Hysteresis Operators in Phase-Field Models of Penrose-Fife Type

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Abstract

Phase-field systems as mathematical models for phase transitions have drawn a considerable interest in recent years. However, while they are capable of capturing many of the experimentally observed phenomena, they are only of restricted value in modelling hysteresis effects occuring during phase transition processes. To overcome this shortcoming of existing phase-field theories, the authors recently proposed a new approach to phase-field models which is based on the mathematical theory of hysteresis operators developed in the past fifteen years. Well-posedness and thermodynamic consistency were proved for a phase-field system with hysteresis which is closely related to the model advanced by Caginalp in a series of papers. In this note the more difficult case of a phase-field system of Penrose-Fife type with hysteresis is investigated. Under slightly more restrictive assumptions than in the Caginalp case it is shown that the system is well-posed and thermodynamically consistent.

1 Introduction

The theory of hysteresis operators developed in the past fifteen years (let us at least refer to the monographs [13], [19], [25], [4], [14] devoted to this subject) has proved to be a powerful tool for solving mathematical problems in various branches of applications such as solid mechanics, material fatigue, ferromagnetism, phase transitions, and many others. In this paper we propose an approach using hysteresis operators to classical *phase-field models* for phase transitions and their generalizations.

For the reader's convenience, let us recall the motivation that we explained already in the previous paper [15]. In nature, many phase transitions are accompanied by hysteresis effects (rather they are driving mechanisms behind their occurence). On the other hand, the nonconvex free energy functionals (typically, double-well potentials) usually considered in phase-field models may induce hysteresis effects by themselves (cf., for instance, Chapter 4 in [4]); however, they are by far too simplistic to give a correct account of the complicated loopings due to the storage and deletion of internal memory that are observed in thermoplastic materials or ferromagnets. An additional motivation comes from the fact that hysteresis operators also arise quite naturally already in simple classical phase-field models. To demonstrate this, let us consider the well-known model for melting and solidification which is usually referred to as the *relaxed Stefan problem with undercooling and overheating* (see [9], [23], [24], for instance).

To fix things, suppose that the phase transition takes place in some open and bounded container $\Omega \subset \mathbb{R}^N$ during the time period $[0,T]$, where $T > 0$ is some final time. Then the mathematical problem consists in finding real-valued functions $\theta = \theta(x,t)$ (absolute temperature) and $\chi = \chi(x,t)$ (phase fraction, the *order parameter* of the phase transition) in $\Omega \times]0,T[$. The function χ is allowed to take values only in the interval $[0,1]$, where $\chi = 1$ corresponds to the liquid phase, $\chi = 0$ to the solid phase and $\chi \in]0,1[$ to the mushy region. The evolution of the system is governed by the balance of internal energy

$$U_t = -\mathrm{div}\; q + \psi\,, \tag{1.1}$$

where $U = U(\theta, \chi)$ is the internal energy, q is the heat flux which we assume here to obey Fourier's law

$$q = -\kappa \nabla \theta \tag{1.2}$$

with a constant heat conduction coefficient $\kappa > 0$, and ψ is the heat source density, and by the melting/solidification law

$$\hat{\mu}(\chi, \theta)\, \chi_t \in -\partial_\chi F(\theta, \chi), \tag{1.3}$$

where $F = F(\theta, \chi)$ is the free energy, ∂_χ is the partial subdifferential with respect to χ and $\hat{\mu} : [0,1] \times]0, \infty[\to]0, \infty[$ is the relaxation coefficient. In order to ensure the thermodynamical consistency of the model, we have to require that

$$\theta(x,t) > 0 \quad \text{a.e. in } \Omega \times]0, T[, \tag{1.4}$$

and that the Clausius-Duhem inequality $S_t \geq -\mathrm{div}\left(\frac{q}{\theta}\right) + \frac{\psi}{\theta}$ holds, which in view of (1.1), (1.2) and (1.4) is certainly the case if only

$$U_t \leq \theta\, S_t \quad \text{a.e.}, \tag{1.5}$$

where $S := \frac{1}{\theta}(U - F)$ denotes the entropy.

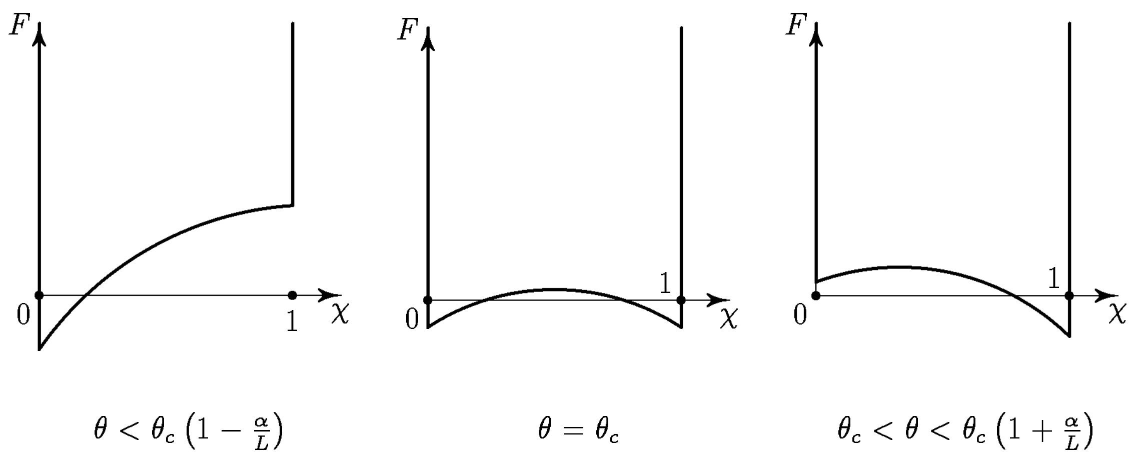

Figure 1: *Free energy F at different temperatures θ.*

A standard choice [9] for F is given by

$$F := F_0(\theta) + \lambda(\chi) + \theta\, I(\chi) - \frac{L}{\theta_c}(\theta - \theta_c)\chi, \tag{1.6}$$

$$U := c_V\, \theta + \lambda(\chi) + L\chi, \tag{1.7}$$

where

$$F_0(\theta) := c_V\, \theta(1 - \log \theta), \tag{1.8}$$

$$\lambda(\chi) := \alpha\, \chi(1 - \chi). \tag{1.9}$$

Here I is the indicator function of the interval $[0,1]$ and L (latent heat), θ_c (melting temperature), c_V (specific heat) and $\alpha < L$ (limit of undercooling/overheating) are positive constants (see Fig. 1). Note that the graph $\Gamma_\theta(\chi) := \theta\left(I(\chi) - L\chi/\theta_c\right) + L\chi + \alpha\,\chi(1-\chi)$ is just the "double-obstacle potential" considered in a number of recent papers. We refer the reader to [2], [3], [8], [12].

The differential inclusion (1.3) then reads

$$\hat{\mu}(\chi,\theta)\,\chi_t + \lambda'(\chi) - \frac{L}{\theta_c}(\theta - \theta_c) \in -\partial_\chi I(\chi)\,, \tag{1.10}$$

or, equivalently (see Fig. 2),

$$\chi \in [0,1]\,, \quad \Big(\hat{\mu}(\chi,\theta)\,\chi_t + \lambda'(\chi) - \frac{L}{\theta_c}(\theta - \theta_c)\Big)(z - \chi) \geq 0 \quad \forall\, z \in [0,1]\,. \tag{1.11}$$

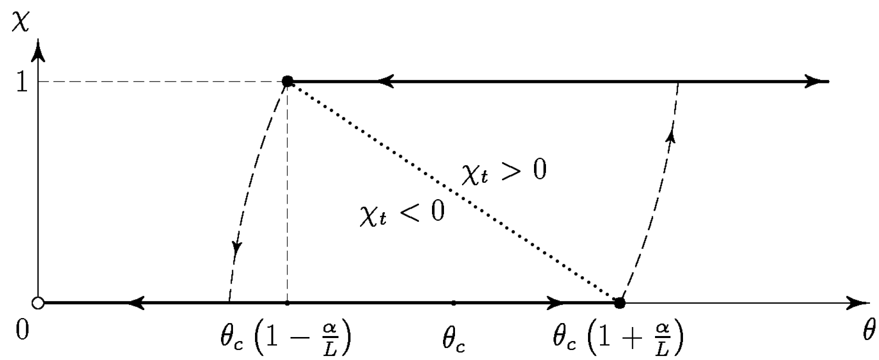

Figure 2: *A $\theta - \chi$ diagram corresponding to (1.11).*

It is easy to see that every solution (θ,χ) of (1.1), (1.2), (1.6)–(1.9), (1.11) for which (1.4) holds, satisfies formally the Clausius-Duhem inequality. Indeed, we have for $\chi \in [0,1]$, $I(\chi) \equiv 0$ and $S = c_V\,\log\theta + \dfrac{L}{\theta_c}\chi$, hence

$$U_t - \theta\,S_t = \Big(\lambda'(\chi) - \frac{L}{\theta_c}(\theta - \theta_c)\Big)\chi_t \leq 0\,, \tag{1.12}$$

according to (1.11).

We now introduce an auxiliary variable

$$w(x,t) := \int_0^t \frac{1}{\hat{\mu}(\chi,\theta)}\Big(\frac{L}{\theta_c}(\theta - \theta_c) - \lambda'(\chi)\Big)(x,\tau)\,d\tau\,. \tag{1.13}$$

Then inequality (1.11) takes the form

$$\chi \in [0,1]\,, \quad (\chi_t - w_t)(z - \chi) \geq 0 \quad \forall\, z \in [0,1]\,. \tag{1.14}$$

At this point, the notion of hysteresis operators comes into play. Variational inequality (1.14) is known to have a unique solution $\chi \in W^{1,1}(0,T)$ for every $w \in W^{1,1}(0,T)$ and initial condition $\chi(0) = \chi^0 \in [0,1]$. According to [13], [25], [4], [14], it is convenient to introduce the solution operator s_Z of (1.14) called *stop*, where the subscript Z stands for the convex constraint $Z = [0,1]$, that is,

$$\chi = s_Z[\chi^0, w]\,. \tag{1.15}$$

The hysteretic input-output behaviour of the stop operator is illustrated in Fig. 3. Along the upper (lower) threshold line $\chi = 1$, ($\chi = 0$), the process is irreversible and can only move to the right (to the left, respectively), while in between, motions in both directions are admissible. This is similar to Prandtl's model of perfect elastoplasticity, where the horizontal parts of the diagram correspond to plastic yielding and the intermediate lines can be interpreted as linearly elastic trajectories.

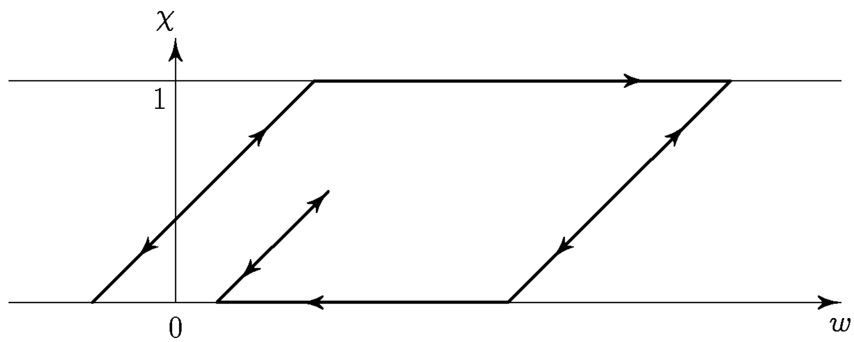

Figure 3: *A diagram of the stop operator (1.15).*

Identity (1.15) enables us to eliminate χ from (1.13) and rewrite the system (1.1) – (1.3) in the form

$$\hat{\mu}(s_Z[\chi^0, w], \theta)\, w_t \;=\; \frac{L}{\theta_c}(\theta \,-\, \theta_c) \,-\, \lambda'\big(s_Z[\chi^0, w]\big)\,, \tag{1.16}$$

$$\Big(c_V\, \theta \,+\, \lambda\,(s_Z[\chi^0, w]) \,+\, L\, s_Z[\chi^0, w]\Big)_t \,-\, \kappa\, \Delta\theta \;=\; \psi\,. \tag{1.17}$$

We thus obtain in a natural way a system of equations for an order parameter w and the absolute temperature θ involving hysteresis operators.

In [15], we have studied a generalization of the field equations (1.16), (1.17), namely an initial-boundary value problem for a system of the form

$$\mu\, w_t \,+\, f_1[w] \,+\, f_2[w]\, \theta \;=\; 0\,, \tag{1.18}$$

$$(c_V\, \theta \,+\, F_1[w])_t \,-\, \kappa\, \Delta\theta \;=\; \psi(x,t,\theta)\,, \tag{1.19}$$

where f_1, f_2, F_1 denote hysteresis operators and $\mu > 0$ is a constant. Note that (1.16), (1.17) with $\hat{\mu}(\chi, \theta) \equiv \mu$ becomes a special case of (1.18), (1.19) if we put $g[w] := s_Z[\chi^0, w]$ and define $f_1[w] := \lambda'(g[w]) + L$, $F_1[w] := \lambda(g[w]) + Lg[w]$, $f_2[w] := -L/\theta_c$. Thinking in terms of classical models, the system (1.18), (1.19) can be regarded as a phase-field model of *Caginalp type* (see [5], [4] and the references cited there) for a free energy of the form

$$F = F_0(\theta) + F_1[w] + F_2[w]\,\theta\,, \tag{1.20}$$

where F_1 and F_2 are so-called *clockwise admissible hysteresis potentials* of f_1 and f_2, respectively (the precise definition of clockwise admissibility will be given below). It is the aim of this paper to investigate a hysteresis counterpart of the so-called *Penrose-Fife model* of phase transitions (cf. [20], [4], [6], [7], [11], [10], [17], [18], [22]) which formally results if we choose $\hat{\mu}(\chi,\theta) = \mu\theta$ with a constant coefficient $\mu > 0$.

We therefore replace here (1.18) by

$$\mu\, w_t + \frac{1}{\theta} f_1[w] + f_2[w] = 0\,. \tag{1.21}$$

It should be clear, however, that we are now dealing with a free energy of the form (1.20), so that (1.3) needs to be properly interpreted. Indeed, in the classical case the relaxation law (1.3), with χ replaced by w, is combined with identities of the form $f_i[w] = \delta_w\, F_i[w]$, $i = 1,2$, where δ_w denotes the variation with respect to w, in order to make the model comply with the Second Principle of Thermodynamics. However, since hysteresis operators are, as a rule, *non-differentiable*, we cannot hope to have these identities, as the variation $\delta_w\, F_i[w]$ of F_i with respect to w does not exist. In this regard, the situation is entirely different from classical phase-field models. What is needed here to guarantee the thermodynamical consistency of the model is a property of *energy dissipation* which is specific for hysteresis operators and leads to the concept of clockwise admissibility as made precise below.

The main mathematical difficulty in the analysis of the system (1.21), (1.19) is, besides the hysteretic nonlinearities, the occurence of the singularity in (1.21) which makes it necessary to prove the positivity of temperature along with the existence of the solution. It will be shown in the following sections that under quite natural conditions on the hysteresis operators involved an initial-boundary value problem for the system (1.21), (1.19) is well-posed and thermodynamically consistent. The technique used in the proof is an extension of a method recently introduced in [8]. It combines a "cutoff"–method with the specific property of energy dissipation of hysteresis operators.

2 Statement of the problem

We put, for the sake of convenience, $c_V = \kappa = 1$ and consider the system of equations in $\Omega \times]0,T[$

$$\mu\, w_t + \frac{1}{\theta} f_1[w] + f_2[w] = 0\,, \tag{2.1}$$

$$(\theta + F_1[w])_t - \Delta\theta = \psi(x,t,\theta)\,, \tag{2.2}$$

coupled with the initial conditions

$$w(x,0) = w^0(x)\,,\quad \theta(x,0) = \theta^0(x)\,,\quad \text{for } x \in \Omega\,, \tag{2.3}$$

and with the Neumann boundary condition

$$\frac{\partial \theta}{\partial n}(x,t) = 0 \quad \text{for } (x,t) \in \partial\Omega \times]0,T[\,, \tag{2.4}$$

where $n(x)$ is the unit outward normal to $\partial\Omega$ at the point $x \in \partial\Omega$. This simple boundary condition has been chosen in order to make the method of hysteresis operators more transparent, which is our main goal here. We assume that $T > 0$, $\mu > 0$ are given numbers and that $\Omega \subset \mathbb{R}^N$ is a given bounded domain with a lipschitzian boundary.

We now formulate precisely the assumptions on the mappings f_1, f_2, F_2, ψ.

(H1) $f_1, f_2 : C[0,T] \to C[0,T]$ are causal, bounded and Lipschitz continuous operators; in other words, there exists a constant K_1 such that for every $w_1, w_2, w \in C[0,T]$ and $t \in [0,T]$ it holds

$$\left|f_i[w_1](t) - f_i[w_2](t)\right| \le K_1 \max_{0 \le s \le t} |w_1(s) - w_2(s)|\,, \quad i = 1,2\,, \tag{2.5}$$

$$\left|f_i[w](t)\right| \le K_1\,, \quad i = 1,2\,. \tag{2.6}$$

(H2) The mapping $F_1 : W^{1,2}(0,T) \to W^{1,2}(0,T)$ is causal, and there exist a constant $K_2 > 0$ and a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\left|\frac{d}{dt} F_1[w](t)\right| \le K_2 |\dot{w}(t)| \quad \textit{a.e. in }]0,T[\,, \quad \forall\, w \in W^{1,2}(0,T)\,, \tag{2.7}$$

$$\begin{aligned} &\left|F_1[w_1](t) - F_1[w_2](t)\right| \le \varphi(M)\, \|w_1 - w_2\|_{W^{1,2}(0,t)} \\ &\quad \forall M > 0,\ \forall\, w_1, w_2 \in W^{1,2}(0,T) : \max\left\{\|w_i\|_{W^{1,2}(0,T)} : i = 1,2\right\} \le M, \end{aligned} \tag{2.8}$$

where we denote

$$\|w\|_{W^{1,p}(0,t)} := |w(0)| + \left(\int_0^t |\dot{w}(s)|^p\, ds\right)^{1/p} \quad \forall\, t \in]0,T]\,, \quad 1 \le p < \infty\,. \tag{2.9}$$

We moreover assume that the function ψ satisfies the condition

$$\psi_0 := \psi(\,\cdot\,,\,\cdot\,,0) \in L^q(\Omega \times]0,T[)\,, \quad |\psi_\theta(x,t,\theta)| \le K_2 \ \textit{a.e.}, \tag{2.10}$$

for some $q > \dfrac{r_N^2}{r_N - 1}$, where $r_N := \max\left\{2, 1 + \frac{N}{2}\right\}$.

(H3) It holds

$$\psi_0(x,t) \ge 0 \quad \textit{a.e. in } \Omega \times]0,T[\,, \tag{2.11}$$

$$F_1[w](t) \ge 0 \quad \forall\ w \in W^{1,2}(0,T)\,, \quad \forall\, t \in [0,T]\,, \tag{2.12}$$

and there exist operators $F_2, g : W^{1,2}(0,T) \to W^{1,2}(0,T)$ and a constant $K_3 > 0$ such that the inequalities

$$0 \le g[w]_t\, w_t \le K_3\, w_t^2\,, \tag{2.13}$$

$$F_i[w]_t \;-\; f_i[w]\,g[w]_t \;\le\; 0\,, \tag{2.14}$$

hold for each $w \in W^{1,2}(0,T)$ and a.e. $t \in]0,T[\,,\ i = 1,2$.

Let us mention that property (2.13) is called *piecewise* ([25]) or *local* ([14]) *monotonicity* .

Remark 2.1. The domains of definition of the operators f_i, F_i, g can be extended in a natural way to functions which depend on both x and t and appear in (2.1), (2.2). It suffices to keep the same symbols and to put

$$f_i[w](x,t) \;:=\; f_i[w(x,\,\cdot\,)](t) \quad \text{for } x \in \Omega\,,\ t \in]0,T[\,, \tag{2.15}$$

and similarly for F_i and g, for every function w such that $w(x\,,\cdot\,)$ belongs to the original domain of definition for a.e. $x \in \Omega$.

Remark 2.2. Inequality (2.14) is a typical condition which guarantees the thermodynamical consistency of hysteresis operators also in other areas of application. It is fulfilled, in particular, for operators of the form

$$f_i[w] \;:=\; \mathcal{P}_i\big[g[w]\big]\,, \quad F_i[w] \;:=\; \mathcal{U}_i\big[g[w]\big]\,, \tag{2.16}$$

where $\mathcal{P}_i$ is a hysteresis operator with a *clockwise admissible hysteresis potential* $\mathcal{U}_i$ in the sense of Section 2.5 in [4]. Note that in this case the “dissipation” over a closed cycle (i.e. $u(t_1) = u(t_2)$, $\mathcal{P}_i[u](t_1) = \mathcal{P}_i[u](t_2)$, $\mathcal{U}_i[u](t_1) = \mathcal{U}_i[u](t_2)$) is positive and equal to the integral

$$\int_{t_1}^{t_2} \mathcal{P}_i[u](t)\frac{du(t)}{dt}\,dt$$

or, in geometrical terms, to the area of the corresponding hysteresis loop, see Fig. 4. A classical example is the Prandtl-Ishlinskii operator

$$\mathcal{P}_i[u] \;:=\; \int_0^\infty h_i(r)\,s_{Z_r}[u]\,dr, \quad \mathcal{U}_i[u] \;:=\; \frac{1}{2}\int_0^\infty h_i(r)\,s_{Z_r}^2[u]\,dr, \tag{2.17}$$

where s_{Z_r} is the stop operator with characteristic $Z_r = [-r,r]$ and h_i are given nonnegative density functions.

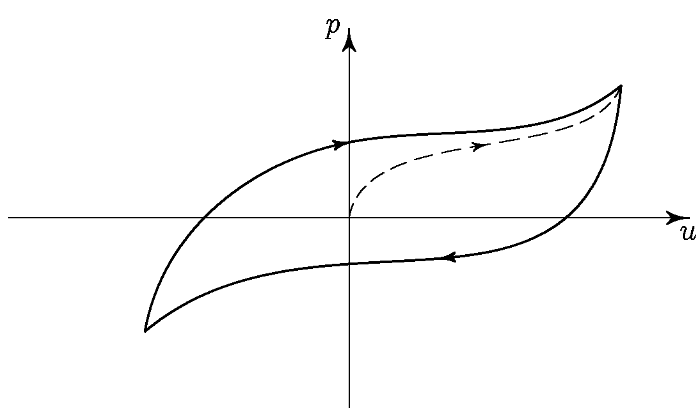

Figure 4: *Clockwise admissibility for* $p = \mathcal{P}_i[u]$

Also here, the condition (1.5) follows from (2.14) provided θ is positive. Indeed, if we define the internal energy $U = U[w,\theta] := \theta + F_1[w]$ and the entropy $S = S[w,\theta] := \log\theta - F_2[w]$, then we obtain formally

$$U_t - \theta S_t = F_1[w]_t + \theta F_2[w]_t \le -\mu\theta w_t g[w]_t \le 0, \tag{2.18}$$

so that (1.5) is satisfied. We shall see below in Theorem 2.3 that hypotheses **(H1)** – **(H3)** ensure also the positivity of θ. In conclusion, inequality (2.14), which reflects the fundamental energy dissipation properties of hysteresis operators f_i, takes over the role of the identity $f_i[w] = \delta_w F_i[w]$ which is meaningless here. We should recall that for constant temperature, (2.18) just means that F decreases in time.

The next three sections are devoted to the proof of the following theorems.

Theorem 2.3 (Existence). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a lipschitzian boundary, let hypotheses* **(H1), (H2), (H3)** *hold, and let $\delta > 0$ be given. Then for every $w^0 \in L^\infty(\Omega)$ and $\theta^0 \in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ such that $\theta^0(x) \ge \delta$ a.e. in Ω, problem* (2.1)–(2.4) *has a solution $(w,\theta) \in (L^\infty(\Omega\times]0,T[))^2$ such that $\theta_t, \Delta\theta \in L^2(\Omega\times]0,T[)$, $w_t \in L^\infty(\Omega\times]0,T[)$, $\theta(x,t) \ge \delta\, e^{-\beta t}$ a.e. in $\Omega\times]0,T[$, where β is explicitly given in terms of the constants K_1, K_2, K_3 from* **(H1), (H2), (H3)**, *namely $\beta := K_2 + K_1^2 K_3/4\mu$, and such that* (2.1), (2.2) *are satisfied almost everywhere.*

Theorem 2.4 (Uniqueness and continuous dependence). *Let the hypotheses of Theorem 2.3 hold. Let $w_i^0 \in L^\infty(\Omega)$, $\theta_i^0 \in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ and $\psi_i : \Omega\times]0,T[\times\mathbb{R} \to \mathbb{R}$, $i = 1,2$, be given functions such that $\theta_i^0(x) \ge \delta$ a.e. in Ω, $i = 1,2$. Let each of the functions $\psi = \psi_1$, $\psi = \psi_2$ satisfy* (2.10), (2.11), *and let there exist a function $d_\psi \in L^2(\Omega\times]0,T[)$ such that for a.e. $(x,t,\vartheta^i) \in \Omega\times]0,T[\times\mathbb{R}$, $i = 1,2$, we have*

$$\left|\psi_1(x,t,\vartheta^1) - \psi_2(x,t,\vartheta^2)\right| \le d_\psi(x,t) + K_2|\vartheta^1 - \vartheta^2|. \tag{2.19}$$

Let (w_1,θ_1), (w_2,θ_2) be solutions to (2.1)–(2.4) *from Theorem 2.3 corresponding to the data w_1^0, θ_1^0, ψ_1 and $w_2^0, \theta_2^0, \psi_2$, respectively. Then there exists a constant $C > 0$ depending only on the norm of the data in their respective spaces such that, for all $t \in [0,T]$,*

$$\int_0^t\int_\Omega \left|\theta_1 - \theta_2\right|^2(x,\tau)\,dx\,d\tau \le C\Big[t\big(\|w_1^0 - w_2^0\|^2_{L^2(\Omega)} + \|\theta_1^0 - \theta_2^0\|^2_{L^2(\Omega)}\big) + \int_0^t\int_\Omega d_\psi^2(x,\tau)\,dx\,d\tau\Big], \tag{2.20}$$

$$\int_\Omega \|w_1 - w_2\|^2_{W^{1,2}(0,T)}(x)\,dx \le C\Big[\|w_1^0 - w_2^0\|^2_{L^2(\Omega)} + \|\theta_1^0 - \theta_2^0\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega d_\psi^2(x,t)\,dx\,dt\Big]. \tag{2.21}$$

3 An auxiliary equation

Instead of (2.1), we first consider the "ordinary" differential equation

$$\mu\,\dot w(t) + \varrho(t)\,f_1[w](t) + f_2[w](t) = 0, \quad w(0) = w^0, \tag{3.1}$$

where the dot denotes the derivative with respect to t and ϱ is a given function of t. We omit the proof of the following two lemmas which are just special cases of Lemmas 3.1, 3.2 in [15].

Lemma 3.1 (Existence). *Let hypothesis* **(H1)** *hold, and let* $\varrho \in L^1(0,T)$ *and* $w^0 \in \mathbb{R}$ *be given. Then there exists a solution* $w \in W^{1,1}(0,T)$ *of* (3.1) *such that* (3.1) *holds a.e., together with the estimate*

$$|\dot{w}(t)| \leq \frac{K_1}{\mu}\big(1 + |\varrho(t)|\big). \tag{3.2}$$

Lemma 3.2 (Uniqueness and continuous dependence). *Let hypothesis* **(H1)** *hold. Then to every* $M > 0$ *there exists a constant* $C_M > 0$ *such that for every* $\varrho_1, \varrho_2 \in L^1(0,T)$, $\|\varrho_i\|_{L^1(0,T)} \leq M$, $i = 1,2$, *the corresponding solutions* w_1, w_2 *of* (3.1) *with initial conditions* w_1^0, w_2^0, *respectively, satisfy for a.e.* $t \in]0,T[$ *the estimates*

$$|w_1(t) - w_2(t)| \leq C_M\Big(|w_1^0 - w_2^0| + \int_0^t |\varrho_1 - \varrho_2|(s)\,ds\Big), \tag{3.3}$$

$$\begin{aligned} |\dot{w}_1(t) - \dot{w}_2(t)| \;\leq\; & C_M\Big(|w_1^0 - w_2^0| + \int_0^t |\varrho_1 - \varrho_2|(s)\,ds\Big) \\ & \cdot \big(1 + |\varrho_1(t)|\big) + \frac{K_1}{\mu}\big|\varrho_1(t) - \varrho_2(t)\big|. \end{aligned} \tag{3.4}$$

Lemmas 3.1 and 3.2 enable us to introduce the solution operator $\mathcal{P}_p : \mathbb{R} \times L^p(0,T) \to W^{1,p}(0,T)$ of equation (3.1) for every $1 \leq p \leq \infty$ through the formula

$$w = \mathcal{P}_p[w^0, \varrho]. \tag{3.5}$$

$\mathcal{P}_p$ is obviously causal, and it satisfies according to Lemmas 3.1, 3.2 for every $t \in [0,T]$ the following inequalities.

Proposition 3.3 *Let hypothesis* **(H1)** *hold. Then there exist a constant* $C_2 > 0$ *and a function* $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for every* $M > 0$ *and every* (w^0, ϱ), (w_1^0, ϱ_1), $(w_2^0, \varrho_2) \in \mathbb{R} \times L^p(0,T)$ *and* $t \in [0,T]$ *satisfying* $\max\big\{|w_i^0|, \|\varrho_i\|_{L^p(0,t)};\, i = 1,2\big\} \leq M$, *we have*

$$\big\|\mathcal{P}_p[w^0, \varrho]\big\|_{W^{1,p}(0,t)} \leq C_2\big(1 + |w^0| + \|\varrho\|_{L^p(0,t)}\big), \tag{3.6}$$

$$\|\mathcal{P}_p[w_1^0, \varrho_1] - \mathcal{P}_p[w_2^0, \varrho_2]\|_{W^{1,p}(0,t)} \leq \gamma(M)\big(|w_1^0 - w_2^0| + \|\varrho_1 - \varrho_2\|_{L^p(0,t)}\big). \tag{3.7}$$

4 Existence, uniqueness and stability

This section is devoted to the proof of Theorems 2.3 and 2.4. Let δ and β be fixed as in Theorem 2.3. We couple (2.2) with the "truncated" equation

$$\mu\, w_t + Q(\theta, t)\, f_1[w] + f_2[w] = 0, \quad w(x,0) = w^0(x), \tag{4.1}$$

where

$$Q(\theta,t) \;:=\; \min\Big\{\frac{1}{\delta}\,e^{\beta t},\,\frac{1}{|\theta|}\Big\}. \tag{4.2}$$

With the notation (3.5) we introduce the operator

$$\mathcal{V}_p[w^0,\theta](x,t) \;:=\; F_1\Big[\mathcal{P}_p[w^0(x),Q(\theta(x,\,\cdot\,),\cdot)]\Big](t)\,. \tag{4.3}$$

This enables us to rewrite the system (4.1), (2.2) – (2.4) as a single equation for θ, namely

$$\big(\theta \,+\, \mathcal{V}_p[w^0\,,\,\theta]\big)_t \,-\, \Delta\theta \;=\; \psi(x,t,\theta)\,, \tag{4.4}$$

coupled with initial and boundary conditions (2.3), (2.4). The natural domains of definition of $\mathcal{V}_p$ are the spaces $\mathcal{D}_p^t := L^p(\Omega)\times L^p(\Omega\times\,]0,t[)$ for $p\in[1,\infty]$ and $t\in\,]0,T]$. From Proposition 3.3 and hypothesis **(H2)** we see that $\mathcal{V}_p$ maps $\mathcal{D}_p^t$ into $L^p(\Omega;W^{1,p}(0,t))$. Moreover, since for every $p>r$ it holds $\mathcal{V}_r\big|_{\mathcal{D}_p^t} = \mathcal{V}_p$, we may simply write $\mathcal{V}$ in place of $\mathcal{V}_p$, with an implicitly given domain of definition. The operator $\mathcal{V}$ has the following properties.

Proposition 4.1 *Let hypotheses* **(H1), (H2)** *hold. Then there exists a function* $\tilde\psi : \mathbb{R}^+\to\mathbb{R}^+$ *such that for every* $M>0$, $\theta\in L^1(\Omega\times\,]0,t[)$, $\theta_1,\theta_2\in L^2(\Omega\times\,]0,t[)$, $w^0,w_1^0,w_2^0\in L^\infty(\Omega)$ *satisfying* $\max\big\{\|w_i^0\|_{L^\infty(\Omega)} : i=1,2\big\}\le M$, *and every* $t\in\,]0,T]$, *it holds*

$$\|\mathcal{V}[w^0\,,\,\theta]_t\|_{L^\infty(\Omega\,\times\,]0,t[)} \;\le\; \frac{K_1K_2}{\mu}\Big(1\,+\,\frac{1}{\delta}e^{\beta t}\Big)\,, \tag{4.5}$$

$$\begin{aligned}\big\|\mathcal{V}[w_1^0\,,\,\theta_1] \;&-\; \mathcal{V}[w_2^0\,,\,\theta_2]\big\|_{L^2(\Omega;\,L^\infty(0,t))} \\ &\le\; \tilde\psi(M)\Big(\|w_1^0\,-\,w_2^0\|_{L^2(\Omega)} \,+\, \|\theta_1\,-\,\theta_2\|_{L^2(\Omega\,\times\,]0,t[)}\Big)\,.\end{aligned} \tag{4.6}$$

Proof. It suffices to use Lemma 3.1, Proposition 3.3, hypothesis **(H2)** and to integrate over Ω. □

The existence result for the truncated system (4.1), (2.2) – (2.4) can be stated as follows. We omit here its proof which is based on an easy successive approximation scheme and is identical to the proof of Theorem 4.2 of [15].

Theorem 4.2 *Let hypotheses* **(H1), (H2)** *hold. Then, for every* $w^0\in L^\infty(\Omega)$ *and* $\theta^0\in L^\infty(\Omega)\cap W^{1,2}(\Omega)$, *there exists* $\theta\in L^\infty(\Omega\times\,]0,T[)$ *such that* $\theta_t,\ \Delta\theta\in L^2(\Omega\times\,]0,T[)$ *and such that the equation*

$$\big(\theta \,+\, \mathcal{V}[w^0\,,\,\theta]\big)_t \,-\, \Delta\theta \;=\; \psi(x,t,\theta) \tag{4.7}$$

is satisfied almost everywhere, together with the initial and boundary conditions (2.3), (2.4).

Note that equation (4.7) does not have the general form considered by Visintin [25], since the operator $\mathcal{V}$ is not piecewise monotone. We now show that the additional hypothesis **(H3)** ensures that the solution from the above theorem satisfies also the original system (2.1) – (2.4).

Theorem 4.3 *Let hypotheses* **(H1)** – **(H3)** *hold, and let* $w^0 \in L^\infty(\Omega)$ *and* $\theta^0 \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ *satisfy the assumptions of Theorem 2.3. Let* θ *be a solution to (4.7), (2.3), (2.4) from Theorem 4.2. Then*

$$\theta(x,t) \geq \delta e^{-\beta t}. \tag{4.8}$$

Proof of Theorem 4.3. Put $w := \mathcal{P}_\infty[w^0, \theta]$. Then w, θ satisfy (4.1), (2.2) – (2.4) almost everywhere. Let us test (2.2) with an arbitrary function $p \in W^{1,2}(\Omega \times]0,T[)$ such that $p \leq 0$ a.e. . This yields, according to hypotheses **(H2)**, **(H3)**,

$$\int_\Omega \big(p\,\theta_t + \langle \nabla p, \nabla\theta\rangle\big)\,dx \leq \int_\Omega |p|\big(F_1[w]\big)_t dx + K_2 \int_\Omega |p|\,|\theta|\,dx. \tag{4.9}$$

Let us estimate the first integral in the right-hand side of (4.9). Using **(H3)** we obtain

$$\begin{aligned} \int_\Omega |p|\big(F_1[w]\big)_t\,dx \quad &\leq \int_\Omega |p|\,f_1[w]\,g[w]_t\,dx \qquad (4.10)\\ &= -\int_\Omega |p|\,\frac{g[w]_t}{\mu\,w_t}\,f_1[w]\big(Q(\theta,t)\,f_1[w] + f_2[w]\big)dx\\ &= -\int_\Omega |p|\,\frac{g[w]_t}{\mu\,w_t}\Big[\Big(\sqrt{Q(\theta,t)}\,f_1[w] + \frac{f_2[w]}{2\sqrt{Q(\theta,t)}}\Big)^2 - \frac{f_2^2[w]}{4\,Q(\theta,t)}\Big]dx\\ &\leq \frac{K_1^2\,K_3}{4\,\mu}\int_\Omega |p|\,\max\big\{|\theta|, \delta\,e^{-\beta t}\big\}\,dx. \end{aligned}$$

Combining (4.9) and (4.10) we obtain

$$\int_\Omega \big(p\,\theta_t + \langle \nabla p, \nabla\theta\rangle\big)\,dx \leq \beta\int_\Omega |p|\,\max\big\{|\theta|, \delta\,e^{-\beta t}\big\}\,dx. \tag{4.11}$$

for every non-positive function $p \in W^{1,2}(\Omega \times]0,T[)$. We now put in the above inequality

$$p(x,t) := -\big(\delta\,e^{-\beta t} - \theta(x,t)\big)^+. \tag{4.12}$$

This yields

$$\int_\Omega \big(p\,(p + \delta\,e^{-\beta t})_t + |\nabla p|^2\big)\,dx \leq \beta\int_\Omega |p|\,\big(|p| + \delta\,e^{-\beta t}\big)\,dx\,, \tag{4.13}$$

hence

$$\int_\Omega \big(p\,p_t + |\nabla p|^2\big)\,dx \leq \beta\int_\Omega p^2\,dx \tag{4.14}$$

with $p(x,0) \equiv 0$, and Gronwall's lemma yields $p(x,t) \equiv 0$. Theorem 4.3 is proved. □

Proof of Theorem 2.3. From Theorems 4.1, 4.2 it follows that under the hypotheses of Theorem 2.3, every solution θ, w of (4.1), (2.2) – (2.4) fulfils $Q(\theta(x,t),t) = 1/\theta(x,t)$, hence also (2.1) is satisfied. □

Proof of Theorem 2.4. Subtracting equations (4.7) for θ_1, θ_2 and integrating with respect to t, we obtain that

$$\begin{aligned} &(\theta_1 - \theta_2)(x,t) - \Delta \int_0^t (\theta_1 - \theta_2)(x,\tau)\, d\tau \qquad (4.15)\\ = \ &(\theta_1^0 - \theta_2^0)(x) + \big(F_1[w_1^0] - F_1[w_2^0]\big)(x,0) - \big(\mathcal{V}[w_1^0, \theta_1] \\ &- \mathcal{V}[w_2^0, \theta_2]\big)(x,t) + \int_0^t \Big(\psi_1\big(x,\tau,\theta_1(x,\tau)\big) - \psi_2\big(x,\tau,\theta_2(x,\tau)\big)\Big)\, d\tau\,. \end{aligned}$$

Multiplication by $(\theta_1 - \theta_2)(x,t)$ and integration over Ω yields

$$\begin{aligned} &\int_\Omega \big|\theta_1 - \theta_2\big|^2(x,t)\, dx + \frac{d}{dt}\int_\Omega \Big|\nabla \int_0^t (\theta_1 - \theta_2)(x,\tau)\, d\tau\Big|^2 dx \qquad (4.16)\\ \leq \ &\int_\Omega \big|\theta_1^0 - \theta_2^0\big|^2(x)\, dx + 4\,T \int_0^t \int_\Omega \Big(d_\psi^2 + K_2\big|\theta_1 - \theta_2\big|^2\Big)\, dx\, d\tau \\ &+ C_8\Big(\big|w_1^0 - w_2^0\big|^2\, dx + \int_0^t \int_\Omega \big|\theta_1 - \theta_2\big|^2(x,\tau)\, dx\, d\tau\Big)\,, \end{aligned}$$

where we used the estimates (2.8), (2.19), (4.5).

To obtain the assertion, it remains to integrate (4.16) from 0 to $\bar{t}$ and to apply a standard Gronwall-type argument. □

References

[1] Besov, O. V., Il'in, V. P., Nikol'skii, S. M. : *Integral representation of functions and embedding theorems* (Russian). Moscow, Nauka, 1975.

[2] Blowey, J. F., Elliott, C. M. : Curvature dependent phase boundary motion and double obstacle problems. In: *Degenerate Diffusion*. W.M. Ni, L.A. Peletier, and J.L. Vázquez eds., pp. 19–60, IMA Vol. Math. Appl., **47**, Springer, New York, 1993.

[3] Blowey, J. F., Elliott, C. M. : A phase-field model with double obstacle potential. In: *Motion by mean curvature and related topics*. G. Buttazzo and A. Visintin eds., pp. 1–22, De Gruyter, Berlin, 1994.

[4] Brokate, M., Sprekels, J. : *Hysteresis and phase transitions*. Appl. Math. Sci., **121**, Springer-Verlag, New York, 1996.

[5] Caginalp, G. : An analysis of a phase field model of a free boundary. *Arch. Rational Mech. Anal.*, **92** (1986), 205–245.

[6] Colli, P., Sprekels, J. : On a Penrose-Fife model with zero interfacial energy leading to a phase-field system of relaxed Stefan type. *Ann. Mat. Pura Appl.* (4), **169** (1995), 269–289.

[7] Colli, P., Sprekels, J. : Stefan problems and the Penrose-Fife phase-field model. *Adv. Math. Sci. Appl.* **7** (1997), 911–934.

[8] Colli, P., Sprekels, J. : Global solutions to the Penrose-Fife phase-field model with zero interfacial energy and Fourier law. *Preprint No. 351,* WIAS Berlin, 1997.

[9] Frémond, M., Visintin, A. : Dissipation dans le changement de phase. Surfusion. Changement de phase irréversible. *C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre,* **301** (1985), 1265–1268.

[10] Kenmochi, N., Niezgódka, M. : Systems of nonlinear parabolic equations for phase change problems. *Adv. Math. Sci. Appl.*, **3** (1993/94), 89–117.

[11] Klein, O. : A semidiscrete scheme for a Penrose-Fife system and some Stefan problems in $\mathbb{R}^3$. *Adv. Math. Sci. Appl.*, **7** (1997), 491–523.

[12] Klein, O. : *Existence and approximation results for phase-field systems of Penrose-Fife type and some Stefan problems.* Ph.D. - thesis, Humboldt University, Berlin, 1997.

[13] Krasnosel'skii, M. A., Pokrovskii, A. V. : *Systems with hysteresis*. Springer-Verlag, Heidelberg, 1989 (Russian edition: Nauka, Moscow, 1983).

[14] Krejčí, P. : *Hysteresis, convexity and dissipation in hyperbolic equations.* Gakuto Int. Series Math. Sci. & Appl., Vol. 8, Gakkōtosho, Tokyo, 1996.

[15] Krejčí, P., Sprekels, J.: A hysteresis approach to phase-field models. Submitted.

[16] Ladyzhenskaya, O. A., Solonnikov, V. A., Ural'tseva, N. N. : *Linear and quasilinear equations of parabolic type*. American Mathematical Society, 1968. (Russian edition: Nauka, Moscow, 1967).

[17] Laurençot, Ph. : Solutions to a Penrose-Fife model of phase-field type. *J. Math. Anal. Appl.*, **185** (1994), 262–274.

[18] Laurençot, Ph. : Weak solutions to a Penrose-Fife model for phase transitions. *Adv. Math. Sci. Appl.*, **5** (1995), 117–138.

[19] Mayergoyz, I. D. : *Mathematical models for hysteresis.* Springer-Verlag, New York, 1991.

[20] Penrose, O., Fife, P.C. : Thermodynamically consistent models of phase field type for the kinetics of phase transitions. *Physica D*, **43** (1990), 44–62.

[21] Protter, M. H., Weinberger, H. F. : *Maximum principle in differential equations.* Prentice Hall, Englewood Cliffs, 1967.

[22] Sprekels, J., Zheng, S. : Global smooth solutions to a thermodynamically consistent model of phase-field type in higher space dimensions. *J. Math. Anal. Appl.*, **176** (1993), 200–223.

[23] Visintin, A. : Stefan problem with phase relaxation. *IMA J. Appl. Math.*, **34** (1985), 225–245.

[24] Visintin, A. : Supercooling and superheating effects in phase transitions. *IMA J. Appl. Math.*, **35** (1985), 233–256.

[25] Visintin, A. : *Differential models of hysteresis.* Springer-Verlag, New York, 1994.