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On bond price dynamics

E. Platen[1,2] , R. Rebolledo[3]

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[1] Australian National University
Institute of Advanced Studies, SRS
GPO Box 4
ACT 2601 Canberra
Australia

[2] Institut für Angewandte Analysis
und Stochastik
Hausvogteiplatz 5-7
D – O 1086 Berlin
Germany

[3] Universidad Católica de Chile
Facultad de Matemática
Casilla 306
Santiago 22
Chile

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Herausgegeben vom
Institut für Angewandte Analysis und Stochastik
Hausvogteiplatz 5-7
D – O 1086 Berlin

Fax: + 49 30 2004975
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ON BOND PRICE DYNAMICS

ECKHARD PLATEN AND ROLANDO REBOLLEDO

ABSTRACT. This article proposes a new approach to bond price dynamics. By means of exponential formulae and a notion of forward derivatives we construct a general theoretical framework, which allows to include most of known bond price properties. In particular, we perform a new analysis of no arbitrage conditions together with their consequences on the corresponding return premium. An expression for the general bond price is obtained which also turns out to be computationally convenient. Finally, we specify our result in a general multifactor bond pricing model.

1. INTRODUCTION

With the development of bond markets the problem of valuation of bonds has become a subject of increasing importance to both practitioners and academics. The lack of a general and satisfactory model has also serious consequences in valuing most of other financial instruments because bonds play the role of the non–risky asset in most portfolios. We are not able here to discuss the wide range of existing literature on bond pricing, but we will mention a few of the papers and approaches to explain our point of view.

Let $P(T) = (P_t(T); 0 \leq t \leq T)$ denote the price process for a default free zero coupon bond which matures at time T with final value 1. We start from the bond price dynamics itself which we describe via a stochastic differential equation:

$$dP_t(T) = \mu_t(T)P_t(T)dt + \sigma_t(T)P_t(T)dW_t, \ (0 \leq t \leq T), \tag{1.1}$$

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where $W = (W_t(T); 0 \leq t \leq T)$ is a standard Wiener process. At the end of the paper in our application for a multifactor model we will consider also multidimensional Wiener processes. The process $\mu(T) = (\mu_t(T); 0 \leq t \leq T)$ is called the *expected rate of return or appreciation rate process* of the bond, and $\sigma(T) = (\sigma_t(T); 0 \leq t \leq T)$ denotes the process of *instantaneous deviation* of the rate of return also called the *volatility process.* No other conditions are required for $\mu(T)$ and $\sigma(T)$ for the moment. So that our deliberate choice of a general geometric Brownian motion does not represent a restriction. The Black and Scholes type dynamics ([3]) with constant $\mu(T)$ and $\sigma(T)$ is still sometimes used for bond pricing by practitioners. This is obviously a very rough approximation to reality. A statistical analysis of bond price data shows that:

(i) *The expected rate of return and the volatility are stochastic processes.*

(ii) *The bond price is strictly positive and approaches its face value at maturity.*

(iii) *The bond price variance becomes zero when time reaches maturity.*

For instance, Ball and Torus ([1]) have proposed a model based on the assumption that the expected rate of return follows a Brownian bridge process. Their bond price dynamics reaches the face value at maturity, thus (ii) is fulfilled, but the volatility process remains a constant, which violates (iii).

There are several papers based on general equilibrium arguments, (see [5], [14] or [10]). The existence of a constant 'market price for risk' λ is assumed together with a specific structure for the instantaneous interest rate $r = (r_t; t \geq 0)$ also called *spot rate process.* The structure we mention for our discussion at the moment corresponds to a stochastic differential equation

$$dr_t = a(t, r_t)dt + b(t, r_t)dW_t, \ (0 \leq t < \infty), \tag{1.2}$$

where $W = (W_t; 0 \leq t < \infty)$ is the above Wiener process. In this framework, the equilibrium proposed by Cox, Ingersoll and Ross [5] refers to the choice of an expected rate of return

$$\mu_t(T) = r_t(1 + \lambda \frac{\sigma_t(T)}{b(t, r_t)}),$$

in the price model. In that case, the dynamics described fullfils our above requests (i) to (iii). But the application of this price does not eliminate arbitrage opportunities. For instance the return $1 - P_t(T)$ obtained at the time interval $[t, T]$ from a $T-$maturity bond is in general different from the expected return obtained by investing in a savings account. Thus, a realistic bond price model should also satisfy the following property:

(iv) *The return on holding a bond to maturity is equal to the expected return on a savings account paying continuously the instantaneous interest rate.*

Other arbitrage approaches were developed in a series of papers e.g. by Brennan and Schwartz [4], Dothan [6], Richard [20], Vasicek [22], Ho and Lee [9], Jamshidian [11], Morton [15], Heath, Jarrow and Morton [8], Black, Derman and Toy [2], Sandmann and Sondermann [21], El Karoui, Myneni and Vishwanathan [12], Platen [16]. Only the last two quoted papers offer the possibility to fulfill also property (iv) above. Indeed this is expressed by a martingale property: in this context any asset discounted by a bond is a martingale if the description of the asset contains only a noise which is independent of W. As we will later see this is related to our condition (iv) and will be requested directly in property (vi) below.

Most of the other mentioned authors use a different approach. They choose a probability measure under which a bond discounted by a savings account becomes a martingale. It is then easy to show that the expected rate of return has to be the same for all bonds and is equal to the spot rate. This does not coincide with reality. Moreover one observes an excess expected rate of return also called *return premium* or *risk premium.* This is larger for long term bonds than for short term ones. Thus, we add the following condition on a good bond pricing model.

(v) *A realistic bond pricing approach should also explain the return premium, possibly appreciating its significance for practitioners as a measure of the change of risk involved.*

To finish the list of required properties for a good bond model, we underline that in most portfolios one needs to discount the risky assets by a bond for hedging purposes. Thus, as we already mentioned, it is desirable to have that

(vi) *Any asset, with expected rate of return equal to the spot rate, should form a martingale when discounted by any bond driven by a Wiener process which is independent of that driving the asset.*

In the following we describe a general approach which provides sufficient conditions under which a bond price dynamics corresponds to all above mentioned properties (i) to (vi).

We will try to keep the model quite general such that restrictions result mainly by fulfilling conditions (i) to (vi). To achieve this we have to introduce extremely helpful mathematical tools such as price generators, exponentials and forward derivatives which we also explain intuitively. We hope the reader will realise the convenience and value of these tools even if it requires some efforts to become familiar with them.

The paper is organised as follows: the second section introduces and solves the main bond price equation; the third, proposes new tools to study the underlying dynamics; the last section discusses consequences of our results. To simplify the presentation of our results we have postponed all proofs to the appendix and consider at first only the case of a one–dimensional driving Wiener process. The results can be extended in an obvious way to the multidimensional case.

2. The bond price equation

We would like to emphasize that our introduction considered a very specific dynamics for bond prices. In the rest of the paper we will use similar notations for convenience but start now to define them properly to have the results in their full generality.

2.1. Preliminary notations on bonds. Our model is based on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The flow of information is represented by a family of σ–fields $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ fulfilling usual conditions. If $\mathbb{F}$ provides all the information up to time t about the actual values of a process $V = (V_t; t \geq 0)$, then we say that V is $\mathbb{F}$–*adapted*. That is, more precisely, for each $t \geq 0$ the random variable V_t is $\mathcal{F}_t$–measurable.

We fix a maturity time T and consider the price process $P(T) = (P_t(T); t \geq 0)$ of a default free zero coupon bond with maturity at time T having the face value 1. Since we will compare later processes with different maturities, for the sake of definition we have extended the time parameter interval for $P(T)$ to the whole positive real line. So that we assume $P_t(T) = 1$ for all $t \geq T$. As one can easily understand, the value of $P_t(T)$ will depend on future information and the knowledge of events up to time t will not be enough to describe its behaviour. That is, $P(T)$ is in general not an adapted process. This represents a technical difficulty which can be overcome by considering the *normalised price* $Z_t(T) = P_t(T)/P_0(T)$ which we will assume to be adapted. (This is not a limitation on the model: indeed, our results in [17] show that the process $Z(T) = (Z_t(T); t \geq 0)$ is adapted when the expected rate of return and the volatility have that property). Thus, according to our condition (ii), the normalised price has to satisfy:

$$Z_t(T) = \frac{1}{P_0(T)}, \text{ for all } t \geq T. \tag{2.1}$$

Now, to fulfill condition (i), we take two continuous and adapted processes $\mu(T) =$

$(\mu_t(T); t \geq 0)$ and $(\sigma(T) = (\sigma_t(T); t \geq 0)$. The first process is the expected rate of return of the bond price; the second, is the volatility. We will see in section 3 that they can be interpreted as

$$\mu_t(T) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}\left(\frac{P_{t+h}(T) - P_t(T)}{P_t(T)} / \mathcal{F}_t\right), \ (t \geq 0), \tag{2.2}$$

and as the second moment of the rate of return

$$\sigma_t^2(T) = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}\left[\left(\frac{P_{t+h}(T) - P_t(T)}{P_t(T)}\right)^2 / \mathcal{F}_t\right], \ (t \geq 0), \tag{2.3}$$

where both limits have to be understood in probability and we assume that $\sigma(T)^2$ and $\mu(T)$ are $\mathbb{P}$–almost surely integrable. These limits will be defined rigorously in the next section.

2.2. The fundamental equation. Realising that in principle bond prices express the market expectations over the future interest rate developments, we characterise the normalised bond price as a solution of a stochastic equation

$$Z_t(T) = \frac{1}{P_0(T)} - \int_t^T \mu_s(T) Z_s(T) ds - \int_t^T \sigma_s(T) Z_s(T) dW_s, (0 \leq t \leq T). \tag{2.4}$$

Notice this equation is coherent with our first two conditions (i), (ii). To solve it one needs a new tool, the price generator, to extend usual methods for solving ordinary linear differential equations. As we will see this mathematical object has also an important interpretation in pricing. It describes in a compact form the dynamics of the bond price process.

2.3. Price Generator. We introduce the concept of the *price generator* as a process which will allow us to generate the whole dynamics of prices by means of an exponential–type operation. Such a kind of generator satisfies a well defined mathematical property: it is a continuous semimartingale. A continuous semimartingale is the sum of a continuous (local) martingale and a continuous drift which is a process with bounded variation on finite intervals.

The mechanism by which a specific semimartingale S generates prices is expressed by a kind of exponential. Given a continuous semimartingale S, which represents the price generator, we denote by $\mathcal{E}(S)$ the Doléans exponential of S (see e.g. [18]), given by

$$\mathcal{E}(S)_t = \exp(S_t - \frac{1}{2}[S,S]_t), \ (t \in [0,T]), \tag{2.5}$$

where $[S,S]$ is the quadratic variation of S. The quadratic variation $[S,S]_t$ of the price generator can be calculated as a limit in probability of sums of the type $\sum(S_{t_{i+1}} - S_{t_i})^2$ where the t_i's are taken in refining partitions of the interval $[0,t]$. This computation is simpler for a semimartingale which is of the form $S_t = \int_0^t \varphi_s dW_s$ (a stochastic integral of the Wiener process): in this case $[S,S]_t = \int_0^t \varphi_s^2 ds$. Moreover all the continuous semimartingales we will consider (besides in the multi–factor model at the end) are of the form $S_t = \int_0^t \varphi_s dW_s + \int_0^t \psi_s ds$ and have quadratic variations like

$$[S,S]_t = \int_0^t \varphi_s^2 ds, \ (t \geq 0).$$

Therefore,

$$\mathcal{E}(S)_t = \exp(\int_0^t \varphi_s dW_s - \frac{1}{2}\int_0^t \varphi_s^2 ds + \int_0^t \psi_s ds), \ (t \geq 0),$$

for the above kind of semimartingale or price generator.

If we take two such continuous semimartingales, $S_t = \int_0^t \varphi_s dW_s + \int_0^t \psi_s ds$ and $S'_t = \int_0^t \varphi'_s dW'_s + \int_0^t \psi'_s ds$, where W' is another Wiener process, then they have a quadratic covariation:

$$[S,S']_t = \int_0^t \varphi_s \varphi'_s d[W,W']_s, \ (t \geq 0), \tag{2.6}$$

where $[W,W']$ is the covariation process between the two Wiener processes: it will be zero if they are independent; if they coincide, then $[W,W']_t = [W,W]_t = t, \ (t \geq 0)$.

The exponential of a continuous semimartingale is a very important and powerful tool because it is connected with linear stochastic differential equations: $\mathcal{E}(S)$ is the unique solution to the equation

$$\mathcal{E}(S)_t = 1 + \int_0^t \mathcal{E}(S)_u dS_u, \ (t \geq 0), \tag{2.7}$$

which is the analog to the deterministic ordinary linear differential equation satisfied by customary exponentials. Thus if S is a price generator, then $\mathcal{E}(S)$ is, up to a constant factor, already the corresponding price process.

It is also useful to recall here Yor's formula for the product of semimartingale exponentials:

$$\mathcal{E}(S)\mathcal{E}(S') = \mathcal{E}(S + S' + [S, S']), \tag{2.8}$$

which gives an easy tool to consider discounted price processes and measure transformations. From the above formula it follows, in particular, that

$$(\mathcal{E}(S))^{-1} = \mathcal{E}(-S + [S, S]).$$

Let us stop here this introduction of the price generator to present a crucial result which is based on it.

2.4. Solving the fundamental equation. Obviously, the solution of our fundamental equation will provide an expression for both, the normalised price and the price itself.

Theorem 2.1. *Given μ and σ which satisfy the above conditions there exists a unique solution to* (2.4) *given by*

$$Z_t(T) = \exp\left(\int_0^t \sigma_s(T) dW_s - \frac{1}{2}\int_0^t \sigma_s(T)^2 ds + \int_0^t \mu_s(T) ds\right) = \mathcal{E}(S)_t, \ (t \geq 0), \tag{2.9}$$

where we denote by S the continuous semimartingale $S_t = \int_0^t \sigma_s(T) dW_s + \int_0^t \mu_s(T) ds$, $(t \geq 0)$, which is the price generator.

Moreover, the price at time t is given by the formula

$$P_t(T) = \exp\left[-\left(\int_t^T \sigma_s(T)dW_s - \frac{1}{2}\int_t^T \sigma_s(T)^2 ds + \int_t^T \mu_s(T)ds\right)\right], \ (t \geq 0). \tag{2.10}$$

The prove of this theorem is given in the appendix.

3. Expected growths

To measure the *expected growth rate* of a process V, we introduce the following definition which generalises ordinary deterministic derivatives to the stochastic framework.

3.1. Forward derivative. The *forward derivative* of a locally integrable continuous process V is defined to be the limit in probability

$$D_t^+ V = \lim_{h>0, h\downarrow 0}.\mathrm{pr}\, \frac{1}{h}\mathbb{E}(V_{t+h} - V_t/\mathcal{F}_t), \tag{3.1}$$

when this limit exists, where $\mathbb{E}(\cdot/\mathcal{F}_t)$ is supposed to be a right–continuous version of the conditional expectation.

A continuous semimartingale with an absolutely continuous drift has such a forward derivative (see e.g. [19]) and it coincides with the derivative of the drift. That is, if $S_t = \int_0^t \varphi_s dW_s + \int_0^t \psi_s ds$, $(t \geq 0)$, then

$$D_t^+ S = \psi_t, \ (t \geq 0).$$

Therefore such a semimartingale reduces to a martingale if and only if its forward derivative is zero. We then obtain

Lemma 3.1. *For a continuous semimartingale S which has a forward derivative its exponential also admits a forward derivative and*

$$D^+\mathcal{E}(S) = \mathcal{E}(S)D^+S. \tag{3.2}$$

It is important to remark that the above lemma shows in particular that the exponential of a continuous semimartingale S is a martingale (forward derivative zero) if and only if $D_t^+ S = 0$, $(t \geq 0)$ since the exponential is never zero. That means S has to be a martingale itself.

3.2. Rate of return and volatility. Now we return to the interpretation of our coefficients $\mu(T)$ and $\sigma(T)$, since they can be interpreted by means of the same kind of limit used to define forward derivatives.

Proposition 3.1. *Under the hypotheses of Theorem* 2.1,

$$\mu_t(T) = \lim_{h>0,h\downarrow 0} .pr \ \frac{1}{h}\mathbb{E}\left(\frac{P_{t+h}(T) - P_t(T)}{P_t(T)}/\mathcal{F}_t\right) \tag{3.3}$$

$$\sigma_t(T)^2 = \lim_{h>0,h\downarrow 0} .pr \ \frac{1}{h}\mathbb{E}\left[\left(\frac{P_{t+h}(T) - P_t(T)}{P_t(T)}\right)^2/\mathcal{F}_t\right], \ (t \geq 0). \tag{3.4}$$

4. No arbitrage condition

We now go into the analysis of no arbitrage according to our condition (vi).

4.1. No arbitrage. This term will mean for us that given any adapted asset A, with expected rate of return equal to the spot rate, will have zero expected growth rate when discounted by the normalised bond price Z, that is

$$D_t^+\left(\frac{A}{Z}\right) = 0, (0 \leq t \leq T). \tag{4.1}$$

This represents a martingale property for the quotient A/Z. Using the normalised price Z instead of $P(T)$ allows us to concentrate first on the dynamics of the price process separated from any measurability requirements on $P_0(T)$.

Let us establish here a lemma which turns out to be very useful in the analysis of no arbitrage

Lemma 4.1. *Assume $Z = \mathcal{E}(S)$ and $Z' = \mathcal{E}(S')$ to be continuous semimartingale exponentials for*

$$S_t = \int_0^t \sigma_s W_s + \int_0^t \mu_s ds, \ (t \geq 0), \text{ and} \tag{4.2}$$

$$S'_t = \int_0^t \sigma'_s dW'_s + \int_0^t \mu'_s ds, \ (t \geq 0), \tag{4.3}$$

respectively where W' is another Wiener process on the same probability space and μ, μ', σ^2, σ'^2 are continuous, adapted, $\mathbb{P}$–almost surely integrable processes. Then the process Z'/Z is a martingale if and only if

$$\int_0^t (\mu'_s - \mu_s + {\sigma_s}^2) ds + \int_0^t \sigma'_s \sigma_s d[W, W']_s = 0, \ (t \geq 0). \tag{4.4}$$

4.2. Assets discounted by bonds. We consider a continuous adapted asset A which is the solution of a linear stochastic differential equation

$$dA_t = \mu'_t A_t dt + \sigma'_t A_t dW'_t, \tag{4.5}$$

where W' is another Wiener process (possibly independent of W) and μ', σ'^2 are adapted, continuous, $\mathbb{P}$–almost surely integrable processes; the initial value A_0 is supposed to be $\mathcal{F}_0$–measurable.

We may interpret A in many different ways, e.g. as a savings account, as a stock, or as another bond. The asset satisfies a linear equation and can be expressed as a semimartingale exponential:

$$\frac{A_t}{A_0} = Z'_t = \mathcal{E}(S'), \ (t \geq 0), \tag{4.6}$$

where the price generator has the form $S'_t = \int_0^t \sigma'_s dW'_s + \int_0^t \mu'_s ds, \ (t \geq 0)$.

We then have

Theorem 4.1. *Under the above conditions on assets and bonds there is no arbitrage if and only if*

$$\int_0^t (\mu'_s - \mu_s(T) + \sigma_s(T)^2) ds + \int_0^t \sigma'_s \sigma_s(T) d[W, W']_s = 0, \ (0 \leq t \leq T). \tag{4.7}$$

Furthermore, in that case the normalised price satisfies:

$$\begin{aligned} Z_t(T) &= \left\{ \mathbb{E}\left[P_0(T) \exp\left(\int_t^T \sigma'_s dW_s - \frac{1}{2}\int_t^T {\sigma'_s}^2 ds + \int_t^T \mu'_s ds \right) / \mathcal{F}_t \right] \right\}^{-1} \\ &= \frac{\mathcal{E}(S')_t}{\mathbb{E}(P_0(T)\mathcal{E}(S')_T/\mathcal{F}_t)}, \quad (0 \leq t \leq T). \end{aligned} \tag{4.8}$$

And the price is

$$\begin{aligned} P_t(T) &= P_0(T) \left\{ \mathbb{E}\left[P_0(T) \exp\left(\int_t^T \sigma'_s dW_s - \frac{1}{2}\int_t^T {\sigma'_s}^2 ds + \int_t^T \mu'_s ds \right) / \mathcal{F}_t \right] \right\}^{-1} \\ &= \frac{P_0(T)\mathcal{E}(S')_t}{\mathbb{E}(P_0(T)\mathcal{E}(S')_T/\mathcal{F}_t)}, \quad (0 \leq t \leq T). \end{aligned} \tag{4.9}$$

Moreover the variance of the price tends to zero as t approaches maturity:

$$\lim_{t \to T} \mathbb{E}\left[|P_t(T) - \mathbb{E}(P_t(T))|^2 \right] = 0. \tag{4.10}$$

If a different maturity time T' is given, then different prices satisfy the balance equation

$$Z_t(T')\mathbb{E}(P_0(T')\mathcal{E}(S')_{T'}/\mathcal{F}_t) = \mathcal{E}(S')_t = Z_t(T)\mathbb{E}(P_0(T)\mathcal{E}(S')_T/\mathcal{F}_t), \quad (t \geq 0). \tag{4.11}$$

Corollary 4.1. *If in addition to the above no arbitrage condition* (4.7), *the initial price $P_0(T)$ is assumed to be $\mathcal{F}_0$-measurable, then*

$$\begin{aligned} P_t(T) &= \left\{ \mathbb{E}\left[\exp\left(\int_t^T \sigma'_s dW_s - \frac{1}{2}\int_t^T {\sigma'_s}^2 ds + \int_t^T \mu'_s ds \right) / \mathcal{F}_t \right] \right\}^{-1} \\ &= \frac{\mathcal{E}(S')_t}{\mathbb{E}(\mathcal{E}(S')_T/\mathcal{F}_t)}, \quad (t \geq 0). \end{aligned} \tag{4.12}$$

I f another maturity time T' is considered, then the balance equation is:

$$\mathbb{E}(P_t(T')\mathcal{E}(S')_{T'}/\mathcal{F}_t) = \mathcal{E}(S')_t = \mathbb{E}(P_t(T)\mathcal{E}(S')_T/\mathcal{F}_t), \quad (t \geq 0).. \tag{4.13}$$

Balance equations give a stronger property than demanded in condition (iv) in the introduction. Indeed, in the corollary above if we choose the asset A as continuously interest r paying savings account, then $S'_t = \int_0^t r_s ds$ and we get after dividing (4.13) by $\mathcal{E}(S')_t$ that

$$\begin{aligned} 1 &= \mathbb{E}(P_t(T)\frac{\mathcal{E}(S')_T}{\mathcal{E}(S')_t}/\mathcal{F}_t) \\ &= \mathbb{E}(P_t(T)\exp(\int_t^T r_s ds)/\mathcal{F}_t), \end{aligned} \qquad (4.14)$$

for all $t \geq 0$, where $P_t(T)\exp(\int_t^T r_s ds)$ represents the amount which one obtains at maturity by investing the amount $P_t(T)$ at time t in such a savings account. The expectation of this random variable corresponds according to (4.14) to the face value 1 of the bond. Thus, our condition (iv) from the introduction is satisfied and we have no arbitrage opportunity between bonds and savings account.

4.3. Examples. Theorem 4.1 and Corollary 4.1 show that our bond price satisfies conditions (iii), (iv) and (vi) discussed in the introduction.

Here we consider some particular cases to illustrate the above results. We derive *sufficient conditions* to obtain the martingale property for the discounted asset A/Z:

(1) The case of independent Wiener processes W, W'. Then $[W, W'] = 0$ and (4.7) reduces to

$$\int_0^t (\mu_s(T) - \mu'_s - \sigma_s(T)^2) ds = 0, \ (0 \leq t \leq T). \qquad (4.15)$$

A sufficient condition to obtain the no arbitrage property is then

$$\mu_t(T) - \sigma_t(T)^2 = \mu'_t, \ (0 \leq t \leq T). \qquad (4.16)$$

(2) As a particular case of the above, consider the asset A to be a savings account which we denote by

$$B_t = \exp\left(\int_0^t r_s ds\right), \ (t \geq 0). \qquad (4.17)$$

In this case there is no noise associated to the asset and $\mu'_t = r_t$, $(t \geq 0)$. Therefore (4.16) gives

$$\mu_t(T) = r_t + \sigma_t(T)^2, \ (t \geq 0). \tag{4.18}$$

The term $\sigma_t(T)^2$ represents then the *return premium.*

One realises that the return premium equals the square of the volatility, that is

$$\frac{\mu_t(T) - r_t}{\sigma_t(T)^2} = 1, \ (t \in [0,T]). \tag{4.19}$$

We can consider the square of the volatility as a measure of the instantaneous actual uncertainty of the bond price. Furthermore the return premium $e_t(T) = \mu_t(T) - r_t$ can be understood as a measure for the change of the remaining risk. Equation (4.19) then says that the relative change of the remaining risk $e_t(T)/\sigma_t(T)^2$ is independent of the maturity time as well as any other parameter since it is constant. That means the relative change of the remaining risk is equally distributed over the whole life time of a bond. This gives an intuitive explanation for the relation between return premium and volatility of a bond requested in property (v) in the introduction.

From (4.9) we obtain

$$P_t(T) = \left\{ \mathbb{E}(\exp(\int_t^T r_s ds)/\mathcal{F}_t) \right\}^{-1}, \ (t \geq 0). \tag{4.20}$$

in the particular case when $P_0(T)$ is $\mathcal{F}_0$–measurable. This generalises the bond price proposed in [16].

(3) Assume W and W' to be dependent, then $[W,W'] \neq 0$ and (4.7) has to be solved with additional assumptions. In particular, when discounting bonds by bonds one may assume $W = W'$ so that $[W,W']_t = t$, $(t \geq 0)$. Therefore, a

sufficient condition to obtain the martingale property in this case is

$$\mu_t(T) - \mu'_t - \sigma_t(T)^2 - \sigma'_t\sigma_t(T) = 0, \ (t \in [0,T]), \tag{4.21}$$

which requires an appropriate measure transformation to be fulfilled.

5. Conclusions

Within this paper we proposed a general approach to no arbitrage bond pricing based on stochastic analytic tools as price generators, exponentials and forward derivatives. We were able to show that our requests (i) to (vi) in the introduction on a realistic bond price model are fulfilled. Starting from an exponential type evolution our no arbitrage condition fixed the bond price dynamics providing the key properties requested.

Important is the consequence that the return premium represents just the squared volatility which interprets it as a measure for the change of the remaining risk.

It is clear that this approach is extendable to the general pricing of other financial instruments on stocks and also price processes like bonds, options, etc. which we will consider in a forthcoming paper.

Finally we remark that the explicit expressions for the obtained general bond price allow an easy computation e.g. as described in [16] for specific examples or by stochastic numerical methods as considered in [13]. The latter becomes inavoidable as soon as one studies multifactor models for the spot rate as we will discuss in the last appendix.

6. Appendix: Proofs of main results

Theorem 2.1. *Proof.* We write equation (2.4) for $t = 0$. That gives

$$1 = Z_0(T) = \frac{1}{P_0(T)} - \int_0^T \mu_s(T)Z_s(T)(s)ds - \int_0^T \sigma_s(T)Z_s(T)dW_s. \tag{6.1}$$

Solve the above equation for $\frac{1}{P_0(T)}$ and replace in (2.4) to obtain:

$$Z_t(T) = 1 + \int_0^t \mu_s(T) Z_s(T) ds + \int_0^t \sigma_s(T) Z_s(T) dW_s, \ (t \geq 0). \tag{6.2}$$

That is

$$Z_t(T) = 1 + \int_0^t Z_s dS_s, \ (t \geq 0). \tag{6.3}$$

Now, (2.4) will have a solution if and only if (6.3) has one. Since (6.3) has a unique solution given by the Doléans exponential $\mathcal{E}(S)$, the theorem is proved. To obtain the expression of $P_t(T)$ it suffices to compute $P_t(T) = P_o(T) Z_t(T) = (Z_T(T))^{-1} Z_t(T) = (\mathcal{E}(S)_T)^{-1} \mathcal{E}(S)_t$, for all $t \geq 0$. □

Lemma 3.1. *Proof.* It suffices to apply the remark which precedes the lemma, since

$$\mathcal{E}(S) = 1 + \mathcal{E}(S) \cdot S, \tag{6.4}$$

and S decomposes as the sum of a martingale and a drift given by $\int_0^t D_u^+ S du$, $(t \geq 0)$. □

Proposition 3.1. *Proof.* We notice that

$$\frac{P_{t+h}(T) - P_t(T)}{P_t(T)} = \frac{Z_{t+h}(T) - Z_t(T)}{Z_t(T)}, \ (h > 0, \ t \geq 0). \tag{6.5}$$

But $Z(T)$ is a semimartingale, hence it is in particular adapted and

$$\begin{aligned} \frac{1}{h}\mathbb{E}(\frac{P_{t+h}(T) - P_t(T)}{P_t(T)}/\mathcal{F}_t) &= \frac{1}{h}\mathbb{E}(\frac{Z_{t+h}(T) - Z_t(T)}{Z_t(T)}/\mathcal{F}_t) \\ &= \frac{1}{Z_t(T)}\frac{1}{h}\mathbb{E}(Z_{t+h}(T) - Z_t(T)/\mathcal{F}_t), \end{aligned}$$

for all $t \geq 0$, $h > 0$. Therefore, the limit in probability of the first member of the above equation exists when $h \downarrow 0$ and it is

$$\lim_{h>0, h\downarrow 0} .\mathrm{pr} \frac{1}{h}\mathbb{E}(\frac{P_{t+h}(T) - P_t(T)}{P_t(T)}/\mathcal{F}_t) = \frac{1}{Z_t(T)} D_t^+ Z(T), \tag{6.6}$$

since $Z(T)$ is a semimartingale. Moreover from Lemma 3.1 it follows

$$D_t^+ Z(T) = Z_t(T) D_t^+ S = Z_t(T)\mu_t(T), \ (t \geq 0), \tag{6.7}$$

so that

$$\lim_{h>0, h\downarrow 0} .\mathrm{pr}\ \frac{1}{h}\mathbb{E}\left(\frac{P_{t+h}(T) - P_t(T)}{P_t(T)} / \mathcal{F}_t\right) = \mu_t(T), \ (t \geq 0). \tag{6.8}$$

To prove the formula for the volatility we use (6.5) and observe from (6.3) that

$$\begin{aligned}\left(\frac{Z_{t+h}(T) - Z_t(T)}{Z_t(T)}\right)^2 &= \left(\int_t^{t+h} \sigma_s(T)\frac{Z_s(T)}{Z_t(T)}dW_s + \int_t^{t+h} \mu_s(T)\frac{Z_s(T)}{Z_t(T)}ds\right)^2 \\ &= \left(\int_t^{t+h} \sigma_s(T)\frac{Z_s(T)}{Z_t(T)}dW_s\right)^2 + \left(\int_t^{t+h} \mu_s(T)\frac{Z_s(T)}{Z_t(T)}ds\right)^2 \\ &+2\left(\int_t^{t+h} \mu_s(T)\frac{Z_s(T)}{Z_t(T)}ds\right)\left(\int_t^{t+h} \sigma_s(T)\frac{Z_s(T)}{Z_t(T)}dW_s\right),\end{aligned} \tag{6.9}$$

for all $t \geq 0$.

Since

$$\frac{1}{h}\int_t^{t+h} \mu_s(T)\frac{Z_s(T)}{Z_t(T)}ds$$

tends to $\mu_t(T)$ almost surely and in L^1 then

$$\frac{1}{h}\left(\int_t^{t+h} \mu_s(T)\frac{Z_s(T)}{Z_t(T)}ds\right)^2$$

goes to zero in L^1, hence in probability, as $h \downarrow 0$. And by the continuity of stochastic integrals with respect to the Wiener process we obtain also that

$$\frac{1}{h}\mathbb{E}\left[\left(\int_t^{t+h} \mu_s(T)\frac{Z_s(T)}{Z_t(T)}ds\right)\left(\int_t^{t+h} \sigma_s(T)\frac{Z_s(T)}{Z_t(T)}dW_s\right)/\mathcal{F}_t\right]$$

goes to zero in L^1 and in probability as $h \downarrow 0$.

Moreover,

$$\frac{1}{h}\mathbb{E}\left[\left(\int_t^{t+h} \sigma_s(T)\frac{Z_s(T)}{Z_t(T)}dW_s\right)^2/\mathcal{F}_t\right] = \frac{1}{h}\mathbb{E}\left[\int_t^{t+h} \sigma_s(T)^2\frac{Z_s(T)^2}{Z_t(T)^2}ds/\mathcal{F}_t\right],$$

and the last expression tends in probability to $\sigma_t(T)^2$ as $h \downarrow 0$. Therefore, taking the conditional expectation with respect to $\mathcal{F}_t$ in (6.9) and dividing by $h > 0$, we obtain

$$\begin{aligned} \lim_{h>0,h\downarrow 0}.\mathrm{pr}\ \frac{1}{h}\mathbb{E}\left[\left(\frac{P_{t+h}(T)-P_t(T)}{P_t(T)}\right)^2/\mathcal{F}_t\right] &= \lim_{h>0,h\downarrow 0}.\mathrm{pr}\ \frac{1}{h}\mathbb{E}\left[\left(\frac{Z_{t+h}(T)-Z_t(T)}{Z_t(T)}\right)^2/\mathcal{F}_t\right] \\ &= \sigma_t(T)^2,\ (t \geq 0). \end{aligned}$$

□

Lemma 4.1. *Proof.* We write Z'/Z as the quotient of Doléans exponentials applying formula (2.8):

$$\begin{aligned} \frac{Z}{Z'} &= \frac{\mathcal{E}(S')}{\mathcal{E}(S)} \\ &= \mathcal{E}(S' - S + [S,S] + [S,S']). \end{aligned}$$

Therefore,

$$D_t^+(\frac{Z'}{Z}) = \mathcal{E}(S' - S + [S,S] + [S,S'])D_t^+(S' - S + [S,S] + [S,S']),\ (t \geq 0). \tag{6.10}$$

To complete the proof, we recall the remark after Lemma 3.1 : the quotient Z'/Z will be a martingale if and only if

$$D_t^+(S' - S + [S,S] + [S,S']) = \int_0^t (\mu_s' - \mu_s + \sigma_s{}^2)ds + \int_0^t \sigma_s'\sigma_s d[W,W']_s = 0,\ (t \geq 0).$$

□

Theorem 4.1. *Proof.* The first part of the theorem follows by a straightforward application of Lemma 4.1 and is omitted. The second part is based on the martingale property of the quotient A/Z:

$$\frac{A_t}{Z_t(T)} = \mathbb{E}\left(\frac{A_T}{Z_T(T)}/\mathcal{F}_t\right),\ (t \geq 0). \tag{6.11}$$

We multiply both sides of the above equality by A_t^{-1} which is an $\mathcal{F}_t$-measurable factor, going inside the conditional expectation. Using the explicit expression of A

that is $A = A_0\mathcal{E}(S')$, one can easily derive that

$$\frac{A_T}{A_t} = \exp(\int_t^T \sigma'_s dW_s - \frac{1}{2}\int_t^T {\sigma'_s}^2 ds + \int_t^T \mu'_s ds),\ (t \geq 0). \tag{6.12}$$

Replacing the above quotient in (6.11) one obtains (4.8). Equation (4.9) follows by multiplication of (4.8) by $P_0(T)$.

To prove the property on the variance, we first notice that $P_0(T)\mathcal{E}(S')_T$ has a second moment so that by the Martingale Limit Theorem:

$$\mathbb{E}(P_0(T)\mathcal{E}(S')_T/\mathcal{F}_t) \to P_0(T)\mathcal{E}(S')_T, \text{ almost surely and in } L^2, \text{ as } t \to T. \tag{6.13}$$

Since $\mathcal{E}(S')$ is continuous it follows that

$$P_t(T) = \frac{P_0(T)\mathcal{E}(S')_T}{\mathbb{E}(P_0(T)\mathcal{E}(S')_T/\mathcal{F}_t)},$$

converges to 1 $\mathbb{P}$–almost surely and also in L^2. But we also have

$$\mathbb{E}(P_t(T)) \to 1.$$

Therefore the variance of $P_t(T)$ tends to zero as $t \to T$.

Finally, the balance equation follows by solving equation (4.8) in terms of $\mathcal{E}(S')_t$: this quantity is maturity independent. □

Corollary 4.1. *Proof.* The first part is a trivial consequence of (4.9) for $P_0(T)$ $\mathcal{F}_0$–measurable. The same holds for the balance equation which follows from (4.11) if we take $P_0(T)$ and $P_0(T')$ out of the conditional expectations. □

7. Appendix: A multifactor bond price model

Let us assume that the spot rate process can be obtained by a d–dimensional diffusion type stochastic diferential equation

$$dX_t^i = a^i(t, X_t)dt + \sum_{j=1}^{m} b^{ij}(t, X_t)dW_t^j, \tag{7.1}$$

for all $0 \leq t < \infty$, $i = 1, \ldots, d$, with $X_t = (X_t^1, \ldots, X_t^d)$, where the first component $X_t^1 = r_t$ represents the spot rate at time t. The other components of X can play the role of the volatility of the spot rate etc. Here the dynamics is driven by an m–dimensional Wiener process $W = (W^j)_{j=1}^m = (W_t^j; t \geq 0)_{j=1}^m$ which is more general as we formulated in our derivations above. But those results are easily generalised to this case and we give here an example. The drift and diffusion coefficients $a^i(t,x)$, $b^{i,j}(t,x)$, $i = 1, \ldots, d$, $j = 1, \ldots, m$, represent real valued functions on $[0,\infty[\times \mathbb{R}^d$ and are assumed to be continuously differentiable with respect to t and twice continuously differentiable with respect to x, having bounded first derivatives. Further, we suppose $\mathbb{E}|X_0|^2 < \infty$. This model is quite flexible and covers most existing spot rate models.

Given this specific structure for the spot rate dynamics we are now going to compute the bond price process $P(T)$ for fixed $T \in [0, \infty[$ and the volatility process $\sigma(T)^2$.

From (4.20) the bond price is easily obtained as

$$P_t(T) = P(t, X_t, T) = \frac{1}{u(t, X_t)} \tag{7.2}$$

via the Feynman-Kac functional

$$u(t, X_t) = \mathbb{E}(\exp(\int_t^T r_s ds)/\mathcal{F}_t), \tag{7.3}$$

where $u(t,x)$ solves the linear partial differential equation

$$\frac{\partial}{\partial t}u + \sum_{i=1}^{d} a^i \frac{\partial}{\partial x^i}u + \frac{1}{2}\sum_{i,l=1}^{d}\sum_{j=1}^{m} b^{ij}b^{lj}\frac{\partial^2}{\partial x^i \partial x^l}u + ru = 0 \tag{7.4}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$ with terminal condition $u(T,x) = 1$. Applying now the Itô formula to $P(t, X_t, T) = 1/u(t, X_t)$ one obtains as stochastic differential equation for the bond price dynamics

$$dP_t(T) = (r_t + \sigma_t(T)^2)P_t(T)dt + \sum_{j=1}^{m} \sigma_t^j(T)P_t(T)dW_t^j, \tag{7.5}$$

with volatility process

$$\sigma_t(T)^2 = \sum_{j=1}^{m} \sigma_t^j(T)^2, \tag{7.6}$$

where

$$\sigma_t^j(T) = -\sum_{i=1}^{d} \frac{b^{ij}(t, X_t)}{P(t, X_t, T)} \frac{\partial}{\partial x^i} P(t, X_t, T). \tag{7.7}$$

In the case $d = m = 1$ this result corresponds to that in [16].

We note that in this multifactor model the noise terms can be quite different for bonds with different maturities, which explains part of the practical observation that the fluctuations of bonds are not always perfectly correlated.

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Australian National University, IAS, SMS, GPO Box 4, ACT 2601 Canberra, Australia and Institut für Angewandte Analysis und Stochastik, Berlin, Germany

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