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Stability of numerical schemes for stochastic differential equations with multiplicative noise

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ABSTRACT. A notion of stability for a special type of test equations is proposed. These are stochastic differential equations with multiplicative noise for which there is a connection between the parameters in the drift and diffusion coefficient. By means of the Euler scheme and two different implicit Euler schemes a method to find the regions of stability is also examined.

1. Introduction

If we want to apply numerical methods to Stratonovich stochastic differential equations of the form $dX_t = a(t, X_t)dt + b(t, X_t) \circ dW_t$ we have to examine their regions of stability. The knowledge about the stability of a numerical method is a crucial point to decide for a given stochastic differential equation whether the method is appropriate or not. The existence of noise in the stochastic case provides a number of difficulties which we have not for ordinary differential equations. Before we consider the situation for stochastic differential equations we give a short overview about the concept of stability for deterministic numerical schemes.
In the deterministic sense talking of numerical stability of a one - step method

$$y_{n+1} = y_n + \Psi(t_n, y_n, \Delta_n)\Delta_n \tag{1.1}$$

with an increment function $\Psi = \Psi(t, x, \Delta)$ means that an error will remain bounded with respect to an initial error for an ordinary differential equation

$$\frac{dx}{dt} = a(t, x) \tag{1.2}$$

where $a(t, x)$ satisfies a Lipschitz condition. We say more precisely that a one - step method (1.1) is called numerically stable if for each time interval $[t_0, T]$ and given differential equation (1.2) there exist positive constants Δ_0 and M such that

$$|y_n - \tilde{y}_n| \leq M|y_0 - \tilde{y}_0| \tag{1.3}$$

for all $n = 0, 1, ..., n_T$ (we have $y_{n_T} = y(T)$) and any two solutions $y_n, \tilde{y}_n$ of (1.1) corresponding to any time discretization with $\max_n \Delta_n < \Delta_0$.
Here the constant M can be quite large. In order to ensure that the error does not grow over an infinite time horizont one introduces the notion of asymptotic numerical stability. A one - step method (1.1) is called asymptotically numerically stable for a given differential equation if there exist positive constants Δ_a and M such that

$$\lim_{n\to\infty} |y_n - \tilde{y}_n| \leq M|y_0 - \tilde{y}_0| \tag{1.4}$$

for any two solutions $y, \tilde{y}$ of (1.1) corresponding to any time discretization with $\max_n \Delta_n < \Delta_a$. From the practical point of view one is not only interested in the problem whether a method is numerically stable or not , but one asks for the step size Δ which one has to choose. For this purpose one considers the class of complex valued test equations

$$\frac{dx}{dt} = \lambda x \tag{1.5}$$

with $\lambda = \lambda_1 + \lambda_2 i$ which is equivalent to the 2 - dimensional differential equations

$$\frac{dx}{dt}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

where $x = x^1 + x^2 i$. To decide which step sizes one can use it is helpful to study the region of stability of the scheme. If one can write a numerical scheme in a recursive form

$$Y_{n+1} = G(\lambda\Delta)Y_n \tag{1.6}$$

then the set of all complex numbers $\lambda\Delta$ with $|G(\lambda\Delta)| < 1$ describes the region of absolute stability of the scheme. For the Euler scheme

$$Y_{n+1} = (1 + \lambda\Delta)Y_n$$

the region of absolute stability is an open unit disc centered at the point $-1 + 0i$. Including additive noise in the test equations (1.5) leads to a simple stochastic generalization of the concept of asymptotic numerical stability. For the resulting class of test equations

$$dX_t = \lambda X_t dt + dW_t \tag{1.7}$$

where the parameter λ is a complex number with Re $(\lambda) < 0$ and W is a real - valued standard Wiener process the regions of stability of some stochastic numerical schemes were considered in [5]. Under the assumption that a given scheme with equidistant step size Δ applied to the test equation (1.7) with Re (λ) < 0 allows a representation in the form

$$Y_{n+1} = G(\lambda\Delta)Y_n + Z_n \tag{1.8}$$

for $n = 0, 1, ...,$ where G is a complex function and the $Z_0, Z_1, ...$ are random variables which do not depend on λ or the $Y_0, ..., Y_{n+1}$ the set of complex numbers $\lambda\Delta$ with $\lambda_1 = \mathrm{Re}(\lambda) < 0$ and $|G(\lambda\Delta)| < 1$ is called the region of absolute stability of the scheme . For example we know from [5] that the region of absolute stability for the explicit Euler scheme

$$Y_{n+1} = Y_n + \left(a(t_n, Y_n) + \frac{1}{2}b(t_n, Y_n)\frac{\delta}{\delta y}b(t_n, Y_n)\right)\Delta + b(t_n, Y_n)\Delta W_n \tag{1.9}$$

is the same as in the deterministic case, namely the interior of an unit circle with the centre in the point $-1+0i$. Similar as in the deterministic case one also has the notion of A- stability for stochastic schemes. One says that a stochastic scheme is A - stable if its region of absolute stability is the whole left half of the complex plane. Of course an A - stable stochastic scheme is also A -stable in the deterministic sense for an ordinary differential equation. If we want to use stochastic numerical schemes to solve applied problems we have to simulate only in a few cases such simple equations as (1.7). The underlying situation is a completely different one if the diffusion coefficient is more complicated. Then it is the first request to introduce a new reasonable notion of stability of stochastic numerical schemes. The aim of this paper is to provide such a notion and to use it to find the regions of stability for given numerical methods with respect to a class of test equations. Here this will be a specific class of stochastic differential equations involving the effect of multiplicative noise.

2. A NOTION OF STABILITY FOR A CLASS OF TEST EQUATIONS

In this section we introduce a concept of stochastic numerical stability for stochastic differential equations with multiplicative noise. For this purpose we consider the class of complex valued test equations

$$dX_t = (1-\alpha)\lambda X_t dt + \sqrt{\alpha}\gamma X_t \circ dW_t \tag{2.1}$$

where λ and γ are complex numbers, W is a real standard Wiener process and the parameter α is a real number which belongs to the interval $[0,1]$. Changing the parameter α shifts the weights between drift and diffusion coefficients in the equation. In order to simplify the descriptions of the regions of stability it will be our aim to look at such equations (2.1) for which there is a suitable connection between the parameters λ and γ. Suppose that we can write a given stochastic scheme with equidistant step size Δ applied to a test equation which belongs to the class (2.1) in the recursive form

$$Y_{n+1} = G(\lambda\Delta, \gamma\sqrt{\Delta}, \omega)Y_n \tag{2.2}$$

where G is a complex valued function which is random and which does not depend on $Y_0, ..., Y_{n+1}$. Then we shall say that the subset Γ of the complex plane with

$$\Gamma = \left\{\lambda\Delta \in \mathbb{C} : \mathrm{Re}(\lambda) < 0, \mathrm{Re}(\gamma^2) < 0, \left|\operatorname*{ess\,sup}_{\omega} G(\lambda\Delta, \gamma\sqrt{\Delta}, \omega)\right| < 1\right\} \tag{2.3}$$

forms the region of stability of the scheme. The main difference between the multiplicative noise case and the additive noise case is that we can not easily express the recursive representation of a given scheme in terms of a deterministic complex mapping and a random variable which is separated from the mapping. That means it remains a complex mapping which involves a random variable. So, in some sense we have to consider all possible realizations of this random variable. By using the essential supremum of the mapping to characterize the region of stability we consider the worst case.

Now, let us investigate whether the choice of our class of test equations is reasonable. For this we have to examine the stability of the test equation itself. Obviously, it is helpful to show that for every t the absolute value of the p th moment of X_t remains bounded. At the beginning we refer to the fact that the explicit solution of (2.1) is

$$X_t = X_0 \exp\{(1-\alpha)\lambda t + \sqrt{\alpha}\gamma W_t\} \tag{2.4}$$

that is

$$X_t^p = X_0^p \exp\{p((1-\alpha)\lambda t + \sqrt{\alpha}\gamma W_t)\}. \tag{2.5}$$

Then we can understand X_t^p as solution of the *Stratonovich* equation

$$X_t^p = X_0^p + \int_0^t p(1-\alpha)\lambda X_s^p ds + \int_0^t p\sqrt{\alpha}\gamma X_s^p \circ dW_s. \tag{2.6}$$

Rewriting (2.6) in the corresponding *Ito* form leads to

$$X_t^p = X_0^p + \int_0^t (p(1-\alpha)\lambda \;+\; \frac{1}{2}p^2\alpha\gamma^2)X_s^p ds \qquad (2.7)$$
$$+ \int_0^t p\sqrt{\alpha}\gamma X_s^p dW_s.$$

By the help of the solution (2.5) of equation (2.7) we can derive the following expression for the p th moment

$$\begin{aligned} E(X_t^p) &= E(X_0^p)\exp\{p((1-\alpha)\lambda + \frac{1}{2}p\alpha\gamma^2)t\} \qquad (2.8) \\ &= E(X_0^p)\exp\{p((1-\alpha)(\lambda_1+\lambda_2 i) + \frac{1}{2}p\alpha(\gamma_1^2 + 2\gamma_1\gamma_2 i - \gamma_2^2))t\}. \end{aligned}$$

From (2.8) follows for the absolute value

$$|E(X_t^p)| \;=\; \left|E(X_0^p)\right|\left|\exp\{p(1-\alpha)\lambda_1 t\}\right| \cdot \left|\exp\{\frac{1}{2}p^2\alpha Re(\gamma^2)t\}\right| \cdot \qquad (2.9)$$

So, we get under the conditions $\lambda_1 = Re(\lambda) < 0$ and $Re(\gamma^2) < 0$ the estimate

$$\left|E(X_t^p)\right| \;<\; \left|E(X_0^p)\right|. \qquad (2.10)$$

That means in this case the test equation is stable for all moments. This shows that the restrictions $Re(\lambda) < 0$ and $Re(\gamma^2) < 0$ in (2.3) are reasonable. The condition $Re(\lambda) < 0$ is sufficient if we consider only test equations satisfying the relation $\gamma^2 = \lambda$.

To be able to clear some questions concerning the stability of stochastic numerical methods we restrict our interest in this paper on test equations with $\gamma^2 = \lambda$. In this case we can hope that it is possible to express the regions of stability in terms of $\lambda\Delta$ only. The fact that the condition $Re(\gamma^2) < 0$ vanishes is also a justification for our decision.

3. Stability of the explicit Euler scheme

In this section by means of the explicit Euler scheme we want to show how we can find a region of the complex $\lambda\Delta$ - plane for which we have reliable information about the step size Δ and the parameter λ to yield a stable behaviour . For reasons we already explained above we assume that $\gamma^2 = \lambda$. Thus we can express γ in terms of λ. Supposing $\gamma^2 = \lambda$ leads to

$$\lambda_1 = Re(\lambda) = \gamma_1^2 - \gamma_2^2$$

and

$$\lambda_2 = Im(\lambda) = 2\gamma_2\gamma_1.$$

The two equations are equivalent to

$$\gamma_1 = Re(\gamma) = \frac{\lambda_2}{2\gamma_2} \tag{3.1}$$

$$\text{and} \qquad \gamma_2^2 = (Im(\gamma))^2 = \gamma_1^2 - \lambda_1 \tag{3.2}$$

By using (3.1) we get from (3.2) the quadratic equation

$$\eta^2 + \lambda_1\eta - \left(\frac{\lambda_2}{2}\right)^2 = 0$$

where $\eta := \gamma_2^2$. The two solutions of this equation are

$$\eta_{1,2} = \frac{1}{2}(\pm|\lambda| - \lambda_1)$$

and hence four different cases are possible . By applying (3.1) to every γ_2 we can find the corresponding γ_1 . We discuss the situation in the following way :

$$(\gamma_2^2)_1 = \frac{1}{2}(|\lambda| - \lambda_1)$$

leads to

$$(\gamma_2)_1 = \pm\sqrt{\frac{1}{2}(|\lambda| - \lambda_1)}$$

that is we can use either

$$\gamma_2 = \sqrt{\frac{1}{2}(|\lambda| - \lambda_1)} \qquad \text{and} \qquad \gamma_1 = \frac{\lambda_2}{\sqrt{2(|\lambda| - \lambda_1)}} \tag{3.3}$$

or

$$\gamma_2 = -\sqrt{\frac{1}{2}(|\lambda| - \lambda_1)} \qquad \text{and} \qquad \gamma_1 = -\frac{\lambda_2}{\sqrt{2(|\lambda| - \lambda_1)}}. \tag{3.4}$$

Furthermore from

$$(\gamma_2^2)_2 = -\frac{1}{2}(|\lambda| + \lambda_1)$$

it follows

$$(\gamma_2)_2 = \pm\sqrt{\frac{1}{2}(|\lambda| + \lambda_1)}\, i$$

such that either

$$\gamma_2 = \sqrt{\frac{1}{2}(|\lambda| + \lambda_1)}\, i \qquad \text{and} \qquad \gamma_1 = -\frac{\lambda_2 i}{\sqrt{2(|\lambda| + \lambda_1)}} \tag{3.5}$$

or

$$\gamma_2 = -\sqrt{\frac{1}{2}(|\lambda| + \lambda_1)}\, i \qquad \text{and} \qquad \gamma_1 = \frac{\lambda_2 i}{\sqrt{2(|\lambda| + \lambda_1)}}\, . \tag{3.6}$$

Now, we are especially interested in the Euler scheme with respect to the following question. In which way can we characterize the region of stability for the Euler scheme for each of the four different possibilities to choose γ ?

The simplified Euler scheme which is suitable for weak approximation applied on the test equation (2.1) has the form

$$\begin{aligned} Y_{n+1} &= Y_n + \left((1-\alpha)\lambda Y_n + \frac{1}{2}\alpha\gamma^2 Y_n\right)\Delta + \sqrt{\alpha}\,\gamma Y_n\sqrt{\Delta}\,\xi \\ &= (1 + \left((1-\alpha)\lambda + \frac{1}{2}\alpha\gamma^2\right)\Delta + \sqrt{\alpha}\,\gamma\sqrt{\Delta}\,\xi)Y_n \end{aligned} \tag{3.7}$$

where ξ is a two - point distributed random variable with $P(\xi = \pm 1) = \frac{1}{2}$. So, we have a recursive representation of the scheme involving a complex mapping G with

$$G(\lambda\Delta, \gamma\sqrt{\Delta}, \omega) = 1 + (1-\alpha)\lambda\Delta + \frac{1}{2}\alpha(\gamma\sqrt{\Delta})^2 + \sqrt{\alpha}\,\gamma\sqrt{\Delta}\,\xi.$$

We choose γ in the form (3.3). It follows

$$\begin{aligned} G(\lambda\Delta, \gamma\sqrt{\Delta}, \omega) &= G(\lambda\Delta, \omega) \\ &= 1 + (1-\alpha)(\lambda_1\Delta + \lambda_2\Delta\, i) + \frac{1}{2}\alpha\left(\frac{\lambda_2}{\sqrt{2(|\lambda| - \lambda_1)}} + \sqrt{\frac{1}{2}(|\lambda| - \lambda_1)}\, i\right)^2 \cdot \\ &\quad \Delta + \sqrt{\alpha}\left(\frac{\lambda_2}{\sqrt{2(|\lambda| - \lambda_1)}} + \sqrt{\frac{1}{2}(|\lambda| - \lambda_1)}\, i\right) \cdot \sqrt{\Delta}\,\xi. \end{aligned} \tag{3.8}$$

Then the third condition on the set Γ in (2.3) leads to

$$\begin{aligned} &\left|\operatorname*{ess\,sup}_{\omega} G(\lambda\Delta, \gamma\sqrt{\Delta}, \omega)\right| = \\ &\quad \Bigg|1 + (1-\alpha)\lambda_1\Delta + \frac{\alpha\lambda_2^2\Delta}{4(|\lambda| - \lambda_1)} - \frac{\alpha\Delta}{4}(|\lambda| - \lambda_1) + \lambda_2\sqrt{\frac{\alpha\Delta}{2(|\lambda| - \lambda_1)}} \\ &\quad + \left[(1-\alpha)\lambda_2\Delta + \frac{1}{2}\alpha\lambda_2\Delta + \sqrt{\frac{\alpha\Delta}{2}(|\lambda| - \lambda_1)}\right] i\Bigg| < 1. \end{aligned} \tag{3.9}$$

Squaring this inequality we obtain

$$
\begin{aligned}
& (1-\alpha)\lambda_1\Delta\Big\{2+(1-\alpha)\lambda_1\Delta-\frac{\alpha\Delta}{2}(|\lambda|-\lambda_1) \\
+ & \lambda_2\sqrt{\frac{2\alpha\Delta}{|\lambda|-\lambda_1}}\Big(\frac{\lambda_2}{2}\sqrt{\frac{\alpha\Delta}{2(|\lambda|-\lambda_1)}}+1\Big)\Big\} \\
+ & (1-\frac{1}{2}\alpha)\lambda_2\Delta\Big\{(1-\frac{1}{2}\alpha)\lambda_2\Delta+\sqrt{2\alpha\Delta(|\lambda|-\lambda_1)}\Big\} \\
+ & \lambda_2^2\frac{\alpha\Delta}{2(|\lambda|-\lambda_1)}\Big\{2+\frac{\lambda_2^2\alpha\Delta}{8(|\lambda|-\lambda_1)}+\lambda_2\sqrt{\frac{\alpha\Delta}{2(|\lambda|-\lambda_1)}}\Big\} \\
+ & \lambda_2\sqrt{\frac{2\alpha\Delta}{|\lambda|-\lambda_1}}-\frac{1}{8}\alpha^2\lambda_2^2\Delta_2^2-\frac{1}{2}\alpha\lambda_2\Delta\sqrt{\frac{\alpha\Delta}{2}(|\lambda|-\lambda_1)} \\
+ & \frac{1}{16}\Big[\alpha\Delta(|\lambda|-\lambda_1)\Big]^2 \;<\; 0\,.
\end{aligned} \tag{3.10}
$$

Setting $\alpha = 0$ reduces (2.1) to the ordinary differential equation $dX_t = \lambda X_t dt$ for which the region of absolute stability of the explicit Euler scheme is the interior of the unit disc centered on $-1+0i$. On the other hand for $\alpha = 1$ we obtain

$$
\begin{aligned}
& \frac{1}{16}(\lambda_2\Delta)^4\frac{1}{(|\lambda|\Delta-\lambda_1\Delta)^2}+\frac{1}{2\sqrt{2}}(\lambda_2\Delta)^3\frac{1}{\sqrt{(|\lambda|\Delta-\lambda_1\Delta)^3}} \\
+ & (\lambda_2\Delta)^2\frac{1}{|\lambda|\Delta-\lambda_1\Delta}+\frac{1}{8}(\lambda_2\Delta)^2 \\
+ & \frac{1}{2\sqrt{2}}\lambda_2\Delta\sqrt{|\lambda|\Delta-\lambda_1\Delta}+\sqrt{2}\lambda_2\Delta\frac{1}{\sqrt{|\lambda|\Delta-\lambda_1\Delta}} \\
+ & \frac{1}{16}(|\lambda|\Delta-\lambda_1\Delta)^2 \;<0\,.
\end{aligned}
$$

Multiplying this inequality with $(|\lambda|\Delta-\lambda_1\Delta)^2$ provides an implicit representation which is appropriate to describe the region of stability. So, we get

$$
\begin{aligned}
& \frac{1}{16}(\lambda_2\Delta)^4+\frac{1}{2\sqrt{2}}(\lambda_2\Delta)^3\sqrt{(|\lambda|\Delta-\lambda_1\Delta)} \\
+ & (\lambda_2\Delta)^2\Big\{\frac{1}{8}(|\lambda|\Delta-\lambda_1\Delta)^2+(|\lambda|\Delta-\lambda_1\Delta)\Big\} \\
+ & \lambda_2\Delta\Big\{\frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta-\lambda_1\Delta)^5}+\sqrt{2}\sqrt{(|\lambda|\Delta-\lambda_1\Delta)^3}\Big\} \\
+ & \frac{1}{16}(|\lambda|\Delta-\lambda_1\Delta)^4 \;<\; 0
\end{aligned} \tag{3.11}
$$

and we obtain the following result for a situation without drift coefficient .
For every complex number $\lambda\Delta = \lambda_1\Delta + \lambda_2\Delta i$ with $\lambda_1 < 0$ which satisfies the inequality (3.11) the explicit Euler scheme is numerically stable if we choose γ as in (3.3).

The other three possibilities to choose γ yield corresponding but other inequalities. More precisely that means we obtain

$$\begin{aligned}
& \frac{1}{16}(\lambda_2\Delta)^4 - \frac{1}{2\sqrt{2}}(\lambda_2\Delta)^3\sqrt{(|\lambda|\Delta - \lambda_1\Delta)} \\
+ \; & (\lambda_2\Delta)^2\left\{\frac{1}{8}(|\lambda|\Delta - \lambda_1\Delta)^2 + (|\lambda|\Delta - \lambda_1\Delta)\right\} \\
- \; & \lambda_2\Delta\left\{\frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta - \lambda_1\Delta)^5} + \sqrt{2}\sqrt{(|\lambda|\Delta - \lambda_1\Delta)^3}\right\} \\
+ \; & \frac{1}{16}(|\lambda|\Delta - \lambda_1\Delta)^4 \; < \; 0
\end{aligned} \tag{3.12}$$

for γ as in (3.4),

$$\begin{aligned}
& \frac{1}{16}(\lambda_2\Delta)^4 + (\lambda_2\Delta)^2\left\{\frac{1}{8}(|\lambda|\Delta + \lambda_1\Delta)^2\right. \\
- \; & \left.\frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^3}\right\} + \frac{1}{16}(|\lambda|\Delta + \lambda_1\Delta)^4 \\
- \; & \frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^7} + (|\lambda|\Delta + \lambda_1\Delta)^3 \\
- \; & \sqrt{2}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^5} \; < \; 0
\end{aligned} \tag{3.13}$$

for γ as in (3.5) and

$$\begin{aligned}
& \frac{1}{16}(\lambda_2\Delta)^4 + (\lambda_2\Delta)^2\left\{\frac{1}{8}(|\lambda|\Delta + \lambda_1\Delta)^2\right. \\
+ \; & \left.\frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^3}\right\} + \frac{1}{16}(|\lambda|\Delta + \lambda_1\Delta)^4 \\
+ \; & \frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^7} + (|\lambda|\Delta + \lambda_1\Delta)^3 \\
+ \; & \sqrt{2}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^5} \; < \; 0
\end{aligned} \tag{3.14}$$

for γ as in (3.6). All these results refer to the fully stochastic case $\alpha = 1$. On one hand we can show that any solution of (3.11) with a negative imaginary part is a solution of (3.13) too. The same conclusion holds for any solution of (3.12) for which the imaginary part is positive. On the other hand we can show that any solution of (3.11) with a positive imaginary part is a solution of (3.14) too. The same conclusion holds for any solution of (3.12) for which the imaginary part is negative. However it is easy to see that the set of complex numbers $\lambda\Delta$ which satisfy the inequality (3.14) is empty. Hence the whole region of stability is described by (3.13). That is in the fully stochastic case $\alpha = 1$ the Euler scheme is stable inside a subset $\tilde{\Gamma}$ of the complex $\lambda\Delta$ - plane where

$$\tilde{\Gamma} = \left\{\lambda\Delta \in \mathbb{C} : Re(\lambda) \; < 0, \; (3.13) \text{ is fulfilled}\right\}. \tag{3.15}$$

In order to obtain regions of stability for other parameter values of α one has to handle the corresponding more general inequality (for example (3.10)) in an appropriate way. To plot the regions of stability one has to evaluate the equation for its boundary numerically. For example, a complex number $\lambda\Delta$ belongs to the boundary of $\widetilde{\Gamma}$ if the relation

$$\begin{aligned} & \frac{1}{16}(\lambda_2\Delta)^4 + (\lambda_2\Delta)^2\Big\{\frac{1}{8}(|\lambda|\Delta + \lambda_1\Delta)^2 \\ - & \frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^3}\Big\} + \frac{1}{16}(|\lambda|\Delta + \lambda_1\Delta)^4 \\ - & \frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^7} + (|\lambda|\Delta + \lambda_1\Delta)^3 \\ - & \sqrt{2}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^5} \; = \; 0 \end{aligned} \tag{3.16}$$

is satisfied.

4. Implicit Euler schemes

Now, we want to investigate under which conditions the application of the drift implicit Euler scheme

$$\begin{aligned} Y_{n+1} &= Y_n + \Big(a(t_{n+1}, Y_{n+1}) + \frac{1}{2}b(t_{n+1}, Y_{n+1}) \cdot \\ &\qquad \frac{\delta}{\delta y}b(t_{n+1}, Y_{n+1})\Big)\Delta + b(t_n, Y_n)\Delta W_n \end{aligned}$$

and the fully implicit Euler scheme

$$\begin{aligned} Y_{n+1} &= Y_n + \Big(a(t_{n+1}, Y_{n+1}) - \frac{1}{2}b(t_{n+1}, Y_{n+1}) \cdot \\ &\qquad \frac{\delta}{\delta y}b(t_{n+1}, Y_{n+1})\Big)\Delta + b(t_{n+1}, Y_{n+1})\Delta W_n \end{aligned}$$

increase the stability. For this purpose we will compare the two different implicit Euler schemes with each other and with the explicit Euler scheme respectively. In one case we introduced implicitness only in the drift coefficient and in the other case we also made the diffusion coefficient implicit. Later we will see that for a suitable choice of the involved random variables the fully implicit Euler scheme with implicit drift and diffusion coefficients is more stable than other schemes of Euler type. At first let us look on the implicit Euler scheme which is implicit only in the drift term. This method applied to equation (2.1) yields

$$Y_{n+1} = Y_n + \Big((1-\alpha)\lambda Y_{n+1} + \frac{1}{2}\alpha\gamma^2 Y_{n+1}\Big)\Delta + \sqrt{\alpha}\gamma Y_n\sqrt{\Delta}\xi.$$

It follows

$$\Big(1 - (1-\alpha)\lambda\Delta - \frac{1}{2}\alpha\gamma^2\Delta\Big)Y_{n+1} = (1 + \sqrt{\alpha}\gamma\sqrt{\Delta}\xi)Y_n$$

that is

$$Y_{n+1} = \Big(1 - (1-\alpha)\lambda\Delta - \frac{1}{2}\alpha\gamma^2\Delta\Big)^{-1} \cdot (1 + \sqrt{\alpha}\,\gamma\sqrt{\Delta}\,\xi)Y_n \tag{4.1}$$

where ξ is two-point distributed with $P(\xi = \pm 1) = \frac{1}{2}$. In the deterministic case $\alpha = 0$ the scheme (4.1) is stable in the whole left half of the complex plane as already explained in [5]. Hence the scheme is A - stable . In the case $\alpha = 1$ we have for γ as in (3.3)

$$\begin{aligned} G(\lambda\Delta, \gamma\sqrt{\Delta}, \omega) &= G(\lambda\Delta, \omega) \\ &= \Big(1 - \frac{1}{2}\big(\frac{\lambda_2^2}{2(|\lambda| - \lambda_1)} + \lambda_2 i - \frac{1}{2}(|\lambda| - \lambda_1)\big)\Delta\Big)^{-1} \cdot \\ &\quad \Big(1 + \big(\frac{\lambda_2}{\sqrt{2(|\lambda| - \lambda_1)}} + \sqrt{\frac{1}{2}(|\lambda| - \lambda_1) \cdot i}\big)\sqrt{\Delta}\,\xi\Big). \end{aligned} \tag{4.2}$$

For (4.2) the condition

$$\Big| \operatorname*{ess\,sup}_{\omega} G(\lambda\Delta, \gamma\sqrt{\Delta}, \omega) \Big| < 1$$

leads to the inequality

$$\Big|1 + \big(\frac{\lambda_2}{\sqrt{2(|\lambda| - \lambda_1)}} + \sqrt{\frac{1}{2}(|\lambda| - \lambda_1) i}\big)\sqrt{\Delta}\Big| <$$

$$\Big|1 - \frac{1}{2}\big(\frac{\lambda_2^2}{2(|\lambda| - \lambda_1)} + \lambda_2 i - \frac{1}{2}(|\lambda| - \lambda_1)\big)\Delta\Big|$$

and we obtain finally

$$\begin{aligned} - &\frac{1}{16}(\lambda_2\Delta)^4 - (\lambda_2\Delta)^2\big\{\frac{1}{8}(|\lambda|\Delta - \lambda_1\Delta)^2 - (|\lambda|\Delta - \lambda_1\Delta)\big\} \\ + &\sqrt{2}\lambda_2\Delta\sqrt{(|\lambda|\Delta - \lambda_1\Delta)^3} - \frac{1}{16}(|\lambda|\Delta - \lambda_1\Delta)^4 < 0. \end{aligned} \tag{4.3}$$

In similar way follows

$$\begin{aligned} - &\frac{1}{16}(\lambda_2\Delta)^4 - (\lambda_2\Delta)^2\big\{\frac{1}{8}(|\lambda|\Delta - \lambda_1\Delta)^2 - (|\lambda|\Delta - \lambda_1\Delta)\big\} \\ - &\sqrt{2}\lambda_2\Delta\sqrt{(|\lambda|\Delta - \lambda_1\Delta)^3} - \frac{1}{16}(|\lambda|\Delta - \lambda_1\Delta)^4 < 0 \end{aligned} \tag{4.4}$$

for γ as in (3.4),

$$\begin{aligned} - &\frac{1}{16}(\lambda_2\Delta)^4 - \frac{1}{8}(\lambda_2\Delta)^2(|\lambda|\Delta + \lambda_1\Delta)^2 - \frac{1}{16}(|\lambda|\Delta + \lambda_1\Delta)^4 \\ + &(|\lambda|\Delta + \lambda_1\Delta)^3 - \sqrt{2}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^5} < 0 \end{aligned} \tag{4.5}$$

for γ as in (3.5) and

$$\begin{aligned} - &\frac{1}{16}(\lambda_2\Delta)^4 - \frac{1}{8}(\lambda_2\Delta)^2(|\lambda|\Delta + \lambda_1\Delta)^2 - \frac{1}{16}(|\lambda|\Delta + \lambda_1\Delta)^4 \\ + &(|\lambda|\Delta + \lambda_1\Delta)^3 + \sqrt{2}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^5} < 0 \end{aligned} \tag{4.6}$$

for γ as in (3.6). It is possible to show that any solution of (4.3) with a negative imaginary part and any solution of (4.4) with a positive imaginary part respectively is a solution of (4.5) too. Just as any solution of (4.3) with a positive imaginary part and any solution of (4.4) with a negative imaginary part respectively is a solution of (4.6) too. Furthermore we can show that (4.5) also follows from (4.6). Therefore for the drift implicit Euler scheme the inequality (4.5) is sufficient to characterize the region of stability. So, we can say that in the case $\alpha = 1$ the drift implicit Euler scheme is stable inside a subset $\tilde{\Gamma}_1$ of the complex $\lambda\Delta$-plane where

$$\tilde{\Gamma} = \Big\{\lambda\Delta \in \mathbb{C} : Re(\lambda) \; < 0, \text{ (4.5) is fulfilled}\Big\}. \tag{4.7}$$

Once again we start with equation (2.1) and now we use the fully implicit Euler scheme to obtain

$$\begin{aligned} Y_{n+1} &= Y_n + \Big((1-\alpha)\lambda Y_{n+1} - \frac{1}{2}\alpha\gamma^2 Y_{n+1}\Big)\Delta \\ &\quad + \sqrt{\alpha}\,\gamma Y_{n+1}\sqrt{\Delta}\,\xi. \end{aligned}$$

So, we have

$$\Big(1-(1-\alpha)\lambda\Delta + \frac{1}{2}\alpha\gamma^2\Delta - \sqrt{\alpha}\gamma\sqrt{\Delta}\,\xi\Big)Y_{n+1} = Y_n$$

that is

$$\begin{aligned} Y_{n+1} &= \Big(1-(1-\alpha)\lambda\Delta + \frac{1}{2}\alpha\gamma^2\Delta - \sqrt{\alpha}\gamma\sqrt{\Delta}\,\xi\Big)^{-1} Y_n \\ &= \frac{1}{\Big(1-(1-\alpha)\lambda\Delta + \frac{1}{2}\alpha\gamma^2\Delta - \sqrt{\alpha}\gamma\sqrt{\Delta}\,\xi\Big)} Y_n\,. \end{aligned} \tag{4.8}$$

Here we also choose ξ as two- point distributed with $P(\xi = {}^+_-1 = \frac{1}{2}$. Under the familiar assumption we proceed in the same manner as above to find the region of stability. That is for each of the four possible choices of γ we try to get the inequality which describes our region of stability. We have for example for γ from (3.3)

$$\begin{aligned} G(\lambda\Delta, \gamma\sqrt{\Delta}, \omega) &= G(\lambda\Delta, \omega) \\ &= \Big(1-(1-\alpha)(\lambda_1\Delta + \lambda_2\Delta i) \\ &\quad + \frac{1}{2}\alpha\Big(\frac{\lambda_2^2}{2(|\lambda|-\lambda_1)} + \lambda_2 i - \frac{1}{2}(|\lambda|-\lambda_1)\Big)\Delta \\ &\quad - \sqrt{\alpha}\Big(\frac{\lambda_2}{\sqrt{2(|\lambda|-\lambda_1)}} + \sqrt{\frac{1}{2}(|\lambda|-\lambda_1)}\,i\Big) \cdot \\ &\quad \sqrt{\Delta}\,\xi\Big)^{-1}. \end{aligned} \tag{4.9}$$

For (4.9) in the case $\alpha = 1$ the condition

$$|\operatorname*{ess\,sup}_{\omega} G(\lambda\Delta, \gamma\sqrt{\Delta}, \omega)| \; < \; 1$$

leads to the inequality

$$
\begin{aligned}
& \frac{1}{16}(\lambda_2\Delta)^4 - \frac{1}{2\sqrt{2}}(\lambda_2\Delta)^3\sqrt{|\lambda|\Delta - \lambda_1\Delta} \\
+ & (\lambda_2\Delta)^2\Big\{\frac{1}{8}(|\lambda|\Delta - \lambda_1\Delta)^2 + (|\lambda|\Delta - \lambda_1\Delta)\Big\} \\
- & \lambda_2\Delta\Big\{\frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta - \lambda_1\Delta)^5} + \sqrt{2}\sqrt{(|\lambda|\Delta - \lambda_1\Delta)^3}\Big\} \\
+ & \frac{1}{16}(|\lambda|\Delta - \lambda_1\Delta)^4 > 0. \qquad (4.10)
\end{aligned}
$$

If we denote the subsets of the complex $\lambda\Delta$- plane which correspond to the inequalities (3.11), (3.12), (3.13) and (3.14) with $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3$ and $\tilde{\Gamma}_4$, respectively, then it is not difficult to see that the set $\hat{\Gamma}_1 := \mathbb{C}\backslash(\tilde{\Gamma}_2 \cup \partial\tilde{\Gamma}_2)$ corresponds to (4.10) where $\partial\tilde{\Gamma}_2$ denotes the boundary of $\tilde{\Gamma}_2$. For γ from (3.4) we obtain in an analogous way the inequality

$$
\begin{aligned}
& \frac{1}{16}(\lambda_2\Delta)^4 + \frac{1}{2\sqrt{2}}(\lambda_2\Delta)^3\sqrt{|\lambda|\Delta - \lambda_1\Delta} \\
+ & (\lambda_2\Delta)^2\Big\{\frac{1}{8}(|\lambda|\Delta - \lambda_1\Delta)^2 + (|\lambda|\Delta - \lambda_1\Delta)\Big\} \\
+ & \lambda_2\Delta\Big\{\frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta - \lambda_1\Delta)^5} + \sqrt{2}\sqrt{(|\lambda|\Delta - \lambda_1\Delta)^3}\Big\} \\
+ & \frac{1}{16}(|\lambda|\Delta - \lambda_1\Delta)^4 > 0. \qquad (4.11)
\end{aligned}
$$

which is fulfilled for every element from $\hat{\Gamma}_2 := \mathbb{C}\backslash(\tilde{\Gamma}_1 \cup \partial\tilde{\Gamma}_1)$. For γ from (3.5) we get the inequality

$$
\begin{aligned}
& \frac{1}{16}(\lambda_2\Delta)^4 + (\lambda_2\Delta)^2\Big\{\frac{1}{8}(|\lambda|\Delta + \lambda_1\Delta)^2 + \frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^3}\Big\} \\
+ & \frac{1}{16}(|\lambda|\Delta + \lambda_1\Delta)^4 + \frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^7} \\
+ & (|\lambda|\Delta + \lambda_1\Delta)^3 + \sqrt{2}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^5} > 0 \qquad (4.12)
\end{aligned}
$$

which holds for every element from $\hat{\Gamma}_3 := \mathbb{C}\backslash\mathbb{R}_-$ and for γ from (3.6) we get the inequality

$$
\begin{aligned}
& \frac{1}{16}(\lambda_2\Delta)^4 + (\lambda_2\Delta)^2\Big\{\frac{1}{8}(|\lambda|\Delta + \lambda_1\Delta)^2 - \frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^3}\Big\} \\
+ & \frac{1}{16}(|\lambda|\Delta + \lambda_1\Delta)^4 - \frac{1}{2\sqrt{2}}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^7} \\
+ & (|\lambda|\Delta + \lambda_1\Delta)^3 - \sqrt{2}\sqrt{(|\lambda|\Delta + \lambda_1\Delta)^5} > 0 \qquad (4.13)
\end{aligned}
$$

which holds for every element from $\hat{\Gamma}_4 := \mathbb{C}\backslash(\tilde{\Gamma}_3 \cup \partial\tilde{\Gamma}_3)$. We notice that the $\lambda\Delta$ with $Im(\lambda) = 0$ and $Re(\lambda) < 0$ are not elements of $\hat{\Gamma}_3$, but they are elements of both $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$. So, we obtain for the fully implicit Euler scheme with $\alpha = 1$ the whole left half of the complex $\lambda\Delta$- plane as region of stability. With other words

we can say that to every complex number λ we can find a suitable γ such that the fully implicit Euler scheme (4.8) is numerically stable.
From the investigations of section 2 and 3 we learned that the applicability of a method is strongly dependent on the choice of the random variable ξ. If we take for instance in the scheme (3.7) ξ as a standard Gaussian random variable then $\operatorname{ess\,sup}_\omega G(\lambda\Delta, \gamma\sqrt{\Delta}, \omega)$ is infinite because of (3.8). We note easily that one of the factors may become extremely large for standard Gaussian ξ if we represent scheme (4.8) in the form

$$Y_n = \left(\frac{1}{1-(1-\alpha)\lambda\Delta + \frac{1}{2}\alpha\gamma^2\Delta - \sqrt{\alpha}\,\gamma\sqrt{\Delta}\,\xi} \right)^n Y_0 . \tag{4.14}$$

This clearly can not be a stable scheme.

5. Summary and outlook

We introduced a suitable test equation to make statements about the stability of stochastic numerical methods applied to stochastic differential equations with multiplicative noise. Similar as in the additive noise case (see [5]) we observed clear advantages of implicit Euler schemes compared with the explicit Euler scheme also in the multiplicative noise case.
Furthermore, we proposed a fully implicit scheme with implicitness in drift and diffusion coefficient.
We noticed the surprising and appealing fact that we obtain in the case $\alpha = 1$ the same region of stability as in the deterministic case $\alpha = 0$. So it makes sense to expect that this scheme is in more general situations a very powerful one. It will be our next aim to evaluate the corresponding regions of stability numerically and to apply our notion of stability in a more general situation. It should be also our interest to get regions of stability for higher order schemes. Another idea is to look for a characterization of the regions of stability by the use of Lyapunov exponents corresponding to the discrete time systems. That means we search for the Lyapunov exponents of the stochastic numerical schemes.

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