NUMERICAL METHODS FOR NONLINEAR PARABOLIC EQUATIONS WITH SMALL PARAMETER BASED ON PROBABILITY APPROACH

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ABSTRACT. The probabilistic approach is used for constructing special layer methods to solve the Cauchy problem for semilinear parabolic equations with small parameter. In spite of the probabilistic nature these methods are nevertheless deterministic. The algorithms are tested by simulating the Burgers equation with small viscosity and the generalized KPP-equation with a small parameter.

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1. Introduction

Nonlinear partial differential equations (PDE) are usually not susceptible of analytic solutions and mostly investigated by means of numerical methods. Numerical methods used for solving PDE are traditionally based on deterministic approaches (see, e.g., [22, 23, 26] and references therein). A class of layer methods intended to solve semilinear parabolic equations is introduced in [18], where the well-known probabilistic representations of solutions to linear parabolic equations and the ideas of weak sense numerical integration of stochastic differential equations (SDE) are used to construct numerical algorithms. In spite of the probabilistic nature these methods are nevertheless deterministic.

Nonlinear parabolic equations with small parameter arise in a variety of applications (see, e.g., [2, 4, 8, 11, 25] and references therein). For instance, they are used in gas dynamics, when one has to take into account small viscosity and small heat conductivity. Some problems of combustion are described by PDE with small parameter. They also arise as the result of introducing artificial viscosity in systems of first-order hyperbolic equations that is one of the popular approaches to numerical solving of inviscid problems of gas dynamics [20, 24, 29].

Here we construct some layer methods for solving the Cauchy problem for semilinear parabolic equations with small parameter of the form

$$\frac{\partial u}{\partial t}+\frac{\varepsilon^2}{2}\sum_{i,j=1}^{d}a^{ij}(t,x,u)\frac{\partial^2 u}{\partial x^i\partial x^j}+\sum_{i=1}^{d}(b^i(t,x,u)+\varepsilon^2 c^i(t,x,u))\frac{\partial u}{\partial x^i}+g(t,x,u)=0, \tag{1.1}$$

$$t\in[t_0,T),\ x\in R^d,$$

$$u(T,x)=\varphi(x). \tag{1.2}$$

The probabilistic representations of the solution to the problem (1.1)-(1.2) are connected with systems of SDE with small noise. Namely for such systems, special weak approximations are proposed in [19]. Applying these special approximations, we get new layer methods intended to solve the Cauchy problem (1.1)-(1.2).

As it turned out, if the solution of (1.1)-(1.2) is regular, errors of the proposed methods have the form of $O(h^p+\varepsilon^l h^q)$, $p>q$, $l>0$, h is a step of time discretization. Owing to the fact that the accuracy order of such methods is equal to a comparatively small q, the methods are not too complicated. But due to the large p and the small factor ε^l at h^q, their errors are fairly low and therefore these methods are highly efficient. The singular case, when derivatives of the solution go to infinity as $\varepsilon\to 0$, requires a special theoretical investigation. For the equations of a particular type, we give the corresponding theoretical results. As to the equation (1.1) of the general type, we restrict ourselves to the analysis of the one-step error. Further theoretical investigations should rest on a stability analysis and on particular properties of the solution. However, we test the methods constructed here on model problems for which shock waves are observed. The tests give quite good results not only in simulations of wave formation, that corresponds to the regular case, but also in simulations of wave propagation, i.e., in the singular case. The reasons of these experimental facts and possible ways to get realistic errors of the methods in the singular case are discussed.

Section 2 gives some results from [18, 19] used in this paper. In Section 3, new implicit and explicit layer methods for the problem (1.1)-(1.2) are proposed. Some theorems on their rates of convergence both in the regular and in the singular cases are proved. For implementation of the layer methods, we need in a space discretization. The numerical algorithms based on the proposed layer methods and on the linear interpolation are constructed in Section 4. For the sake of simplicity in writing, Sections 3 and 4 deal with the one-dimensional case of the problem (1.1)-(1.2). In Section 5, extensions for the multi-dimensional case and for a system of reaction-diffusion equations are given. In Sections 6 and 7, we propose layer methods in two particular cases of the problem (1.1)-(1.2). Together with two-layer methods, some three-layer methods are obtained in Section 6. For constructing a layer method in Section 7, we attract the exact simulation of the Brownian motion [17] instead of the weak simulation used in Sections 2-6 to approximate SDE arising in the probabilistic representations.

All the numerical algorithms presented in the paper are tested through computer experiments. Some results of numerical tests on the Burgers equation with small viscosity and on the generalized KPP-equation with a small parameter are given in Section 8.

This paper is devoted to initial value problems. Boundary value problems for nonlinear parabolic equations with small parameter will be considered in a separate work. The probability approach to linear boundary value problems is treated in [15, 16, 17].

2. Preliminaries

Here we give, in the required form, some results from [18, 19] which are used in the next sections.

2.1. Probabilistic approach to constructing numerical methods for semilinear PDE. Let the Cauchy problem (1.1)-(1.2) have the unique solution $u = u(t,x)$ which is sufficiently smooth and satisfies some needed conditions of boundedness (see the corresponding theoretical results, e.g., in [13, 28]). If we substitute $u = u(t,x)$ in the coefficients of (1.1), we obtain a linear parabolic equation with small parameter. The solution to this linear equation has the following probabilistic representation

$$u(t,x) = E(\varphi(X_{t,x}(T)) + Z_{t,x,0}(T)),\ t \le T,\ x \in R^d, \tag{2.1}$$

where $X_{t,x}(s)$, $Z_{t,x,z}(s)$, $s \ge t$, is the solution to the Cauchy problem for the system of stochastic differential equations

$$dX = (b(s,X,u(s,X)) + \varepsilon^2 c(s,X,u(s,X)))ds \tag{2.2}$$
$$+\varepsilon\sigma(s,X,u(s,X))dw,\ X(t) = x,$$

$$dZ = g(s,X,u(s,X))ds,\ Z(t) = z. \tag{2.3}$$

Here $w(s) = (w^1(s), \dots, w^d(s))^\top$ is a d-dimensional standard Wiener process, $b(s,x,u)$ and $c(s,x,u)$ are d-dimensional column-vectors compounded from the coefficients $b^i(s,x,u)$ and $c^i(s,x,u)$ of (1.1), $\sigma(s,x,u)$ is a $d\times d$-matrix obtained from the equation $a(s,x,u) = \sigma(s,x,u)\sigma^\top(s,x,u)$, where $a = \{a^{ij}\}$; the equation is solvable with respect to σ (for instance, by a lower triangular matrix) at least in the case of a positively definite a.

Note that to simplify the notation we write $u(t,x)$ instead of $u(t,x;\varepsilon)$ and $X_{t,x}(s)$ instead of $X^{\varepsilon}_{t,x}(s)$ in the whole paper.

Introduce a discretization, for definiteness the equidistant one:

$$T = t_N > t_{N-1} > \cdots > t_0 = t, \quad h := \frac{T - t_0}{N}.$$

Note that all the methods given in the paper can be easily adapted for a nonequidistant discretization. For instance, we use a variable discretization step h in some our numerical tests (see Section 8.1).

Because the right-hand side of the equation (2.3) is independent of Z, we have from (2.1):

$$u(t_k, x) = E(\varphi(X_{t_k,x}(T)) + Z_{t_k,x,0}(T)) \tag{2.4}$$

$$= E(\varphi(X_{t_{k+1},X_{t_k,x}(t_{k+1})}(T)) + Z_{t_{k+1},X_{t_k,x}(t_{k+1}),Z_{t_k,x,0}(t_{k+1})}(T))$$

$$= EE(\varphi(X_{t_{k+1},X_{t_k,x}(t_{k+1})}(T)) + Z_{t_{k+1},X_{t_k,x}(t_{k+1}),Z_{t_k,x,0}(t_{k+1})}(T) \diagup X_{t_k,x}(t_{k+1})), Z_{t_k,x,0}(t_{k+1}))$$

$$= E(u(t_{k+1}, X_{t_k,x}(t_{k+1})) + Z_{t_k,x,0}(t_{k+1})).$$

In accordance with the probabilistic approach to constructing numerical methods for semilinear PDE from [18], the ideas of weak sense numerical integration of SDE [12, 14, 21] are attracted to obtain some approximate relations from (2.2)-(2.4). The relations allow us to express approximations $\bar{u}(t_k, x)$ of the solution $u(t_k, x)$ in terms of $\bar{u}(t_{k+1}, x)$ recurrently, i.e., to construct some layer methods which are discrete in the variable t only. To make clear the approach, it is relevant to derive one of the methods from [18] which is used in this paper in a broad fashion. For simplicity in writing, we restrict ourselves to the case $d = 1$.

Applying the explicit weak Euler scheme with the simplest noise simulation [12, 14, 21] to the system (2.2)-(2.3), we get

$$X_{t_k,x}(t_{k+1}) \simeq \bar{X}_{t_k,x}(t_{k+1}) = x + hb(t_k, x, u(t_k, x)) + \varepsilon^2 hc(t_k, x, u(t_k, x)) \tag{2.5}$$

$$+\varepsilon h^{1/2}\sigma(t_k, x, u(t_k, x))\xi_k,$$

$$Z_{t_k,x,z}(t_{k+1}) \simeq \bar{Z}_{t_k,x,z}(t_{k+1}) = z + hg(t_k, x, u(t_k, x)),$$

where $\xi_{N-1}, \xi_{N-2}, \ldots, \xi_0$ are i.i.d. random variables with the law $P(\xi = \pm 1) = 1/2$.

Using (2.4)-(2.5), we obtain

$$u(t_k, x) \simeq E(u(t_{k+1}, \bar{X}_{t_k,x}(t_{k+1})) + \bar{Z}_{t_k,x,0}(t_{k+1})) \tag{2.6}$$

$$= \frac{1}{2}u(t_{k+1}, x + hb(t_k, x, u(t_k, x)) + \varepsilon^2 hc(t_k, x, u(t_k, x)) + \varepsilon h^{1/2}\sigma(t_k, x, u(t_k, x)))$$

$$+\frac{1}{2}u(t_{k+1}, x + hb(t_k, x, u(t_k, x)) + \varepsilon^2 hc(t_k, x, u(t_k, x)) - \varepsilon h^{1/2}\sigma(t_k, x, u(t_k, x)))$$

$$+hg(t_k, x, u(t_k, x)).$$

Thus, we can calculate the approximations $\bar{u}(t_k, x)$ layerwise:

$$\bar{u}(t_N, x) = \varphi(x), \tag{2.7}$$

$$\bar{u}(t_k, x) =$$

$$= \frac{1}{2}\bar{u}(t_{k+1}, x + hb(t_k, x, \bar{u}(t_k, x)) + \varepsilon^2 hc(t_k, x, \bar{u}(t_k, x)) + \varepsilon h^{1/2}\sigma(t_k, x, \bar{u}(t_k, x)))$$

$$+\frac{1}{2}\bar{u}(t_{k+1}, x + hb(t_k, x, \bar{u}(t_k, x)) + \varepsilon^2 hc(t_k, x, \bar{u}(t_k, x)) - \varepsilon h^{1/2}\sigma(t_k, x, \bar{u}(t_k, x)))$$

$$+hg(t_k, x, \bar{u}(t_k, x)), \quad k = N-1, \dots, 1, 0.$$

The method (2.7) is an implicit layer method for solving the Cauchy problem (1.1)-(1.2). This method is deterministic though the probabilistic approach is used for its construction.

Applying the method of simple iteration to (2.7) with $\bar{u}(t_{k+1}, x)$ as a null iteration, we get the first iteration (we denote it as $\bar{u}(t_k, x)$ again):

$$\bar{u}(t_N, x) = \varphi(x), \tag{2.8}$$

$$\bar{u}(t_k, x) =$$

$$= \frac{1}{2}\bar{u}(t_{k+1}, x + hb(t_k, x, \bar{u}(t_{k+1}, x)) + \varepsilon^2 hc(t_k, x, \bar{u}(t_{k+1}, x)) + \varepsilon h^{1/2}\sigma(t_k, x, \bar{u}(t_{k+1}, x)))$$

$$+\frac{1}{2}\bar{u}(t_{k+1}, x + hb(t_k, x, \bar{u}(t_{k+1}, x)) + \varepsilon^2 hc(t_k, x, \bar{u}(t_{k+1}, x)) - \varepsilon h^{1/2}\sigma(t_k, x, \bar{u}(t_{k+1}, x)))$$

$$+hg(t_k, x, \bar{u}(t_{k+1}, x)), \quad k = N-1, \dots, 1, 0.$$

According to the propositions proved in [18], this explicit layer method has the one-step error estimated by $O(h^2)$ and the global error $O(h)$.

Of course, all the results from [18] can be applied to solving the problem with small parameter (1.1)-(1.2). But since the probabilistic representation of the solution to (1.1)-(1.2) is connected with the system of differential equations with small noise (2.2)-(2.3), one can expect that use of weak approximations for SDE with small noise [19] leads to new effective methods for solving (1.1)-(1.2).

2.2. Some weak approximations for SDE with small noise. Here we recall some weak approximations for SDE with small noise

$$dX = b(t, X)dt + \varepsilon^2 c(t, X)dt + \varepsilon\sigma(t, X)dw(t), \; X(t_0) = x, \tag{2.9}$$

$$t \in [t_0, T], \quad 0 \le \varepsilon \le \varepsilon_0,$$

where X, $b(t,x)$, and $c(t,x)$ are d-dimensional column-vectors, $\sigma(t,x)$ is a $d \times m$-matrix, $w(s) = (w^1(t), \dots, w^m(t))^\top$ is an m-dimensional standard Wiener process, ε_0 is a positive number. The coefficients are supposed to satisfy the corresponding conditions of smoothness and boundedness.

The one-step error ρ and the global error R of a weak approximation $X_k = \bar{X}_{t,x}(t_k)$ at the point (t, x) are defined as

$$\rho = |Ef(X_{t,x}(t+h)) - Ef(\bar{X}_{t,x}(t+h))|,$$

$$R = |Ef(X_{t,x}(T)) - Ef(\bar{X}_{t,x}(T))|,$$

where $f(x)$ is a function belonging to a sufficiently wide class (see details, e.g., in [12, 14, 21]).

Below we write down a number of weak schemes from [14] and [19] which are used in the next sections.

The Euler scheme:

$$X_{k+1} = X_k + hb(t_k, X_k) + \varepsilon^2 hc(t_k, X_k) + \varepsilon h^{1/2}\sigma(t_k, X_k)\xi_k, \tag{2.10}$$

$$\rho = O(h^2),\ R = O(h),$$

where $\xi_k = (\xi_k^1, \dots, \xi_k^m)$ are i.i.d. m-dimensional vectors with i.i.d. components and each component is distributed by the law $P(\xi = \pm 1) = 1/2$.

The Runge-Kutta scheme with error $O(h^2 + \varepsilon^2 h)$:

$$X_{k+1} = X_k + \frac{1}{2}hb(t_k, X_k) + \frac{1}{2}hb(t_{k+1}, X_k + hb(t_k, X_k)) \tag{2.11}$$

$$+\varepsilon^2 hc(t_k, X_k) + \varepsilon h^{1/2}\sigma(t_k, X_k)\xi_k,$$

$$\rho = O(h^3 + \varepsilon^2 h^2),\ R = O(h^2 + \varepsilon^2 h),$$

where ξ_k are the same as in (2.10).

The special second-order Runge-Kutta scheme (for SDE with small additive noise, i.e., σ is a constant matrix; $c \equiv 0$):

$$X_{k+1} = X_k + \frac{1}{2}hb(t_k, X_k) + \frac{1}{2}hb(t_{k+1}, X_k + hb(t_k, X_k) + \varepsilon h^{1/2}\sigma\xi_k)) + \varepsilon h^{1/2}\sigma\xi_k, \tag{2.12}$$

$$\rho = O(h^3),\ R = O(h^2),$$

where $\xi_k = (\xi_k^1, \dots, \xi_k^m)$ are i.i.d. m-dimensional vectors with i.i.d. components and each component is distributed by the law $P(\xi = 0) = 2/3$, $P(\xi = \pm\sqrt{3}) = 1/6$.

The special Runge-Kutta scheme with error $O(h^4 + \varepsilon^2 h^2)$ (for SDE with small additive noise, i.e., σ is a constant matrix; $c \equiv 0$):

$$X_{k+1} = X_k + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + \varepsilon h^{1/2}\sigma\xi_k, \tag{2.13}$$

$$k_1 = hb(t_k, X_k),\ k_2 = hb(t_{k+1/2}, X_k + k_1/2),$$

$$k_3 = hb(t_{k+1/2}, X_k + k_2/2 + \varepsilon h^{1/2}\sigma\xi_k),\ k_4 = hb(t_{k+1}, X_k + k_3 + \varepsilon h^{1/2}\sigma\xi_k),$$

$$\rho = O(h^5 + \varepsilon^2 h^3),\ R = O(h^4 + \varepsilon^2 h^2),$$

where ξ_k are the same as in (2.12).

Note that in Sections 6 and 7 we also attract some other special weak approximations.

3. Methods for a general semilinear parabolic equation with small parameter

Here we construct new layer methods for semilinear parabolic equations with small parameter (1.1)-(1.2) using the probabilistic approach from [18] and applying specific weak approximations from [19] to the system with small noise (2.2)-(2.3).

3.1. Implicit layer method. For the sake of simplicity in writing, let us consider the Cauchy problem (1.1)-(1.2) under $d = 1$:

$$\frac{\partial u}{\partial t} + \frac{\varepsilon^2}{2}\sigma^2(t, x, u)\frac{\partial^2 u}{\partial x^2} + (b(t, x, u) + \varepsilon^2 c(t, x, u))\frac{\partial u}{\partial x} + g(t, x, u) = 0, \tag{3.1}$$

$$t \in [t_0, T),\ x \in R^1,$$

$$(3.2) \qquad u(T,x) = \varphi(x).$$

The probabilistic representation of the solution $u(t,x)$ to this problem has the form (2.1)-(2.3) with $d=1$.

Applying the Runge-Kutta scheme (2.11) to the system (2.2)-(2.3), we get

$$(3.3) \qquad X_{t_k,x}(t_{k+1}) \simeq \bar{X}_{t_k,x}(t_{k+1}) = x + \frac{1}{2}hb_k + \frac{1}{2}hb(t_{k+1}, x+hb_k, u(t_{k+1}, x+hb_k))$$

$$+\varepsilon^2 hc_k + \varepsilon h^{1/2}\sigma_k\xi_k,$$

$$Z_{t_k,x,z}(t_{k+1}) \simeq \bar{Z}_{t_k,x,z}(t_{k+1}) = z + \frac{1}{2}hg_k + \frac{1}{2}hg(t_{k+1}, x+hb_k, u(t_{k+1}, x+hb_k)),$$

where b_k, c_k, σ_k, and g_k are the coefficients b, c, σ, and g calculated at the point $(t_k, x, u(t_k,x))$.

Using the probabilistic representation (2.4), we obtain

$$u(t_k,x) \simeq E(u(t_{k+1}, \bar{X}_{t_k,x}(t_{k+1})) + \bar{Z}_{t_k,x,0}(t_{k+1}))$$

$$= \frac{1}{2}u(t_{k+1}, x + h[b_k + b(t_{k+1}, x+hb_k, u(t_{k+1}, x+hb_k))]/2 + \varepsilon^2 hc_k + \varepsilon h^{1/2}\sigma_k)$$

$$+\frac{1}{2}u(t_{k+1}, x + h[b_k + b(t_{k+1}, x+hb_k, u(t_{k+1}, x+hb_k))]/2 + \varepsilon^2 hc_k - \varepsilon h^{1/2}\sigma_k)$$

$$+\frac{1}{2}hg_k + \frac{1}{2}hg(t_{k+1}, x+hb_k, u(t_{k+1}, x+hb_k)).$$

We can approximate $u(t_k,x)$ by $v(t_k,x)$ found from

$$(3.4) \qquad v(t_k,x) =$$

$$= \frac{1}{2}u(t_{k+1}, x + h[\tilde{b}_k + b(t_{k+1}, x+h\tilde{b}_k, u(t_{k+1}, x+h\tilde{b}_k))]/2 + \varepsilon^2 h\tilde{c}_k + \varepsilon h^{1/2}\tilde{\sigma}_k)$$

$$+\frac{1}{2}u(t_{k+1}, x + h[\tilde{b}_k + b(t_{k+1}, x+h\tilde{b}_k, u(t_{k+1}, x+h\tilde{b}_k))]/2 + \varepsilon^2 h\tilde{c}_k - \varepsilon h^{1/2}\tilde{\sigma}_k)$$

$$+\frac{1}{2}h\tilde{g}_k + \frac{1}{2}hg(t_{k+1}, x+h\tilde{b}_k, u(t_{k+1}, x+h\tilde{b}_k)),$$

where $\tilde{b}_k$, $\tilde{c}_k$, $\tilde{\sigma}_k$, and $\tilde{g}_k$ are the coefficients b, c, σ, and g calculated at the point $(t_k, x, v(t_k,x))$.

The corresponding implicit layer method has the form

$$(3.5) \qquad \bar{u}(t_N,x) = \varphi(x),$$

$$\bar{u}(t_k,x) = \frac{1}{2}\bar{u}(t_{k+1}, x + h[\bar{b}_k + b(t_{k+1}, x+h\bar{b}_k, \bar{u}(t_{k+1}, x+h\bar{b}_k))]/2 + \varepsilon^2 h\bar{c}_k + \varepsilon h^{1/2}\bar{\sigma}_k)$$

$$+\frac{1}{2}\bar{u}(t_{k+1}, x + h[\bar{b}_k + b(t_{k+1}, x+h\bar{b}_k, \bar{u}(t_{k+1}, x+h\bar{b}_k))]/2 + \varepsilon^2 h\bar{c}_k - \varepsilon h^{1/2}\bar{\sigma}_k)$$

$$+\frac{1}{2}h\bar{g}_k + \frac{1}{2}hg(t_{k+1}, x+h\bar{b}_k, \bar{u}(t_{k+1}, x+h\bar{b}_k)),\ k = N-1, \ldots, 0,$$

where $\bar{b}_k$, $\bar{c}_k$, $\bar{\sigma}_k$, and $\bar{g}_k$ are the coefficients b, c, σ, and g calculated at the point $(t_k, x, \bar{u}(t_k,x))$.

3.2. Convergence theorem in the regular case. Let us assume that

(i) The coefficients $b(t,x,u)$ and $g(t,x,u)$ and their first and second derivatives are continuous and uniformly bounded, the coefficients $c(t,x,u)$ and $\sigma(t,x,u)$ and their first derivatives are continuous and uniformly bounded:

$$
\begin{aligned}
&\left|\frac{\partial^{i+j+l} b}{\partial t^i \partial x^j \partial u^l}\right| \le K, \ \left|\frac{\partial^{i+j+l} g}{\partial t^i \partial x^j \partial u^l}\right| \le K, \ \ 0 \le i+j+l \le 2, \\
&\left|\frac{\partial^{i+j+l} c}{\partial t^i \partial x^j \partial u^l}\right| \le K, \ \left|\frac{\partial^{i+j+l} \sigma}{\partial t^i \partial x^j \partial u^l}\right| \le K, \ \ 0 \le i+j+l \le 1, \\
&t_0 \le t \le T, \ x \in R^1, \ u_\circ < u < u^\circ,
\end{aligned}
\tag{3.6}
$$

where $-\infty \le u_\circ, \ u^\circ \le \infty$ are some constants.

(ii) There exists the only bounded solution $u(t,x)$ to the problem (3.1)-(3.2) such that

$$
u_\circ < u_* \le u(t,x) \le u^* < u^\circ, \tag{3.7}
$$

where $u_*, \ u^*$ are some constants, and there exist the uniformly bounded derivatives:

$$
\begin{aligned}
\left|\frac{\partial^{i+j} u}{\partial t^i \partial x^j}\right| &\le K, \ i=0, \ j=1,2,3,4; \ i=1, \ j=0,1,2; \ i=2, \ j=0,1; \\
i &= 3, \ j=0; \ \ t_0 \le t \le T, \ x \in R^1, \ 0 < \varepsilon \le \varepsilon^*.
\end{aligned}
\tag{3.8}
$$

Below, in Section 3.4, we consider the singular case, when the condition (3.8) is not fulfilled.

In Lemma 3.1 and Theorem 3.1 we use the letters K and C without any index for various constants which do not depend on $h, \ k, \ x, \ \varepsilon$.

Lemma 3.1. *Under the assumptions (i) - (ii), the one-step error of the implicit layer method* (3.5) *is estimated by* $O(h^3 + \varepsilon^2 h^2)$, *i.e.,*

$$
|v(t_k,x) - u(t_k,x)| \le C \cdot (h^3 + \varepsilon^2 h^2),
$$

where $v(t_k,x)$ *is found from* (3.4), C *does not depend on* $h, \ k, \ x, \ \varepsilon$.

Proof. Introduce the function

$$
U_{t_k,x}(v) :=
$$

$$
\begin{aligned}
:= &\frac{1}{2} u(t_{k+1}, x + h[\breve{b}_k + b(t_{k+1}, x + h\breve{b}_k, u(t_{k+1}, x + h\breve{b}_k))]/2 + \varepsilon^2 h \breve{c}_k + \varepsilon h^{1/2} \breve{\sigma}_k) \\
+ &\frac{1}{2} u(t_{k+1}, x + h[\breve{b}_k + b(t_{k+1}, x + h\breve{b}_k, u(t_{k+1}, x + h\breve{b}_k))]/2 + \varepsilon^2 h \breve{c}_k - \varepsilon h^{1/2} \breve{\sigma}_k) \\
&+ \frac{1}{2} h \breve{g}_k + \frac{1}{2} h g(t_{k+1}, x + h\breve{b}_k, u(t_{k+1}, x + h\breve{b}_k)),
\end{aligned}
$$

where $\breve{b}_k, \ \breve{c}_k, \ \breve{\sigma}_k$, and $\breve{g}_k$ are the coefficients b, c, σ, and g calculated at (t_k, x, v).

To prove the lemma, we make use of the method of simple iteration. Define the sequence

$$
v^{(i)}(t_k,x) := U_{t_k,x}(v^{(i-1)}(t_k,x)), \ \ i = 1,2,\dots,
$$

and take $u(t_k,x)$ as a null iteration:

$$
v^{(0)}(t_k,x) = u(t_k,x).
$$

Firstly we prove that

$$|v^{(1)}(t_k,x)-v^{(0)}(t_k,x)|=|v^{(1)}(t_k,x)-u(t_k,x)|\leq C\cdot(h^3+\varepsilon^2h^2). \tag{3.9}$$

We have

$$v^{(1)}(t_k,x)= \tag{3.10}$$

$$=\frac{1}{2}u(t_{k+1},x+h[b_k+b(t_{k+1},x+hb_k,u(t_{k+1},x+hb_k))]/2+\varepsilon^2hc_k+\varepsilon h^{1/2}\sigma_k)$$

$$+\frac{1}{2}u(t_{k+1},x+h[b_k+b(t_{k+1},x+hb_k,u(t_{k+1},x+hb_k))]/2+\varepsilon^2hc_k-\varepsilon h^{1/2}\sigma_k)$$

$$+\frac{1}{2}hg_k+\frac{1}{2}hg(t_{k+1},x+hb_k,u(t_{k+1},x+hb_k)),$$

where $b_k,\ c_k,\ \sigma_k$, and g_k are the coefficients b,c,σ, and g calculated at the point $(t_k,x,u(t_k,x))$.

Using the assumptions (i)-(ii), we expand the functions u and g :

$$u(t_{k+1},x+hb_k/2+hb(t_{k+1},x+hb_k,u(t_{k+1},x+hb_k))/2+\varepsilon^2hc_k\pm\varepsilon h^{1/2}\sigma_k)$$

$$=u(t_k,x)+\frac{\partial u}{\partial t}h+\frac{\partial u}{\partial x}\cdot(b_kh+\frac{1}{2}\frac{\partial b}{\partial t}h^2+\frac{1}{2}\frac{\partial b}{\partial x}b_kh^2+\frac{1}{2}\frac{\partial b}{\partial u}\frac{\partial u}{\partial t}h^2$$

$$+\frac{1}{2}\frac{\partial b}{\partial u}\frac{\partial u}{\partial x}b_kh^2+\varepsilon^2c_kh\pm\varepsilon\sigma_kh^{1/2})+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}h^2+\frac{\partial^2 u}{\partial t\partial x}\cdot(b_kh^2\pm\varepsilon\sigma_kh^{3/2})$$

$$+\frac{\partial^2 u}{\partial x^2}\cdot(\frac{1}{2}\varepsilon^2\sigma_k^2h+\frac{1}{2}b_k^2h^2\pm\varepsilon h^{1/2}\sigma_k\cdot(b_kh+\varepsilon^2c_kh))$$

$$\pm\frac{1}{6}\frac{\partial^3 u}{\partial x^3}\varepsilon^3\sigma_k^3h^{3/2}+O(h^3+\varepsilon^2h^2),$$

$$g(t_{k+1},x+hb_k,u(t_{k+1},x+hb_k))=g_k+\frac{\partial g}{\partial t}h+\frac{\partial g}{\partial x}b_kh+\frac{\partial g}{\partial u}\frac{\partial u}{\partial t}h$$

$$+\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}b_kh+O(h^2),$$

where the derivatives of u are calculated at the point (t_k,x), the derivatives of the coefficients b and g are calculated at the point $(t_k,x,u(t_k,x))$, and

$$|O(h^3+\varepsilon^2h^2)|\leq C\cdot(h^3+\varepsilon^2h^2),\ |O(h^2)|\leq C\cdot h^2. \tag{3.11}$$

Substituting these expansions in (3.10), we get

$$v^{(1)}(t_k,x)=u(t_k,x)+h\cdot(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\cdot(b_k+\varepsilon^2c_k)+\frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial x^2}\sigma_k^2+g_k)$$

$$+\frac{1}{2}h^2\cdot(\frac{\partial^2 u}{\partial t^2}+\frac{\partial b}{\partial t}\frac{\partial u}{\partial x}+\frac{\partial b}{\partial u}\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}+b_k\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial g}{\partial t}+\frac{\partial g}{\partial u}\frac{\partial u}{\partial t})$$

$$+\frac{1}{2}b_kh^2\cdot(\frac{\partial^2 u}{\partial t\partial x}+\frac{\partial b}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial b}{\partial u}\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+b_k\frac{\partial^2 u}{\partial x^2}+\frac{\partial g}{\partial x}+\frac{\partial g}{\partial u}\frac{\partial u}{\partial x})$$

$$+O(h^3+\varepsilon^2h^2).$$

Then, adding and taking off the appropriate terms of the order $O(\varepsilon^2 h^2)$ in the above expression, we obtain

$$v^{(1)}(t_k,x) = u(t_k,x) + h\cdot(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\cdot(b_k+\varepsilon^2 c_k) + \frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial x^2}\sigma_k^2 + g_k) \tag{3.12}$$

$$+\frac{1}{2}h^2\frac{\partial}{\partial t}(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\cdot(b+\varepsilon^2 c) + \frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2 + g)$$

$$+\frac{1}{2}b_k h^2\frac{\partial}{\partial x}(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\cdot(b+\varepsilon^2 c) + \frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial x^2}\sigma^2 + g) + O(h^3+\varepsilon^2 h^2),$$

where, after differentiation, all the expressions are calculated at the point (t_k,x).

Taking into account that $u(t,x)$ is the solution of the problem (3.1)-(3.2), the relation (3.12) implies

$$v^{(1)}(t_k,x) = u(t_k,x) + O(h^3+\varepsilon^2 h^2).$$

The estimate (3.9) is proved.

Clearly, under a sufficiently small h we obtain

$$u_\circ < -Ch^2(h+\varepsilon) + u_* \le v^{(1)}(t_k,x) \le u^* + Ch^2(h+\varepsilon) < u^\circ,\ x\in R^1.$$

Using the assumptions (i)-(ii), we get

$$|v^{(2)}(t_k,x) - v^{(1)}(t_k,x)| = |U_{t_k,x}(v^{(1)}(t_k,x)) - U_{t_k,x}(v^{(0)}(t_k,x))|$$

$$\le Kh|v^{(1)}(t_k,x) - v^{(0)}(t_k,x)|,$$

whence it follows that

$$|v^{(2)}(t_k,x) - v^{(0)}(t_k,x)| \le (1+Kh)\,|v^{(1)}(t_k,x) - v^{(0)}(t_k,x)|.$$

It is not difficult to show that there exists a sufficiently small h such that the procedure can be continued infinitely (i.e., $u_\circ < v^{(i)}(t_k,x) < u^\circ,\ i=2,3,\dots$) and

$$|v^{(n)}(t_k,x) - v^{(n-1)}(t_k,x)| \le (Kh)^{n-1}|v^{(1)}(t_k,x) - v^{(0)}(t_k,x)|,$$

$$|v^{(n)}(t_k,x) - v^{(0)}(t_k,x)| \le \frac{1-(Kh)^n}{1-Kh}\,|v^{(1)}(t_k,x) - v^{(0)}(t_k,x)|,\ n=1,2,3,\dots.$$

Further, we prove by the usual arguments that there is a unique root of the equation

$$v(t_k,x) = U_{t_k,x}(v(t_k,x))$$

such that

$$|v(t_k,x) - v^{(0)}(t_k,x)| \le \frac{1}{1-Kh}|v^{(1)}(t_k,x) - v^{(0)}(t_k,x)|. \tag{3.13}$$

Substituting (3.9) in (3.13), we come to the statement of the lemma. Lemma 3.1 is proved.

Let us prove the following theorem on global convergence.

Theorem 3.1. *Under the assumptions (i) - (ii), the global error of the implicit layer method (3.5) is estimated by $O(h^2+\varepsilon^2 h)$:*

$$|u(t_k,x) - \bar{u}(t_k,x)| \le K\cdot(h^2+\varepsilon^2 h),$$

where the constant K does not depend on h, k, x, ε.

Proof. We follow the proof of the corresponding theorem in [18].

Denote the error of the method (3.5) to the k-th step ($(N-k)$-th layer) as

$$R(t_k,x) := \bar{u}(t_k,x) - u(t_k,x). \tag{3.14}$$

Introduce the notation

$$X_{k+1}^{(\pm)} := x + \frac{1}{2}h\bar{b}_k + \frac{1}{2}hb(t_{k+1}, x + h\bar{b}_k, \bar{u}(t_{k+1}, x + h\bar{b}_k)) + \varepsilon^2 h\bar{c}_k \pm \varepsilon h^{1/2}\bar{\sigma}_k,$$

where (we remember) $\bar{b}_k$, $\bar{c}_k$, and $\bar{\sigma}_k$ are the coefficients $b(t,x,u)$, $c(t,x,u)$, and $\sigma(t,x,u)$ calculated at $t=t_k$, $x=x$, $u=\bar{u}(t_k,x)=u(t_k,x)+R(t_k,x)$.

Using this notation, (3.5), and (3.14), we get

$$u(t_k,x) + R(t_k,x) = \bar{u}(t_k,x) \tag{3.15}$$

$$= \frac{1}{2}\bar{u}(t_{k+1}, X_{k+1}^{(+)}) + \frac{1}{2}\bar{u}(t_{k+1}, X_{k+1}^{(-)}) + \frac{1}{2}h\bar{g}_k + \frac{1}{2}hg(t_{k+1}, x + h\bar{b}_k, \bar{u}(t_{k+1}, x + h\bar{b}_k))$$

$$= \frac{1}{2}u(t_{k+1}, X_{k+1}^{(+)}) + \frac{1}{2}u(t_{k+1}, X_{k+1}^{(-)}) + \frac{1}{2}R(t_{k+1}, X_{k+1}^{(+)}) + \frac{1}{2}R(t_{k+1}, X_{k+1}^{(-)})$$

$$+\frac{1}{2}h\bar{g}_k + \frac{1}{2}hg(t_{k+1}, x + h\bar{b}_k, \bar{u}(t_{k+1}, x + h\bar{b}_k)).$$

Clearly, $R(t_N,x)=0$. Below we prove recurrently that $R(t_k,x)$, $k=N-1,\dots,0$, is sufficiently small under a small h. Using the assumption (3.7), we shall be able to justify the following suggestion, in which we need now: the value $u(t_k,x)+R(t_k,x)$ remains in the interval $(u_\circ, u^\circ)$ under h small enough.

We have

$$\bar{b}_k = b(t_k, x, \bar{u}(t_k,x)) = b(t_k, x, u(t_k,x) + R(t_k,x))$$

$$= b(t_k, x, u(t_k,x)) + \Delta b = b_k + \Delta b,$$

where $b_k := b(t_k,x,u(t_k,x))$ and due to the assumption (i) Δb satisfies the inequality

$$|\Delta b| \le K|R(t_k,x)|. \tag{3.16}$$

Analogously,

$$\bar{c}_k = c_k + \Delta c,\ |\Delta c| \le K|R(t_k,x)|,\ \bar{\sigma}_k = \sigma_k + \Delta\sigma,\ |\Delta\sigma| \le K|R(t_k,x)|, \tag{3.17}$$

$$\bar{g}_k = g_k + \Delta g,\ |\Delta g| \le K|R(t_k,x)|.$$

Using (3.16)-(3.17) and (3.14), we get

$$X_{k+1}^{(\pm)} = x + \frac{1}{2}h(b_k + \Delta b) + \tag{3.18}$$

$$+\frac{1}{2}hb(t_{k+1}, x + h[b_k + \Delta b], u(t_{k+1}, x + h[b_k + \Delta b]) + R(t_{k+1}, x + h\bar{b}_k))$$

$$+\varepsilon^2 h[c_k + \Delta c] \pm \varepsilon h^{1/2}[\sigma_k + \Delta\sigma]$$

$$= x + \frac{1}{2}hb_k + \frac{1}{2}hb(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k)) + \varepsilon^2 hc_k \pm \varepsilon h^{1/2}\sigma_k$$

$$+h[\Delta b/2 + \varepsilon^2\Delta c] \pm \varepsilon h^{1/2}\Delta\sigma + h^2\Delta_1 + h\Delta_2$$

and

$$\frac{1}{2}h\bar{g}_k + \frac{1}{2}hg(t_{k+1}, x + h\bar{b}_k, \bar{u}(t_{k+1}, x + h\bar{b}_k)) \tag{3.19}$$

$$= \frac{1}{2}hg_k + \frac{1}{2}h\Delta g$$

$$+\frac{1}{2}hg(t_{k+1}, x + h[b_k + \Delta b], u(t_{k+1}, x + h[b_k + \Delta b]) + R(t_{k+1}, x + h\bar{b}_k))$$

$$= \frac{1}{2}hg_k + \frac{1}{2}hg(t_{k+1}, x + hb_k, u(t_{k+1}, x + hb_k)) + \frac{1}{2}h\Delta g + h^2\Delta_3 + h\Delta_4,$$

where $|\Delta_1|, |\Delta_3| \le K|R(t_k, x)|$ and $|\Delta_2|, |\Delta_4| \le K|R(t_{k+1}, x + h\bar{b}_k)|$.

Substituting (3.18) and (3.19) in (3.15) and expanding the functions $\frac{1}{2}u(t_{k+1}, X_{k+1}^{(\pm)})$, it is not difficult to obtain:

$$u(t_k, x) + R(t_k, x) = v^{(1)}(t_k, x) + \frac{1}{2}R(t_{k+1}, X_{k+1}^{(+)}) + \frac{1}{2}R(t_{k+1}, X_{k+1}^{(-)}) + r(t_k, x) + \tilde{r}(t_k, x),$$

where $v^{(1)}(t_k, x)$ is defined in (3.10) and

$$|r(t_k, x)| \le Kh|R(t_{k+1}, x + h\bar{b}_k)|, \ |\tilde{r}(t_k, x)| \le Kh|R(t_k, x)|.$$

Then by (3.9) we get

$$R(t_k, x) = \frac{1}{2}R(t_{k+1}, X_{k+1}^{(+)}) + \frac{1}{2}R(t_{k+1}, X_{k+1}^{(-)}) + r(t_k, x) + \tilde{r}(t_k, x) + O(h^3 + \varepsilon^2 h^2). \tag{3.20}$$

Introduce the notation

$$R_k := \max_{-\infty < x < +\infty} |R(t_k, x)|.$$

We have from (3.20):

$$R_N = 0, \quad R_k \le R_{k+1} + KhR_k + KhR_{k+1} + C \cdot (h^3 + \varepsilon^2 h^2), \ k = N-1, \dots, 0.$$

Hence

$$R_k \le \frac{C}{2K}\left(\left(\frac{1 + Kh}{1 - Kh}\right)^N - 1\right) \cdot (h^2 + \varepsilon^2 h),$$

and therefore (remember $N = (T - t_0)/h$)

$$R_k \le \frac{C}{2K}(e^{4K(T - t_0)} - 1) \cdot (h^2 + \varepsilon^2 h).$$

Theorem 3.1 is proved.

Remark 3.1. For linear parabolic equations, i.e., when the coefficients of (3.1) do not depend on u, the method (3.5) becomes the explicit one with the global error $O(h^2 + \varepsilon^2 h)$ and can be applied to solving linear parabolic equations with small parameter. Note also that if the dimension d of the linear problem is high ($d > 3$ in practice) and it is enough to find the solution in a few points only, the Monte-Carlo technique is preferable.

3.3. Explicit layer methods. For implementation of the implicit method (3.5), one can use the method of simple iteration. If we take $u(t_{k+1}, x)$ as a null iteration, in the case of $b(t, x, u) \neq b(t, x)$ or $g(t, x, u) \neq g(t, x)$ the first iteration provides the one-step error $O(h^2)$ only. One can show that applying the second iteration we get $O(h^3 + \varepsilon^2 h^2)$ as the one-step error. However it is possible to reach the same one-step accuracy by some modification of the first iteration that reduces the number of recalculations. The explicit layer method obtained on this way has the form (we use the same notation $\bar{u}(t_k, x)$ again)

$$\bar{u}(t_N, x) = \varphi(x), \tag{3.21}$$

$$\hat{b}_k = b(t_k, x, \bar{u}(t_{k+1}, x)),\ \hat{c}_k = c(t_k, x, \bar{u}(t_{k+1}, x)),\ \hat{\sigma}_k = \sigma(t_k, x, \bar{u}(t_{k+1}, x)),$$

$$\bar{u}^{(1)}(t_k, x) = \bar{u}(t_{k+1}, x + h\hat{b}_k) + hg(t_k, x, \bar{u}(t_{k+1}, x)),$$

$$\bar{u}(t_k, x) = \frac{1}{2}\bar{u}(t_{k+1}, x + h[b(t_k, x, \bar{u}^{(1)}(t_k, x)) + b(t_{k+1}, x + h\hat{b}_k, \bar{u}(t_{k+1}, x + h\hat{b}_k))]/2$$

$$+\varepsilon^2 h\hat{c}_k + \varepsilon h^{1/2}\hat{\sigma}_k)$$

$$+\frac{1}{2}\bar{u}(t_{k+1}, x + h[b(t_k, x, \bar{u}^{(1)}(t_k, x)) + b(t_{k+1}, x + h\hat{b}_k, \bar{u}(t_{k+1}, x + h\hat{b}_k))]/2$$

$$+\varepsilon^2 h\hat{c}_k - \varepsilon h^{1/2}\hat{\sigma}_k)$$

$$+\frac{1}{2}hg(t_k, x, \bar{u}^{(1)}(t_k, x)) + \frac{1}{2}hg(t_{k+1}, x + h\hat{b}_k, \bar{u}(t_{k+1}, x + h\hat{b}_k)),$$

$$k = N-1, \dots, 0.$$

The following theorem can be proved by the arguments like that in Lemma 3.1 and Theorem 3.1.

Theorem 3.2. *Under the assumptions (i) - (ii), the global error of the explicit layer method (3.21) is estimated by $O(h^2 + \varepsilon^2 h)$:*

$$|u(t_k, x) - \bar{u}(t_k, x)| \le K \cdot (h^2 + \varepsilon^2 h),$$

where the constant K does not depend on h, k, x, ε.

Remark 3.2. Naturally, we can take other (more accurate than we use above) weak approximations of SDE with small noise [19] to construct the corresponding high-order (with respect to h and ε) methods for the problem (3.1)-(3.2). In Section 6 we give high-order methods in some particular cases of the equation (3.1).

3.4. Singular case. The estimates of errors for the methods proposed in this section (Theorems 3.1 and 3.2) are obtained provided the bounds of derivatives of the solution to the considered problems are uniform with respect to $x \in R^d$, $t \in [t_0, T]$, and $0 < \varepsilon \le \varepsilon^*$ (see (3.8)). This assumption is ensured, e.g., in the following case. Consider the first-order partial differential equation, obtained from (3.1) under $\varepsilon = 0$:

$$\frac{\partial u^0}{\partial t} + b(t, x, u^0)\frac{\partial u^0}{\partial x} + g(t, x, u^0) = 0,\ t \in [t_0, T),\ x \in R^1, \tag{3.22}$$

$$u^0(T, x) = \varphi(x). \tag{3.23}$$

If the coefficients of the equation (3.22) and the initial condition (3.23) are such that the solution $u^0(t, x),\ x \in R^1$, is sufficiently smooth under $t_0 \le t \le T$, then the derivatives of the solution $u(t, x)$ to (3.1)-(3.2) can be uniformly bounded with respect to $0 \le \varepsilon \le \varepsilon^*$ under $t \in [t_0, T]$ (see [7, 27]). Note that it is generally not enough to assume that the coefficients of (3.22) and the initial condition $\varphi(x)$ are bounded and smooth functions to ensure the regular behavior of $u^0(t, x)$ at any $t < T$ [7].

A lot of physical phenomena (e.g., formation and propagation of shock waves) having singular behavior is described by equations with small parameter. The derivatives of their solutions go to infinity as $\varepsilon \to 0$ and, rigorously speaking, the results of Theorems 3.1 and 3.2 become inapplicable.

After change of variables $t = \varepsilon^2 t'$, $x = \varepsilon^2 x'$, the problem (3.1)-(3.2) is rewritten for $v(t',x') := u(\varepsilon^2 t', \varepsilon^2 x')$ in the form:

$$\frac{\partial v}{\partial t'} + \frac{1}{2}\sigma^2(\varepsilon^2 t', \varepsilon^2 x', v)\frac{\partial^2 v}{\partial x'^2} + (b(\varepsilon^2 t', \varepsilon^2 x', v) + \varepsilon^2 c(\varepsilon^2 t', \varepsilon^2 x', v))\frac{\partial v}{\partial x'} \tag{3.24}$$

$$+\varepsilon^2 g(\varepsilon^2 t', \varepsilon^2 x', v) = 0,\ t' \in [t_0/\varepsilon^2, T/\varepsilon^2),\ x' \in R^1,\ 0 < \varepsilon \le \varepsilon^*,$$

$$v(T/\varepsilon^2, x') = \varphi(\varepsilon^2 x'). \tag{3.25}$$

If the assumptions like (ii) take place for the solution $v(t',x')$ to (3.24)-(3.25) (we pay attention that the problem (3.24)-(3.25) is considered on long time intervals), then the derivatives of the solution $u(t,x)$ to (3.1)-(3.2) are estimated as:

$$\left|\frac{\partial^{i+j} u}{\partial t^i \partial x^j}\right| \le \frac{K}{\varepsilon^{2(i+j)}},\ t \in [t_0, T],\ x \in R^1,\ 0 < \varepsilon \le \varepsilon^*. \tag{3.26}$$

These bounds are natural ones for the problem (3.1)-(3.2) in the singular case.

If one followed the arguments of Lemma 3.1 and Theorem 3.1 in the singular case (i.e., taking the assumptions (i), (3.7), (3.26) instead of (i)-(ii)), the estimate of the form $\frac{K}{C}\frac{h}{\varepsilon^2}(e^{K(T-t_0)/\varepsilon^2} - 1)$ would be obtained for the proposed methods. Due to the big factor $1/\varepsilon^2$ in the exponent, this estimate is meaningless for practical purposes. Our numerical tests (see Section 8.1) demonstrate that the proposed methods possess essentially better quality than it can be predicted by this estimate. Apparently, the methods work fairly good in the singular case because the derivatives are large only in a small domain known as internal layer (see, e.g., [8]) that is attributable to the majority of interesting applications. The further theoretical investigation, to prove a realistic estimate for the errors of the proposed methods, should rest on a stability analysis and on more extensive properties of the considered solution. Recently, the similar problem for finite elements methods is considered in a few papers (see [3] and references therein).

However, in some particular, but important, singular cases of the problem (3.1)-(3.2) we get reasonable estimates (without $1/\varepsilon^2$ in the exponent) for the errors of the proposed methods by the arguments of Lemma 3.1 and Theorem 3.1.

Theorem 3.3. *Assume the coefficients b and σ in (3.1) be independent of u. Let the conditions (i) and (3.7) fulfill and the derivatives $\left|\frac{\partial^{i+j} u}{\partial t^i \partial x^j}\right|$, $i = 0, j = 1,2,3,4$; $i = 1, j = 0,1,2$; $i = 2, j = 0,1$; $i = 3, j = 0$, satisfy (3.26). Then under a sufficiently small h/ε^2, the global error of the explicit layer method (3.21) is estimated as*

$$|u(t_k,x) - \bar{u}(t_k,x)| \le \frac{K}{C}\frac{h}{\varepsilon^4}(e^{K(T-t_0)} - 1),$$

where the constants C and K do not depend on h, k, x, ε.

Proof. Denote the one-step approximation of $u(t_k,x)$ corresponding to the method (3.21) as $v(t_k,x)$. By the arguments of Lemma 3.1, we get

$$u(t_k,x) - v(t_k,x) = \varepsilon^2 h^2 C_1(t_k,x;\varepsilon,h) + h^3 C_2(t_k,x;\varepsilon,h). \tag{3.27}$$

Here $C_1(t_k,x;\varepsilon,h)$ has the form

$$C_1(t_k,x;\varepsilon,h)=\frac{1}{2}(-\frac{\partial c}{\partial u}\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}-\frac{\partial c}{\partial t}\frac{\partial u}{\partial x}-c\frac{\partial^2 u}{\partial x\partial t}-bc\frac{\partial^2 u}{\partial x^2}-b\frac{\partial c}{\partial x}\frac{\partial u}{\partial x}-b\frac{\partial c}{\partial u}(\frac{\partial u}{\partial x})^2$$

$$-\sigma\frac{\partial \sigma}{\partial t}\frac{\partial^2 u}{\partial x^2}-\sigma b\frac{\partial \sigma}{\partial x}\frac{\partial^2 u}{\partial x^2}-\frac{\sigma^2}{2}\frac{\partial^3 u}{\partial x^2\partial t}+b\frac{\sigma^2}{2}\frac{\partial^3 u}{\partial x^3}-\frac{\partial g}{\partial u}(c\frac{\partial u}{\partial x}+\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}))_k$$

$$+\frac{\partial c}{\partial u}(t_k,x,\zeta)(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x})_k+\tilde{c}_k b_k(\frac{\partial^2 u}{\partial x^2})_k+\tilde{c}_k(\frac{\partial^2 u}{\partial x\partial t})_k+\frac{\varepsilon^2}{2}\tilde{c}_k\sigma_k^2(\frac{\partial^3 u}{\partial x^3})_k+\frac{\varepsilon^2}{2}\tilde{c}_k^2(\frac{\partial^2 u}{\partial x^2})_k$$

$$+\frac{\sigma_k}{4}(\sigma_k-\frac{2h^{1/2}}{\varepsilon}b_k-2\varepsilon h^{1/2}\tilde{c}_k)\frac{\partial^3 u}{\partial x^2\partial t}(\tilde{\xi},\tilde{\eta}_-)-\frac{\sigma_k}{4}\frac{h^{1/2}}{\varepsilon}\frac{\partial^3 u}{\partial x\partial t^2}(\breve{\xi},\breve{\eta}_-)$$

$$+\frac{\sigma_k}{4}(\sigma_k+\frac{2h^{1/2}}{\varepsilon}b_k+2\varepsilon h^{1/2}\tilde{c}_k)\frac{\partial^3 u}{\partial x^2\partial t}(\tilde{\xi},\tilde{\eta}_+)+\frac{\sigma_k}{4}\frac{h^{1/2}}{\varepsilon}\frac{\partial^3 u}{\partial x\partial t^2}(\breve{\xi},\breve{\eta}_+)$$

$$+\frac{\varepsilon\sigma_k^3}{12}((\frac{\varepsilon\sigma_k}{4}-h^{1/2}(b_k+\varepsilon^2\tilde{c}_k))\frac{\partial^4 u}{\partial x^4}(\xi,\eta_-)+(\frac{\varepsilon\sigma_k}{4}+h^{1/2}(b_k+\varepsilon^2\tilde{c}_k))\frac{\partial^4 u}{\partial x^4}(\xi,\eta_+)),$$

where the index k means calculation at (t_k,x), $\tilde{c}_k := c(t_k,x,u(t_{k+1},x))$, ζ is a point between $u(t_k,x)$ and $u(t_{k+1},x)$, $t_k<\tilde{\xi},\xi,\breve{\xi}<t_{k+1}$, and $\tilde{\eta}_\pm,\eta_\pm,\breve{\eta}_\pm$ are points between x and $x+\frac{1}{2}hb_k+\frac{1}{2}hb(t_{k+1},x+hb_k)+\varepsilon^2h\tilde{c}_k\pm\varepsilon h^{1/2}\sigma_k$.

It is not difficult to write down the corresponding expression for $C_2(t_k,x;\varepsilon,h)$ as well.

Due to the assumptions (i), (3.7), (3.26), we obtain

$$|C_1(t_k,x;\varepsilon,h)|\le\frac{C}{\varepsilon^6},\ |C_2(t_k,x;\varepsilon,h)|\le\frac{C}{\varepsilon^6}. \tag{3.28}$$

Under a sufficiently small h/ε^2 the one-step error of the method (3.21) is estimated as

$$|u(t_k,x)-v(t_k,x)|\le C\frac{h^2}{\varepsilon^4}.$$

By the arguments of Theorem 3.1, it is not difficult to get the following analog of (3.20):

$$R(t_k,x)=\frac{1}{2}R(t_{k+1},X_{k+1}^{(+)})+\frac{1}{2}R(t_{k+1},X_{k+1}^{(-)})+r(t_k,x)+\varepsilon^2h^2C_1(t_k,x;\varepsilon,h) \tag{3.29}$$

$$+h^3C_2(t_k,x;\varepsilon,h),$$

$$X_{k+1}^{(\pm)}=x+\frac{1}{2}hb(t_k,x)+\frac{1}{2}hb(t_{k+1},x+hb(t_k,x))+\varepsilon^2hc(t_k,x,\bar{u}(t_{k+1},x))\pm\varepsilon h^{1/2}\sigma(t_k,x),$$

$$|r(t_k,x)|\le KhR_{k+1},$$

where

$$R_{k+1}:=\max_{-\infty<x<+\infty}|R(t_{k+1},x)|.$$

(Unlike (3.20) the formula (3.29) does not contain $\tilde{r}(t_k,x)$ because the method (3.21) is explicit.)

From (3.29) and (3.28) we obtain the statement of the theorem by the usual arguments. Theorem 3.3 is proved.

In a lot of applications (e.g., in shock waves) the derivatives are significant only in a small interval (internal layer) $(x_*(t), x^*(t))$ with the width $|x_*(t) - x^*(t)| \sim \varepsilon^2$:

$$(3.30) \qquad |\frac{\partial^{i+j} u}{\partial t^i \partial x^j}| \leq \frac{K}{\varepsilon^{2(i+j)}},\ t \in [t_0, T],\ x \in (x_*(t), x^*(t)),\ 0 < \varepsilon \leq \varepsilon^*,$$

and

$$(3.31) \qquad |\frac{\partial^{i+j} u}{\partial t^i \partial x^j}| \leq K,\ t \in [t_0, T],\ x \notin (x_*(t), x^*(t)),\ 0 < \varepsilon \leq \varepsilon^*,$$

$$\int\limits_{x \notin (x_*(t), x^*(t))} |\frac{\partial^{i+j} u}{\partial t^i \partial x^j}| dx \leq K,\ i + j \neq 0,\ t \in [t_0, T],\ 0 < \varepsilon \leq \varepsilon^*.$$

Theorem 3.4. *Assume the coefficients b and σ in (3.1) be independent of u. Let the conditions (i) and (3.7) fulfill and the derivatives $|\frac{\partial^{i+j} u}{\partial t^i \partial x^j}|$, $i = 0, j = 1, 2, 3, 4$; $i = 1, j = 0, 1, 2$; $i = 2, j = 0, 1$; $i = 3, j = 0$, satisfy (3.30) and (3.31). Then under a sufficiently small h/ε^2, the global error of the explicit layer method (3.21) is estimated in l^1-norm as*

$$(3.32) \qquad \int_{R^1} |u(t_k, x) - \bar{u}(t_k, x)| dx \leq \frac{K}{C} \frac{h}{\varepsilon^2} (e^{K(T - t_0)} - 1),$$

where the constants C and K do not depend on h, k, x, ε.

Proof. Introduce the notation

$$F_k := \int_{R^1} |R(t_k, x)| dx.$$

We have from (3.29):

$$F_k \leq F_{k+1} + KhF_{k+1} + \int_{R^1} |\varepsilon^2 h^2 C_1(t_k, x; \varepsilon, h) + h^3 C_2(t_k, x; \varepsilon, h)| dx.$$

Due to the assumptions (i), (3.7), (3.30) (cf. (3.28))

$$|C_1(t_k, x; \varepsilon, h)| \leq \frac{C}{\varepsilon^6},\ |C_2(t_k, x; \varepsilon, h)| \leq \frac{C}{\varepsilon^6} \text{ for } x \in (x_*(t_k), x^*(t_k)),$$

and due to (i), (3.7), (3.31)

$$\int\limits_{x \notin (x_*(t_k), x^*(t_k))} |\varepsilon^2 h^2 C_1(t_k, x; \varepsilon, h) + h^3 C_2(t_k, x; \varepsilon, h)| dx \leq C \cdot (\varepsilon^2 h^2 + h^3).$$

Therefore, under a sufficiently small $\frac{h}{\varepsilon^2}$

$$F_k \leq F_{k+1} + KhF_{k+1} + C\frac{h^2}{\varepsilon^2},$$

whence (3.32) follows. Theorem 3.4 is proved.

The analogous theorems for the more simple method (2.8) give the same estimates of its error. However, in our experiments the layer method (3.21) gives better results than (2.8). To show the advantages of the method (3.21) in the singular case theoretically, further investigation is required. Seemingly, a more accurate analysis of the error of the method (3.21) should rest on more extensive properties of the solution $u(t, x)$.

See also Section 7 and Remarks 6.1 and 8.1, where in the singular situation we give reasonable estimates of the errors for some other particular cases of the problem (3.1)-(3.2).

4. Numerical algorithms based on interpolation

To calculate $u(t_k, x) \simeq \bar{u}(t_k, x)$ at a certain point x by one of the above written explicit layer methods, one can use the recursive procedure. But it is evident that if the number of steps $N = (T - t_0)/h$ is relatively large, the recursive procedure is practically unrealizable due to a huge volume of needed calculations. In [18] another way is proposed, which is based on a discretization in the variable x and on an interpolation of $\bar{u}(t_k, x)$.

Introduce an equidistant space discretization: $\{x_j = x_0 + jh_x,\ j = 0, \pm 1, \pm 2, \dots\}$, $x_0 \in R^1$, h_x is a sufficiently small positive number.

While it does not lead to any misunderstanding, we use the old notation $\bar{u}$, $\bar{u}^{(1)}$, etc. for new values here.

Theorem 4.1. *Under the assumptions* (*i*)-(*ii*), *the numerical algorithm based on the explicit method* (3.21) *and on the linear interpolation:*

$$\bar{u}(t_N, x) = \varphi(x), \tag{4.1}$$

$$\hat{b}_{k,j} = b(t_k, x_j, \bar{u}(t_{k+1}, x_j)),\ \hat{c}_{k,j} = c(t_k, x_j, \bar{u}(t_{k+1}, x_j)),\ \hat{\sigma}_{k,j} = \sigma(t_k, x_j, \bar{u}(t_{k+1}, x_j)),$$

$$\bar{u}^{(1)}(t_k, x_j) = \bar{u}(t_{k+1}, x_j + h\hat{b}_{k,j}) + hg(t_k, x_j, \bar{u}(t_{k+1}, x_j)),$$

$$\bar{u}(t_k, x_j) =$$

$$= \frac{1}{2}\bar{u}(t_{k+1}, x_j + h[b(t_k, x_j, \bar{u}^{(1)}(t_k, x_j)) + b(t_{k+1}, x_j + h\hat{b}_{k,j}, \bar{u}(t_{k+1}, x_j + h\hat{b}_{k,j}))]/2$$

$$+\varepsilon^2 h\hat{c}_{k,j} + \varepsilon h^{1/2}\hat{\sigma}_{k,j})$$

$$+\frac{1}{2}\bar{u}(t_{k+1}, x_j + h[b(t_k, x_j, \bar{u}^{(1)}(t_k, x_j)) + b(t_{k+1}, x_j + h\hat{b}_{k,j}, \bar{u}(t_{k+1}, x_j + h\hat{b}_{k,j}))]/2$$

$$+\varepsilon^2 h\hat{c}_{k,j} - \varepsilon h^{1/2}\hat{\sigma}_{k,j})$$

$$+\frac{1}{2}hg(t_k, x_j, \bar{u}^{(1)}(t_k, x_j)) + \frac{1}{2}hg(t_{k+1}, x_j + h\hat{b}_{k,j}, \bar{u}(t_{k+1}, x_j + h\hat{b}_{k,j})),$$

$$\bar{u}(t_k, x) = \frac{x_{j+1} - x}{h_x}\bar{u}(t_k, x_j) + \frac{x - x_j}{h_x}\bar{u}(t_k, x_{j+1}),\quad x_j < x < x_{j+1},$$

$$j = 0, \pm 1, \pm 2, \dots,\quad k = N-1, \dots, 0,$$

has the global error estimated by $O(h^2 + \varepsilon^2 h)$ *if the value of* h_x *is selected as* $h_x = \alpha\min(h^{3/2}, \varepsilon h)$, *where* α *is a positive constant.*

Proof. Here we follow the proof of the corresponding theorem in [18].

Introduce the notation

$$X^{(\pm)}_{k+1,j} = x_j + \frac{1}{2}hb(t_k, x_j, \bar{u}^{(1)}(t_k, x_j)) + \frac{1}{2}hb(t_{k+1}, x_j + h\hat{b}_{k,j}, \bar{u}(t_{k+1}, x_j + h\hat{b}_{k,j}))$$

$$+\varepsilon^2 h\hat{c}_{k,j} \pm \varepsilon h^{1/2}\hat{\sigma}_{k,j}.$$

Just as in Theorem 3.1 for the implicit method (3.5), it is possible to obtain the expression like (3.20) for the algorithm (4.1) at the nodes x_j:

$$R(t_k, x_j) = \frac{1}{2}R(t_{k+1}, X^{(+)}_{k+1,j}) + \frac{1}{2}R(t_{k+1}, X^{(-)}_{k+1,j}) + r(t_k, x_j) + O(h^3 + \varepsilon^2 h^2),$$

$$|r(t_k, x_j)| \le KhR_{k+1},$$

where

$$R_{k+1} := \max_{-\infty<x<+\infty} |R(t_{k+1}, x)|.$$

(unlike (3.20) this formula does not contain $\tilde{r}(t_k, x_j)$ because the algorithm (4.1) is explicit and, for instance, Δb satisfies the inequality $|\Delta b| < K|R(t_{k+1}, x_j)|$).

Hence

$$|R(t_k, x_j)| \le R_{k+1} + KhR_{k+1} + C \cdot (h^3 + \varepsilon^2 h^2). \tag{4.2}$$

We have

$$u(t_k, x) = \frac{x_{j+1} - x}{h_x} u(t_k, x_j) + \frac{x - x_j}{h_x} u(t_k, x_{j+1}) + O(h_x^2), \quad x_j < x < x_{j+1}, \tag{4.3}$$

where the interpolation error $O(h_x^2)$ satisfies the inequality $|O(h_x^2)| \le Ch_x^2$ with C independent of h, k, h_x, j, x, ε.

From the last relation of (4.1) and from (4.3), we get

$$R(t_k, x) = \frac{x_{j+1} - x}{h_x} R(t_k, x_j) + \frac{x - x_j}{h_x} R(t_k, x_{j+1}) + O(h_x^2), \quad x_j < x < x_{j+1},$$

whence due to (4.2):

$$|R(t_k, x)| \le R_{k+1} + KhR_{k+1} + C \cdot (h^3 + \varepsilon^2 h^2 + h_x^2)$$

with a new constant C.

As $h_x = \alpha \min(h^{3/2}, \varepsilon h)$, we obtain from here the statement of the theorem by the usual arguments. Theorem 4.1 is proved.

Remark 4.1. The way of proving Theorem 4.1 gives us the restriction on the type of interpolation procedure, which we can use for constructing the numerical algorithm. The restriction is such that the sum of absolute values of the coefficients staying at $\bar{u}(t_k, \cdot)$ in the interpolation procedure must be not greater than 1. We can make use of B-splines of the order $O(h_x^2)$ for which this restriction takes place. The cubic interpolation of the order $O(h_x^4)$ does not satisfy the restriction. However, our numerical tests give good results in the case of the algorithm based on the cubic interpolation. See also Section 8.1 and some details and theoretical explanations in [18].

Remark 4.2. One can use a nonequidistant space discretization. For instance, one can take small h_x in the intervals of x, where derivatives of the solution are big, and take relatively large h_x outside these intervals.

5. Extensions to multi-dimensional case and to reaction-diffusion systems with small parameter

It is easy to generalize the proposed here algorithms to the multi-dimensional case ($d > 1$). For instance, consider the algorithm like (4.1) in the case of $d = 2$. Introduce the equidistant space discretization: $x_j^1 = x_0^1 + jh_{x^1}$, $x_l^2 = x_0^2 + lh_{x^2}$, $j, l = 0, \pm 1, \pm 2, \ldots$, $(x_j^1, x_l^2)^\top \in R^2$, $h_{x^i} = \alpha^i \min(h^{3/2}, \varepsilon h)$, $i = 1, 2$, α^i are positive constants. The algorithm with the global error $O(h^2 + \varepsilon^2 h)$ has the form

$$\bar{u}(t_N, x^1, x^2) = \varphi(x^1, x^2), \tag{5.1}$$

$$\hat{b}_{k,j,l} = b(t_k, x_j^1, x_l^2, \bar{u}(t_{k+1}, x_j^1, x_l^2)), \;\; \hat{c}_{k,j,l} = c(t_k, x_j^1, x_l^2, \bar{u}(t_{k+1}, x_j^1, x_l^2)),$$

$$\hat{\sigma}_{k,j,l} = \sigma(t_k, x_j^1, x_l^2, \bar{u}(t_{k+1}, x_j^1, x_l^2)),$$

$$\bar{u}^{(1)}(t_k, x_j^1, x_l^2) = \bar{u}(t_{k+1}, x_j^1 + h\hat{b}^1_{k,j,l}, x_l^2 + h\hat{b}^2_{k,j,l}) + hg(t_k, x_j^1, x_l^2, \bar{u}(t_{k+1}, x_j^1, x_l^2)),$$

$$(\tilde{X}^1_{k+1,j,l}, \tilde{X}^2_{k+1,j,l})^\top = (x_j^1, x_l^2)^\top + \frac{1}{2}hb(t_k, x_j^1, x_l^2, \bar{u}^{(1)}(t_k, x_j^1, x_l^2))$$

$$+\frac{1}{2}hb(t_{k+1}, x_j^1 + h\hat{b}^1_{k,j,l}, x_l^2 + h\hat{b}^2_{k,j,l}, \bar{u}(t_{k+1}, x_j^1 + h\hat{b}^1_{k,j,l}, x_l^2 + h\hat{b}^2_{k,j,l})) + \varepsilon^2 h\hat{c}_{k,j,l},$$

$$\bar{u}(t_k, x_j^1, x_l^2) =$$

$$= \frac{1}{4}\bar{u}(t_{k+1}, \tilde{X}^1_{k+1,j,l} + \varepsilon h^{1/2}\hat{\sigma}^{11}_{k,j,l} + \varepsilon h^{1/2}\hat{\sigma}^{12}_{k,j,l}, \tilde{X}^2_{k+1,j,l} + \varepsilon h^{1/2}\hat{\sigma}^{21}_{k,j,l} + \varepsilon h^{1/2}\hat{\sigma}^{22}_{k,j,l})$$

$$+\frac{1}{4}\bar{u}(t_{k+1}, \tilde{X}^1_{k+1,j,l} - \varepsilon h^{1/2}\hat{\sigma}^{11}_{k,j,l} + \varepsilon h^{1/2}\hat{\sigma}^{12}_{k,j,l}, \tilde{X}^2_{k+1,j,l} - \varepsilon h^{1/2}\hat{\sigma}^{21}_{k,j,l} + \varepsilon h^{1/2}\hat{\sigma}^{22}_{k,j,l})$$

$$+\frac{1}{4}\bar{u}(t_{k+1}, \tilde{X}^1_{k+1,j,l} + \varepsilon h^{1/2}\hat{\sigma}^{11}_{k,j,l} - \varepsilon h^{1/2}\hat{\sigma}^{12}_{k,j,l}, \tilde{X}^2_{k+1,j,l} + \varepsilon h^{1/2}\hat{\sigma}^{21}_{k,j,l} - \varepsilon h^{1/2}\hat{\sigma}^{22}_{k,j,l})$$

$$+\frac{1}{4}\bar{u}(t_{k+1}, \tilde{X}^1_{k+1,j,l} - \varepsilon h^{1/2}\hat{\sigma}^{11}_{k,j,l} - \varepsilon h^{1/2}\hat{\sigma}^{12}_{k,j,l}, \tilde{X}^2_{k+1,j,l} - \varepsilon h^{1/2}\hat{\sigma}^{21}_{k,j,l} - \varepsilon h^{1/2}\hat{\sigma}^{22}_{k,j,l})$$

$$+\frac{1}{2}hg(t_k, x_j^1, x_l^2, \bar{u}^{(1)}(t_k, x_j^1, x_l^2))$$

$$+\frac{1}{2}hg(t_{k+1}, x_j^1 + h\hat{b}^1_{k,j,l}, x_l^2 + h\hat{b}^2_{k,j,l}, \bar{u}(t_{k+1}, x_j^1 + h\hat{b}^1_{k,j,l}, x_l^2 + h\hat{b}^2_{k,j,l})),$$

$$\bar{u}(t_k, x^1, x^2) = \frac{x^1_{j+1} - x^1}{h_{x^1}} \cdot \frac{x^2_{l+1} - x^2}{h_{x^2}}\bar{u}(t_k, x_j^1, x_l^2) + \frac{x^1_{j+1} - x^1}{h_{x^1}} \cdot \frac{x^2 - x^2_l}{h_{x^2}}\bar{u}(t_k, x_j^1, x^2_{l+1})$$

$$+\frac{x^1 - x^1_j}{h_{x^1}} \cdot \frac{x^2_{l+1} - x^2}{h_{x^2}}\bar{u}(t_k, x^1_{j+1}, x_l^2) + \frac{x^1 - x^1_j}{h_{x^1}} \cdot \frac{x^2 - x^2_l}{h_{x^2}}\bar{u}(t_k, x^1_{j+1}, x^2_{l+1}),$$

$$x_j^1 \le x^1 \le x^1_{j+1},\ x_l^2 \le x^2 \le x^2_{l+1},\ j, l = 0, \pm 1, \pm 2, \dots,$$

$$k = N-1, \dots, 0.$$

The proposed methods are applicable to the Cauchy problem for systems of reaction-diffusion equations with small parameter as well. For instance, in the case of the following system (we take $d = 1$ for simplicity in writing)

$$\frac{\partial u_q}{\partial t} + \frac{\varepsilon^2}{2}\sigma_q^2(t, x, u)\frac{\partial^2 u_q}{\partial x^2} + (b_q(t, x, u) + \varepsilon^2 c_q(t, x, u))\frac{\partial u_q}{\partial x} + g_q(t, x, u) = 0, \tag{5.2}$$

$$t \in [t_0, T),\ x \in R^1,\ q = 1, \dots n,\ u := (u_1, \dots, u_n),$$

$$u_q(T, x) = \varphi_q(x),$$

the method like (3.21) has the form:

$$\bar{u}_q(t_N, x) = \varphi_q(x), \tag{5.3}$$

$$\hat{b}_{q,k} = b_q(t_k, x, \bar{u}(t_{k+1}, x)),\ \hat{c}_{q,k} = c_q(t_k, x, \bar{u}(t_{k+1}, x)),\ \hat{\sigma}_{q,k} = \sigma_q(t_k, x, \bar{u}(t_{k+1}, x)),$$

$$\bar{u}_i^{(1)}(t_k, x) = \bar{u}_i(t_{k+1}, x + h\hat{b}_{i,k}) + hg_i(t_k, x, \bar{u}(t_{k+1}, x)),\ i = 1, \dots, n,$$

$$\bar{u}_q(t_k, x) = \frac{1}{2}\bar{u}_q(t_{k+1}, x + h[b_q(t_k, x, \bar{u}^{(1)}(t_k, x))$$

$$+ b_q(t_{k+1}, x + h\hat{b}_{q,k}, \bar{u}(t_{k+1}, x + h\hat{b}_{q,k}))]/2 + \varepsilon^2 h\hat{c}_{q,k} + \varepsilon h^{1/2}\hat{\sigma}_{q,k})$$

$$+ \frac{1}{2}\bar{u}_q(t_{k+1}, x + h[b_q(t_k, x, \bar{u}^{(1)}(t_k, x))$$

$$+ b_q(t_{k+1}, x + h\hat{b}_{q,k}, \bar{u}(t_{k+1}, x + h\hat{b}_{q,k}))]/2 + \varepsilon^2 h\hat{c}_{q,k} - \varepsilon h^{1/2}\hat{\sigma}_{q,k})$$

$$+ \frac{1}{2}hg_q(t_k, x, \bar{u}^{(1)}(t_k, x)) + \frac{1}{2}hg_q(t_{k+1}, x + h\hat{b}_{q,k}, \bar{u}(t_{k+1}, x + h\hat{b}_{q,k})),$$

$$q = 1, \dots, n,\ k = N - 1, \dots, 0.$$

It is easy to write down the corresponding algorithm like (4.1) on the base of this method.

See [18] to obtain such methods and algorithms for another type of reaction-diffusion systems.

6. High-order methods for semilinear equation with small constant diffusion and zero advection

Here we restrict ourselves to the case of $d = 1$ for simplicity in writing again.

Consider the Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial x^2} + g(t, x, u) = 0,\ t \in [t_0, T),\ x \in R^1, \tag{6.1}$$

$$u(T, x) = \varphi(x). \tag{6.2}$$

We assume that the coefficient $g(t, x, u)$ is a uniformly bounded and sufficiently smooth function and conditions like (ii) from Section 3 are fulfilled for the solution $u(t, x)$ to (6.1)-(6.2). Note, to construct high-order methods we need in uniform boundedness of derivatives of $u(t, x)$ with higher orders than in the assumption (3.8). To realize the methods of this section, we can avoid any interpolation. It occurs so because under $\sigma \equiv 1$, $b \equiv 0, c \equiv 0$ we are able to choose the special space discretization. The methods of this section are tested by simulation of the generalized KPP-equation with a small parameter (see Section 8.2).

The probabilistic representation of the solution to (6.1)-(6.2) has the form (see (2.1)-(2.3))

$$u(t, x) = E(\varphi(X_{t,x}(T)) + Z_{t,x,0}(T)), \tag{6.3}$$

where $X_{t,x}(s)$, $Z_{t,x,z}(s)$, $s \geq t$, satisfies the system

$$\begin{aligned} dX &= \varepsilon dw(s),\ X(t) = x, \\ dZ &= g(s, X, u(s, X))ds,\ Z(t) = z. \end{aligned} \tag{6.4}$$

Note that the system (6.4) is a system of differential equations with small *additive* noise.

6.1. Two-layer methods. For completeness of presentation, let us write down the layer methods (2.8) and (3.21) and the second-order layer method from [18] in the case of the problem (6.1)-(6.2).

The explicit layer method with error $O(h)$ (2.8) has the form

$$\bar{u}(t_N, x_j) = \varphi(x_j), \tag{6.5}$$

$$\bar{u}(t_k, x_j) = \frac{1}{2}\bar{u}(t_{k+1}, x_j + \varepsilon h^{1/2}) + \frac{1}{2}\bar{u}(t_{k+1}, x_j - \varepsilon h^{1/2}) + hg(t_{k+1}, x_j, \bar{u}(t_{k+1}, x_j)),$$

$$x_j = x_0 + j\varepsilon h^{1/2},\ j = 0, \pm 1, \pm 2, \dots,\ k = N-1, \dots, 0.$$

Note that it coincides with the well-known finite-difference scheme under the special relation of time and space steps ($h_x = \varepsilon h_t^{1/2}$) in the scheme.

In the case of the problem (6.1)-(6.2) *the explicit layer method* (3.21) *with error* $O(h^2 + \varepsilon^2 h)$ takes the form

$$\bar{u}(t_N, x_j) = \varphi(x_j), \tag{6.6}$$

$$\bar{u}^{(1)}(t_k, x_j) = \bar{u}(t_{k+1}, x_j) + hg(t_k, x_j, \bar{u}(t_{k+1}, x_j)),$$

$$\bar{u}(t_k, x_j) = \frac{1}{2}\bar{u}(t_{k+1}, x_j + \varepsilon h^{1/2}) + \frac{1}{2}\bar{u}(t_{k+1}, x_j - \varepsilon h^{1/2})$$

$$+\frac{1}{2}h[g(t_k, x_j, \bar{u}^{(1)}(t_k, x_j)) + g(t_{k+1}, x_j, \bar{u}(t_{k+1}, x_j))],$$

$$x_j = x_0 + j\varepsilon h^{1/2},\ j = 0, \pm 1, \pm 2, \dots,\ k = N-1, \dots, 0.$$

Using the second-order Runge-Kutta scheme (2.12), the implicit second-order layer method for the semi-linear parabolic equation with constant diffusion is constructed in [18]. In the case of the problem (6.1)-(6.2) *the implicit layer method with error* $O(h^2)$ has the form

$$\bar{u}(t_N, x_j) = \varphi(x_j), \tag{6.7}$$

$$\bar{u}(t_k, x_j) = \frac{1}{6}\bar{u}(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2}) + \frac{2}{3}\bar{u}(t_{k+1}, x_j) + \frac{1}{6}\bar{u}(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2})$$

$$+\frac{h}{2}g(t_k, x_j, \bar{u}(t_k, x_j)) + \frac{h}{3}g(t_{k+1}, x_j, \bar{u}(t_{k+1}, x_j))$$

$$+\frac{h}{12}g(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2}))$$

$$+\frac{h}{12}g(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2})),$$

$$x_j = x_0 + j\sqrt{3}\varepsilon h^{1/2},\ j = 0, \pm 1, \pm 2, \dots,\ k = N-1, N-2, \dots, 0.$$

To solve the algebraic equations obtained at each step of the method (6.7), one can use the Newton method or the method of simple iteration.

Remark 6.1. In the singular case the natural bounds for derivatives of the solution to (6.1)-(6.2) have the form

$$|\frac{\partial^{i+j}u}{\partial t^i \partial x^j}| \leq \frac{K}{\varepsilon^j},\ t \in [t_0, T],\ x \in R^1,\ 0 < \varepsilon \leq \varepsilon^*. \tag{6.8}$$

These bounds are obtained after the following change of variables: $t = t'$, $x = \varepsilon^2 x'$ (cf. Section 3.4).

By the same arguments as in Theorem 3.3 one can prove under (6.8) that the errors of both methods (6.5) and (6.6) are estimated as

$$|u(t_k, x) - \bar{u}(t_k, x)| \leq Kh,$$

where the constant K does not depend on x, k, h, ε.

Nevertheless, the method (6.6) gives better results than (6.5) in our experiments. One can explain this by the fact that the constant K of (6.6) is essentially less than the K of (6.5).

Under (6.8) the error of the method (6.7) remains $O(h^2)$.

Note that in Section 8.2 we present results of testing these methods (instead of (6.5) we use some its modification) on the equation, the coefficient g of which depends on ε, and the derivatives of its solution have other bounds than (6.8) (see Remark 8.1 and other details in Section 8.2).

6.2. Three-layer methods. Here we obtain two three-layer methods. Their one-step errors can be estimated by the same arguments as in Lemma 3.1. We do not prove their convergence that requires stability analysis of multi-layer methods. We test these methods in our experiments and they give fairly good results.

To calculate $\bar{u}(t_{k+1}, x)$ by a three-layer method, two previous layers are used. So, to start simulations we should know $\bar{u}(t_N, x)$ and $\bar{u}(t_{N-1}, x)$. To simulate $\bar{u}(t_{N-1}, x)$ one can use, e.g., the two-layer method (6.7) under a sufficiently small step. Below we consider this layer to be known and denote

$$\psi(x) := \bar{u}(t_{N-1}, x).$$

Apply the special Runge-Kutta scheme (2.13) to approximate (6.4):

$$X_{t_k,x}(t_{k+1}) \simeq \bar{X}_{t_k,x}(t_{k+1}) = x + \varepsilon h^{1/2}\xi_k, \tag{6.9}$$

$$\begin{aligned} Z_{t_k,x,z}(t_{k+1}) \ \simeq \ & \bar{Z}_{t_k,x,z}(t_{k+1}) = z + \frac{h}{6}(g(t_k, x, u(t_k, x)) + 2g(t_{k+1/2}, x, u(t_{k+1/2}, x)) \\ & +2g(t_{k+1/2}, x + \varepsilon h^{1/2}\xi_k, u(t_{k+1/2}, x + \varepsilon h^{1/2}\xi_k)) \\ & +g(t_{k+1}, x + \varepsilon h^{1/2}\xi_k, u(t_{k+1}, x + \varepsilon h^{1/2}\xi_k))), \end{aligned}$$

where ξ_k are i.i.d. variables with the law $P(\xi = 0) = 2/3$, $P(\xi = \pm\sqrt{3}) = 1/6$.

The implicit method with the one-step error $O(h^5 + \varepsilon^2 h^3)$ has the form (to get the method we use the scheme (6.9) with the time step $2h$)

$$\bar{u}(t_N, x_j) = \varphi(x_j),\ \bar{u}(t_{N-1}, x_j) = \psi(x_j), \tag{6.10}$$

$$\begin{aligned} \bar{u}(t_k, x_j) = & \frac{1}{6}\bar{u}(t_{k+2}, x_j + \sqrt{6}\varepsilon h^{1/2}) + \frac{2}{3}\bar{u}(t_{k+2}, x_j) + \frac{1}{6}\bar{u}(t_{k+2}, x_j - \sqrt{6}\varepsilon h^{1/2}) \\ & + \frac{h}{3}g(t_k, x_j, \bar{u}(t_k, x_j)) + \frac{10h}{9}g(t_{k+1}, x_j, \bar{u}(t_{k+1}, x_j)) \end{aligned}$$

$$+\frac{h}{9}g(t_{k+1},x_j+\sqrt{6}\varepsilon h^{1/2},\bar{u}(t_{k+1},x_j+\sqrt{6}\varepsilon h^{1/2}))$$

$$+\frac{h}{9}g(t_{k+1},x_j-\sqrt{6}\varepsilon h^{1/2},\bar{u}(t_{k+1},x_j-\sqrt{6}\varepsilon h^{1/2}))$$

$$+\frac{h}{18}g(t_{k+2},x_j+\sqrt{6}\varepsilon h^{1/2},\bar{u}(t_{k+2},x_j+\sqrt{6}\varepsilon h^{1/2}))+\frac{2h}{9}g(t_{k+2},x_j,\bar{u}(t_{k+2},x_j))$$

$$+\frac{h}{18}g(t_{k+2},x_j-\sqrt{6}\varepsilon h^{1/2},\bar{u}(t_{k+2},x_j-\sqrt{6}\varepsilon h^{1/2})),$$

$$x_j=x_0+j\sqrt{6}\varepsilon h^{1/2},\ j=0,\pm1,\pm2,\dots,\ k=N-2,N-3,\dots,0.$$

Let us look at the stability properties of this method in the simple case when u and g in (6.1) do not depend on x, i.e., apply the method (6.10) to the ordinary differential equation

$$\frac{du}{dt}+g(t,u)=0,\ \ t\leq T,\ \ u(T)=\varphi. \tag{6.11}$$

Recall (see, e.g., [10]) that a linear n-step method for (6.11)

$$\alpha_n u_k+\alpha_{n-1}u_{k+1}+\dots+\alpha_0 u_{k+n}=h\cdot(\beta_n g_k+\dots+\beta_0 g_{k+n}),$$

$$g_i=g(t_i,u_i),\ \ \alpha_n\neq0,\ |\alpha_0|+|\beta_0|>0,$$

is zero-stable (D-stable) if the generating polynomial

$$\alpha_n\lambda^n+\alpha_{n-1}\lambda^{n-1}+\dots+\alpha_0=0 \tag{6.12}$$

satisfies the root condition: the roots of (6.12) lie on or within the unit circle, and the roots on the unit circle are simple.

In the case of (6.11) the method (6.10) coincides with the Milne two-step method which is of the order $O(h^4)$ and is zero-stable. Its generating polynomial has two roots: 1 and -1. As is known [10], the root -1 can be dangerous for some differential equations. The method (6.10) has unstable behavior in our numerical tests on the generalized KPP-equation with a small parameter (8.11)-(8.13) (Section 8.2). One can see that the method (6.10) does not preserve the property $u\leq1$ of the problem (8.11)-(8.13) that leads to an unstable behavior of the approximate solutions. We modify the method (6.10) in the experiments: if $\bar{u}(t_k,x_j)>1$, we put $\bar{u}(t_k,x_j)=1$. Because locally, at one step, the arising difference $0<\bar{u}(t_k,x_j)-1$ is not greater than the one-step error of this method, this modification does not change the one-step accuracy order of the method. The modified method turned out to be fairly good in applying to the generalized KPP-equation. However, the modification is based on the knowledge of the properties of the solution and it may be difficult to find such a modification for another problem. Fortunately, we are able to approximate the system (6.4) by another weak scheme and obtain a method for (6.1)-(6.2) with better stability properties in the sense considered above (see the method (6.13) below) but with the one-step error of lower order. To reach both the same one-step accuracy $O(h^5+\varepsilon^2h^3)$ and the better stability properties is possible by a four-layer method.

Approximate (6.4) by the special scheme with one-step order $O(h^4+\varepsilon^2h^3)$:

$$X_{t_k,x}(t_{k+1})\simeq\bar{X}_{t_k,x}(t_{k+1})=x+\varepsilon h^{1/2}\xi_k,$$

$$Z_{t_k,x,z}(t_{k+1}) \simeq \bar{Z}_{t_k,x,z}(t_{k+1}) = z + \frac{h}{12}(5g(t_k, x, u(t_k, x)) + g(t_{k+1}, x, u(t_{k+1}, x))$$

$$+7g(t_{k+1}, x + \varepsilon h^{1/2}\xi_k, u(t_{k+1}, x + \varepsilon h^{1/2}\xi_k))$$

$$-g(t_{k+2}, x + \varepsilon h^{1/2}\xi_k, u(t_{k+2}, x + \varepsilon h^{1/2}\xi_k))),$$

where ξ_k are i.i.d. variables with the law $P(\xi = 0) = 2/3$, $P(\xi = \pm\sqrt{3}) = 1/6$.

The three-layer implicit method with one-step error $O(h^4 + \varepsilon^2 h^3)$ has the form

$$\bar{u}(t_N, x_j) = \varphi(x_j),\ \bar{u}(t_{N-1}, x_j) = \psi(x_j), \tag{6.13}$$

$$\bar{u}(t_k, x_j) = \frac{1}{6}\bar{u}(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2}) + \frac{2}{3}\bar{u}(t_{k+1}, x_j) + \frac{1}{6}\bar{u}(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2})$$

$$+\frac{5h}{12}g(t_k, x_j, \bar{u}(t_k, x_j)) + \frac{17h}{36}g(t_{k+1}, x_j, \bar{u}(t_{k+1}, x_j))$$

$$+\frac{7h}{72}g(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+1}, x_j + \sqrt{3}\varepsilon h^{1/2}))$$

$$+\frac{7h}{72}g(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+1}, x_j - \sqrt{3}\varepsilon h^{1/2}))$$

$$-\frac{h}{72}g(t_{k+2}, x_j + \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+2}, x_j + \sqrt{3}\varepsilon h^{1/2}))$$

$$-\frac{h}{18}g(t_{k+2}, x_j, \bar{u}(t_{k+2}, x_j)) - \frac{h}{72}g(t_{k+2}, x_j - \sqrt{3}\varepsilon h^{1/2}, \bar{u}(t_{k+2}, x_j - \sqrt{3}\varepsilon h^{1/2})),$$

$$x_j = x_0 + j\sqrt{3}\varepsilon h^{1/2},\ j = 0, \pm 1, \pm 2, \ldots,\ k = N-2, N-3, \ldots, 0.$$

For (6.11) this method coincides with one of the implicit two-step Adams methods of order $O(h^3)$ and the roots of its generating polynomial are 1 and 0. One can expect that in the case of the problem (6.1)-(6.2) the method (6.13) also possesses better stability properties than (6.10). In our numerical tests on the generalized KPP-equation with a small parameter (Section 8.2) the method (6.13) has stable behavior.

To solve the algebraic equations, obtained at each step of the methods (6.10) and (6.13), one can use the Newton method or the method of simple iteration.

Remark 6.2. The methods of this section can be extended for a problem of a higher dimension or for a system of reaction-diffusion equations. In addition using some other weak approximations to SDE with small additive noise, new layer methods can be constructed. For instance, three- and four-layer methods with the one-step error $O(h^5 + \varepsilon^2 h^2)$ can be obtained. It is also not difficult to get an implicit four-layer method with the one-step error $O(h^5 + \varepsilon^2 h^3)$ for (6.1)-(6.2) possessing good stability properties in the sense as above or an explicit four-layer method with the one-step error $O(h^4 + \varepsilon^2 h^3)$ and so on.

7. Method based on exact simulation of the Brownian motion

In this section we construct a layer method for a model nonlinear problem using the probabilistic approach as above but to approximate SDE, arising in the probabilistic representation of the solution to the problem, we attract a numerical method with *exact simulation* of the Brownian motion instead of the weak schemes used in Sections 2 - 6. In [17] a few methods with exact simulation of some components of SDE are proposed for linear problems. It was shown that these methods are preferable to weak schemes in some situations. One can expect that in the nonlinear case layer methods used the exact simulation of the Brownian motion possess some preferable properties as well.

Let us consider the model problem

$$\frac{\partial u}{\partial t} + \frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial x^2} + g_0(t,x)u + \varepsilon^2 g_1(t,x,u) = 0,\ t \in [t_0,T),\ x \in R^1, \tag{7.1}$$

$$u(T,x) = \varphi(x;\varepsilon),\ 0 < \varepsilon \le \varepsilon^*. \tag{7.2}$$

As the initial condition $\varphi(x;\varepsilon)$, one can take, for instance, a smooth function being close to the step function (e.g., $0.5 - 0.5\tanh(x/\varepsilon)$). Such a problem may be the result of regularization (after introducing the artificial viscosity) of the first-order hyperbolic problem with weak nonlinearity:

$$\frac{\partial u^0}{\partial t} + g_0(t,x)u^0 + \varepsilon^2 g_1(t,x,u^0) = 0,\ t \in [t_0,T),\ x \in R^1,$$

$$u^0(T,x) = \begin{cases} 1,\ x < 0, \\ 1/2,\ x = 0, \\ 0,\ x > 0. \end{cases}$$

The introduction of the artificial viscosity is a common way for numerical simulations of first-order hyperbolic problems with discontinuous solutions [4, 20, 24, 29].

The solution $u(t,x)$ to the problem (7.1)-(7.2) has the probabilistic representation

$$u(t,x) = E(\varphi(X_{t,x}(T);\varepsilon)Y_{t,x,1}(T) + Z_{t,x,1,0}(T)), \tag{7.3}$$

where $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s \ge t$, satisfies the system of SDE

$$\begin{aligned} dX &= \varepsilon dw(s),\ X(t) = x, \\ dY &= g_0(s,X)Y ds,\ Y(t) = y, \\ dZ &= \varepsilon^2 g_1(s,X,u(s,X))Y ds,\ Z(t) = z. \end{aligned} \tag{7.4}$$

We have

$$u(t_k,x) = E(u(t_{k+1},X_{t_k,x}(t_{k+1}))Y_{t_k,x,1}(t_{k+1}) + Z_{t_k,x,1,0}(t_{k+1})). \tag{7.5}$$

Let us assume that:

(a) The coefficient $g_0(t,x) \in C^{2,4}([t_0,T] \times R^1)$, and both it and its derivatives are uniformly bounded with respect to $t \in [t_0,T],\ x \in R^1$:

$$\left|\frac{\partial^{i+j} g_0}{\partial t^i \partial x^j}\right| \le K,\ 0 \le 2i + j \le 4,\ t \in [t_0,T],\ x \in R^1.$$

The coefficient $g_1(t,x,u)$ and its first derivatives (with respect to t,x,u) and second derivatives (with respect to x and u) are continuous and uniformly bounded:

$$\left|\frac{\partial^{i+j+l} g_1}{\partial t^i \partial x^j \partial u^l}\right| \le K,\ 0 \le 2i + j + l \le 2,\ t \in [t_0,T],\ x \in R^1,\ u_\circ < u(t,x) < u^\circ,$$

where $-\infty \le u_\circ, u^\circ \le \infty$ are some constants.

(b) There exists the only bounded solution $u(t,x)$ to the problem (7.1)-(7.2) such that

$$u_\circ < u_* \le u(t,x) \le u^* < u^\circ, \tag{7.6}$$

where u_*, u^* are some constants, and there exist the bounded derivatives:

$$|\frac{\partial u}{\partial t}| \le K,\ |\frac{\partial u}{\partial x}| \le \frac{K}{\varepsilon},\ |\frac{\partial^2 u}{\partial x^2}| \le \frac{K}{\varepsilon^2},\ t\in[t_0,T],\ x\in R^1,\ 0<\varepsilon\le\varepsilon^*. \tag{7.7}$$

Note that because $u(T,x)=\varphi(x;\varepsilon)$, the function $\varphi(x;\varepsilon)$ is also assumed to satisfy the inequalities (7.7). In addition, this set of assumptions is natural by the same reasons as it is pointed out in Remark 6.1 and in Section 3.4.

We are able to simulate $X(t_{k+1})$ exactly:

$$X_{t_k,x}(t_{k+1}) := X_{k+1} = x + \varepsilon\Delta w_k, \tag{7.8}$$

where $\Delta w_k := w(t_{k+1}) - w(t_k)$.

We are going to construct a method with the one-step error $O(h^3+\varepsilon^2h^2)$ based on the exact simulation of $X(t_{k+1})$. Taking into account the results of [17], we approximate $Y_{t_k,x,y}(t_{k+1})$ and $Z_{t_k,x,y,z}(t_{k+1})$ by

$$Y_{t_k,x,y}(t_{k+1}) \simeq \bar{Y}_{t_k,x,y}(t_{k+1}) = y + hg_0(t_k,x)y + \frac{\varepsilon h}{2}\frac{\partial g_0}{\partial x}(t_k,x)y\Delta w_k + \frac{h^2}{2}(\frac{\partial g_0}{\partial t}(t_k,x) + g_0^2(t_k,x))y \tag{7.9}$$

$$Z_{t_k,x,y,z}(t_{k+1}) \simeq \bar{Z}_{t_k,x,y,z}(t_{k+1}) = z + \varepsilon^2 hg_1(t_k,x,u(t_{k+1},x))y. \tag{7.10}$$

Using (7.5), (7.4), and (7.8)-(7.10), we construct the one-step approximation

$$u(t_k,x) \simeq v(t_k,x) = Eu(t_{k+1},X_{k+1})Y_{k+1} + \varepsilon^2 hg_1(t_k,x,u(t_{k+1},x)) \tag{7.11}$$

with X_{k+1} from (7.8) and

$$Y_{k+1} := \bar{Y}_{t_k,x,1}(t_{k+1}) = 1 + hg_0(t_k,x) + \frac{\varepsilon h}{2}\frac{\partial g_0}{\partial x}(t_k,x)\Delta w_k + \frac{h^2}{2}(\frac{\partial g_0}{\partial t}(t_k,x) + g_0^2(t_k,x)). \tag{7.12}$$

Below, in Lemma 7.1, Theorems 7.1 and 7.2, we use the letters C and K without any index for various constants, which do not depend on h, k, x, and ε.

Lemma 7.1. *Under the assumptions (a)-(b), the error of the one-step approximation (7.11) is estimated as*

$$|u(t_k,x) - v(t_k,x)| \le C\cdot(h^3+\varepsilon^2h^2), \tag{7.13}$$

where C is independent of h, k, x, and ε.

Proof. We have (see (7.5), (7.8), and (7.11))

$$\begin{aligned}|u(t_k,x) - v(t_k,x)| &= \\ &|E(u(t_{k+1},X_{t_k,x}(t_{k+1}))Y_{t_k,x,1}(t_{k+1}) + Z_{t_k,x,1,0}(t_{k+1})) \\ &-(Eu(t_{k+1},X_{k+1})Y_{k+1} + \varepsilon^2hg_1(t_k,x,u(t_{k+1},x)))| \\ &\le |Eu(t_{k+1},X_{k+1})(Y_{t_k,x,1}(t_{k+1}) - Y_{k+1})| \\ &+|E(Z_{t_k,x,1,0}(t_{k+1}) - \varepsilon^2hg_1(t_k,x,u(t_{k+1},x)))|.\end{aligned} \tag{7.14}$$

Introduce the notation

$$\Lambda = \frac{\partial}{\partial x},\ L = \frac{\partial}{\partial t} + g_0(t,x)y\frac{\partial}{\partial y} + \frac{\varepsilon^2}{2}\frac{\partial^2}{\partial x^2}.$$

By the Ito formula, we obtain

$$\begin{aligned} Y_{t_k,x,y}(t_{k+1}) &= y + \int_{t_k}^{t_{k+1}} g_0(s,X(s))Y(s)ds = y + hg_0(t_k,x)y \\ &+\varepsilon(\frac{\partial g_0}{\partial x})_k y\int_{t_k}^{t_{k+1}}(w(s)-w(t_k))ds + \frac{h^2}{2}(\frac{\partial g_0}{\partial t}+g_0^2)_k y + \rho(t_k,x,y), \end{aligned}$$

where

$$\begin{aligned} \rho(t_k,x,y) &= \frac{\varepsilon^2 h^2}{4}(\frac{\partial^2 g_0}{\partial x^2})_k y + \varepsilon^2\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\int_{t_k}^{s_1}\Lambda\Lambda g_0(s_2,X(s_2))Y(s_2)dw(s_2)dw(s_1)ds \\ &+\varepsilon\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\int_{t_k}^{s_1} L\Lambda g_0(s_2,X(s_2))Y(s_2)ds_2dw(s_1)ds \\ &+\varepsilon\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\int_{t_k}^{s_1} \Lambda L g_0(s_2,X(s_2))Y(s_2)dw(s_2)ds_1ds \\ &+\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\int_{t_k}^{s_1} L^2 g_0(s_2,X(s_2))Y(s_2)ds_2ds_1ds. \end{aligned}$$

Here the operators $\Lambda\Lambda$, $L\Lambda$, ΛL, L^2 are applied to the function $g_0(t,x)y$.

Denote $\rho_k := \rho(t_k,x,1)$.

We have

$$|Eu(t_{k+1},X_{k+1})(Y_{t_k,x,1}(t_{k+1})-Y_{k+1})| =$$

$$= |Eu(t_{k+1},X_{k+1})(\rho_k + \varepsilon(\frac{\partial g_0}{\partial x})_k(\int_{t_k}^{t_{k+1}}(w(s)-w(t_k))ds - \frac{h}{2}\Delta w_k))|$$

$$\begin{aligned} &= |Eu(t_{k+1},X_{k+1})\rho_k + \varepsilon(\frac{\partial g_0}{\partial x})_k E(u(t_{k+1},X_{k+1})E((\int_{t_k}^{t_{k+1}}(w(s)-w(t_k))ds \\ &-\frac{h}{2}\Delta w_k)/X_{k+1}))|. \end{aligned}$$

The formula

$$E(\int_{t_k}^{t_{k+1}}(w(s)-w(t_k))ds/X_{k+1}) = E(\int_{t_k}^{t_{k+1}}(w(s)-w(t_k))ds/\Delta w_k) = \frac{h}{2}\Delta w_k$$

implies

$$|Eu(t_{k+1},X_{k+1})(Y_{t_k,x,1}(t_{k+1})-Y_{k+1})| = |Eu(t_{k+1},X_{k+1})\rho_k|.$$

By the Taylor expansion, we get

$$u(t_{k+1},X_{k+1}) = u(t_{k+1},x) + \varepsilon\Delta w_k\frac{\partial u}{\partial x}(t_{k+1},\xi),$$

where ξ is a point between x and $x+\varepsilon\Delta w_k$.

Then

$$|Eu(t_{k+1},X_{k+1})\rho_k| \le |u(t_{k+1},x)||E\rho_k| + \varepsilon|E\Delta w_k\frac{\partial u}{\partial x}(t_{k+1},\xi)\rho_k|.$$

By the assumption (a) we obtain

$$|E\rho_k| \le C\cdot(h^3+\varepsilon^2h^2).$$

Besides, one can prove

$$(E\rho_k^2)^{1/2} \le C\cdot(h^3+\varepsilon^2h^2).$$

Using the Cauchy-Bunyakovskii inequality and the assumption (b), we get

$$\varepsilon|E\Delta w_k\frac{\partial u}{\partial x}(t_{k+1},\xi)\rho_k| \le C\cdot(h^{7/2}+\varepsilon^2h^{5/2}).$$

Thus,

$$|Eu(t_{k+1},X_{k+1})(Y_{t_k,x,1}(t_{k+1})-Y_{k+1})| \le C\cdot(h^3+\varepsilon^2h^2). \tag{7.15}$$

Now estimate the second term of (7.14).

Applying the Ito formula, we obtain

$$\rho_z := |E(Z_{t_k,x,1,0}(t_{k+1})-\varepsilon^2hg_1(t_k,x,u(t_{k+1},x)))|$$

$$= |E\int_{t_k}^{t_{k+1}}\varepsilon^2g_1(s,X,u(s,X))Y(s)ds-\varepsilon^2hg_1(t_k,x,u(t_{k+1},x))|$$

$$= \varepsilon^2|hg_1(t_k,x,u(t_k,x))-hg_1(t_k,x,u(t_{k+1},x))$$

$$+E(\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\varepsilon\Lambda g_1(s_1,X,u(s_1,X))Y(s_1)dw(s_1)ds$$

$$+\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}Lg_1(s_1,X,u(s_1,X))Y(s_1)ds_1ds)|$$

$$\le \varepsilon^2h|g_1(t_k,x,u(t_k,x))-g_1(t_k,x,u(t_{k+1},x))|$$

$$+\varepsilon^2|E\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}Lg_1(s_1,X,u(s_1,X))Y(s_1)ds_1ds)|.$$

Then due to the assumptions (a) and (b)

$$\rho_z \le C\varepsilon^2h^2. \tag{7.16}$$

The inequalities (7.15) and (7.16) together with (7.14) lead to the statement of the lemma. Lemma 7.1 is proved.

Remark 7.1. If we omit the term $\frac{\varepsilon h}{2}\frac{\partial g_0}{\partial x}(t_k,x)y\Delta w_k$ in the approximation (7.9), the corresponding method has the one-step error $O(h^3+\varepsilon h^2)$.

The layer method, based on the one-step approximation (7.11), has the form

$$\bar u(t_N,x) = \varphi(x), \tag{7.17}$$

$$\begin{aligned}\bar u(t_k,x) &= E\bar u(t_{k+1},x+\varepsilon\Delta w_k)(1+hg_0(t_k,x)+\frac{\varepsilon h}{2}\frac{\partial g_0}{\partial x}(t_k,x)\Delta w_k\\ &\quad +\frac{h^2}{2}(\frac{\partial g_0}{\partial t}(t_k,x)+g_0^2(t_k,x)))+\varepsilon^2hg_1(t_k,x,\bar u(t_{k+1},x)),\\ &\qquad k=N-1,\dots,1,0.\end{aligned}$$

Theorem 7.1. *Under the assumptions (a)-(b), the global error of the layer method* (7.17) *is estimated by*

$$|u(t_k,x)-\bar{u}(t_k,x)| \le K(h^2+\varepsilon^2 h), \tag{7.18}$$

where K does not depend on h, k, x, and ε.

Proof. We follow the proof of the corresponding convergence theorem of [18] again. Denote the error of the method (7.17) to the k-th step ($(N-k)$-th layer) as

$$R(t_k,x) := \bar{u}(t_k,x) - u(t_k,x).$$

Then

$$u(t_k,x)+R(t_k,x)=\bar{u}(t_k,x)=$$

$$=E\bar{u}(t_{k+1},x+\varepsilon\Delta w_k)Y_{k+1}+\varepsilon^2 h g_1(t_k,x,\bar{u}(t_{k+1},x))$$

$$=Eu(t_{k+1},x+\varepsilon\Delta w_k)Y_{k+1}+ER(t_{k+1},x+\varepsilon\Delta w_k)Y_{k+1}+\varepsilon^2 h g_1(t_k,x,\bar{u}(t_{k+1},x)),$$

where Y_{k+1} is from (7.12).

Clearly, $R(t_N,x)=0$. Below we prove recurrently that $R(t_k,x)$, $k=N-1,\ldots,0$, is sufficiently small under a small h. Using the assumption (7.6), we shall be able to justify the following suggestion, in which we need now: the value $u(t_k,x)+R(t_k,x)$ remains in the interval $(u_\circ,u^\circ)$ under h small enough.

We have

$$g_1(t_k,x,\bar{u}(t_{k+1},x))=g_1(t_k,x,u(t_{k+1},x)+R(t_{k+1},x)) \tag{7.19}$$

$$=g_1(t_k,x,u(t_{k+1},x))+\Delta g_1.$$

Due to the assumption (a), Δg_1 satisfies the inequality

$$|\Delta g_1| \le K|R(t_{k+1},x)|.$$

Taking into account the assumption (a), it is not difficult to obtain:

$$(E|Y_{k+1}-1|^2)^{1/2} \le Kh.$$

Now we have

$$u(t_k,x)+R(t_k,x)=\bar{u}(t_k,x)=$$

$$=Eu(t_{k+1},x+\varepsilon\Delta w_k)Y_{k+1}+ER(t_{k+1},x+\varepsilon\Delta w_k)$$

$$+ER(t_{k+1},x+\varepsilon\Delta w_k)(Y_{k+1}-1)+\varepsilon^2 h g_1(t_k,x,u(t_{k+1},x))+\varepsilon^2 h\Delta g_1$$

$$=v(t_k,x)+ER(t_{k+1},x+\varepsilon\Delta w_k)+r(t_k,x),$$

where

$$|r(t_k,x)| \le Kh(\varepsilon^2|R(t_{k+1},x)|+(ER^2(t_{k+1},x+\varepsilon\Delta w_k))^{1/2}).$$

Then Lemma 7.1 implies

$$R(t_k,x)=ER(t_{k+1},x+\varepsilon\Delta w)+r(t_k,x)+O(h^3+\varepsilon^2h^2). \tag{7.20}$$

Introduce the notation

$$R_k := \max_{-\infty<x<+\infty}|R(t_k,x)|.$$

We have from (7.20):

$$R_N = 0, \quad R_k \le R_{k+1} + KhR_{k+1} + C \cdot (h^3 + \varepsilon^2 h^2), \ k = N-1, \dots, 0,$$

whence (7.18) follows. Theorem 7.1 is proved.

To realize the method (7.17), we need at each step k in an approximation of $u(t_{k+1}, x)$ at all points x. As in Section 4, we attract a space discretization here.

Introduce an equidistant space discretization: $\{x_j = x_0 + jh_x, \ j = 0, \pm 1, \pm 2, \dots\}$, $x_0 \in R^1$. Using the linear interpolation, we construct the algorithm:

(7.21)
$$\bar{u}(t_N, x) = \varphi(x),$$

$$\bar{u}(t_k, x_j) = E\bar{u}(t_{k+1}, x_j + \varepsilon \Delta w_k)(1 + hg_0(t_k, x_j) + \frac{\varepsilon h}{2} \frac{\partial g_0}{\partial x}(t_k, x_j)\Delta w_k$$

$$+ \frac{h^2}{2}(\frac{\partial g_0}{\partial t}(t_k, x_j) + g_0^2(t_k, x_j))) + \varepsilon^2 h g_1(t_k, x_j, \bar{u}(t_{k+1}, x_j)),$$

$$\bar{u}(t_k, x) = \frac{x_{j+1} - x}{h_x} \bar{u}(t_k, x_j) + \frac{x - x_j}{h_x} \bar{u}(t_k, x_{j+1}),$$

$$x_j < x < x_{j+1}, \ j = 0, \pm 1, \pm 2, \dots, \ k = N-1, \dots, 0.$$

Calculate the expectation in (7.21). Firstly, let us calculate $E\bar{u}(t_{k+1}, x_j + \varepsilon \Delta w_k)$:

$$E\bar{u}(t_{k+1}, x_j + \varepsilon \Delta w_k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{u}(t_{k+1}, x_j + \varepsilon h^{1/2}\xi) \exp(-\frac{\xi^2}{2})\, d\xi$$

$$= \frac{\varepsilon h^{1/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{u}(t_{k+1}, x) \exp(-\frac{(x - x_j)^2}{2\varepsilon^2 h})\, dx$$

$$= \frac{\varepsilon h^{1/2}}{\sqrt{2\pi}} \sum_{i=-\infty}^{\infty} \int_{x_i}^{x_{i+1}} (\frac{x_{i+1} - x}{h_x} \bar{u}(t_{k+1}, x_i) + \frac{x - x_i}{h_x} \bar{u}(t_{k+1}, x_{i+1})) \cdot \exp(-\frac{(x - x_j)^2}{2\varepsilon^2 h})\, dx$$

$$= \sum_{i=-\infty}^{\infty} \alpha_i^{(j)} \bar{u}(t_{k+1}, x_i),$$

where

$$\alpha_i^{(j)} = \frac{\varepsilon h^{1/2}}{\sqrt{2\pi} h_x} (\exp(-\frac{(x_{i-1} - x_j)^2}{2\varepsilon^2 h}) - 2\exp(-\frac{(x_i - x_j)^2}{2\varepsilon^2 h}) + \exp(-\frac{(x_{i+1} - x_j)^2}{2\varepsilon^2 h}))$$

$$+ \frac{1}{2h_x}(x_j - x_{i-1})(\operatorname{erfc} \frac{x_{i-1} - x_j}{\sqrt{2}\varepsilon h^{1/2}} - \operatorname{erfc} \frac{x_i - x_j}{\sqrt{2}\varepsilon h^{1/2}})$$

$$+ \frac{1}{2h_x}(x_{i+1} - x_j)(\operatorname{erfc} \frac{x_i - x_j}{\sqrt{2}\varepsilon h^{1/2}} - \operatorname{erfc} \frac{x_{i+1} - x_j}{\sqrt{2}\varepsilon h^{1/2}}).$$

Analogously,

$$E\bar{u}(t_{k+1}, x_j + \varepsilon \Delta w_k)\Delta w_k = \sum_{i=-\infty}^{\infty} \beta_i^{(j)} \bar{u}(t_{k+1}, x_i),$$

where

$$\beta_i^{(j)} = \frac{\varepsilon h}{2h_x}(\operatorname{erfc}\frac{x_{i-1}-x_j}{\sqrt{2}\varepsilon h^{1/2}} - 2\operatorname{erfc}\frac{x_i-x_j}{\sqrt{2}\varepsilon h^{1/2}} + \operatorname{erfc}\frac{x_{i+1}-x_j}{\sqrt{2}\varepsilon h^{1/2}}).$$

Therefore, the formula for $\bar{u}(t_k, x_j)$ in (7.21) has the following form

$$\bar{u}(t_k, x_j) = \sum_{i=-\infty}^{\infty} \alpha_i^{(j)} \bar{u}(t_{k+1}, x_i) f(t_k, x_j; h) + \sum_{i=-\infty}^{\infty} \beta_i^{(j)} \bar{u}(t_{k+1}, x_i) \frac{\varepsilon h}{2} \frac{\partial g_0}{\partial x}(t_k, x_j) \tag{7.22}$$

$$+\varepsilon^2 h g_1(t_k, x_j, \bar{u}(t_{k+1}, x_j))$$

with

$$f(t_k, x_j; h) = 1 + h g_0(t_k, x_j) + \frac{h^2}{2}(\frac{\partial g_0}{\partial t}(t_k, x_j) + g_0^2(t_k, x_j)).$$

Of course, under calculations, one has to take into account that the expressions for $\alpha_i^{(j)}$ and $\beta_i^{(j)}$ contain the differences of close numbers.

Theorem 7.2. *Under the assumptions* (a)-(b), *the global error of the numerical algorithm* (7.21) *is estimated by* $O(h^2 + \varepsilon^2 h)$ *if the value of* h_x *is selected as* $h_x = \alpha \min(\varepsilon h^{3/2}, \varepsilon^2 h)$, α *is a positive number.*

Proof. We follow the proof of the corresponding theorem in [18].

Denote the error of the numerical algorithm (7.21) to the k-th step as

$$R(t_k, x) := \bar{u}(t_k, x) - u(t_k, x).$$

Just as in Theorem 7.1, we obtain the expression like (7.20) for the algorithm (7.21) at the nodes x_j:

$$R(t_k, x_j) = ER(t_{k+1}, x_j + \varepsilon \Delta w_k) + r(t_k, x_j) + O(h^3 + \varepsilon^2 h^2),$$

$$|r(t_k, x_j)| \le K h R_{k+1},$$

where

$$R_k := \max_{-\infty < x < +\infty} |R(t_k, x)|.$$

Hence

$$|R(t_k, x_j)| \le R_{k+1} + K h R_{k+1} + C \cdot (h^3 + \varepsilon^2 h^2). \tag{7.23}$$

We have

$$u(t_k, x) = \frac{x_{j+1} - x}{h_x} u(t_k, x_j) + \frac{x - x_j}{h_x} u(t_k, x_{j+1}) + \rho_{int}, \quad x_j < x < x_{j+1}, \tag{7.24}$$

where the interpolation error ρ_{int} satisfies the inequality (see the assumption (7.7))

$$|\rho_{int}| \le \frac{1}{8} \max_{-\infty < x < \infty} |\frac{\partial^2 u}{\partial x^2}| \, h_x^2 \le K \frac{h_x^2}{\varepsilon^2}.$$

From (7.21) and (7.24), we get

$$R(t_k, x) = \frac{x_{j+1} - x}{h_x} R(t_k, x_j) + \frac{x - x_j}{h_x} R(t_k, x_{j+1}) + O(\frac{h_x^2}{\varepsilon^2}), \quad x_j < x < x_{j+1},$$

whence due to (7.23) we obtain for all x :

$$|R(t_k, x)| \le R_{k+1} + K h R_{k+1} + C \cdot (h^3 + \varepsilon^2 h^2 + \frac{h_x^2}{\varepsilon^2}). \tag{7.25}$$

As $h_x = \alpha \min(\varepsilon h^{3/2}, \varepsilon^2 h)$, we come to the statement of the theorem. Theorem 7.2 is proved.

Remark 7.2. The approach described in this section may be applied to a more general equation than (7.1), for instance, to the problem (1.1)-(1.2) under the constant diffusion. Using the Girsanov transformation, one can obtain the following probabilistic representation of the solution to this problem

$$u(t,x) = E(\varphi(X_{t,x}(T))Y_{t,x,1}(T) + Z_{t,x,1,0}(T)),$$

where $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s \geq t$, satisfies the system of SDE

$$dX = b_0 dt + \varepsilon \sigma dw(s),\ X(t) = x,$$

$$dY = \frac{1}{\varepsilon}(\sigma^{-1}(b(s,X,u(s,X)) - b_0))^\top Y dw(s),\ Y(t) = y,$$

$$dZ = g(s,X,u(s,X))Y ds,\ Z(t) = z,$$

which is suitable for construction of layer methods based on the exact simulation of $X(t_{k+1})$. Here b_0 is a constant vector.

Since we use the probabilistic representation to construct only a one-step approximation of the problem, it is possible to select b_0 at each step, e.g., as $b(t_k, x, u(t_{k+1}, x))$. Such a reception allows to decrease the influence of the big factor $1/\varepsilon$ staying in the equation for Y.

Method based on weak simulation of the Brownian motion. For completeness of presentation, we construct a layer method with the global error $O(h^2 + \varepsilon^2 h)$ (i.e., with the error like that of the algorithm (7.21)) using a weak simulation of the Brownian motion now. To construct the algorithm (7.21), we need in the bounds (7.7) for the derivatives $\dfrac{\partial^{i+j}u}{\partial t^i \partial x^j}$ with $i = 0, j = 1,2$ and $i = 1, j = 0$. The new method, which we intend to construct, is simpler but we need in additional assumptions on the derivatives of the solution, namely:

$$|\frac{\partial^{i+j}u}{\partial t^i \partial x^j}| \leq \frac{K}{\varepsilon^j},\ 0 \leq 2i + j \leq 6,\ t \in [t_0, T],\ x \in R^1,\ 0 < \varepsilon \leq \varepsilon^*. \tag{7.26}$$

We also suppose that $g_0(t,x)$ and $g_1(t,x,u)$ are uniformly bounded and sufficiently smooth functions.

For approximating (7.4), we use the weak scheme:

$$X_{t_k,x}(t_{k+1}) \simeq X_{k+1} = x + \varepsilon h^{1/2}\xi_k, \tag{7.27}$$

$$\begin{aligned} Y_{t_k,x,y}(t_{k+1}) \ &\simeq\ \bar{Y}_{t_k,x,y}(t_{k+1}) = y + hg_0(t_k,x)y + \varepsilon h^{3/2}\frac{\partial g_0}{\partial x}(t_k,x)y\eta_k \\ &+ \frac{h^2}{2}(\frac{\partial g_0}{\partial t}(t_k,x) + g_0^2(t_k,x))y, \end{aligned}$$

$$Z_{t_k,x,y,z}(t_{k+1}) \simeq \bar{Z}_{t_k,x,y,z}(t_{k+1}) = z + \varepsilon^2 h g_1(t_k, x, u(t_{k+1}, x))y,$$

where ξ_k are i.i.d. variables with the law $P(\xi_k = 0) = \dfrac{2}{3}$, $P(\xi_k = \pm\sqrt{3}) = \dfrac{1}{6}$ and $\eta_k = \dfrac{1}{2}\xi_k$.

The corresponding method for (7.1)-(7.2) has the form

$$\bar{u}(t_N, x_j) = \varphi(x_j; \varepsilon), \tag{7.28}$$

$$
\begin{aligned}
\bar{u}(t_k, x_j) \;\; = \;\; & \frac{1}{6}\bar{u}(t_{k+1}, x_j + \varepsilon h^{1/2}\sqrt{3})(f(t_k, x_j; h) + \frac{\varepsilon h^{3/2}}{2}\frac{\partial g_0}{\partial x}(t_k, x_j)\sqrt{3}) \\
& + \frac{2}{3}\bar{u}(t_{k+1}, x_j) f(t_k, x_j; h) \\
& + \frac{1}{6}\bar{u}(t_{k+1}, x_j - \varepsilon h^{1/2}\sqrt{3})(f(t_k, x_j; h) - \frac{\varepsilon h^{3/2}}{2}\frac{\partial g_0}{\partial x}(t_k, x_j)\sqrt{3}) \\
& + \varepsilon^2 h g_1(t_k, x_j, \bar{u}(t_{k+1}, x_j)),
\end{aligned}
$$

$$
x_j = x_0 + j\sqrt{3}\varepsilon h^{1/2}, \; j = 0, \pm 1, \pm 2, \dots, \;\; k = N-1, N-2, \dots, 0,
$$

with

$$
f(t_k, x_j; h) = 1 + h g_0(t_k, x_j) + \frac{h^2}{2}(\frac{\partial g_0}{\partial t}(t_k, x_j) + g_0^2(t_k, x_j)).
$$

By the arguments like that in Section 3, the global error of this method is estimated under (7.26) as $O(h^2 + \varepsilon^2 h)$. Note that to realize this method we do not need in any interpolation under the special selection of space discretization.

Let us underline that the method (6.6) of Section 6.1 has the same error $O(h^2 + \varepsilon^2 h)$ under the assumption on uniform boundedness of derivatives of the solution. However, under the assumption (7.7), its error is estimated as $O(h)$ only (see Remark 6.1 as well).

8. Numerical tests

In the previous sections we deal with semilinear parabolic equations with *negative* direction of time t : the equations are considered under $t < T$ and the "initial" conditions are given at $t = T$. This form of equations is suitable for the probabilistic approach which we use to construct numerical methods. Of course, the proposed methods are adaptable to semilinear parabolic equations with *positive* direction of time and it can be done especially easy in the autonomous case.

Consider the Cauchy problem for autonomous semilinear parabolic equations with positive direction of time

$$
\frac{\partial u}{\partial t} = \frac{\varepsilon^2}{2}\sigma^2(x, u)\frac{\partial^2 u}{\partial x^2} + (b(x, u) + \varepsilon^2 c(x, u))\frac{\partial u}{\partial x} + g(x, u), \; t > 0, \; x \in R^1, \tag{8.1}
$$

$$
u(0, x) = \varphi(x). \tag{8.2}
$$

Note that if we substitute the solution $u(t, x)$ of this problem in the coefficients σ, b, c, and g, the equation (8.1) becomes nonautonomous. Nevertheless, it is not difficult to obtain numerical procedures with positive direction of time which correspond to the algorithms given in the previous sections. For instance, the algorithm for (8.1)-(8.2), which follows from the algorithm (4.1) for (3.1)-(3.2), has the form

$$
\bar{u}(0, x) = \varphi(x), \tag{8.3}
$$

$$
\hat{b}_{k,j} = b(x_j, \bar{u}(t_k, x_j)), \; \hat{c}_{k,j} = c(x_j, \bar{u}(t_k, x_j)), \; \hat{\sigma}_{k,j} = \sigma(x_j, \bar{u}(t_k, x_j)),
$$

$$
\bar{u}^{(1)}(t_{k+1}, x_j) = \bar{u}(t_k, x_j + h\hat{b}_{k,j}) + h g(x_j, \bar{u}(t_k, x_j)),
$$

$$
\bar{u}(t_{k+1}, x_j) =
$$

$$
\begin{aligned}
= \quad & \frac{1}{2}\bar{u}(t_k, x_j + h[b(x_j, \bar{u}^{(1)}(t_{k+1}, x_j)) + b(x_j + h\hat{b}_{k,j}, \bar{u}(t_k, x_j + h\hat{b}_{k,j}))]/2 \\
& + \varepsilon^2 h\hat{c}_{k,j} + \varepsilon h^{1/2}\hat{\sigma}_{k,j}) \\
& + \frac{1}{2}\bar{u}(t_k, x_j + h[b(x_j, \bar{u}^{(1)}(t_{k+1}, x_j)) + b(x_j + h\hat{b}_{k,j}, \bar{u}(t_k, x_j + h\hat{b}_{k,j}))]/2 \\
& + \varepsilon^2 h\hat{c}_{k,j} - \varepsilon h^{1/2}\hat{\sigma}_{k,j}) \\
& + \frac{1}{2}h[g(x_j, \bar{u}^{(1)}(t_{k+1}, x_j)) + g(x_j + h\hat{b}_{k,j}, \bar{u}(t_k, x_j + h\hat{b}_{k,j}))],
\end{aligned}
$$

$$
\bar{u}(t_k, x) = \frac{x_{j+1} - x}{h_x}\bar{u}(t_k, x_j) + \frac{x - x_j}{h_x}\bar{u}(t_k, x_{j+1}), \quad x_j < x < x_{j+1},
$$

$$
j = 0, \pm 1, \pm 2, \dots, \quad k = 0, \dots, N-1.
$$

The algorithms used in our numerical tests below can be written analogously.

We note in Remark 4.1 that the algorithms rested on the *cubic interpolation* give quite good results. We use the advantage of the cubic interpolation in our numerical tests on the Burgers equation. Let us recall that a sufficiently smooth function $f(x)$, $x \in R^1$, can be interpolated by the cubic interpolation as

$$
f(x) \simeq \bar{f}(x) = \sum_{i=0}^{3} \Phi_{j,i}(x) f(x_{j+i}), \quad x_{j+1} < x < x_{j+2}, \tag{8.4}
$$

$$
\Phi_{j,i}(x) = \prod_{m=0, m\neq i}^{3} \frac{x - x_{j+m}}{x_{j+i} - x_{j+m}},
$$

where $x_j = x_0 + j \cdot h_x$, $x_0 \in R^1$, $j = 0, \pm 1, \pm 2, \dots$, h_x is a positive number.

The error of the cubic interpolation (8.4) is estimated by

$$
|\bar{f}(x) - f(x)| \leq \frac{3}{128} \max_{x_j < x < x_{j+3}} |\frac{\partial^4 u}{\partial x^4}| \cdot h_x^4, \; x_{j+1} < x < x_{j+2}.
$$

Remind (see Theorem 4.1) that the algorithm (4.1), based on the layer method (3.21) and on the linear interpolation, has the error estimated by $O(h^2 + \varepsilon^2 h)$ provided $h_x = \min(h^{3/2}, \varepsilon h)$. One can expect that under the assumptions (i)-(ii) from Section 3 the algorithm based on the layer method (3.21) and on the cubic interpolation (8.4) can achieve the same accuracy $O(h^2 + \varepsilon^2 h)$ with h_x taken equal to $\min(h^{3/4}, \sqrt{\varepsilon h})$ only. Our numerical tests on the Burgers equation approve this supposition. See some theoretical explanations in [18] as well.

As it is mentioned in Introduction, all the methods (including those from Section 7) have been tested through computer experiments. Some of them are presented below.

8.1. The Burgers equation with small viscosity. The one-dimensional Burgers equation with small viscosity has the form

$$
\frac{\partial u}{\partial t} = \frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}, \quad t > 0, \; x \in R^1, \tag{8.5}
$$

$$
u(0, x) = \varphi(x). \tag{8.6}
$$

By means of the Cole-Hopf transformation, it can be found the explicit solution of the problem (8.5)-(8.6):

$$u(t,x) = \frac{\int_{-\infty}^{\infty} K(t,x,y)\varphi(y)\exp(-\frac{1}{\varepsilon^2}\int_0^y \varphi(\xi)d\xi)dy}{\int_{-\infty}^{\infty} K(t,x,y)\exp(-\frac{1}{\varepsilon^2}\int_0^y \varphi(\xi)d\xi)dy},$$

$$K(t,x,y) = \frac{1}{\sqrt{2\pi\varepsilon^2 t}}\exp(-\frac{(x-y)^2}{2\varepsilon^2 t}).$$

Let us take the initial condition $\varphi(x)$ of the form

$$\varphi(x) = \begin{cases} c, & x < l_0, \\ \lambda(x), & l_0 \le x \le l_0 + l, \\ d, & x > l_0 + l, \end{cases} \tag{8.7}$$

where c, d, l_0, l are some numbers, $c > d$, $l \ge 0$; $\lambda(x)$ is a bounded measurable function, and $d \le \lambda(x) \le c$.

Recall some theoretical facts concerning the problem (8.5)-(8.6), (8.7) (see details, e.g., in [9, 27]).

The solution $u(t,x)$ to (8.5)-(8.6), (8.7) is uniformly bounded:

$$d \le u(t,x) \le c,\ x \in R^1,\ 0 \le t,\ 0 \le \varepsilon \le \varepsilon^*.$$

Let the initial condition $\varphi(x)$ be a sufficiently smooth function. Introduce the time moment T such that the solution of the hyperbolic problem obtained from (8.5)-(8.6), (8.7) under $\varepsilon = 0$ is smooth at $t < T$ and discontinuous at $t \ge T$. The solution $u(t,x)$ to (8.5)-(8.6), (8.7) is regular under $t \le t_* < T$:

$$\left|\frac{\partial^{i+j}u}{\partial t^i \partial x^j}(t,x)\right| \le K,\ x \in R^1,\ 0 \le t \le t_*,\ 0 \le \varepsilon \le \varepsilon_*.$$

If $t \ge T$ then the solution is singular in an interval $(x_*(t), x^*(t))$ with the width $|x^*(t) - x_*(t)| \sim \varepsilon^2$:

$$\left|\frac{\partial^{i+j}u}{\partial t^i \partial x^j}(t,x)\right| \le \frac{K}{\varepsilon^{2(i+j)}},\ x \in (x_*(t), x^*(t)),\ t \ge T,\ 0 < \varepsilon \le \varepsilon_*,$$

$$\left|\frac{\partial^{i+j}u}{\partial t^i \partial x^j}(t,x)\right| \le K,\ x \notin (x_*(t), x^*(t)),\ t \ge T,\ 0 < \varepsilon \le \varepsilon_*.$$

In our experiments we take $\lambda(x)$ equal to

$$\lambda(x) = a - b\sin\frac{\pi x}{\mu},\ \mu > 0,\ b > 0,\ \text{and}\ c = a + b,\ d = a - b,\ l = \mu,\ l_0 = -\frac{\mu}{2}. \tag{8.8}$$

Under this $\lambda(x)$ the moment T can easily be found: $T = \dfrac{\mu}{\pi b}$.

We compare the behavior of two algorithms. The first one is based on the layer method (3.21) with the cubic interpolation (8.4). In the case of the problem (8.5)-(8.6) it has the following form:

$$\bar{u}(0,x) = \varphi(x), \tag{8.9}$$

$$\bar{u}(t_{k+1}, x_j) = \frac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j)) + \varepsilon h^{1/2})$$

TABLE 1. The Burgers equation (regular solution). Dependence of the errors $err^c(t_*)$ and $err^l(t_*)$ in h and ε for the algorithms (8.9) and (8.10) under $a = b = 0.5$, $\mu = 8$, and $t_* = 4$ ($T \approx 5.09$).

ε	h	algorithm (8.9)		algorithm (8.10)	
		$err^c(t_*)$	$err^l(t_*)$	$err^c(t_*)$	$err^l(t_*)$
0.3	0.3	$0.1351 \cdot 10^{-1}$	$0.1531 \cdot 10^{-1}$	0.1130	0.1397
	0.1	$0.2146 \cdot 10^{-2}$	$0.3347 \cdot 10^{-2}$	$0.3978 \cdot 10^{-1}$	$0.4628 \cdot 10^{-1}$
	0.01	$0.2295 \cdot 10^{-3}$	$0.3874 \cdot 10^{-3}$	$0.4221 \cdot 10^{-2}$	$0.4799 \cdot 10^{-2}$
	0.001	$0.2265 \cdot 10^{-4}$	$0.3947 \cdot 10^{-4}$	$0.4244 \cdot 10^{-3}$	$0.4814 \cdot 10^{-3}$
0.1	0.3	$0.2325 \cdot 10^{-1}$	$0.2051 \cdot 10^{-1}$	0.1539	0.1519
	0.1	$0.4255 \cdot 10^{-2}$	$0.2287 \cdot 10^{-2}$	$0.6084 \cdot 10^{-1}$	$0.5007 \cdot 10^{-1}$
	0.03	$0.3489 \cdot 10^{-3}$	$0.2396 \cdot 10^{-3}$	$0.2029 \cdot 10^{-1}$	$0.1553 \cdot 10^{-1}$
	0.01	$0.4444 \cdot 10^{-4}$	$0.5442 \cdot 10^{-4}$	$0.6751 \cdot 10^{-2}$	$0.5169 \cdot 10^{-2}$
	0.001	$0.5529 \cdot 10^{-5}$	$0.6374 \cdot 10^{-5}$	$0.6806 \cdot 10^{-3}$	$0.5189 \cdot 10^{-3}$

$$+\frac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j)) - \varepsilon h^{1/2}),$$

$$\bar{u}(t_k, x) = \sum_{i=0}^{3} \Phi_{j,i}(x)\bar{u}(t_k, x_{j+i}), \quad x_{j+1} < x < x_{j+2},$$

$$\Phi_{j,i}(x) = \prod_{m=0, m\neq i}^{3} \frac{x - x_{j+m}}{x_{j+i} - x_{j+m}},$$

$$j = 0, \pm 1, \pm 2, \ldots\,, \quad k = 0, \ldots, N-1,$$

where $x_j = x_0 + j \cdot h_x$.

The second algorithm is based on the layer method (2.8) proposed in [18] and on the cubic interpolation (8.4):

$$\bar{u}(0, x) = \varphi(x), \tag{8.10}$$

$$\bar{u}(t_{k+1}, x_j) = \frac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j) + \varepsilon h^{1/2}) + \frac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j) - \varepsilon h^{1/2}),$$

$$\bar{u}(t_k, x) = \sum_{i=0}^{3} \Phi_{j,i}(x)\bar{u}(t_k, x_{j+i}), \quad x_{j+1} < x < x_{j+2},$$

$$\Phi_{j,i}(x) = \prod_{m=0, m\neq i}^{3} \frac{x - x_{j+m}}{x_{j+i} - x_{j+m}},$$

$$j = 0, \pm 1, \pm 2, \ldots\,, \quad k = 0, \ldots, N-1.$$

Table 1 gives the results of simulation of the problem (8.5)-(8.6) with $\varphi(x)$ from (8.7), (8.8) in the case of the regular solution. In this case the assumptions (i)-(ii) from Section 3 are fulfilled, and the algorithm (8.9) has the error estimated by $O(h^2 + \varepsilon^2 h)$ and the algorithm (8.10) has the error estimated by $O(h)$. The value of h_x is taken equal to $h^{3/4}$.

TABLE 2. The Burgers equation (singular solution). The errors $err^c(t)$ and $err^l(t)$ under $t = 8$ ($T \approx 5.09$). Other parameter values are the same as in Table 1. The time steps h and h_* are used under $t \leq t_*$ and $t > t_*$ correspondingly.

ε	h	h_*	algorithm (8.9)		algorithm (8.10)	
			$err^c(t)$	$err^l(t)$	$err^c(t)$	$err^l(t)$
	0.1	0.01	$0.6322 \cdot 10^{-2}$	$0.2713 \cdot 10^{-2}$	0.1693	$0.6555 \cdot 10^{-1}$
0.3	0.01	0.001	$0.4036 \cdot 10^{-3}$	$0.2482 \cdot 10^{-3}$	$0.1771 \cdot 10^{-1}$	$0.6782 \cdot 10^{-2}$
	0.001	0.0001	$0.5977 \cdot 10^{-4}$	$0.3760 \cdot 10^{-4}$	$0.1776 \cdot 10^{-2}$	$0.6931 \cdot 10^{-3}$
	0.1	0.001	$0.5553 \cdot 10^{-1}$	$0.2351 \cdot 10^{-2}$	> 0.5	$0.6594 \cdot 10^{-1}$
0.1	0.03	0.001	$0.1219 \cdot 10^{-1}$	$0.4699 \cdot 10^{-3}$	> 0.5	$0.3189 \cdot 10^{-1}$
		0.0001	$0.3955 \cdot 10^{-2}$	$0.1718 \cdot 10^{-3}$	0.4029	$0.1716 \cdot 10^{-1}$
	0.01	0.0001	$0.7047 \cdot 10^{-3}$	$0.3007 \cdot 10^{-4}$	0.1687	$0.6828 \cdot 10^{-2}$
	0.001	0.0001	$0.4139 \cdot 10^{-3}$	$0.2312 \cdot 10^{-4}$	$0.5461 \cdot 10^{-1}$	$0.2185 \cdot 10^{-2}$

We present the errors of the approximate solutions $\bar{u}$ in the discrete Chebyshov norm and in l^1-norm:

$$err^c(t) = \max_{x_i} |\bar{u}(t, x_i) - u(t, x_i)|,$$

$$err^l(t) = \sum_i |\bar{u}(t, x_i) - u(t, x_i)| \cdot h_x \, .$$

One can infer from Table 1 that the proposed special algorithm (8.9) with error $O(h^2 + \varepsilon^2 h)$ requires less computational effort than the algorithm (8.10) with error $O(h)$ and that the experimental data are conformed to the orders of accuracy of the algorithms given by the theoretical results.

To find the solution $u(t, x)$ to the problem (8.5)-(8.6), (8.7) under $t > T$, when the solution is singular, we realize the following numerical procedure: we simulate the problem by the algorithms (8.9) and (8.10) with a sufficiently big time step h and with $h_x = h^{3/4}$ up to the time moment $t_* < T$, then we change the time step h to a smaller one h_*, take $h_x = h_*$, and continue simulations.

Table 2 gives the results of simulation of the problem (8.5)-(8.6) with $\varphi(x)$ from (8.7), (8.8) under $t > T$. One can see that in the singular case the behavior of the algorithm (8.9) is also better than the behavior of (8.10).

In connection with this example, see the numerical experiments in [1] and [18] as well.

8.2. The generalized KPP-equation with a small parameter. Consider the problem

$$\frac{\partial u}{\partial t} = \frac{\varepsilon^2}{2} \frac{\partial^2 u}{\partial x^2} + g(x, u; \varepsilon), \quad t > 0, \; x \in R^1, \tag{8.11}$$

$$u(0, x) = \chi_-(x) = \begin{cases} 1, \; x < 0 \\ 1/2, \; x = 0 \\ 0, \; x > 0, \end{cases} \tag{8.12}$$

and take

$$g(x, u; \varepsilon) = \frac{1}{\varepsilon^2} c(x) u(1 - u), \tag{8.13}$$

$$c(x) = c + \frac{a}{\pi} \operatorname{arctg} \alpha(x - b).$$

Here $\varepsilon > 0$ is a small parameter, $\alpha > 0$ is a big number, c, a, and b are positive constants, and $\frac{a}{2} < c < \frac{3a}{2}$.

The problem (8.11)-(8.13) is a generalization of the KPP-equation. The theoretical results for this problem obtained in [5] give the following. For $t < T_0 \approx \frac{b\sqrt{2a}}{c + 0.5a}$, the wave propagates to the right of the domain $G_0 = \{x < 0\}$ with the velocity $\sqrt{2c - a}$, "taking no notice" of the fact that after $x = b$ the coefficient $c(x)$ takes a larger value

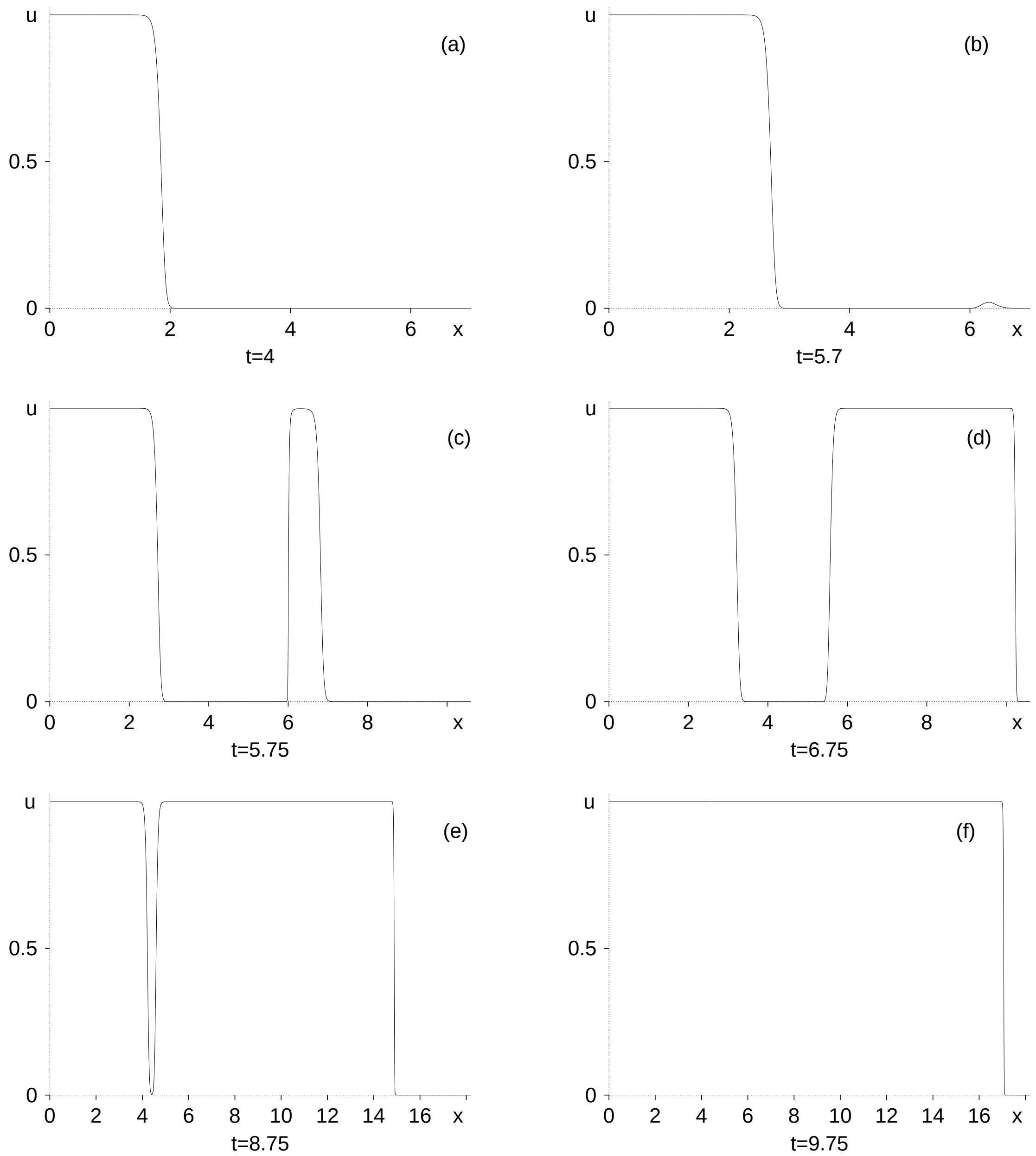

FIGURE 1. The KPP-equation. Evolution of the solution $u(t, x)$ to the problem (8.11)-(8.13) under $\varepsilon = 0.1$, $c = 1.125$, $a = 2$, $b = 6$, $\alpha = 150$ simulated by the method (6.13) with $h = 0.0001$.

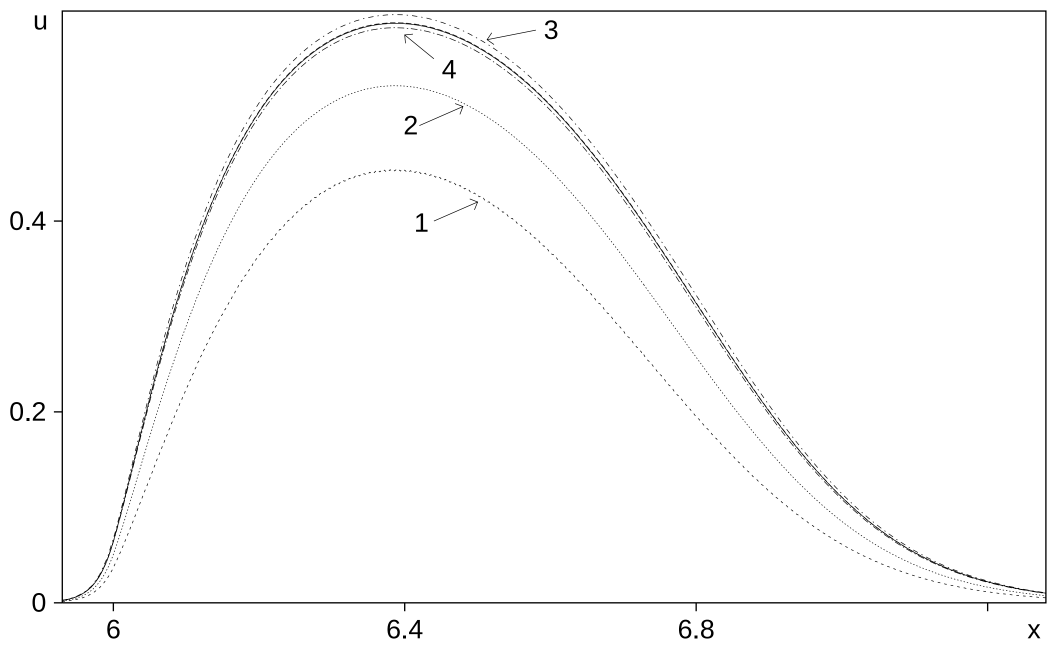

FIGURE 2. The KPP-equation (new source appearance). Comparison of the methods. The solid curve is simulated by (6.7) and (6.13) with $h = 0.0001$ and it visually coincides with the exact solution. The curves 1, 2 are simulated by (8.14) and (6.6) with $h = 0.0001$; the curves 3, 4 - by (6.7) and (6.13) with $h = 0.001$. Here $\varepsilon = 0.2$, $t = 5.8$, other parameter values are the same as in Figure 1.

$c + \dfrac{a}{2}$. But at the time T_0, a new "source" arises at the point $x = b$, away from which the front starts propagating in both directions: to the left with the velocity close to $\sqrt{2c - a}$ and to the right with the velocity close to $\sqrt{2c + a}$.

Figure 1 obtained in our numerical experiments demonstrates this phenomenon. Under the taken parameters (see the figure caption) $T_0 \approx 5.65$. Under the time t being close to T_0 the velocity of the new front to the right is greater than $\sqrt{2c + a} \approx 2.06$ (see Figures 1c and 1d) and with an increase of time the velocity tends to $\sqrt{2c + a}$ (see Figures 1e and 1f). One can explain this by the fact that when the new "source" arises the value of the solution u before the front is greater than the corresponding value of u when the shape of the wave takes its limit form.

Let us note that under the taken parameters ($\varepsilon = 0.1$, see the caption to Figure 1)

$$\min_{-\infty < x < b} \bar{u}(5.75, x) \approx 10^{-71}$$

while there is already the new front at $x = b$ (see Figure 1c). So, the "channel", through which the new "source" is initialized, is very narrow. This fact has to be taken into account for realizing numerical procedures at a computer. For instance, under $\varepsilon = 0.04$

$$\min_{-\infty < x < b} \bar{u}(5.75, x) \approx 10^{-439}$$

that is less than the smallest positive number ($\sim 10^{-308}$) realized by many compilers. To observe the phenomenon in this case, one has to compose a special numerical procedure or use a special compiler.

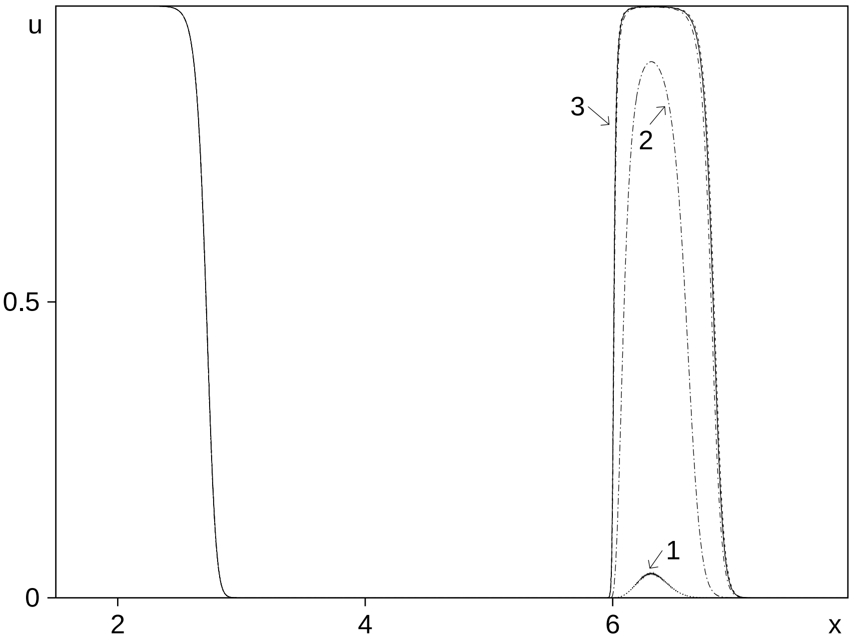

FIGURE 3. The KPP-equation. Comparison of the methods. The curve 3 is simulated by (6.7) and (6.13) with $h = 0.0005$ and $h = 0.0001$ and it visually coincides with the exact solution. The curves 1,2 are simulated by (8.14) and (6.6) with $h = 0.0001$. Here $t = 5.75$, other parameter values are the same as in Figure 1.

An additional confirmation of the high sensitiveness of the considered model is given, for instance, by the following experiment. If we put $u(0,x) = \mu$ for $x > 0$ with a small positive μ, e.g. 10^{-15}, in the initial condition (8.12) and take the other parameters as in Figure 1, the new "source" arises at a moment $t \ll 1$.

Here we compare five numerical methods: the methods (6.6), (6.7), (6.10), (6.13) given in Section 6 (of course, we take their versions adapted to problems with positive direction of time) and the first-order method written below.

The first-order method:

$$\bar{u}(0,x_j) = \chi_{\_}(x_j), \tag{8.14}$$

$$\bar{u}(t_{k+1},x_j) = \frac{1}{2}\bar{u}(t_k, x_j + \varepsilon h^{1/2}) + \frac{1}{2}\bar{u}(t_k, x_j - \varepsilon h^{1/2})$$

$$+\frac{h}{2}(g(x_{j-1}, \bar{u}(t_k, x_{j-1})) + g(x_{j+1}, \bar{u}(t_k, x_{j+1}))),$$

$$x_j = x_0 + j\varepsilon h^{1/2},\ j = 0, \pm 1, \dots,\ k = 0, \dots, N-1.$$

It can be checked that under a sufficiently small h this method preserves the monotonicity property of the solution. The first-order method (6.5), which has $hg(x_j, \bar{u}(t_k, x_j))$ instead of $h(g(x_{j-1}, \bar{u}(t_k, x_{j-1})) + g(x_{j+1}, \bar{u}(t_k, x_{j+1})))/2$, does not preserve the monotonicity property and has unstable behavior in the case of considered problem.

The algebraic equations, arising in the implementation of the methods (6.7), (6.10), and (6.13) at each step, are quadratic ones and are solved exactly. The results of testing of the three-layer method (6.10) are discussed in Section 6.2.

The results of the numerical tests are given on Figures 2 and 3.

Remark 8.1. Derivatives of the solution to (8.11)-(8.12) can be estimated as

$$|\frac{\partial^{i+j}u}{\partial t^i \partial x^j}| \leq \frac{K}{\varepsilon^{2(i+j)}},\ t \in [t_0, T],\ x \in R^1,\ 0 < \varepsilon \leq \varepsilon^*. \tag{8.15}$$

By the same arguments as in Theorem 3.3 one can prove that under (6.8) and a sufficiently small h/ε^2 the errors of both methods (6.5) and (6.6) are estimated as

$$|u(t_k, x) - \bar{u}(t_k, x)| \leq K\frac{h}{\varepsilon^4},$$

and the error of the method (6.7) is estimated as

$$|u(t_k, x) - \bar{u}(t_k, x)| \leq K\frac{h^2}{\varepsilon^6},$$

where the constant K does not depend on x, k, h, ε.

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