varepsilon}{2\Delta}\Big[\sigma_w^2 u - \overline{uw}w\Big] + \frac{1}{2}\frac{d\overline{uw}}{dz} \\ &+ \frac{w}{2\Delta}\left\{\frac{d\sigma_u^2}{dz}\Big(\sigma_w^2 u - \overline{uw}w\Big) + \frac{d\overline{uw}}{dz}\Big(-\overline{uw}u + \sigma_u^2 w\Big)\right\}, \\ a'_w(z,u,w) &= -\frac{C_0\varepsilon}{2\Delta}\Big[\sigma_u^2 w - \overline{uw}u\Big] + \frac{1}{2}\frac{d\sigma_w^2}{dz} \\ &+ \frac{w}{2\Delta}\left\{\frac{d\overline{uw}}{dz}\Big(\sigma_w^2 u - \overline{uw}w\Big) + \frac{d\sigma_w^2}{dz}\Big(-\overline{uw}u + \sigma_u^2 w\Big)\right\}. \end{aligned}$$

Here $\Delta = \sigma_u^2\sigma_w^2 - \overline{uw}^2$.

For the NSSL, the coefficients in this model can be taken as follows [1], [6]

$$\varepsilon(z) = \frac{u_*^3}{\kappa z}, \quad \bar{u}(z) = \frac{u_*}{\kappa}\ln\Big(z/z_0\Big),$$

and σ_u and σ_w are given by (4.1) with $u_*^2 = -\overline{uw}$ =const; $\kappa = 0.4$, z_0 is the roughnes height.

Hence Thomson's 2D model in this case is specified by

$$a'_u(z,u,w) = -\frac{C_0\varepsilon(z)}{2\Delta}\Big(\sigma_w^2 u + u_*^2 w\Big), \quad a'_w(z,u,w) = -\frac{C_0\varepsilon(z)}{2\Delta}\Big(\sigma_u^2 w + u_*^2 u\Big).$$

Our model (3.1), (3.8) in the case of NSSL is specified by

$$a'_u(z,u,w) = -\frac{C_0\varepsilon(z)(1+\rho^2)}{2\sigma^2_{u/w}}(u-\rho w) + \frac{\rho C_0 \varepsilon(z)}{2\sigma^2_w} w, \; a'_w(z,w) = -\frac{C_0\varepsilon(z)}{2\sigma^2_w} w. \qquad (4.7)$$

In the comparative calculations, we have calculated the following dimensionless Lagrangian characteristics:

$$A(t) = \frac{\sqrt{\langle Z^2(t)\rangle}}{u_* t}, \; B(t) = \frac{\langle Z(t)\rangle}{u_* t}, \; C(t) = \frac{z_0}{u_* t} \exp\left\{\frac{\kappa\langle X(t)\rangle}{u_* t} + 1\right\}. \qquad (4.8)$$

It is known (e.g., see [1], p.77) that these functions tend, as $t \to \infty$, to some constant values a, b and c, respecively, provided h_s and z_0 are much less than $u_* t$. Here h_s is the height at which the Lagrangian trajectory starts. The experimenatal measurements of the constants a, b and c are scattered to a certain amount. However as can be extracted from [1], (see the Tables 3.6 and 3.8 therein) we conclude that the values of a, b and c lie in the intervals $(0.32, 0.58)$, $(0.28, 0.49)$ and $(0.14, 0.30)$, respectively.

In our calculations, we have chosen $u_* = 0.4 m/sec$, $z_0 = 0.1$m, $b_u = 2.3$, and two variants of b_w: $b_w = 1.2$ and $b_w = 1.3$.

The results of calculations are given in the tables 4.1 and 4.2.

Table 4.1 Steady-state values of the functions (4.8), for $b_w = 1.2$.

Model	a	b	c
Thomson's	0.65± 0.01	0.48± 0.02	0.21± 0.03
ours	0.52± 0.02	0.38± 0.01	0.14± 0.01

Table 4.2 Steady-state values of the functions (4.8), for $b_w = 1.3$.

Model	a	b	c
Thomson's	0.78± 0.01	0.59± 0.02	0.25± 0.04
ours	0.66± 0.02	0.49± 0.02	0.18± 0.01

Here we present the results which correspond to $b_w = 1.2$ in the table 1, and $b_w = 1.3$ in the table 2. The tables show that in both cases, the results for a and b obtained by our model agree slightly better with the experiments as compared to those obtained by Thomson's model. As to the quantity c, both models agree satisfactorily with the measurements.

In conclusion we stress again that even in the case of homogeneous but anisotropic turbulence, the well-mixed condition (1.7) does not define the one-particle model uniquely. Here we compared our model against Thomson's model which give two significantly different values of the eddy diffusivity coefficient (4.4). To choose between these models, one requires more accurate measurements.

Acknowledgement. This work is partially supported by Grants INTAS-RFBR N 95-IN-RU-726 and the Nato Linkage Grant N 971664.

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