Some analytical properties of the multidimensional continuous Mróz model of plasticity.

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Abstract

We study the geometrical structure of memory induced by the continuous multidimensional Mróz model of plasticity. The results are used for proving the thermodynamic consistency of the model and composition and inversion formulas for input – memory state – output operators. We also show an example of nonuniqueness of solutions to a simple initial value problem involving the Mróz operator.

1 Introduction

One-dimensional mathematical models of plasticity are now fairly well understood. The theory of hysteresis operators seems to be an appropriate tool for solving dynamical problems in uniaxial plasticity [22, 8, 17], in thermodynamics of temperature-dependent models [18, 19, 10] and for developing a mathematical formalism for the material fatigue analysis [4, 16].

The multiaxial situation is much less simple. Models described by variational inequalities with convex shaped yield surfaces and corresponding to various rheological combinations of elastic and rigid – perfectly plastic elements are of generalized standard type [14, 20] and are accessible via the theory of monotone operators. This approach is however sensitive with respect to small perturbations of the model, and modifications of rheological models aiming at a more accurate description of experimentally observed phenomena (ratchetting, nonlinear hardening) require different techniques. The theory of multidimensional hysteresis operators initiated in the pioneering book [15] makes it possible to formulate and solve mathematical problems related to more complex situations, like for instance the nonlinear kinematic hardening models due to Armstrong and Frederick, Bower, Chaboche [1, 2, 9, 5, 6, 7], and the multiyield model of Mróz [21].

The original idea of Mróz was to decompose the stress-strain law into a superposition of the stress-memory state mapping (hardening rule) and the memory state-strain mapping (flow rule). The memory state is characterized by the position of infinitely many moving spherical yield surfaces in the deviatoric stress space which are *included within each other* in contrast with rheological hardening models, where the yield surfaces are independent. The Mróz hardening rule then consists in defining the interior yield surface motion. It turns out that it is given by the same equation as the Armstrong-Frederick model, but with a different physical interpretation (see [5]). Analogously to rheological models, the flow rule is defined in such a way that the plastic strain rate be orthogonal to the largest active (that is, currently moving) yield surface.

In a series of papers [12, 13], Chu considered the Mróz model with a continuous family of moving spheres $S_r(t)$ of all radii $r > 0$ in a time interval $t \in [0, T]$, centered at a point $\varphi(r, t)$ in the deviatoric stress space. For a given stress deviator evolution $\sigma(t)$, the hardening rule is required to satisfy the following hypotheses.

(H1) *For each $t \in [0, T]$, the tensor $\sigma(t)$ lies on or in the interior of $S_r(t)$, i.e.*

$$|\sigma(t) - \varphi(r, t)| \leq r \quad \text{for all} \quad r > 0, \ t \in [0, T], \tag{1.1}$$

where $|\cdot|$ denotes the norm in the space of deviatoric tensors.

(H2) *Under arbitrary piecewise linear loading, the surface S_r moves only if σ moves, lies on the boundary of S_r and its derivative points outward. More precisely, the implication*

$$\dot\sigma(t) = \hat\sigma \text{ is constant in }]t^*, t_*[, \ \exists t_n \downarrow t^* : \varphi(r, t_n) \neq \varphi(r, t^*) \ \Rightarrow$$

$$\hat\sigma \neq 0, \ |\varphi(r, t^*) - \sigma(t^*)| = r, \ \langle \hat\sigma, \varphi(r, t^*) - \sigma(t^*) \rangle \leq 0$$

holds for every $r > 0$.

(H3) *The nonintersection condition holds, that is*

$$|\varphi(r_1, t) - \varphi(r_2, t)| \leq |r_1 - r_2| \quad \text{for all} \quad r_1, r_2 > 0, \ t \in [0, T]. \tag{1.2}$$

The *memory state at time t* is described here by the spatial distribution of the spheres $S_r(t)$ or, which is the same, by the function $r \to \varphi(r, t)$, see Figure 1.

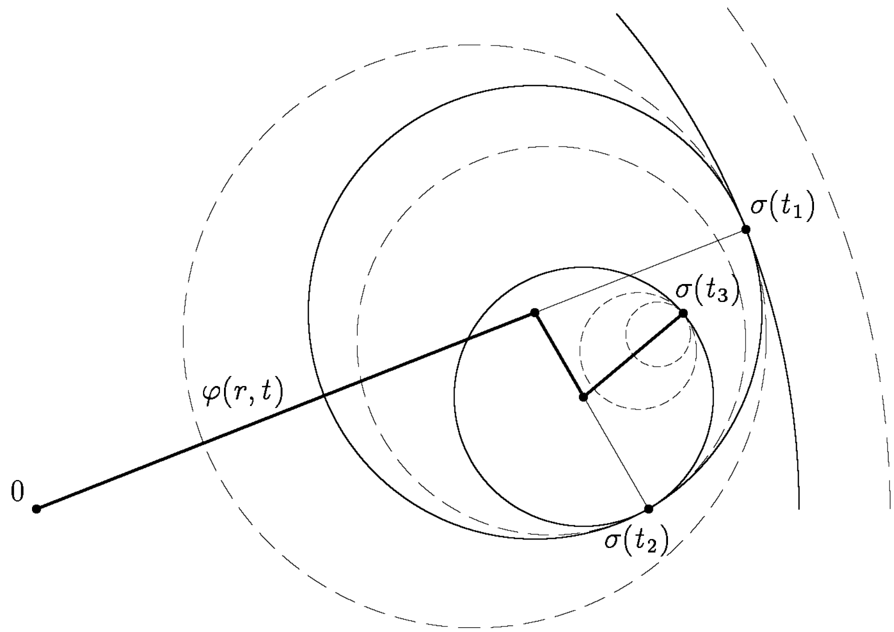

Figure 1: *Yield surfaces for a piecewise linear evolution $t_1 \to t_2 \to t_3$ of σ.*

We shall see below in Propositions 2.3, 2.4 that hypotheses (H1) – (H3) determine in a unique way the evolution of φ for each piecewise linear input σ. From the continuity Theorem 2.1 it then follows that (H1) – (H3) admit a unique continuous extension to arbitrary continuous inputs σ.

Mathematical properties of the input-state mapping $\sigma \to \varphi$ were studied in [3], in particular its continuity and regularity. It was also shown that in this case, the orthogonality rule of the plastic flow is no longer compatible with the second principle of thermodynamics and a different flow rule was proposed satisfying a thermodynamically correct energy inequality.

The aim of this paper is to derive further properties of the continuous Mróz stress-state-strain law defined in [3]. We exploit here the advantage of the simple memory structure of the Mróz hardening rule which is close to the scalar case, and derive explicit superposition and inversion formulas for the input-output mappings. This enables us to give a new interpretation of the energy inequality of [3]. The geometrical simplicity of the Mróz model in comparison with multiyield rheological models (let us note that in the uniaxial case, these two constructions coincide) is compensated by the fact that the time evolution of the Mróz outputs is less regular. This fact has already been pointed out in [3]. Here we present an even more striking evidence by showing the example of a simple evolution equation containing the Mróz input-output operator which admits multiple solutions for given initial data. Indirectly, this means that Mróz operators are not locally Lipschitz in spaces of absolutely continuous functions, while rheological models are, cf. [5].

2 The hardening rule

For mathematical considerations, the geometrical nature of the space where the evolution takes place is not relevant. We therefore fix an arbitrary separable real Hilbert space X endowed with a scalar product $\langle\cdot,\cdot\rangle$ and the norm $|\cdot| := \langle\cdot,\cdot\rangle^{\frac{1}{2}}$, $2 \le \dim X \le \infty$, which will play the role of the space of stress deviators, and consider continuous input functions $\sigma : [0,T] \to X$. We denote by $C([0,T];X)$ the space of such functions endowed with a family of seminorms $\|\sigma\|_{[0,t]} := \max\{|\sigma(\tau)|;\ \tau \in [0,t]\}$ for $t \in [0,T]$, where $\|\cdot\|_{[0,T]}$ turns out to be a norm in $C([0,T];X)$, indeed.

2.1 Discrete inputs

We first define the input-state mapping for finite input sequences $(\sigma_1, \cdots, \sigma_n) \in X^n$. The corresponding sequence $\{\varphi_k : [0,\infty[\to X;\ k = 0,1,\cdots,n\}$ of state functions is constructed by induction as follows:

$$\varphi_0(r) := 0, \qquad r \ge 0, \tag{2.1}$$

$$a_k := \max\{r \ge 0;\ |\varphi_{k-1}(r) - \sigma_k| = r\}, \quad k = 1,\cdots,n, \tag{2.2}$$

$$\varphi_k(r) := \begin{cases} \varphi_{k-1}(r), & r \ge a_k, \\ \sigma_k + \frac{r}{a_k}\big(\varphi_{k-1}(a_k) - \sigma_k\big), & 0 \le r < a_k. \end{cases} \tag{2.3}$$

Figure 2 represents the trajectories of φ_k in X.

We immediately see that for all k, the function φ_k is piecewise affine, $\varphi_k(r) = 0$ for $r \ge R_k := \max\{|\sigma_j|;\ j = 1,\cdots,k\}$, $\varphi_k(0) = \sigma_k$ and $|\frac{d}{dr}\varphi_k(r)| = 1$ a.e. in $]0,R_k[$. Introducing the convex sets

$$\Phi_{R^*}(\sigma^*) := \{\varphi : [0,\infty[\to X \quad \text{absolutely continuous};\ \varphi(0) = \sigma^*, \tag{2.4}$$

$$\varphi(r) = 0 \ \text{ for } \ r \ge R^*,\ \left|\frac{d\varphi}{dr}(r)\right| \le 1 \ \text{ a.e.}\}$$

for arbitrary $R^* > 0$ and $\sigma^* \in X$, we can simply write $\varphi_k \in \Phi_{R_k}(\sigma_k)$. In particular, the function $r \mapsto r - |\varphi_{k-1}(r) - \sigma_k|$ is nondecreasing, hence $|\varphi_{k-1}(r) - \sigma_k| < r$ for $r > a_k$, $|\varphi_{k-1}(r) - \sigma_k| \geq r$ for $r \leq a_k$.

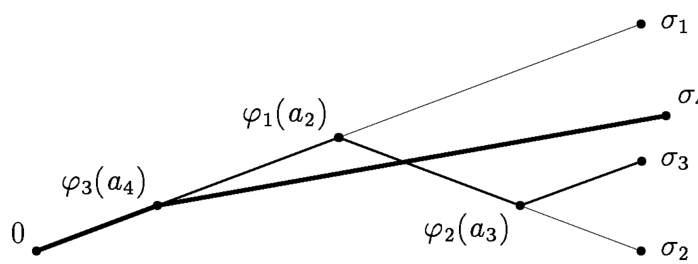

Figure 2: *Update of the memory state.*

For the proof of the following two properties of the memory state sequences we refer to [3], Lemmas 4.2 and 3.7.

Theorem 2.1 (Continuity) *Let $\{\sigma_k^1, \sigma_k^2;\ k = 1, \cdots, n\}$ be two input sequences in X and let $\{\varphi_k^1, \varphi_k^2;\ k = 0, 1, \cdots, n\}$ be the corresponding sequences of state functions defined by (2.1)-(2.3). Put*

$$R := \max\{|\sigma_k^1|, |\sigma_k^2|;\ k = 1, \cdots, n\}, \tag{2.5}$$

$$\delta := \max\{|\sigma_k^1 - \sigma_k^2|;\ k = 1, \cdots, n\}. \tag{2.6}$$

Then for every $k = 1, \cdots, n$ and every $r, s > 0$ we have

$$|\varphi_k^1(r) - \varphi_k^2(s)|^2 \leq 2R\delta + (r - s)^2. \tag{2.7}$$

Theorem 2.2 (Energy inequality) *Let $\{\sigma_k;\ k = 1, \cdots, n\}$ be an input sequence in X and let $\{\varphi_k;\ k = 0, 1, \cdots, n\}$ be the corresponding sequence of state functions. Then for every $k = 1, \cdots, n$ and every $r \geq 0$ we have*

$$\big\langle \varphi_k(r) - \varphi_{k-1}(r), \varphi_k(r) - \sigma_k \big\rangle \leq 0. \tag{2.8}$$

The "energy" interpretation of inequality (2.8) will be given in Section 4. We first pass to the continuous time evolution case.

2.2 The continuous hardening rule

Let us consider the situation where the input moves linearly in a fixed direction, that is

$$\sigma(t) = \sigma(t_0) + (t - t_0)\hat{\sigma}, \quad t \in [t_0, t_1], \tag{2.9}$$

where $\hat{\sigma} \in X$ is a given vector, and assume that $R_0 > 0$ and $\varphi^0 \in \Phi_{R_0}(\sigma(t_0))$ are given. Analogously to (2.1)-(2.3) we define for $t \in [t_0, t_1]$

$$a(t) := \max\{r \geq 0;\ |\varphi^0(r) - \sigma(t)| = r\}, \tag{2.10}$$

$$\varphi(r, t) := \begin{cases} \varphi^0(r), & r \geq a(t), \\ \sigma(t) + \frac{r}{a(t)}\big(\varphi^0(a(t)) - \sigma(t)\big), & 0 \leq r < a(t). \end{cases} \tag{2.11}$$

By construction, we have $\varphi(\cdot, t_0) = \varphi^0$ and $\varphi(\cdot, t) \in \Phi_{\max\{R^0, |\sigma(t)|\}}(\sigma(t))$ for all $t \in [t_0, t_1]$.

Proposition 2.3 *Let* $\varphi^0 \in \Phi_{R_0}(\sigma(t_0))$ *for some* $R_0 > 0$, *and let* σ, a *and* φ *be given by (2.9)-(2.11). Then* a *is increasing in* $]t_0, t_1]$, φ *is continuous in both variables and satisfies (H1) – (H3).*

Proof. Inequality (1.1) is an immediate consequence of (2.11). To check that (1.2) holds, it suffices to consider the case $r_1 < a(t) < r_2$. Then

$$\varphi(r_2,t) - \varphi(r_1,t) = \frac{r_1}{a(t)}\big(\varphi^0(r_2) - \varphi^0(a(t))\big) + \left(1 - \frac{r_1}{a(t)}\right)\big(\varphi^0(r_2) - \sigma(t)\big)$$

and (1.2) follows.

We may assume $\hat\sigma \neq 0$, since otherwise $\varphi(\cdot,t) = \varphi^0$ and the remainder of the proof is trivial. For $t \in]t_0, t_1]$ we have by definition

$$a^2(t) = \big\langle \varphi^0(a(t)) - \sigma(t_0), \varphi^0(a(t)) - \sigma(t)\big\rangle - (t - t_0)\big\langle \hat\sigma, \varphi^0(a(t)) - \sigma(t)\big\rangle,$$

hence

$$\big\langle \hat\sigma, \varphi^0(a(t)) - \sigma(t)\big\rangle \leq 0 \quad \forall\, t \in]t_0, t_1]. \tag{2.12}$$

For arbitrary $s > t > t_0$ it holds

$$|\varphi^0(a(t)) - \sigma(s)|^2 - a^2(t) = |\varphi^0(a(t)) - \sigma(s)|^2 - |\varphi^0(a(t)) - \sigma(t)|^2$$

$$= (s-t)^2|\hat\sigma|^2 - 2\,(s-t)\big\langle \hat\sigma, \varphi^0(a(t)) - \sigma(t)\big\rangle > 0, \tag{2.13}$$

consequently

$$a(s) > a(t) \qquad \forall\, t_0 < t < s \leq t_1. \tag{2.14}$$

To prove the continuity with respect to t, we fix some $s > t \geq t_0$. For $r \geq a(s)$ we have by (2.14) and by definition $\varphi(r,t) = \varphi(r,s) = \varphi^0(r)$. Let $r \in [0, a(s)[$ and put $J := 1/a(s)\big(\varphi^0(a(s)) - \sigma(s)\big)$, $\delta := |\sigma(s) - \sigma(t)|$. Then

$$\begin{aligned}
(r+\delta)^2 \;&\geq\; |\varphi(r,t) - \sigma(s)|^2 = |\varphi(r,t) - \varphi(r,s)|^2 \\
&\quad + r^2 + 2\,r\big\langle \varphi(r,t) - \varphi(r,s), J\big\rangle, &(2.15)\\
(a(s) - r)^2 \;&\geq\; |\varphi(r,t) - \varphi^0(a(s))|^2 = |\varphi(r,t) - \varphi(r,s)|^2 \\
&\quad + (a(s) - r)^2 - 2\,(a(s) - r)\,\big\langle \varphi(r,t) - \varphi(r,s), J\big\rangle. &(2.16)
\end{aligned}$$

Combining (2.15), (2.16) we obtain

$$|\varphi(r,t) - \varphi(r,s)|^2 \leq \left(1 - \frac{r}{a(s)}\right)\big((r+\delta)^2 - r^2\big) \leq 2\,a(t_1)\,\delta + \delta^2, \tag{2.17}$$

hence $\varphi(r,\cdot)$ is continuous for all r.

It remains to prove the implication (H2). We first show that

$$\begin{aligned}
|\varphi(r,t) - \sigma(t)| < r \qquad &\Rightarrow \quad \exists \delta_r > 0,\ \varphi(r,\tau) = \varphi^0(r) \\
&\quad\ \forall\, \tau \in [t, t+\delta_r],\ r > 0,\ t \in [t_0, t_1], &(2.18)
\end{aligned}$$

$$|\varphi^0(r) - \sigma(t_0)| = r, \quad \langle \hat\sigma, \varphi^0(r) - \sigma(t_0)\rangle > 0 \quad \Rightarrow$$
$$\exists \delta_r > 0, \ \varphi(r,\tau) = \varphi^0(r) \quad \forall \tau \in [t_0, t_0 + \delta_r]. \tag{2.19}$$

The implication (2.19) is a consequence of the identity

$$r^2 - |\varphi^0(r) - \sigma(\tau)|^2 = (\tau - t_0)\big(2\,\langle \hat\sigma, \varphi^0(r) - \sigma(t_0)\rangle - (\tau - t_0)|\hat\sigma|^2\big)$$

which entails $|\varphi^0(r) - \sigma(\tau)| < r$ for τ close to t_0. Both (2.19) and (2.18) therefore follow directly from (2.10) and (2.11).

To conclude, assume that for some $r > 0$ and $t^* \in [t_0, t_1[$ there exists a sequence $t_n \downarrow t^*$, $\varphi(r, t_n) \neq \varphi(r, t^*)$. By (2.18) – (2.19), we either have $t^* = t_0$ and

$$|\varphi(r,t^*) - \sigma(t^*)| = r, \quad \langle \hat\sigma, \varphi(r,t^*) - \sigma(t^*)\rangle \le 0, \tag{2.20}$$

or $t^* > t_0$ and $r \le a(t^*)$. By (2.11) we then have

$$\varphi(r,t^*) - \sigma(t^*) = \frac{r}{a(t^*)}\left(\varphi^0(a(t^*)) - \sigma(t^*)\right)$$

and (2.20) follows from (2.12). Proposition 2.3 is proved. □

Proposition 2.4 *Let σ, a and φ^0 be as in (2.9), (2.10), and let a continuous function φ satisfy hypotheses (H1) – (H3), $\varphi(\cdot, t_0) = \varphi^0$. Then φ has the form (2.11).*

P r o o f. Assume first that for some $t \in [t_0, t_1]$ and $r \ge a(t)$ we have $\varphi(r,t) \neq \varphi^0(r)$. Put $t^* := \inf\{\tau \in [t_0, t]\,;\ \varphi(r,\tau) \neq \varphi^0(r)\}$. From (H2) it follows that $\hat\sigma \neq 0$, $|\varphi^0(r) - \sigma(t^*)| = r$, $\langle \hat\sigma, \varphi^0(r) - \sigma(t^*)\rangle \le 0$. By definition of $a(t)$, we therefore have

$$\begin{aligned} 0 &\le r^2 - |\varphi^0(r) - \sigma(t)|^2 = |\varphi^0(r) - \sigma(t^*)|^2 - |\varphi^0(r) - \sigma(t^*) - (t - t^*)\hat\sigma|^2 \\ &= 2\,(t - t^*)\,\langle \hat\sigma, \varphi^0(r) - \sigma(t^*)\rangle - (t - t^*)^2\,|\hat\sigma|^2 \ < \ 0, \end{aligned}$$

which is a contradiction. Consequently, $\varphi(r,t) = \varphi^0(r)$ for all $r \ge a(t)$, $t \in [t_0, t_1]$.

By (H1) and (H3), for all $t \in [t_0, t_1]$ and $r \in [0, a(t)[$ we have

$$|\varphi(r,t) - \sigma(t)| \le r, \quad |\varphi^0(a(t)) - \varphi(r,t)| = |\varphi(a(t),t) - \varphi(r,t)| \le a(t) - r.$$

Since

$$\begin{aligned} a^2(t) &= |\varphi^0(a(t)) - \sigma(t)|^2 = |\varphi^0(a(t)) - \varphi(r,t)|^2 + |\varphi(r,t) - \sigma(t)|^2 \\ &+ 2\,\langle \varphi^0(a(t)) - \varphi(r,t), \varphi(r,t) - \sigma(t)\rangle, \end{aligned}$$

it follows that

$$|\varphi^0(a(t)) - \varphi(r,t)| = a(t) - r, \quad |\varphi(r,t) - \sigma(t)| = r,$$

$$\langle \varphi^0(a(t)) - \varphi(r,t), \varphi(r,t) - \sigma(t)\rangle = (a(t) - r)r$$

for $0 \le r < a(t)$, hence (2.11) holds. □

Let σ, φ be given by (2.9)-(2.11) and let $\tau \in [t_0, t_1]$. Denote by $\psi_\tau(r,t)$, where $r \geq 0, \tau \leq t \leq t_1$, the state function corresponding to the initial state $\psi_\tau^0 = \varphi(\cdot, \tau)$ and the input $\sigma_\tau(t) := \sigma(t)$, $t \in [\tau, t_1]$. By definition, $\psi_{t_0} = \varphi$. Since the function (2.10) is increasing, from (2.11) it follows easily the identity

$$\psi_\tau(\cdot, t) = \varphi(\cdot, t) \qquad \text{for every} \qquad t_0 \leq \tau \leq t \leq t_1,$$

which means that the Mróz input – memory state operator has the semigroup property for linear inputs.

Using this fact, we can define the Mróz hardening rule for every piecewise linear input function $\sigma \in C([0,T];X)$ of the form

$$\sigma(t) = \sigma_k + \frac{t - t_k}{t_{k+1} - t_k}(\sigma_{k+1} - \sigma_k), \quad t \in [t_k, t_{k+1}], \; k = 1, \cdots, n, \tag{2.21}$$

where

$$0 = t_1 < t_2 < \cdots < t_{n+1} = T \tag{2.22}$$

is a given partition and $\{\sigma_1, \cdots, \sigma_{n+1}\}$ is a given sequence in X. In each interval $[t_k, t_{k+1}]$ we define the value of $\varphi(r,t)$ by (2.10), (2.11), where we replace $[t_0, t_1]$ by $[t_k, t_{k+1}]$ and $\varphi^0(r)$ by $\varphi_k(r)$ obtained by the recursive formulas (2.1)-(2.3). Theorem 2.1 immediately yields the following continuity result.

Theorem 2.5 *Let $\sigma^1, \sigma^2 \in C([0,T];X)$ be piecewise linear functions of the form (2.21), and let φ^1, φ^2 be the corresponding state functions. Then φ^1, φ^2 are continuous in both variables and for every $r, s \geq 0$, $t \in [0,T]$ we have*

$$|\varphi^1(r,t) - \varphi^2(s,t)|^2 \leq 2 \max\{\|\sigma^1\|_{[0,t]}, \|\sigma^2\|_{[0,t]}\} \|\sigma^1 - \sigma^2\|_{[0,t]} + (r-s)^2. \tag{2.23}$$

Theorem 2.5 enables us to extend the definition of the Mróz state function to an arbitrary continuous input, since piecewise linear functions form a dense subset of $C([0,T];X)$. This extension is unique and inequality (2.23) holds for all $\sigma^1, \sigma^2 \in C([0,T];X)$. As a consequence of Theorem 2.5 we also obtain

$$\varphi(\cdot, t) \in \Phi_{\|\sigma\|_{[0,t]}}(\sigma(t)) \quad \text{for every} \;\; \sigma \in C([0,T];X) \;\; \text{and} \;\; t \in [0,T]. \tag{2.24}$$

3 Flow rule

We slightly generalize the state-output mapping or the *flow rule* introduced in [3] by considering the set $\mathcal{H}$ of admissible density functions given by

$$\mathcal{H} := \{h : [0, \infty[\to [0, \infty[; \quad h(0) = 0, \quad h \text{ is nondecreasing and} \\ \text{absolutely continuous in } [0, \infty[, \; \frac{dh}{dr} \in BV_{\mathrm{loc}}(0, \infty)\}.$$

For a given density function $h \in \mathcal{H}$ and a given input $\sigma \in C([0,T];X)$ we define the strain ε by the Stieltjes integral

$$\varepsilon(t) := \frac{dh}{dr}(0)\,\sigma(t) + \int_0^\infty \varphi(r,t)\,d\left(\frac{dh}{dr}(r)\right), \tag{3.1}$$

where φ is the state function corresponding to σ. Integrating by parts in (3.1) we can write the input-output operator

$$\varepsilon = M_h(\sigma) \tag{3.2}$$

in the form

$$M_h(\sigma) = -\int_0^\infty \frac{\partial\varphi}{\partial r}(r,t)\frac{dh}{dr}(r)\,dr. \tag{3.3}$$

Clearly, M_h maps $C([0,T];X)$ into $C([0,T];X)$ and from (2.23), (2.24) and (3.3) we obtain

$$|M_h(\sigma)(t)| \leq h(\|\sigma\|_{[0,t]}), \tag{3.4}$$

$$|M_h(\sigma^1) - M_h(\sigma^2)|(t) \leq \frac{dh}{dr}(0)|\sigma^1 - \sigma^2|(t) + C_t\,\|\sigma^1 - \sigma^2\|_{[0,t]}^{1/2} \tag{3.5}$$

for all σ, σ^1, $\sigma^2 \in C([0,T];X)$ and $t \in [0,T]$, where

$$C_t := \sqrt{2R_t}\,\mathop{\mathrm{Var}}_{[0,R_t]}\left(\frac{dh}{dr}\right), \qquad R_t := \max\{\|\sigma^1\|_{[0,t]}, \|\sigma^2\|_{[0,t]}\}.$$

The function h can be interpreted as a counterpart of the *initial loading curve* in uniaxial plasticity.

The following theorem, which is the main result of this section, is in fact a multidimensional version of Corollary II.3.4 of [17].

Theorem 3.1 (Superposition and inversion of Mróz operators)
For every $h_1, h_2 \in \mathcal{H}$ we have

$$M_{h_1} \circ M_{h_2} = M_{h_1 \circ h_2}. \tag{3.6}$$

If moreover $h \in \mathcal{H}$ is such that $h^{-1} \in \mathcal{H}$, then

$$(M_h)^{-1} = M_{h^{-1}}. \tag{3.7}$$

The proof of Theorem 3.1 is based on the following "discrete" lemma.

Lemma 3.2 *Let $\{\sigma_1, \cdots, \sigma_n\}$ be a sequence in X and let $\{\varphi_0, \cdots, \varphi_n\}$ be the sequence of state functions defined by (2.1)-(2.3). For a given function $h \in \mathcal{H}$ put*

$$\varepsilon_k = -\int_0^\infty \frac{d\varphi_k}{dr}(r)\frac{dh}{dr}(r)\,dr, \qquad k = 1, \cdots, n. \tag{3.8}$$

Let $\{\psi_0, \cdots, \psi_n\}$ be the sequence of the state functions corresponding to $\{\varepsilon_1, \cdots, \varepsilon_n\}$ according to (2.1)-(2.3). Then for every $s \in [0, h(\infty)[$ and $k = 0, 1, \cdots, n$ we have

$$\psi_k(s) = -\int_{h^{-1}(s)}^\infty \frac{d\varphi_k}{dr}(r)\frac{dh}{dr}(r)\,dr, \tag{3.9}$$

where $h^{-1}(s) := \inf\{r\,;\;\; h(r) = s\}$.

Proof of Lemma 3.2. For $s = 0$ there is nothing to prove, since (3.9) coincides with (3.8). For $s > 0$ we proceed by induction over k. The assertion is trivial for $k = 0$. Assume now that (3.9) holds for $k - 1$ and that $a_k \neq 0$ (for $a_k = 0$ we have indeed $\sigma_k = \sigma_{k-1}$, $\varphi_k = \varphi_{k-1}$). By (2.2), (2.3) we have

$$\varepsilon_k = -\int_{a_k}^\infty \frac{d\varphi_{k-1}}{dr}(r)\frac{dh}{dr}(r)\,dr - \frac{h(a_k)}{a_k}(\varphi_{k-1}(a_k) - \sigma_k),$$

that is, $\varepsilon_k = \psi_{k-1}(h(a_k)) - h(a_k)J_k$, where

$$J_k := \frac{\varphi_{k-1}(a_k) - \sigma_k}{a_k}, \qquad |J_k| = 1. \tag{3.10}$$

This yields

$$|\psi_{k-1}(h(a_k)) - \varepsilon_k| = h(a_k). \tag{3.11}$$

We may assume $h(a_k) > 0$; otherwise $h(r) = 0$ for all $r \in [0, a_k]$, $\varepsilon_k = \varepsilon_{k-1}$, $\psi_k = \psi_{k-1}$ and (3.9) follows. Put $b_k := \max\{s > 0; |\psi_{k-1}(s) - \varepsilon_k| = s\} \geq h(a_k)$. For $s \in [h(a_k), b_k]$ we have

$$\begin{aligned} s^2 &= |\psi_{k-1}(s) - \varepsilon_k|^2 = |\psi_{k-1}(s) - \psi_{k-1}(h(a_k))|^2 + h^2(a_k) \\ &+ 2\,h(a_k)\left\langle J_k, \psi_{k-1}(s) - \psi_{k-1}(h(a_k))\right\rangle \\ &\leq (s - h(a_k))^2 + h^2(a_k) + 2\,h(a_k)\,(s - h(a_k)) = s^2, \end{aligned}$$

hence

$$\psi_{k-1}(s) = \psi_{k-1}(h(a_k)) + (s - h(a_k))\,J_k = \varepsilon_k + s\,J_k\ .$$

We conclude that

$$\psi_k(s) = \begin{cases} \psi_{k-1}(s), & s \geq h(a_k), \\ \varepsilon_k + sJ_k, & 0 \leq s < h(a_k), \end{cases} \tag{3.12}$$

which is precisely (3.9). The induction step is complete and Lemma 3.2 is proved. □

Proof of Theorem 3.1. Let $h_1, h_2 \in \mathcal{H}$ and $\sigma \in C([0,T];X)$ be arbitrarily given. For the partition (2.22) we construct the linear interpolate $\sigma^{(n)}$ of σ by the formulae (2.21), where $\sigma_k := \sigma(t_k)$, $k = 1, \cdots, n+1$. Put $\varepsilon^{(n)} := M_{h_2}(\sigma^{(n)})$, $\xi^{(n)} := M_{h_1 \circ h_2}(\sigma^{(n)})$ and let $\tilde{\varepsilon}^{(n)}$ be the piecewise linear interpolation of $\varepsilon^{(n)}$, that is,

$$\tilde{\varepsilon}^{(n)}(t) := \varepsilon^{(n)}(t_k) + \frac{t - t_k}{t_{k+1} - t_k}\big(\varepsilon^{(n)}(t_{k+1}) - \varepsilon^{(n)}(t_k)\big), \quad t \in [t_k, t_{k+1}], \tag{3.13}$$

where $k = 1, \cdots, n$. Let $\varphi^{(n)}$, $\psi^{(n)}$ denote the state functions corresponding to $\sigma^{(n)}$, $\tilde{\varepsilon}^{(n)}$, respectively, and for $k = 1, \cdots, n+1$ put $\varphi_k(r) := \varphi^{(n)}(r, t_k)$, $\psi_k(r) := \psi^{(n)}(r, t_k)$, $\varepsilon_k := \varepsilon^{(n)}(t_k) = \tilde{\varepsilon}^{(n)}(t_k)$, $\xi_k := \xi^{(n)}(t_k)$. Then for all k we have

$$\varepsilon_k = -\int_0^\infty \frac{d\varphi_k}{dr}(r) \frac{dh_2}{dr}(r)\, dr, \tag{3.14}$$

$$\xi_k = -\int_0^\infty \frac{d\varphi_k}{dr}(r) \frac{d}{dr}(h_1 \circ h_2)(r)\, dr, \tag{3.15}$$

and, by Lemma 3.2,

$$\psi_k(s) = -\int_{h_2^{-1}(s)}^\infty \frac{d\varphi_k}{dr}(r) \frac{dh_2}{dr}(r)\, dr, \qquad s \geq 0. \tag{3.16}$$

An elementary substitution then yields

$$\xi_k = -\int_0^\infty \frac{d\psi_k}{ds}(s) \frac{dh_1}{ds}(s)\, ds. \tag{3.17}$$

Putting $\tilde{\xi}^{(n)} := M_{h_1}(\tilde{\varepsilon}^{(n)})$ we see from (3.17) that $\tilde{\xi}^{(n)}(t_k) = \xi^{(n)}(t_k) = \xi_k$ for all k. By refining the partition and passing to the limit as $n \to \infty$ we obtain (3.6).

For the function $h(r) \equiv r$, the definition (3.1) of the flow rule immediately yields

$$\varepsilon(t) = -\int_0^\infty \frac{\partial \varphi}{\partial r}(r,t)\, dr = \varphi(0,t) = \sigma(t), \tag{3.18}$$

hence (3.7) follows from (3.6). Theorem 3.1 is proved. □

4 Thermodynamic consistency

In this section we find sufficient conditions on the density function h in (3.1) such that the constitutive law (3.2) is thermodynamically consistent. In other words, we look for a nonnegative potential energy operator U_h such that for every regular input function $\sigma : [0,T] \to X$ we have

$$\frac{d}{dt} U_h(\sigma) \le \left\langle \sigma, \frac{d}{dt} M_h(\sigma) \right\rangle \quad \text{in} \quad]0,T[\tag{4.1}$$

in an appropriate sense. In fact, the main problem consists in interpreting the time derivative properly. Let us first recall the regularity results of [3].

Proposition 4.1 *For every $\sigma \in C([0,T];X) \cap BV(0,T;X)$, the state function φ satisfies the estimate*

$$\operatorname*{Var}_{[0,T]} \varphi(r,\cdot) \le 3 \operatorname*{Var}_{[0,T]} \sigma \qquad \forall r > 0. \tag{4.2}$$

Moreover, there exists $\sigma \in W^{1,\infty}(0,T;X)$ such that $\varphi(r,\cdot)$ does not belong to the space $W^{1,p}(0,T;X)$ for any $p > 1$ and for all r in a set of positive measure.

In particular, the question whether $M_h(\sigma)$ is differentiable even if σ is smooth remains open. Nevertheless, from (4.2) it follows that the output $M_h(\sigma)$ belongs to $C([0,T];X) \cap BV(0,T;X)$ if $\sigma \in C([0,T];X) \cap BV(0,T;X)$ and the estimate

$$\operatorname*{Var}_{[0,T]} M_h(\sigma) \le \left(\frac{dh}{dr}(0) + 3 \operatorname*{Var}_{[0,\|\sigma\|_{[0,T]}]} \left(\frac{dh}{dr} \right) \right) \operatorname*{Var}_{[0,T]} \sigma \tag{4.3}$$

holds. Inequality (4.1) can therefore be interpreted in the Stieltjes integral sense

$$U_h(\sigma)(t) - U_h(\sigma)(s) \le \int_s^t \langle \sigma(\tau), d(M_h(\sigma)(\tau)) \rangle \tag{4.4}$$

for every $\sigma \in C([0,T];X) \cap BV(0,T;X)$ and every $0 \le s < t \le T$.

The situation is slightly more favourable if instead of (3.2), we consider the inverse constitutive law

$$\sigma = M_{\bar{h}}(\varepsilon). \tag{4.5}$$

In fact, by Theorem 3.1, (4.5) is equivalent to (3.2) provided h is invertible and $\bar{h} = h^{-1} \in \mathcal{H}$. This leads us to the following definition.

Definition 4.2 *The constitutive law (4.5) is called* thermodynamically consistent, *if there exists a potential energy operator $\bar{U}_{\bar{h}} : C([0,T];X) \to C([0,T];R^+)$ such that for every $\varepsilon \in W^{1,1}(0,T;X)$ and every $0 \le s < t \le T$ we have*

$$\bar{U}_{\bar{h}}(\varepsilon)(t) - \bar{U}_{\bar{h}}(\varepsilon)(s) \le \int_s^t \langle \dot{\varepsilon}(\tau), M_{\bar{h}}(\varepsilon)(\tau) \rangle \, d\tau, \tag{4.6}$$

where the dot denotes derivative with respect to t.

A hint how to construct the operators U_h, $\bar{U}_{\bar{h}}$ comes from the inequality (2.8). More precisely, we recall Proposition 3.6 of [3] which we state here in the following form.

Proposition 4.3 *Let $u \in W^{1,1}(0,T;X)$ be a given function and let φ be its memory state function. We then have*

$$\frac{1}{2}\left(|\varphi(r,t)|^2 - |\varphi(r,s)|^2\right) - \langle \varphi(r,t), u(t)\rangle + \langle \varphi(r,s), u(s)\rangle + \int_s^t \langle \varphi(r,\tau), \dot{u}(\tau)\rangle \, d\tau \le 0 \tag{4.7}$$

for every $0 \le s < t \le T$ and every $r \ge 0$.

It enables us to prove here the next result.

Theorem 4.4 (Thermodynamic consistency)
Let $h, \bar{h} \in \mathcal{H}$ be given functions. Then

(i) *inequality (4.4) holds provided h is convex and we put*

$$U_h(\sigma)(t) := \frac{1}{2}\left(\frac{dh}{dr}(0)|\sigma(t)|^2 + \int_0^\infty |\varphi(r,t)|^2 \, d\left(\frac{dh}{dr}(r)\right)\right); \tag{4.8}$$

(ii) *inequality (4.6) holds provided $\bar{h}$ is concave and we put*

$$\bar{U}_{\bar{h}}(\varepsilon)(t) := \frac{1}{2}\left(\frac{d\bar{h}}{dr}(\infty)|\varepsilon(t)|^2 - \int_0^\infty |\varepsilon(t) - \varphi(r,t)|^2 \, d\left(\frac{d\bar{h}}{dr}(r)\right)\right),$$

where φ in each case is the memory state function corresponding to the given input function.

Proof. To prove (i), we integrate (4.7) with σ instead of u with respect to $d\left(\frac{dh}{dr}(r)\right)$ (note that $\frac{dh}{dr}$ is nondecreasing) and obtain

$$U_h(\sigma)(t) - U_h(\sigma)(s) - \langle M_h(\sigma)(t), \sigma(t)\rangle + \langle M_h(\sigma)(s), \sigma(s)\rangle \le -\int_s^t \langle M_h(\sigma)(\tau), \dot{\sigma}(\tau)\rangle \, d\tau.$$

Integrating by parts we obtain (4.4).

Part (ii) is obtained similarly. We have indeed

$$M_{\bar{h}}(\varepsilon)(t) = \frac{d\bar{h}}{dr}(\infty)\varepsilon(t) - \int_0^\infty (\varepsilon(t) - \varphi(r,t)) \, d\left(\frac{d\bar{h}}{dr}(r)\right),$$

hence, integrating (4.7) with ε instead of u with respect to $d\left(\frac{d\bar{h}}{dr}(r)\right)$, where, in this case, $\frac{d\bar{h}}{dr}$ is nonincreasing, we immediately obtain (4.6). □

Remark 4.5 The formal difference $\langle \dot\varepsilon, \sigma \rangle - \dot U$ in (4.4) or (4.6) represents the dissipation rate. One particularity of the Mróz model consists in the fact that there exist cyclic motions in the plastic regime that dissipate no energy (and therefore are perfectly reversible). These are so-called *neutral motions* characterized by input functions $u = \sigma$ or $u = \varepsilon$ of the form

$$u(t) = u_0 + r_0\, e(t), \qquad t \in [t_0, t_1], \tag{4.9}$$

where $r_0 > 0$ and $u_0 \in X$ are fixed, $e(t)$ is a smooth vector function such that $|e(t)| = 1$ in $[t_0, t_1]$ and the memory state function φ satisfies $\varphi(r_0, t_0) = u_0$. Indeed, we then have for $t \in [t_0, t_1]$

$$\varphi(r,t) = \begin{cases} \varphi(r, t_0), & r \geq r_0, \\ u_0 + (r_0 - r)\, e(t), & 0 \leq r < r_0, \end{cases}$$

hence we have equality in (4.7), which means no dissipation. In the next section we show another peculiar property of the neutral motions.

5 Example of ill-posedness

We give here the example of an ordinary differential equation coupled with a Mróz constitutive operator which admits multiple neutral motion solutions for given initial data. The construction is much simpler than in the scalar case (see [11]), where no neutral motions exist. We choose here for X the two-dimensional space identified with the complex plane $\mathcal{C}$ endowed with the natural scalar product

$$\langle \xi, \eta \rangle := \mathrm{Re}(\xi \bar\eta), \tag{5.1}$$

where $\bar\eta$ is the complex conjugate of η. Let $h \in \mathcal{H}$ be globally Lipschitz continuous and let M_h be the Mróz operator defined by (3.3). We look for functions $u : [0, \infty[\to \mathcal{C}$ solving the equation

$$i\dot u(t) + M_h(u)(t) = \gamma u(t) \tag{5.2}$$

with the initial condition

$$u(0) = m\, e_0, \tag{5.3}$$

where $\gamma \geq 0$, $m > 0$ and $e_0 \in \mathcal{C}$, $|e_0| = 1$ are given. Note that the operator M_h is continuous and causal in $C([0,T]; \mathcal{C})$ for every $T > 0$. One can therefore prove by a standard retarded argument method that problem (5.2)-(5.3) has a local solution. Using the fact that by (3.4), the operator M_h has sublinear growth, we conclude that each local solution can be extended to a global one. Nevertheless, the following example shows that it may not be unique.

Example 5.1 Put $\omega := h(m) - \gamma m$ and assume that (see Figure 3)

(i) $\omega \neq 0$;

(ii) the equation $h(\rho) - \gamma\rho = \omega$ admits at least one solution $\rho \in]0, m[$.

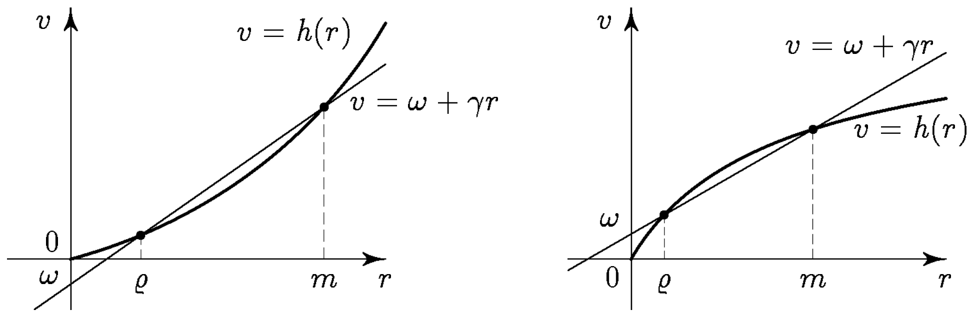

Figure 3: *The convex case* *The concave case*

Let us consider the function

$$u_c(t) := \left(m - c + c e^{i\frac{\omega}{c}t}\right) e_0, \quad t \geq 0, \tag{5.4}$$

with a parameter $c \in]0, m]$. Clearly, its memory state function φ_c has the form

$$\varphi_c(r,t) = \begin{cases} 0, & r \geq m, \\ (m - r)\, e_0, & c \leq r < m, \\ (m - c)\, e_0 + (c - r)\, e^{i\frac{\omega}{c}t}\, e_0, & 0 \leq r < c, \end{cases} \tag{5.5}$$

hence

$$M_h(u_c)(t) = \left(h(m) - h(c) + h(c) e^{i\frac{\omega}{c}t}\right) e_0 \tag{5.6}$$

for all $t \geq 0$. The function u_c satisfies

$$i\dot{u}_c(t) + M_h(u_c)(t) - \gamma u_c(t) = \left(\omega - h(c) + \gamma c\right)\left(1 - e^{i\frac{\omega}{c}t}\right) e_0$$

and fulfils (5.2), (5.3) for both $c = m$ and $c = \rho$. Indeed, this construction leads to a continuum of solutions $\{u^s;\, s > 0\}$ obtained by shifting the trajectory of u_ρ along u_m (see Figure 4), that is, $u^s(t) = u_m(t)$ for $0 \leq t \leq s$, $u^s(t) = \frac{1}{m} u_\rho(t - s)\, \bar{e}_0\, u_m(s)$ for $t > s$.

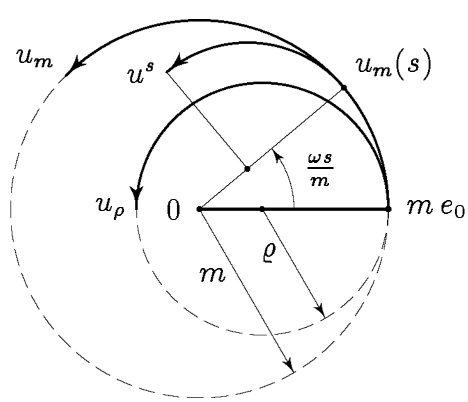

Figure 4: *Trajectories of distinct solutions of (5.2), (5.3) for* $\omega > 0$.

References

[1] Armstrong, P.J., Frederick, C.O.: *A mathematical representation of the multiaxial Bauschinger effect*, C.E.G.B., Report RD/B/N 731.

[2] Bower, A.F.: Cyclic hardening properties of the hard-drawn copper and rail steel, *J. Mech. Phys. Solids* **37** (1989), 455 – 470.

[3] Brokate, M., Dreßler, K., Krejčí, P.: On the Mróz model, *Eur. J. Appl. Math.* **7** (1996), 473 – 497.

[4] Brokate M., Dreßler, K., Krejčí P.: Rainflow counting and energy dissipation for hysteresis models in elastoplasticity, *Eur. J. Mech. A/Solids* **15** (1996), 705 – 737.

[5] Brokate, M., Krejčí, P.: Wellposedness of kinematic hardening models in elastoplasticity *(to appear)*.

[6] Brokate, M., Krejčí, P.: On the wellposedness of the Chaboche model. *Proceedings of the 1996 Vorau Conference on Control and Estimation of Distributed Parameter Systems*, Birkhäuser *(to appear)*.

[7] Brokate, M., Krejčí, P.: Maximum norm wellposedness of nonlinear kinematic hardening models *(to appear)*.

[8] Brokate, M., Sprekels, J.: *Hysteresis and phase transitions*, Springer, New York 1996.

[9] Chaboche, J.-L.: Constitutive equations for cyclic plasticity and cyclic viscoplasticity, *Int. J. Plasticity* **5** (1989), 247 – 302.

[10] Chernorutskii, V.V., Krasnosel'skii, M.A.: Hysteresis systems with variable characteristics. *Nonlinear Analysis. TMA*, **18**(6) (1992), 543 – 557.

[11] Chernorutskii, V.V., Rachinskii D.: On uniqueness of an initial-value problem for ODE with hysteresis. *Nonlin. Diff. Equ. Appl.*, **4** (1997), 391 – 399.

[12] Chu, C.C.: A three-dimensional model of anisotropic hardening in metals and its application to the analysis of sheet metal formability, *J. Mech. Phys. Solids* **32** (1984), 197 – 212.

[13] Chu., C.C.: The analysis of multiaxial cyclic problems with an anisotropic hardening model, *Int. J. Solids Structures* **23** (1987), 567 – 579.

[14] Halphen, B., Nguyen, Q.S.: Sur les matériaux standard généralisés. *J. Mécanique* **14** (1975), 39 – 63.

[15] Krasnosel'skii, M.A., Pokrovskii A.V.: *Systems with hysteresis*, Springer-Verlag, Berlin 1989. Russian edition: Nauka, Moscow 1983.

[16] Krejčí, P.: Modelling of singularities in elastoplastic materials with fatigue, *Appl. Math.* **39** (1994), 137–160.

[17] Krejčí, P.: *Hysteresis, convexity and dissipation in hyperbolic equations*, Gakkōtosho, Tokyo 1996.

[18] Krejčí, P., Sprekels, J.: On a system of nonlinear PDE's with temperature-dependent hysteresis in one-dimensional thermoplasticity. *J. Math. Anal. Appl.* **209** (1997), 25–46.

[19] Krejčí, P., Sprekels, J.: Temperature-dependent hysteresis in one-dimensional thermovisco-elastoplasticity. *Appl. Math.* (to appear).

[20] Le Tallec, P. *Numerical analysis of viscoelastic problems*. Masson, Paris, 1990.

[21] Mróz, Z.: On the description of anisotropic workhardening, *J. Mech. Phys. Solids* **15** (1967), 163 – 175.

[22] Visintin, A.: *Differential models of hysteresis*, Springer-Verlag, Berlin 1994.