Stability of solutions to chance constrained stochastic programs *†‡

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Abstract

Perturbations of convex chance constrained stochastic programs are considered the underlying probability distributions of which are r-concave. Verifiable sufficient conditions are established guaranteeing Hölder continuity properties of solution sets with respect to variations of the original distribution. Examples illustrate the potential, sharpness and limitations of the results.

1 Introduction

Many applied optimization problems under uncertainty in constraints have the form

$$\min\{g(x) \mid x \in X, h(x) \geq \xi\}, \tag{1}$$

where the objective function g is real-valued and convex on $\mathbb{R}^m$, X is a closed convex subset of $\mathbb{R}^m$ expressing all deterministic constraints, the 'production function' $h = (h_1, \ldots, h_s)$ from $\mathbb{R}^m$ to $\mathbb{R}^s$ has concave components h_i $(i = 1, \ldots, s)$ and ξ is an s-dimensional random vector playing e.g. the role of an uncertain demand, load or force etc. In case that it appears to be difficult or even impossible to introduce compensation costs for violations of the stochastic constraint $'h(x) \geq \xi'$, one might be led to fix a certain probability or reliability level $p \in (0,1)$ subject to which the constraint has to be satisfied. Denoting by μ the (Borel) probability distribution of ξ on $\mathbb{R}^s$ and by F_μ the corresponding probability distribution function, this idea leads to the probabilistic or chance constraint

$$\mu(\{\xi \in \mathbb{R}^s \mid h(x) \geq \xi\}) = F_\mu(h(x)) \geq p. \tag{2}$$

Inserting (2) rather than the stochastic constraint $'h(x) \geq \xi'$ into the model (1), leads to the stochastic program

$$\min\{g(x) \mid x \in X, F_\mu(h(x)) \geq p\}. \tag{3}$$

Stochastic programming models of the form (3) represent nonlinear programs which are often nonconvex and nonsmooth due to the properties of the multivariate distribution function F_μ. Compared to the convexity features of the original model (1), the loss of convexity in model (3) or its perturbations (i.e. when replacing μ by an approximate probability distribution ν) appears to be disappointing. On the one hand, concavity properties of measures are well-known (cf. Appendix A for a brief exposition) that lead to convex constraint sets in (3) and cover many practical probability distributions. On the other hand, our analysis has to include nonconvex perturbed models.

In most practical applications of the stochastic programming methodology only incomplete information on the probability distribution is available. This fact motivates a stability or perturbation analysis of (3) with respect to variations of μ in the space $\mathcal{P}(\mathbb{R}^s)$ of all Borel probability measures on $\mathbb{R}^s$. Here we equip this space with the uniform or Kolmogorov distance

$$d_K(\mu, \nu) = \|F_\mu - F_\nu\|_\infty = \sup_{y \in \mathbb{R}^s} |F_\mu(y) - F_\nu(y)|.$$

Stability issues for chance constrained programs are adressed in a number of papers (see e.g. [1],[4],[5],[6],[13], [15] and references therein). A typical question in this respect is the continuity behaviour of optimal values

$$\varphi(\mu) = \inf\{g(x) \mid x \in X,\ F_\mu(h(x)) \geq p\}$$

and solution sets

$$\Psi(\mu) = \mathrm{argmin}\,\{g(x) \mid x \in X,\ F_\mu(h(x)) \geq p\}$$

of problem (3) when the measure μ is subjected to variations in $(\mathcal{P}(I\!\!R^s), d_K)$.

In the present paper, we look at conditions on model (3) implying quantitative continuity properties of solution sets with respect to the metric d_K. Our main result (Theorem 2.5) extends our earlier work (Theorem 4.3 in [6]) for the linear-quadratic case (i.e. g convex quadratic, h linear and X convex polyhedral) considerably. It provides conditions implying upper Hölder continuity of solution sets at the original measure μ with some rate that depends essentially on the data g, h and X. Our stability results are complemented by several examples illustrating their validity and limitations.

Our results allow applications to exponential bounds or convergence rates for solutions in case of nonparametric estimations of the (unknown) measure μ. We do not pursue these ideas here and refer instead to Section 5 in [6] and to [5].

2 Quantitative stability

We study the behaviour of the solution set $\Psi(\mu)$ of the stochastic program (3) when perturbing the original probability distribution μ in the metric space $(\mathcal{P}(I\!\!R^s), d_K)$. In addition to the general assumptions on g, h, p and X in Section 1, we assume throughout that $\mu \in \mathcal{P}(I\!\!R^s)$ is r-concave for some $r \in (-\infty, \infty]$. This implies that F_μ is quasi-concave (cf. Appendix A) and, hence, that (3) has both a convex objective function and a convex constraint set. Since perturbations of (3) may be nonconvex, we also need concepts of localized solutions. Given $V \subseteq I\!\!R^m$, we put for each $\nu \in \mathcal{P}(I\!\!R^s)$

$$\begin{aligned}\varphi_V(\nu) &= \inf\{g(x) \mid x \in X \cap \mathrm{cl}\, V, F_\nu(h(x)) \geq p\}\\ \Psi_V(\nu) &= \{x \in X \cap \mathrm{cl}\, V \mid g(x) = \varphi_V(\nu)\}\end{aligned}$$

where $\mathrm{cl}\, V$ denotes the closure of V. Later the set V plays the role of an open neighbourhood to $\Psi(\mu)$. Consistently with our previous notation, we have $\Psi(\nu) = \Psi_V(\nu)$ if $\Psi(\nu) \subseteq V$. Now we are ready to state our first stability result.

Proposition 2.1 *In addition to the general assumptions, let $\Psi(\mu)$ be nonempty and bounded, and $V \subseteq I\!\!R^m$ be an open, bounded neighbourhood of $\Psi(\mu)$. Furthermore, assume that there exists an $\bar{x} \in X$ such that $F_\mu(h(\bar{x})) > p$ (Slater condition).*

Then, the set-valued mapping Ψ_V from $(\mathcal{P}(I\!\!R^s), d_K)$ to $I\!\!R^m$ is upper semicontinuous at μ, i.e., for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\sup\limits_{x \in \Psi_V(\nu)} d(x, \Psi(\mu)) \leq \varepsilon$ holds *whenever* $d_K(\mu,\nu) < \delta$*. Furthermore, there exist constants* $L > 0, \eta > 0$ *such that* $\Psi_V(\nu)$ *is a nonempty set of local minimizers to the perturbed problem, and it holds that*

$$|\varphi(\mu) - \varphi_V(\nu)| \le L\, d_K(\mu,\nu) \quad \text{whenever} \quad d_K(\mu,\nu) < \eta.$$

Proof: Apply Corollary 3.7 in [13] with $d = 1, H_1(x) = \{\xi \in I\!R^s \mid h(x) \ge \xi\}$ $(x \in I\!R^m), p_1 = p$. □

In the next step of our stabilty analysis we intend to quantify the semicontinuity behaviour of Ψ, i.e., to derive an explicit representation of the function $\delta(\varepsilon)$ (e.g. $\delta(\varepsilon) = (\varepsilon/C)^k$ with some constants $k > 0$ and $C > 0$). The following example illustrates the fact that this quantifying requires further assumptions.

Example 2.2 *In (3) put* $m = 2$, $s = 1, X = [0, 1/2] \times I\!R$, $g(x_1, x_2) = x_1$, $h(x_1, x_2) = -x_2^{2q} + x_1 + 1/2$, $\forall (x_1, x_2) \in I\!R^2$, *and* $q \in I\!N$, $p = 1/2$, *and* μ *be the uniform distribution on* $[0, 1]$, *i.e.,*

$$F_\mu(\xi) = \begin{cases} 0 & , \quad \xi < 0 \\ \xi & , \quad \xi \in [0,1] \\ 1 & , \quad \xi > 1 \end{cases}$$

Then the assumptions of Proposition 2.1 are satisfied, and we have $\Psi(\mu) = \{(0,0)\}$*. Consider the sequence* (μ_n) *of uniform distributions on* $[-n^{-1}, 1-n^{-1}]$, $n \in I\!N$*. Then the constraint* $F_{\mu_n}(x_1, x_2)) \ge p$ *is equivalent to* $x_1 + n^{-1} \ge x_2^{2q}$ *and it holds that* $\Psi(\mu_n) = \{0\} \times [-n^{-(2q)^{-1}}, n^{-(2q)^{-1}}]$*. Hence, we obtain*

$$\sup_{x \in \Psi(\mu_n)} d(x, \Psi(\mu)) = \sup_{x \in \Psi(\mu_n)} \|x\| = n^{-(2q)^{-1}} \quad \text{and} \quad d_K(\mu, \mu_n) = \sup_{\xi \in I\!R} |F_\mu(\xi) - F_{\mu_n}(\xi)| = n^{-1}.$$

Since $q \in I\!N$ *was arbitrary, there is no rate* $k > 0$ *such that* $\delta(\varepsilon) = (\varepsilon/C)^k$ *for some* $C > 0$*.*

A similar example with $X = \{(x_1, x_2) \in I\!R^2 \mid x_2^{2q} \le x_1 \le 1/2\}$, $h(x_1, x_2) = x_1 + 1/2$ and the sequence $(\tilde{\mu}_n)$ of uniform distributions on $[n^{-1}, 1 + n^{-1}]$, $n \in I\!N$, leads to the same effect.

The following reduction argument decomposes the original problem (3) into two auxiliary problems and provides some insight into the structure of the solution set to (3). It also leads us closer to the essential properties of g, h and X needed for quantitative stability and extends Lemma 4.1 in [6].

Lemma 2.3 *In addition to the general assumptions, let* $\nu \in \mathcal{P}(I\!R^s)$ *and* $V \subseteq I\!R^m$ *be convex and bounded. Then we have*

$$\varphi_V(\nu) = \inf\{\pi_V(y) \mid y \in Y_V, F_\nu(y) \ge p\} \quad \text{and} \quad \Psi_V(\nu) = \sigma_V(Y_V(\nu)),$$

where

$$\begin{aligned} Y_V &= \{y \in I\!R^s \mid \exists x \in X \cap \operatorname{cl} V \text{ with } h(x) \ge y\}, \\ Y_V(\nu) &= \operatorname{argmin}\{\pi_V(y) \mid y \in Y_V, F_\nu(y) \ge p\}, \\ \pi_V(y) &= \inf\{g(x) \mid x \in X \cap \operatorname{cl} V,\ h(x) \ge y\}, \\ \sigma_V(y) &= \operatorname{argmin}\{g(x) \mid x \in X \cap \operatorname{cl} V,\ h(x) \ge y\}\ (y \in Y_V). \end{aligned}$$

Moreover, π_V is convex on the closed convex set $Y_V = \operatorname{dom}\sigma_V$.

Proof:
Since the constraint set $\{x \in X \cap \operatorname{cl} V \mid F_\nu(h(x)) \geq p\}$ is compact, the set $\Psi_V(\nu)$ is nonempty. Let $x \in \Psi_V(\nu)$. Then

$$\varphi_V(\nu) = g(x) \geq \pi_V(h(x)) \geq \inf\{\pi_V(y) \mid y \in Y_V, F_\nu(y) \geq p\}.$$

Conversely, let $y \in Y_V$ with $F_\nu(y) \geq p$. Since $\sigma_V(y)$ is nonempty, there exists $x \in \sigma_V(y)$. Hence $x \in X \cap \operatorname{cl} V$ and $F_\nu(h(x)) \geq F_\nu(y) \geq p$, thus $\pi_V(y) = g(x) \geq \varphi_V(\nu)$. This implies

$$\varphi_V(\nu) = \inf\{\pi_V(y) \mid y \in Y_V, F_\nu(y) \geq p\} \quad \text{and} \quad g(x) = \pi_V(h(x)) \ \forall x \in \Psi_V(\nu),$$

and hence $\Psi_V(\nu) = \sigma_V(Y_V(\nu))$. The convexity properties of Y_V and π_V are immediate. The closedness of Y_V follows from the compactness of $X \cap \operatorname{cl} V$. □

The lemma suggests to study the stability behaviour of Ψ_V at μ by looking at the stability properties of two programs that are different by nature. The first program contains the somewhat simpler chance constraint $F_\nu(y) \geq p$ and its decisions belong to the support of the measure ν, while the second one is a convex parametric program having a finite-dimensional parameter in the right-hand side of a constraint. Later we impose conditions implying that the solution set $Y_V(\mu)$ is a singleton and a quadratic growth condition holds near $Y_V(\mu)$. We conclude Hölder stability of Y_V at μ and combine this with Hölder or (even) Lipschitz stability results of the solution set mapping σ_V of the convex parametric program in order to obtain quantitative stability of Ψ_V. After the view of our strategy, we first recall some stability results for the convex parametric program

$$\min\{g(x) \mid x \in X \cap \operatorname{cl} V, h(x) \geq y\}.$$

Proposition 2.4 *In addition to the general assumptions, let $V \subseteq \mathbb{R}^m$ be convex and bounded, and Y_V, π_V and σ_V be defined as in Lemma 2.3.*
a) Let the following conditions be satisfied at some $\bar{y} \in Y_V$:

(i) There exists an element $\bar{x} \in X \cap \operatorname{cl} V$ such that $h(\bar{x}) > \bar{y}$ holds componentwise (Slater condition).

(ii) There exist constants $c > 0$, $k \geq 1$ and an open, convex and bounded set V_0 containing $\sigma_V(\bar{y})$ such that it holds $g(x) \geq \pi_V(\bar{y}) + c\, d(x, \sigma_V(\bar{y}))^k$ for all $x \in X \cap \operatorname{cl} V \cap \operatorname{cl} V_0$ with $h(x) \geq \bar{y}$ (growth condition of order k).

Then σ_V is upper Hölder continuous at $\bar{y}$ with rate k^{-1}, i.e., there exist constants $L > 0$, $\delta > 0$ such that

$$\sup_{x \in \sigma_V(y)} d(x, \sigma_V(\bar{y})) \leq L \|y - \bar{y}\|^{k^{-1}} \quad \text{whenever } y \in Y_V \text{ and } \|y - \bar{y}\| < \delta.$$

b) Let g be convex quadratic, h be linear, X be convex polyhedral and $\operatorname{cl} V$ be a polytope.
Then σ_V is Hausdorff Lipschitz continuous on the convex polyhedral set $\operatorname{dom}\sigma_V = Y_V$, i.e., there exist a constant $L > 0$ such that $d_H(\sigma_V(y), \sigma_V(\tilde{y})) \leq L \|y - \tilde{y}\|$ for all $y, \tilde{y} \in Y_V$ (d_H denoting the Hausdorff distance on subsets of $\mathbb{R}^m$).

Proof:
For b), apply Theorem 4.2 in [8]. For a), note that the set-valued mapping $y \mapsto M(y) := \{x \in X \cap \mathrm{cl}\, V \mid h(x) \geq y\}$ with closed convex graph is pseudo-Lipschitzian at each pair (y, x), $x \in M(y)$, $y \in Y_V$ [12]. Then Theorem 2.2 in [7] applies and provides that for $U := V_0 \cap V$ the solution set mapping σ_U is upper Hölder continuous at $\bar{y}$ with rate k^{-1}. Since $\sigma_V(\bar{y})$ is contained in V_0 and σ_V is upper semicontinuous at $\bar{y}$, we have that $\sigma_V(y) \subseteq V_0$ for all y close to $\bar{y}$. Hence, $\sigma_V(y) = \sigma_U(y)$ for all $y \in Y_V$ close to $\bar{y}$. □

Complementing part a), we note that in our applications the set V is an open neighbourhood of $\sigma_V(\bar{y})$ for some specific $\bar{y} \in Y_V$. Hence, the set V_0 in (ii) may be chosen as a subset of V. In this case, $\sigma_V(y)$ is contained in V for all y close to $\bar{y}$ and the proposition provides Hölder or Lipschitz continuity results for $y \mapsto \sigma(y) := \mathrm{argmin}\, \{g(x) \mid x \in X,\ h(x) \geq y\} = \sigma_V(y)$ at $\bar{y}$.

Growth conditions of the type used in (ii) are discussed in Section 4 of [9]. Corollary 16 of [9] states that growth conditions of some order $k \geq 1$ are available in case that the constraint set can be described by finitely many analytic functions and that the objective function is analytic as well. For more specific models, it is possible to characterize the growth order k more explicitly. For instance, in case of a quadratic objective function and polyhedral constraints one has $k = 2$ (Corollary 12 in [9] and Lemma 4.1 in [6]). Another instance with convex quadratic objective and (finitely many) quadratic constraints can be derived from Theorem 11 in [9] using the technique in the proof of Lemma 4.1 in [6].

Next we state our main result on quantitative stability of solution sets to (3).

Theorem 2.5 *In addition to the general assumptions, assume that*

(i) $\Psi(\mu)$ *is nonempty, and there exists an open, convex and bounded set* V *containing* $\Psi(\mu)$*;*

(ii) there exists an $\bar{x} \in X$ *such that* $F_\mu(h(\bar{x})) > p$ *(Slater condition);*

(iii) $\Psi(\mu) \cap \mathrm{argmin}\, \{g(x) \mid x \in X\} = \emptyset$ *(strict complementarity);*

(iv) F_μ^r *is strongly convex on some open, convex neighbourhood* U *of* $Y_V(\mu)$ *where* $r \in (-\infty, 0)$ *is chosen such that* μ *is* r*-concave;*

(v) σ_V *is upper Hölder continuous at some* $\bar{y} \in Y_V(\mu)$ *with rate* k^{-1} *for some* $k \geq 1$.

Then there exist constants $L > 0$ *and* $\delta > 0$ *such that*

$$\sup_{x \in \Psi_V(\nu)} d(x, \Psi(\mu)) \leq L\, d_K(\mu, \nu)^{(2k)^{-1}} \quad \text{whenever} \quad d_K(\mu, \nu) < \delta,$$

i.e., Ψ_V *is upper Hölder continuous at* μ *with rate* $(2k)^{-1}$ *(as a set-valued mapping from* $(\mathcal{P}(\mathbb{R}^s), d_K)$ *to* $\mathbb{R}^m$*).*

Proof:
With the notations from Lemma 2.3 we consider the problem

$$\min\{\pi_V(y) \mid y \in Y_V,\ F_\mu(y) \geq p\}$$

or, equivalently, with $b(y) := F_\mu^r(y) - p^r$

$$\min\{\pi_V(y) \mid y \in Y_V,\ b(y) \leq 0\}. \tag{4}$$

From Lemma 2.3 we know for the solution set $Y_V(\mu)$ of this problem that $\Psi(\mu) = \Psi_V(\mu) = \sigma_V(Y_V(\mu))$. Let $y_* \in Y_V(\mu)$ and $x_* \in \sigma_V(y_*)$. Then we have $x_* \in V$, $h(x_*) \geq y_*$ and

$$\begin{aligned} b(h(\lambda\bar{x} + (1-\lambda)x_*)) &= F_\mu^r(h(\lambda\bar{x} + (1-\lambda)x_*)) - p^r \leq F_\mu^r(\lambda h(\bar{x}) + (1-\lambda)h(x_*)) - p^r \\ &\leq \lambda(F_\mu^r(h(\bar{x})) - p^r) + (1-\lambda)(F_\mu^r(h(x_*)) - p^r) \\ &\leq \lambda(F_\mu^r(h(\bar{x})) - p^r) < 0 \end{aligned}$$

for all $\lambda \in (0,1]$. Here we used the concavity of the components of h, the monotonicity of F_μ and the r-concavity of μ. Now, we select $\hat{\lambda} \in (0,1]$ such that $\hat{\lambda}\bar{x} + (1-\hat{\lambda})x_* \in V$ and, hence $\hat{y} := h(\hat{\lambda}\bar{x} + (1-\hat{\lambda})x_*)$ belongs to Y_V and has the property $b(\hat{y}) < 0$. This Slater condition implies the existence of a Kuhn-Tucker multiplier $\lambda_* \geq 0$ for y_* in (4) such that

$$\pi_V(y_*) = \min\{\pi_V(y) + \lambda_* b(y) \mid y \in Y_V\} \quad \text{and} \quad \lambda_* b(y_*) = 0.$$

In case $\lambda_* = 0$, this would imply $y_* \in \operatorname{argmin}\{\pi_V(y) \mid y \in Y_V\}$ and, hence, we obtain for $x_* \in \sigma_V(y_*) \subseteq \Psi(\mu) \subset V$ that

$$\begin{aligned} g(x_*) &= \inf\{g(x) \mid x \in X \cap \operatorname{cl} V,\ h(x) \geq y_*\} = \pi_V(y_*) \\ &= \inf\{\pi_V(y) \mid y \in Y_V\} = \inf\{g(x) \mid x \in X \cap \operatorname{cl} V\} = \inf\{g(x) \mid x \in X\}, \end{aligned}$$

which contradicts *(iii)*. Here we have used that any minimizer $\hat{x} \in X \cap \operatorname{cl} V$ of g has the property $g(\hat{x}) \geq \pi_V(h(\hat{x})) \geq \pi_V(y_*) = g(x_*)$ and that x_* belongs to the open set V.

Thus $\lambda_* > 0$ and $\pi_V + \lambda_* b$ is strongly convex on $Y_V \cap U$ by *(iv)*. This implies that y_* is the unique minimizer of $\pi_V + \lambda_* b$ and that there exists a constant $\rho > 0$ such that

$$\rho\|y - y_*\|^2 \leq \pi_V(y) + \lambda_* b(y) - \pi_V(y_*) \tag{5}$$

for all $y \in Y_V \cap U$.

Let $\delta_0 \in (0,p)$ and $\nu \in \mathcal{P}(I\!R^s)$ such that $d_K(\mu,\nu) < \delta_0$. Then the constraint set $\{y \in Y_V \mid F_\nu(y) \geq p\}$ is contained in $\{y \in Y_V \mid F_\mu(y) \geq p - \delta_0\}$ and the latter set is bounded. Indeed, supposing unboundedness, there would exist a sequence (y_n) such that $y_n \in Y_V$, $F_\mu(y_n) \geq p - \delta_0$ and $\|y_n\| \to \infty$. Hence, there is a sequence (x_n) in $X \cap \operatorname{cl} V$ such that $h(x_n) \geq y_n$ and, since each component of h is bounded on bounded sets, each component of y_n is bounded from above. On the other hand, the condition $F_\mu(y_n) \geq p - \delta_0 > 0$, for each $n \in I\!N$, implies all components of y_n to be bounded from below due to F_μ being a distribution function. This contradicts $\|y_n\| \to \infty$.

Now, we appeal to Corollary 3.7 of [13] applied to problem (4) with an open bounded neighbourhood that contains the set $\{y \in Y_V \mid F_\mu(y) \geq p - \delta_0\}$ and conclude that the solution set mapping $Y_V(\cdot)$ is upper semicontinuous at μ (as a mapping from $(\mathcal{P}(\mathbb{R}^s), d_K)$ to $\mathbb{R}^s$). Hence, there exists a constant $\delta \in (0, \delta_0)$ such that the perturbed solution set $Y_V(\nu)$ is contained in the neighbourhood U of $Y_V(\mu) = \{y_*\}$ whenever $\nu \in \mathcal{P}(\mathbb{R}^s)$ with $d_K(\mu, \nu) < \delta$.

With the notations from Lemma 2.3 and using that $\Psi(\mu) = \sigma_V(y_*)$ we obtain for any $\nu \in \mathcal{P}(\mathbb{R}^s)$ with $d_K(\mu, \nu) < \delta$,

$$\sup_{x \in \Psi_V(\nu)} d(x, \Psi(\mu)) = \sup_{x \in \sigma_V(Y_V(\nu))} d(x, \sigma_V(y_*)) \leq \hat{L} \sup_{y \in Y_V(\nu)} \|y - y_*\|^{k^{-1}}, \tag{6}$$

where $\hat{L}$ is the Hölder constant of σ_V from *(v)*. Since $Y_V(\nu) \subseteq Y_V \cap U$, we may continue the estimate using (5) and obtain

$$\begin{aligned}
\sup_{x \in \Psi_V(\nu)} d(x, \Psi(\mu)) &\leq \hat{L} \sup_{y \in Y_V(\nu)} [\rho^{-1}(\pi_V(y) + \lambda_* b(y) - \pi_V(y_*))]^{(2k)^{-1}} \\
&= \hat{L}\rho^{-(2k)^{-1}} \sup_{y \in Y_V(\nu)} [\varphi_V(\nu) - \varphi(\mu) + \lambda_*(F_\mu^r(y) - p^r)]^{(2k)^{-1}} \\
&\leq \hat{L}\rho^{-(2k)^{-1}} \sup_{y \in Y_V(\nu)} [\varphi_V(\nu) - \varphi(\mu) + \lambda_*(F_\mu^r(y) - F_\nu^r(y))]^{(2k)^{-1}} \\
&\leq \hat{L}\rho^{-(2k)^{-1}} \sup_{y \in Y_V(\nu)} [|\varphi_V(\nu) - \varphi(\mu)| \\
&\quad + \lambda_*|r|(p - \delta_0)^{r-1}|F_\mu(y) - F_\nu(y)|]^{(2k)^{-1}} \\
&\leq \hat{L}\rho^{-(2k)^{-1}}[(L + \lambda_*|r|(p - \delta_0)^{r-1})d_K(\mu, \nu)]^{(2k)^{-1}},
\end{aligned}$$

where $L > 0$ is the constant from Proposition 2.1 and we used that $F_\nu^r(y) \leq p^r$ for any $y \in Y_V(\nu)$ and that the inequality

$$|u^r - v^r| \leq |r| \max\{u^{r-1}, v^{r-1}\}|u - v|$$

holds for any $u, v \in (0, 1]$. This completes the proof. □

Corollary 2.6 *Adopt the setting of the previous theorem, but replace condition (v) by the stronger assumption*

(v') σ_V is Hausdorff Hölder continuous at some $\bar{y} \in Y_V(\mu)$ with rate k^{-1} $(k \geq 1)$.

Then Ψ_V is Hausdorff Hölder continuous at μ with rate $(2k)^{-1}$.

Proof:
The only change in the proof of the theorem concerns the estimate

$$d_H(\Psi_V(\nu), \Psi(\mu)) = d_H(\sigma_V(Y_V(\nu)), \sigma_V(y_*)) \leq \hat{L} \sup_{y \in Y_V(\nu)} \|y - y_*\|^{k^{-1}}.$$

The rest of the proof remains unchanged. □

Corollary 2.7 *Let g be convex quadratic, h be linear, X be convex polyhedral and assume that $\Psi(\mu)$ is nonempty and bounded. Moreover, let the conditions (ii), (iii), (iv) of Theorem 2.5 be satisfied. Then, for any open convex bounded neighbourhood V of $\Psi(\mu)$ the closure $\mathrm{cl}\, V$ of which is polyhedral, the set-valued mapping Ψ_V is Hausdorff Hölder continuous at μ with rate 1/2.*

Proof:
Let V be open, convex, bounded and such that it contains $\Psi(\mu)$ and its closure is polyhedral. Then σ_V is Hausdorff Lipschitz continuous on $\mathrm{dom}\, \sigma_V = Y_V$ (Proposition 2.4) and, hence, the result is a consequence of Corollary 2.6 (for $k = 1$). $\square$

Corollary 2.7 essentially recovers Theorem 4.3 in [6] in a slightly improved formulation. In particular, it provides Hausdorff Hölder stability of (global) solution sets in case that X is bounded.

The conditions *(i)-(iv)* imposed in Theorem 2.5 concern the original problem (3). Conditions *(ii)* and *(iii)* reflect the significance of an appropriate choice of the probability level p. They represent natural conditions from a modelling point of view. The strong convexity condition *(iv)* of F_μ^r forms a local property around the singleton $Y_V(\mu) = \{y_*\}$. Since condition *(iii)* implies that $F_\mu(y_*) = p$, a sufficient condition for *(iv)* is the strong convexity of F_μ^r on any convex bounded subset of the interior of the support of μ. Although no general result in this direction is available so far, it is worth noting that the uniform distribution on rectangles and the one-dimensional normal distribution satisfy this sufficient condition for *(iv)*. Condition *(v)* contains in a condensed form the assumptions on the (deterministic) data g, h and X of (3).

The following example shows that the result of Corollary 2.7 is lost if *(iv)* is violated.

Example 2.8 *In (3) put $m = s = 2$, $g(x_1, x_2) = x_1 + x_2$, $h(x_1, x_2) = (x_1, x_2)$, $X = [0,1]^2$, $p = 1/4$ and $\mu \in \mathcal{P}(I\!R^2)$ be the uniform distribution on the triangle $\mathrm{conv}\,\{(1,0), (0,1), (1,1)\}$. The distribution function F_μ of μ has the following form*

$$F_\mu(x_1,x_2) = \begin{cases} 1 & , \ x_1, x_2 \geq 1 \\ (x_1 + x_2 - 1)^2 & , \ x_1 + x_2 \geq 1 \text{ and } x_1, x_2 \in [0,1], \\ x_1^2 & , \ x_2 \geq 1 \text{ and } x_1 \in [0,1], \\ x_2^2 & , \ x_1 \geq 1 \text{ and } x_2 \in [0,1] \\ 0 & , \ \text{else} \end{cases}$$

Hence, F_μ is constant on the line segments $\{(x_1,x_2) \in [0,1]^2 \mid x_1 + x_2 = a\}$ with $a \in [0,1]$ (see Fig. 1). Therefore, F_μ^r is not strongly convex on any convex subset of the interior of D for any $r < 0$, although μ is r-concave for any such r (cf. Appendix A). Furthermore, one easily checks that all the remaining assumptions of Theorem 2.5 are satisfied. We have $\Psi(\mu) = \{(x_1,x_2) \in [0,1]^2 \mid x_1 + x_2 = 3/2\}$ and $\varphi(\mu) = 3/2$. Let $\tilde{\mu}$ be the uniform distribution on $[1/2,1]^2$ and consider the perturbed probability measures $\mu_\lambda = (1-\lambda)\mu + \lambda\tilde{\mu}$, $\lambda \in [0,1]$. Then we obtain $\Psi(\mu_\lambda) = \{(3/4, 3/4)\}$ (emphasized points in Fig. 1) and $d_K(\mu, \mu_\lambda) = \lambda\, d(\mu, \tilde{\mu}) \leq \lambda$ for each $\lambda \in (0,1]$. Evidently, we have

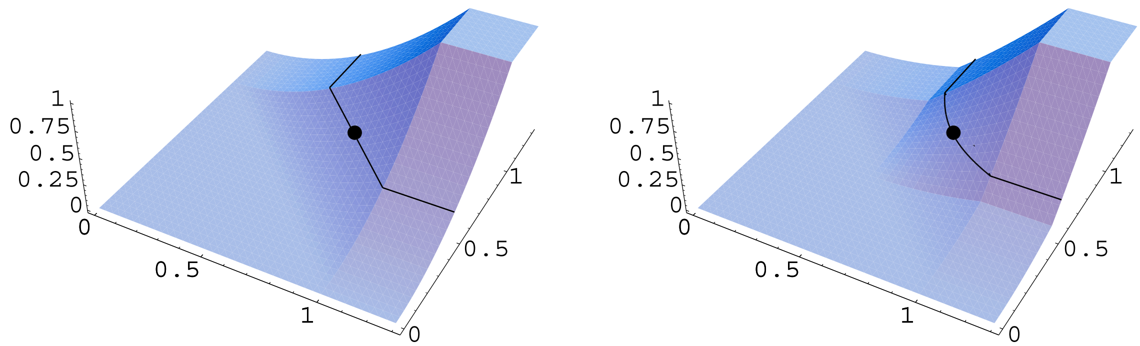

Figure 1: Distribution function F_μ for the uniform distribution on the right upper triangle conv $\{(1,0),(0,1),(1,1)\}$ (left) and distribution function F_{μ_λ} ($\lambda = 0.5$) for the perturbed measure. The level lines $F_\mu(x_1, x_2) = p$ and $F_{\mu_\lambda}(x_1, x_2) = p$ are indicated on both graphs.

$\Psi(\mu_\lambda) \subseteq \Psi(\mu)$ for each $\lambda \in (0, 1]$, but

$$d_H(\Psi(\mu), \Psi(\mu_\lambda)) = \sup_{x \in \Psi(\mu)} \|x - (3/4, 3/4)\| = \sqrt{2}/4 \quad \textit{for any } \lambda \in (0, 1],$$

and indeed the result of Corollary 2.7 gets lost.

The next result seems to support the conjecture that the upper Hölder continuity rate in Theorem 2.5 might be improved. However, in Example 2.10, we provide a counterexample showing that the rates in Theorem 2.5 and its corollaries are the best possible.

Proposition 2.9 *Adopt the setting of Theorem 2.5 and let $s = 1$ (the case of a one-dimensional random variable). Then, Ψ_V is upper Hölder continuous at μ with rate k^{-1}.*

Proof:
Referring back to the proof of Theorem 2.5, we see that (y_* being a minimizer of (4))

$$\pi_V(y) \geq \pi_V(y_*) \quad \forall y \in Y_V, b(y) \leq 0. \tag{7}$$

From the strict complementarity ($\lambda_* > 0$) it follows that $b(y_*) = 0$. For the 'Slater point' $\hat{y} \in Y_V$ with $b(\hat{y}) < 0$, one may suppose without loss of generality that $\hat{y} > y_*$ due to the one-dimensionality of the y-variables assumed above. Then the convexity of b implies

$$b(y) \leq 0 \;\Rightarrow\; y \geq y_* \tag{8}$$

and, furthermore, $\pi_V(\hat{y}) \geq \pi_V(y_*)$ due to (7). Consequently, $\pi_V(y) \geq \pi_V(y_*)$ for all $y \geq y_*$, $y \in Y_V$ by convexity of π_V. Now, there must exist some $y' \in Y_V$ such that

$y' < y_*$ and $\pi_V(y') < \pi_V(y_*)$, since otherwise one would arrive at $y_* \in \operatorname{argmin}\{\pi_V(y) \mid y \in Y_V\}$ contradicting assumption *(iii)* of Theorem 2.5 (see proof). Finally, we consider an arbitrary $y \in Y_V$ with $y > y_*$. From the convexity of π_V and from $y' < y_* < y$, one derives that

$$\pi_V(y_*) = \pi_V\left(\frac{y - y_*}{y - y'}y' + \frac{y_* - y'}{y - y'}y\right) \leq \frac{y - y_*}{y - y'}\pi_V(y') + \frac{y_* - y'}{y - y'}\pi_V(y),$$

which gives

$$\pi_V(y) \geq \pi_V(y_*) + \frac{\pi_V(y_*) - \pi_V(y')}{y_* - y'}(y - y_*) \quad \forall y \in Y_V, y > y_*.$$

Now, (8) allows to write this as

$$\pi_V(y) \geq \pi_V(y_*) + \rho\|y - y_*\| \quad \forall y \in Y_V, b(y) \leq 0$$

with some $\rho > 0$. Using this global linear growth of π_V in contrast to its local quadratic growth in (5), one may directly apply Theorem 2.2 in [7] to the parametrization of problem (4)

$$\min\{\pi_V(y) \mid y \in Y_V,\, b_\nu(y) \leq 0\}, \quad b_\nu(y) := F_\nu^r(y) - p^r \tag{9}$$

accompanied by the same upper-semicontinuity argument as in the proof of Theorem 2.5 (following (5)), Hence, $Y_V(\cdot)$ is upper Lipschitz continuous at μ. Appealing to *(v)*, the result follows from (6). □

In the last proposition, the one-dimensionality of the random variable was substantially exploited. Below, we construct a two-dimensional counterexample showing that, in general, one cannot expect a Lipschitz-like behavior of the solution set under the assumptions of Corollary 2.7, hence the result stated there is sharp. This example even lives in the class of probability measures having a density and a (globally) Lipschitzian distribution function (both the original and the perturbed measures). It has to be noted that such a counterexample is easily constructed in a non-probabilistic setting. To find a counterexample, where in particular assumptions *(iii)* and *(iv)* have to be fulfilled, requires a more sophisticated construction. The details of verification in this example are therefore left to the Appendix B.

Example 2.10 *In (3) put* $m = s = 2$, $h(x_1, x_2) = (x_1, x_2)$, $g(x_1, x_2) = x_1 + x_2$, $X = [0,1]^2$, *μ is the uniform distribution on X and* $p = 1/4$. *Then,* $F_\mu(x_1, x_2) = x_1 x_2$ *(for* $(x_1, x_2) \in X$*),* $\Psi(\mu) = \{(0.5, 0.5)\} \neq \operatorname{argmin}\{g(x) \mid x \in X\} = \{(0,0)\}$, *$\mu$ is an r-concave measure for $r < 0$ and F_μ^{-1} is strongly convex on an open convex neighbourhood of $\Psi(\mu)$. Finally, there exists a Slater point (e.g. $\bar{x} = (1,1)$), hence, all assumptions of Corollary 2.7 are satisfied. We define a perturbed measure $\nu_\varepsilon \in \mathcal{P}(\mathbb{R}^2)$ depending on $\varepsilon > 0$ via the* *following density:*

$$f_\varepsilon(x_1,x_2) = \begin{cases} 1-\varepsilon & (x_1,x_2) \in A := [0,a_\varepsilon] \times [0,a_\varepsilon] \\ \eta_\varepsilon(x_1) & (x_1,x_2) \in B := [a_\varepsilon,b_\varepsilon] \times [0,a_\varepsilon] \\ \eta_\varepsilon(x_2) & (x_1,x_2) \in C := [0,a_\varepsilon] \times [a_\varepsilon,b_\varepsilon] \\ 1 & (x_1,x_2) \in D := (b_\varepsilon,1] \times [0,a_\varepsilon] \\ 1 & (x_1,x_2) \in E := [0,a_\varepsilon] \times (b_\varepsilon,1] \\ \frac{5(1-\varepsilon)-4\sqrt{1-\varepsilon}}{(2\sqrt{1-\varepsilon}-1)^2} & (x_1,x_2) \in F := (a_\varepsilon,1] \times (a_\varepsilon,1] \\ 0 & (x_1,x_2) \notin X \end{cases}$$

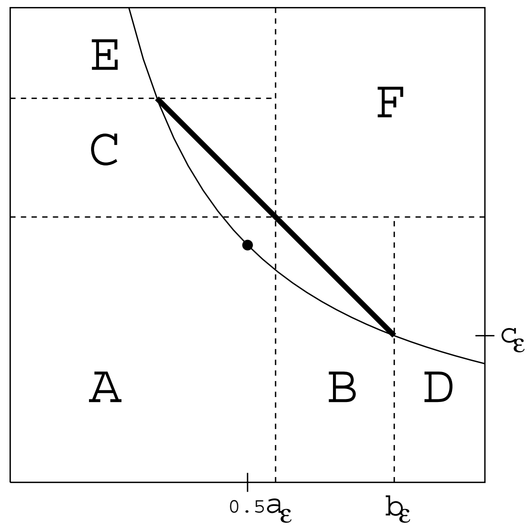

Figure 2: Partition of the unit square in the counterexample. The curve represents the level line of the unperturbed probabilistic constraint, and the dot corresponds to the optimal solution. After perturbation, the level line is deformed such as to contain a linear piece which becomes the optimal solution set then.

Here,

$$a_\varepsilon = \frac{1}{2\sqrt{1-\varepsilon}},\ b_\varepsilon = a_\varepsilon(1+\sqrt{\varepsilon}),\ c_\varepsilon = a_\varepsilon(1-\sqrt{\varepsilon}),\ \eta_\varepsilon(t) = \frac{1}{4(2a_\varepsilon - t)^2}\ (t \in [a_\varepsilon, b_\varepsilon]).$$

First, the correctness of the definition has to be checked: it is easily seen, that for $\varepsilon < 9/25$, *it holds that* $0.5 \le a_\varepsilon \le b_\varepsilon \le 1$, *that* η_ε *is well-defined and non-negative on* $[a_\varepsilon, b_\varepsilon]$ *and that* f_ε *is non-negative on the domains* A *and* F. *It is shown in Appendix B, that the integral of* f_ε *over* X *equals one for all these* ε*-values, hence the* f_ε *may be indeed considered as densities for perturbed probability measures, and evidently, for* $\varepsilon \downarrow 0$, *the* f_ε *converge pointwise towards the density* f *(being the characteristic function of* X*) of the original measure* μ.
In Appendix B, the following relations are verified:

$$\|F_\mu - F_{\nu_\varepsilon}\|_\infty < \rho\varepsilon \quad \text{for some } \rho > 0 \text{ and all } \varepsilon < \varepsilon_0 \tag{10}$$

$$\Psi_{\nu_\varepsilon} = [(c_\varepsilon, b_\varepsilon), (b_\varepsilon, c_\varepsilon)] \quad \text{(line segment in Fig. 2)} \tag{11}$$

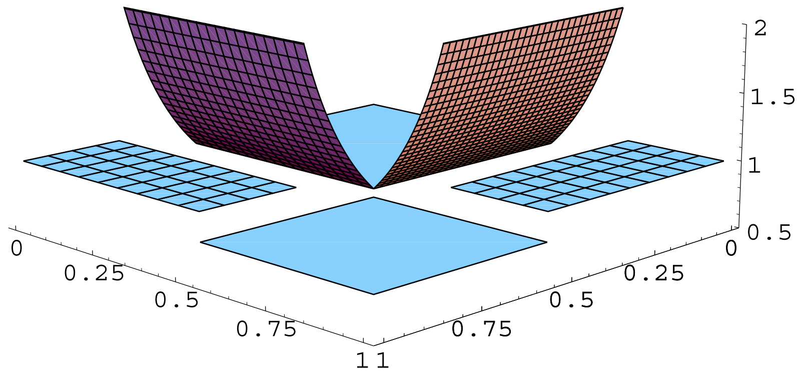

Figure 3: Perturbed density f_ε viewed at from the right upper angle of the unit square

From (11) it follows that

$$d_H(\Psi(\mu), \Psi(\nu_\varepsilon)) = \|(0.5, 0.5) - (b_\varepsilon, c_\varepsilon)\| = \sqrt{\frac{1-\sqrt{1-\varepsilon}}{1-\varepsilon}}$$

In particular, one has $\lim_{\varepsilon \downarrow 0} d_H(\Psi(\mu), \Psi(\nu_\varepsilon)) = 0$*, hence, for each open neighbourhood* V *of* $\Psi(\mu)$ *it holds that* $\Psi_V(\nu_\varepsilon) = \Psi(\nu_\varepsilon)$ *with sufficiently small* ε*. Supposed the stability result of Corollary 2.7 would hold with rate 1. Then, for* $\varepsilon < \min\{\delta/\rho, \varepsilon_0\}$*, (10) would yield the contradiction*

$$\sqrt{\frac{1-\sqrt{1-\varepsilon}}{1-\varepsilon}} = d_H(\Psi(\mu), \Psi(\nu_\varepsilon)) = d_H(\Psi(\mu), \Psi_V(\nu_\varepsilon)) \le L\|F_\mu - F_{\nu_\varepsilon}\|_\infty \le L\rho\varepsilon.$$

Appendix A: r-concave probability measures

Here we introduce the notion of an r-concave probability measure for some $r \in [-\infty, \infty]$ and discuss some essential properties. We start with the definition of the generalized mean function m_r on $I\!R_+ \times I\!R_+ \times [0,1]$:

$$m_r(a,b;\lambda) = \begin{cases} (\lambda a^r + (1-\lambda)b^r)^{1/r} & \text{if } \; r \in (0,\infty) \text{ or } r \in (-\infty, 0), ab > 0 \\ 0 & \text{if } \; ab = 0, r \in (-\infty, 0) \\ a^\lambda b^{1-\lambda} & \text{if } \; r = 0 \\ \max\{a,b\} & \text{if } \; r = \infty \\ \min\{a,b\} & \text{if } \; r = -\infty \end{cases} \tag{12}$$

The measure $\mu \in \mathcal{P}(I\!R^s)$ is called *r-concave* ([2]) for some $r \in [-\infty, \infty]$, if the inequality

$$\mu(\lambda B_1 + (1-\lambda)B_2) \ge m_r(\mu(B_1), \mu(B_2); \lambda) \tag{13}$$

holds for all $\lambda \in [0,1]$ and all convex subsets B_1, B_2 of $I\!R^s$. For $r = 0$ and $r = -\infty$, μ is also called logarithmic concave and quasi-concave, respectively ([10]). Since $m_r(a,b;\lambda)$

is increasing in r if all the other variables are fixed, the sets $\mathcal{M}_r(I\!R^s)$ of all r-concave probability measures are increasing if r is decreasing, i.e., we have for all $-\infty < r_1 \leq r_2 < \infty$ that

$$\mathcal{M}_{-\infty}(I\!R^s) \supseteq \mathcal{M}_{r_1}(I\!R^s) \supseteq \mathcal{M}_{r_2}(I\!R^s) \supseteq \mathcal{M}_{\infty}(I\!R^s). \tag{14}$$

For the particular case of cells $B = y + I\!R^s_-$, $y \in I\!R^s$, and for $r \in (-\infty, 0)$ the inequality implies that the distribution function F_μ has the property that the extended real-valued function F^r_μ is convex on $I\!R^s$. Moreover, (13) and (14) imply that F_μ is quasi-concave on $I\!R^s$.

A useful criterion for r-concavity is known from [2], [3], [10] (for $r = 0$) and [11]. It says that a measure $\mu \in \mathcal{P}(I\!R^s)$ is r-concave for some $r \in [-\infty, s^{-1}]$ if μ has a density f_μ such that

$$f_\mu(\lambda y + (1-\lambda)\tilde{y}) \geq m_{r(s)}(f_\mu(y), f_\mu(\tilde{y}); \lambda),$$

holds for all $\lambda \in [0,1]$ and $y, \tilde{y} \in I\!R^s$ where $r(s) = r(1-rs)^{-1}$. For example, the uniform distribution (on any bounded convex subset of $I\!R^s$), the (nondegenerate) multivariate normal distribution, the Dirichlet distribution, the multivariate Student and Pareto distributions belong to $\mathcal{M}_r(I\!R^s)$ for some $r \in (-\infty, \infty]$ (cf. [2], [11]). For more information on all this, proofs and details we refer to Chapter 4 of [11].

Appendix B: Verification of Example 2.10

Estimation of the maximal difference between F_μ and F_{ν_ε} (see (10))

We assume that $\varepsilon < 9/25$ according to the remarks in Example 2.10.

ad A: Over A, both F_μ and F_{ν_ε} have constant densities, hence the maximal deviation occurs at the right upper corner $(a_\varepsilon, a_\varepsilon)$:

$$\|F_\mu - F_{\nu_\varepsilon}\|^A_\infty = F_\mu(a_\varepsilon, a_\varepsilon) - F_{\nu_\varepsilon}(a_\varepsilon, a_\varepsilon) = a_\varepsilon^2 - a_\varepsilon^2(1-\varepsilon) = \frac{\varepsilon}{4(1-\varepsilon)} \leq \varepsilon \quad \text{for } \varepsilon \leq 3/4 \tag{15}$$

ad B: For $(x_1, x_2) \in B$ one has

$$\begin{aligned} F_{\nu_\varepsilon}(x_1, x_2) &= x_2 a_\varepsilon(1-\varepsilon) + \int_0^{x_2}\int_{a_\varepsilon}^{x_1} \eta_\varepsilon(\xi) d\xi_1 d\xi_2 = x_2\left(\sqrt{1-\varepsilon}/2 + \left[\frac{1}{4(2a_\varepsilon - \xi_1)}\right]_{a_\varepsilon}^{x_1}\right) \\ &= \frac{x_2}{4\left(\frac{1}{\sqrt{1-\varepsilon}} - x_1\right)} \end{aligned} \tag{16}$$

For $x_1 \in [a_\varepsilon, b_\varepsilon]$ it follows

$$x_1 - \frac{1}{4\left(\frac{1}{\sqrt{1-\varepsilon}} - x_1\right)} \geq 0,$$

where equality occurs exactly at $x_1 = b_\varepsilon$. Consequently, due to $x_2 \geq 0$, it holds that

$$F_\mu(x_1, x_2) - F_{\nu_\varepsilon}(x_1, x_2) =$$
$$x_2 \left(x_1 - \frac{1}{4\left(\frac{1}{\sqrt{1-\varepsilon}} - x_1\right)} \right) \begin{cases} \geq 0 & (x_1, x_2) \in B \\ = 0 & (x_1, x_2) \in B, (x_1 = b_\varepsilon \text{ or } x_2 = 0) \end{cases} \tag{17}$$

In particular, the maximal deviation over B computes as the maximum of the above (nonnegative) difference. This maximum is assumed over B at the point $(1/\sqrt{1-\varepsilon} - 1/2, a_\varepsilon)$, and it realizes the value $(1 - \sqrt{1-\varepsilon})/(2(1-\varepsilon))$, which for $\varepsilon = 0$ equals zero and the derivative w.r.t. ε of which equals $1/4$ at $\varepsilon = 0$. Thus, there exists an ε_1 with

$$\|F_\mu - F_{\nu_\varepsilon}\|_\infty^B \leq \varepsilon \quad \text{for } \varepsilon < \varepsilon_1. \tag{18}$$

ad C: symmetric with B

ad D: For $(x_1, x_2) \in D$ the definition of the density and (17) imply:

$$F_\mu(x_1, x_2) - F_{\nu_\varepsilon}(x_1, x_2) = x_1 x_2 - (F_{\nu_\varepsilon}(b_\varepsilon, x_2) + (x_1 - b_\varepsilon)x_2) = x_1 x_2 - b_\varepsilon x_2 - (x_1 - b_\varepsilon)x_2 = 0.$$

ad E: symmetric with D. In particular, F_μ and F_{ν_ε} coincide on D and E (see Fig. 4).

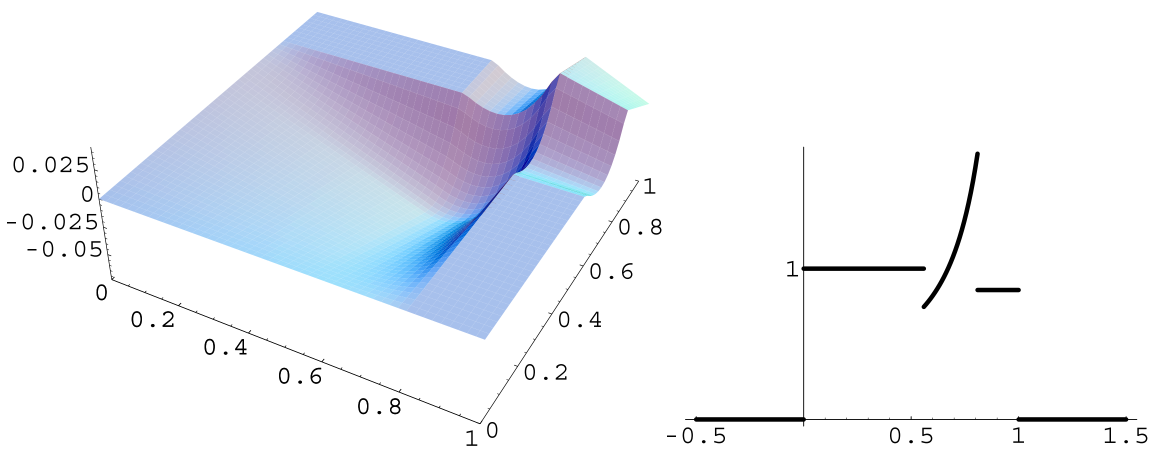

Figure 4: Graph of $F_{\nu_\varepsilon} - F_\mu$ (left) and marginal density (right) for the perturbed measure ν_ε

ad F: For $(x_1, x_2) \in F$, one gets

$$F_\mu(x_1, x_2) - F_{\nu_\varepsilon}(x_1, x_2) =$$
$$F_\mu(x_1, a_\varepsilon) + F_\mu(a_\varepsilon, x_2) - F_\mu(a_\varepsilon, a_\varepsilon) + (x_1 - a_\varepsilon)(x_2 - a_\varepsilon)$$
$$-F_{\nu_\varepsilon}(x_1, a_\varepsilon) - F_{\nu_\varepsilon}(a_\varepsilon, x_2) + F_{\nu_\varepsilon}(a_\varepsilon, a_\varepsilon) - p_\varepsilon(x_1 - a_\varepsilon)(x_2 - a_\varepsilon),$$

where p_ε briefly denotes the constant density of ν_ε on F. By comparison of terms located on top of each other, the previous estimations on A,B,C,D and E provide:

$$\|F_\mu - F_{\nu_\varepsilon}\|_\infty^F \leq 3\varepsilon + |1-p_\varepsilon| \quad \text{for } \varepsilon < \min\{3/4, \varepsilon_1\}$$

Since $p_\varepsilon \leq 1$ and by $1-p_0 = 0$ along with $(dp_\varepsilon/d\varepsilon)(0) = -1$, one arrives at $(1-p_\varepsilon) \leq 2\varepsilon$ for $\varepsilon < \varepsilon_2$. Consequently,

$$\|F_\mu - F_{\nu_\varepsilon}\|_\infty^F \leq 5\varepsilon \quad \text{for } \varepsilon < \varepsilon_3 := \min\{3/4, \varepsilon_1, \varepsilon_2\} \tag{19}$$

ad $(x_1,x_2) \notin X$: We distinguish the four cases $x_1 < 0$ or $x_2 < 0$, $x_1 > 1$ and $x_2 \in [0,1]$, $x_2 > 1$ and $x_1 \in [0,1]$, $x_1 > 1$ and $x_2 > 1$. Exploiting the fact that both the original and the perturbed densities are zero here, one concludes that $F_\mu = F_{\nu_\varepsilon}$ in the first case and that $F_\mu(x_1,x_2) - F_{\nu_\varepsilon}(x_1,x_2) = F_\mu(1,x_2) - F_{\nu_\varepsilon}(1,x_2)$ in the second case, where now the results concerning the regions D and F may be applied. The third case is symmetric to the second one. Finally, we have $F_\mu = F_{\nu_\varepsilon}$ in the fourth case again, once we know that $F_{\nu_\varepsilon}(1,1) = 1$. This last property would simultaneously confirm that ν_ε is a probability measure for all admissible $\varepsilon < 9/25$. Indeed, from the previous representations it follows that

$$\begin{aligned} F_{\nu_\varepsilon}(1,1) &= F_{\nu_\varepsilon}(1,a_\varepsilon) + F_{\nu_\varepsilon}(a_\varepsilon,1) - F_{\nu_\varepsilon}(a_\varepsilon,a_\varepsilon) + p_\varepsilon(1-a_\varepsilon)^2 \\ &= 2a_\varepsilon - (1-\varepsilon)a_\varepsilon^2 + \frac{5(1-\varepsilon) - 4\sqrt{1-\varepsilon}}{(2\sqrt{1-\varepsilon}-1)^2}(1-a_\varepsilon)^2 = -1/4 + 5/4 = 1, \end{aligned}$$

Summarizing, the combination of the estimations (15), (18) and (19) leads to 1. with $\rho = 5$ and $\varepsilon_0 := \min\{\varepsilon_3, 9/25\}$.

Characterization of the perturbed chance constraint (see (11))

In order to verify (11), we define the continuous function

$$\theta(t) := \begin{cases} \frac{1}{4t} & t \in (0, c_\varepsilon] \cup [b_\varepsilon, \infty) \\ 2a_\varepsilon - t & t \in [c_\varepsilon, b_\varepsilon] \end{cases},$$

the graph of which is the thick line joined with the curve over E and D in Fig. 2. We claim that $F_{\nu_\varepsilon}(x_1,x_2) = p = 0.25\ \forall (x_1,x_2) \in X \cap \operatorname{Gph}\theta$. By $b_\varepsilon c_\varepsilon = 0.25$, one gets $x_2 = \theta(x_1) = (4x_1)^{-1} \geq b_\varepsilon$ for $x_1 \leq c_\varepsilon$ and analogously $x_2 = \theta(x_1) \leq c_\varepsilon$ for $x_1 \geq b_\varepsilon$. Consequently, for such x_1, the $(x_1,x_2) \in X \cap \operatorname{Gph}\theta$ belong to the regions D or E, where, according to the previous section F_μ and F_{ν_ε} coincide. Therefore, these points fulfill: $F_{\nu_\varepsilon}(x_1,x_2) = x_1x_2 = x_1\theta(x_1) = 0.25$.

In the case $x_1 \in [c_\varepsilon, b_\varepsilon]$ the $(x_1,x_2) \in X \cap \operatorname{Gph}\theta$ belong to the regions C or B. For $(x_1,x_2) \in B$ one has according to (16):

$$F_{\nu_\varepsilon}(x_1,x_2) = F_{\nu_\varepsilon}(x_1,\theta(x_1)) = F_{\nu_\varepsilon}(x_1, 2a_\varepsilon - x_1) = \frac{2a_\varepsilon - x_1}{4(\frac{1}{\sqrt{1-\varepsilon}} - x_1)} = \frac{2a_\varepsilon - x_1}{4(2a_\varepsilon - x_1)} = 0.25.$$

The case $(x_1, x_2) \in C$ follows analogously for symmetry reasons.

For $(x_1, x_2) \in X \setminus \mathrm{Gph}\,\theta$ the strict positivity of the density (or alternatively the previous statements on F_{ν_ε}) imply that $F_{\nu_\varepsilon}(x_1, x_2) > 0.25$ for $(x_1, x_2) \in X \cap \mathrm{int\,epi}\,\theta$ (interior of the epigraph), whereas $F_{\nu_\varepsilon}(x_1, x_2) < 0.25$ for $(x_1, x_2) \in X \setminus \mathrm{epi}\,\theta$. Summarizing, one obtains that the perturbed chance constraint $X \cap \{(x_1, x_2) \mid F_{\nu_\varepsilon}(x_1, x_2) \geq 0.25\}$ coincides with $X \cap \mathrm{epi}\,\theta$. This immediately entails the correctness of the representation of the perturbed solution set in (11), since the line segment mentioned there has the same direction as the level sets of the linear goal function g.

Properties of the approximating densities and distribution functions

According to the definition of the perturbed density, the maximum deviation between perturbed and original density occurs (among others) at the point $(b_\varepsilon, 0)$ where it calculates as

$$\eta_\varepsilon(b_\varepsilon) - 1 = \frac{2\sqrt{\varepsilon}}{(1 - \sqrt{\varepsilon})^2}$$

This shows that, for $\varepsilon \downarrow 0$, the densities converge uniformly with rate 1/2.

Concerning the perturbed distribution functions, it is important to note that they are globally Lipschitzian as is the original distribution function. This follows from the perturbed marginal densities being bounded (cf. Proposition 3.8 in [14]). Indeed, for an admissible value of $\varepsilon > 0$, an upper bound of the marginal density (which is the same for x_1 and x_2 due to symmetry) is given by $(1/4)a_\varepsilon(2a_\varepsilon - b_\varepsilon)^2 + (1 - a_\varepsilon)p_\varepsilon$. This is the peak-value of the curvilinear part of the perturbed marginal density illustrated in Fig. 4 which for $\varepsilon \downarrow 0$ converges towards one.

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