Minimax detection of a signal for l_q^n-balls with l_p^n-balls removed [1] [2] [3]

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Abstract

In this paper we continue the researches of hypothesis testing problems leading to infinitely divisible distributions which have been started in the papers by Ingster, 1996a, 1997.

Let the n-dimensional Gaussian random vector $x = \xi + v$ is observed where ξ is a standard n-dimensional Gaussian vector and $v \in R_n$ is an unknown mean. We consider the minimax hypothesis testing problem $H_0 : v = 0$ versus alternatives $H_1 : v \in V_n$, where V_n is l_q^n-ball of radius $R_{1,n}$ with l_p^n-balls of radius $R_{2,n}$ removed. We are interesting in the asymptotics (as $n \to \infty$) of the minimax second kind error probability $\beta_n(\alpha) = \beta_n(\alpha; p, q, R_{1,n}, R_{2,n})$ where $\alpha \in (0,1)$ is a level of the first kind error probability. Close minimax estimation problem had been studied by Donoho and Johnstone (1994).

We show that the asymptotically least favorably priors in the problem of interest are of the product type: $\pi^n = \pi_n \times \cdots \times \pi_n$. Here $\pi_n = (1-h_n)\delta_0 + \frac{h_n}{2}(\delta_{-b_n} + \delta_{b-n})$ are the three-point measures with some $h_n = h_n(p, q, R_{1,n}, R_{2,n}$ and $b_n = b_n(p, q, R_{1,n}, R_{2,n}$. This reduces the problem of interest to Bayssian hypothesis testing problems where the asymptotics of error probabilities had been studied by Ingster, 1996a, 1997.

In particularly, if $p \leq q$, then the asymptotics of $\beta_n \alpha$ are of Gaussian type, but if $p > q$ then its are either Gaussian or degenerate or belong to a special class of infinitely divisible distributions which was described in Ingster, 1996a, 1997.

1 Introduction

1.1 Setting

Let n-dimensional Gaussian random vector $x = \xi + v$ is observed where ξ is a standard n-dimensional Gaussian vector with zero mean and unit covariance matrix and $v \in R_n$ is an unknown mean. We test null hypothesis $H_0 : v = 0$ versus the alternative $H_1 : v \in V_n$, here V_n is l_q^n-ball of radius $R_{1,n}$ with l_p^n-balls of radius $R_{2,n}$ removed:

$$V_n = V_n^{p,q}(R_{1,n}, R_{2,n}) = \{v = (v_1, \dots, v_n) \in R^n : \sum_{i=1}^n |v_i|^p \geq R_{1,n}^p, \sum_{i=1}^n |v_i|^q \leq R_{1,n}^q\} \tag{1.1}$$

where $p \in (0, \infty), q \in (0, \infty]$ are given values and $R_{1,n} > 0, R_{2,n} > 0$ are given sequences of radiuses (with evident modifications for $q = \infty$ or $p = \infty$; we consider the case $p < \infty$ in this paper). We assume

$$R_{1,n} \leq R_{2,n} \quad \text{for} \quad p > q; \quad R_{1,n} n^{-1/p} \leq R_{2,n} n^{-1/q} \quad \text{for} \quad p \leq q \tag{1.2}$$

which imply that the sets V_n are nonempty.

We deal with asymptotically minimax hypothesis testing problem (see Ingster, 1993). Let $\Psi_{n,\alpha}$ be the set of tests of level α, $\alpha \in (0,1)$ (the set of measurable functions $\psi : R^n \to [0,1]$) such that $\alpha(\psi) \leq \alpha$ where $\alpha(\psi) = E_{n,0}\psi$ is the first kind error probability. Here and below $E_{n,v}$ means the expectation with respect to Gaussian measure $P_{n,v}$ with the mean v and unit covariance matrix.

Let $\beta_n(\psi, v) = E_{n,v}(1 - \psi)$ be the second kind error probability and let

$$\beta_n(\psi, V_n) = \sup_{v \in V_n} \beta_n(\psi, v)$$

be the maximum value of the second kind error probability for test ψ. Let

$$\beta_n(\alpha) = \beta_n(\alpha, V_n) = \inf_{\psi \in \Psi_(n,\alpha)} \beta_n(\psi, V_n) \tag{1.3}$$

be the minimax second kind error probability. It is clear that following inequalities hold:

$$0 \leq \beta_n(\alpha) \leq 1 - \alpha.$$

We are interested in the dependence of the asymptotics of $\beta_n(\alpha)$ on p, q and on the behavior of $R_{1,n}$ and $R_{2,n}$ as $n \to \infty$ for any $\alpha \in (0,1)$ and in the structure of asymptotically minimax tests $\psi_{n,\alpha}$ such that

$$\alpha_n(\psi_{n,\alpha}) \leq \alpha + o(1), \quad \beta_n(\psi_{n,\alpha}) \leq \beta_n(\alpha, V_n) + o(1).$$

Here and below the asymptotic relations are understood as $n \to \infty$.

1.2 Discussion

The problem under consideration seems to be the most natural minimax hypothesis testing problem of increasing dimension. Analogous minimax estimation problem has been studied by Donoho and Johnstone (1994).

Close infinite dimensional problems of hypothesis testing were considered by Ermakov (1990), by Ingster (1990, 1993, 1996b) and by Suslina (1993, 1996). It was assumed that infinite dimensional Gaussian vector $x = \xi + v$ is observed where $v \in l_2$ and ξ is a sequence of independent standard Gaussian variables. Alternatives $H_1 : v \in V_\epsilon$ correspond to a family of subsets $V_\epsilon \in l_2$ with asymptotic parameter $\epsilon \to 0$.

These papers deal with alternatives of the form l_q-ellipsoids with l_p-balls removed:

$$V_\epsilon = \{v \in l_2 : \sum_{i=1}^{\infty} |v_i|^p \geq (\rho_\epsilon/\epsilon)^p, \sum_{i=1}^{\infty} |v_i/a_i|^q \leq \epsilon^{-q}\}. \tag{1.4}$$

Here $\{a_i\}$ is a given sequence of semi-exes of l_q-ellipsoid for fixed orthonormal basis in L_2 which corresponds to the problem of detection of a signal in Gaussian white noise; the factor $\epsilon > 0$ in (1.4) corresponds to normalization.

It is clear that the problem under consideration in this paper is the same as in (1.4) for "series scheme" with $\rho_\epsilon/\epsilon = R_{1,n}$ and $a_i/\epsilon = R_{2,n}$ for $i = 1, \ldots, n$, $a_i = 0$ for $i > n$ and we can try to use methods of Ingster (1996b) and Suslina (1996) to this problem.

These methods are used for $p < \infty$ and reduce the considerable problem to the extreme problem:

$$u_\epsilon^2 = \inf\{\| \overline{\pi} \|^2 : \overline{\pi} \in \Pi_\epsilon\} \tag{1.5}$$

where Π_ϵ is the set of sequences $\overline{\pi} = (\pi_1, \ldots, \pi_n, \ldots)$ of probability measures π_i on the real line such that

$$\sum_{i=1}^{\infty} E_{\pi_i} \mid u \mid^p \geq (\rho_\epsilon/\epsilon)^p, \quad \sum_{i=1}^{\infty} E_{\pi_i} \mid u/a_i \mid^q \leq \epsilon^{-p}$$

and

$$\| \overline{\pi} \|^2 = \sum_{i=1}^{\infty} \| \pi_i \|^2 = \sum_{i=1}^{\infty} \int_{R^1} \int_{R^1} (e^{uv} - 1)\pi_i(du)\pi_i(dv).$$

Under some assumptions (they are formulated in terms of the sequence $\overline{\pi}_\epsilon = (\pi_{\epsilon,1}, \ldots, \pi_{\epsilon,n}, \ldots)$ which minimizes (1.5)) it is shown in Ingster (1996b) that analogous to (1.3) values $\beta_\epsilon(\alpha)$ satisfy to the relation

$$\beta_\epsilon(\alpha) = \beta_\epsilon(\alpha, P_{\pi^\epsilon}) + o(1) = \Phi(t_\alpha - u_\epsilon) + o(1), \epsilon \to 0. \tag{1.6}$$

Here $\beta_\epsilon(\alpha, P_{\pi^\epsilon})$ is minimum second kind error probability of tests of the level α for simple Bayesian alternative which corresponds to a mixture

$$P_{\pi^\epsilon} = \int P_v \pi^\epsilon(dv) = \prod_{i=1}^{\infty} \int_{R^1} P_{1,v}\pi_{\epsilon,i}(dv)$$

over product-measure $\pi^\epsilon = \pi_{\epsilon,1} \times \cdots \times \pi_{\epsilon,i} \times \cdots$ and u_ϵ is defined by (1.5). Here and below Φ stands for distribution function of standard Gaussian low and t_α for this $(1-\alpha)$-quantile.

The relation (1.6) is based on the asymptotical normality of the log-likelihood ratio $\log(dP_{\pi^\epsilon}/dP_0)$

The extreme problem (1.5) may be separated in "one-dimensional" problem

$$f_{p,q}(\lambda,\nu) = \inf\{\ \ \| \pi \|^2: \quad E_\pi \mid u \mid^p \geq \lambda^p, \quad E_\pi \mid u \mid^q \leq \nu^q\}, \tag{1.7}$$

and in "two-sequence" problem

$$u_\epsilon^2 = \inf\{\ \ \sum_{i=1}^{\infty} f_{p,q}(\lambda_i,\nu_i) : \quad \sum_{i=1}^{\infty} \lambda_i^p = (\rho_\epsilon/\epsilon)^p, \quad \sum_{i=1}^{\infty} \nu_i^q = \epsilon^{-p}\} \tag{1.8}$$

(or "one-sequence" problem if $p = q$).

These problems had been studied by Ingster, 1990, 1993 (the case $p = q$) and by Suslina, 1996 (the case $p \neq q$ and the power sequence of semi-axes $a_n \asymp n^{-t}, t > 0$).

In particular, it was shown that the solution of one-dimensional problem (1.7) is the symmetrical three-point measure

$$\pi(b,h) = (1-h)\delta_0 + \frac{h}{2}(\delta_b + \delta_{-b}) \tag{1.9}$$

for some $b = b(p,q,\lambda,\nu) > 0, h = h(p,q,\lambda,\nu) \in (0,1]$; here δ_b is Dirac mass at the point $b \in R^1$.

More exactly, if $\lambda < \nu$ and $p \leq q$, then the set under constraints is empty. If $\lambda \geq \nu$ or $p > q$, then there are the three possible equalities:

$$\begin{array}{ll} (i) & h = 1, \quad b = \lambda; \\ (ii) & h = (\lambda/b_p)^p, \quad b = b_p \quad (\text{for} \quad p > 2); \\ (iii) & hb^p = \lambda^p, \quad hb^q = \nu^q. \end{array}$$

Here the value $b_p > 0$ for $p > 2$ is the root of the equation $p \tanh b^2/2 = b^2$; this value minimizes the function $b^{-p}\sinh(b^2/2)$.

Note that these relations imply equality in the first inequality (1.7). However if the relation (iii) does not hold, then it is not possible the equality in the second inequality in (1.7 and this inequality is not essential in the problem).

The relations between p, q, λ, ν and equalities (i), (ii) and (iii) are described by following

Lemma 1.1 *(Suslina, 1996)*

1. Let $p \leq 2$ and $p \leq q$. Then the relation (i) holds.

2. Let $p \leq 2$ and $p > q$. Then the relation (i) holds, if $\lambda < \nu$ and the relation (iii) holds, if $\lambda \geq \nu$.

3. Let $\infty > p > 2$ and $p \leq q$. Then the relation (i) holds, if $\lambda > b_p$; the relation (ii) holds, if $\lambda \leq b_p \leq (\nu^q/\lambda^p)^{1/(q-p)}$ for $p < q$ or $\lambda \leq b_p$ for $p = q$; the relation (iii) holds, if $\lambda \leq b_p,\ p < q,\ (\nu^q/\lambda^p)^{1/(q-p)} \leq b_p$.

4. Let $\infty > p > 2$ and $p > q$. Then the relation (i) holds, if $b_p < \lambda < \nu$; the relation (ii) holds, if $\lambda \geq \nu$ and $(\lambda/b_p)^p \leq (\nu/b_p)^q$ or $\nu > \lambda$ and $\lambda < b_p$; the relation (iii) holds, if $\lambda \geq \nu$ and $(\lambda/b_p)^p > (\nu/b_p)^q$.

These results imply that

$$f_{p,q}(\lambda, \nu) = 2h^2 \sinh^2(b^2/2)$$

where $b = b(p, q, \lambda, \nu)$ and $h = h(p, q, \lambda, \nu)$ are described in Lemma 1 and if $p \leq q$, then $b_n = O(1)$ for any sequences λ_n and ν_n such that $f_{p,q}(\lambda_n, \nu_n) = O(1)$ as $n \to \infty$.

The required to (1.6) assumptions may be formulated in terms of extreme sequences $b_\epsilon = b_{\epsilon,i}$ and $h_\epsilon = h_{\epsilon,i}$ and they are checked for a power sequence of semi-exes: $a_n \asymp n^{-t}$ as $n \to \infty$, $t > 0$. In particular, one of sufficient conditions is that $\sup_i b_{\epsilon,i} = O(1)$ as $\epsilon \to \infty$ and it is fulfilled for $p \leq q$ if $u_\epsilon = O(1)$.

If we use these methods to the problem under consideration, then we obtain the problem

$$u_n^2 = \inf\{ \sum_{i=1}^n f_{p,q}(\lambda_i, \nu_i) : \quad \sum_{i=1}^n \lambda_i^p = R_{1,n}^p, \quad \sum_{i=1}^n \nu_i^q = R_{2,n}^q \},$$

which is analogous to (1.8). One can easily see that, by the symmetry,

$$u_n^2 = n f_{p,q}(\lambda_n, \nu_n) = 2n h_n^2 \sinh^2(b_n^2/2) \tag{1.10}$$

where

$$\lambda_n = R_{1,n}/n^{1/p}, \quad \nu_n = R_{2,n}/n^{1/q}, \quad b_n = b(p, q, \lambda_n, \nu_n), \quad h_n = h(p, q, \lambda_n \nu_n).$$

If $b_n = O(1)$ here, then the assumptions of Ingster (1996b) hold and we obtain the relation which is analogous to (1.6):

$$\beta_n(\alpha) = \beta_n(\alpha, P_{\pi^n}) + o(1) = \Phi(t_\alpha - u_n) + o(1), \quad n \to \infty. \tag{1.11}$$

Here u_n is defined in (1.10) and π^n is the product measure:

$$\pi^n = \pi^n(b_n, h_n) = \prod_{i=1}^n \pi_{n,i}, \quad \pi_{n,i} = \pi_n = \pi(b_n, h_n) \tag{1.12}$$

where $\pi(b_n, h_n)$ are defined by (1.9).

In particularly, the relation (1.11) holds, if $p \leq q$. In fact, if $u_n = O(1)$, then one can easily check, that $b_n = O(1)$. If $b_n \to \infty$, then, by making $R_{1,n}$ smaller, one can reduce the consideration to the case of arbitrarily large $b_n = O(1)$ and $u_n = O(1)$ which implies $\beta_n(\alpha) \to 0$. However if $\infty > p > q$, then it is possible that $b_n \to \infty$ and $u_n = O(1)$.

The goal of this paper is to study this case. We can use the results by Ingster (1996b) in this case also, if $b_n = o(\sqrt{\log n})$. However, if $b_n \asymp \sqrt{\log n}$, then, as we show below, the asymptotics (1.11) may not hold. If $nh_n \to \infty$, then the product priors (1.12) are the asymptotically least favorable also, however, as it follows from Ingster, 1996a, 1997, the asymptotics of distributions of the log-likelihood ratios $\log(dP_{\pi^n}/dP_{0,n})$ are not Gaussian, but degenerate or infinite divisible of the special type. Note that analogous asymptotically least favorable product priors (1.12) arise in Donoho and Johnstone (1994) for minimax estimation problem also.

The main considerations later assume $\infty > p > q,\ b_n \to \infty$ which by Lemma 1.1 correspond to

$$nh_n b_n^p = R_{n,1}^p,\ nh_n b_n^q = R_{n,2}^q \tag{1.13}$$

or

$$b_n = \left(\frac{R_{1,n}^p}{R_{2,n}^q}\right)^{1/(p-q)},\ nh_n = \left(\frac{R_{2,n}}{R_{1,n}}\right)^{pq/(p-q)}.$$

In fact, as it was noted above, the case $p < q \leq \infty$ can be considered by the methods of Ingster, 1993, 1996b and Suslina, 1996 and corresponds to the Gaussian case. The case $\infty = p \geq q$ can be considered by the methods of Ingster, 1993, nn. 4.4, 5.4 and corresponds to degenerate case with $b_n = R_{1,n},\ nh_n = 1$ (see the Theorem 3 later).

In the next section we remind the results in Bayesian problem and formulate the analogous results for the problem under consideration. In sections 3 - 8 we give the proofs.

2 Main results

Remind the results of Ingster, 1997 on the distributions of the log-likelihood ratios for the product priors (1.12)

$$l_n = l_n(b_n, h_n) = \log(dP_{\pi^n}/dP_{0,n}) = \sum_{i=1}^{n} \log(1 + h_n \xi(x_i, b_n)) \tag{2.14}$$

where

$$\xi(x, b) = \exp(-b^2/2)\cosh bx - 1$$

and on the asymptotics of the values $\beta_n(\alpha, P_{\pi^n})$ in Bayesian problem .

If $b_n \to \infty$ and $h_n \to 0$, then define two sequences T_n and τ_n. Let T_n is defined by the relation

$$h_n \xi(T_n, b_n) = 1 + o(1).$$

One can easily see that

$$T_n = \frac{b_n}{2} + \frac{\log 2h_n^{-1}}{b_n} + o(b_n^{-1}). \tag{2.15}$$

Put

$$\tau_n = \frac{T_n}{b_n} = \frac{1}{2} + \frac{\log h_n^{-1}}{b_n^2} + o(1) \tag{2.16}$$

and assume without loss of generality that

$$\tau_n \to \tau \in [1/2, \infty].$$

Also if $b_n = O(1)$ or $h_n \asymp 1$, then put $\tau = \infty$.

It was shown in Ingster, 1996a, 1997, that there are three different types of the limit distributions of the log-likelihood statistics (2.14) and three types of the asymptotics of the second kind error probabilities $\beta_n(\alpha, P_{\pi^n})$ which correspond to the three intervals of τ: $\tau \in [2, \infty]$ (Gaussian type), $\tau \in (1, 2)$ (infinite-divisible type) and $\tau \in (1/2, 1]$ (degenerate type).

2.1 Gaussian case: $\tau \in [2, \infty]$

Put $u_n = +\sqrt{u_n^2}$ where

$$u_n^2 = \begin{cases} 2nh_n^2(\sinh(b_n^2/2))^2, & \text{if } \tau \in (2, \infty] \ , \\ \frac{1}{2} nh_n^2 e^{b_n^2} \Phi(T_n - 2b_n), & \text{if } \tau = 2 \ . \end{cases} \tag{2.17}$$

Then (Ingster, 1997, Theorem 1)

$$\beta_n(\alpha, P_{\pi_n}) = \Phi(t_\alpha - u_n) + o(1)$$

and if $u_n \asymp 1$, then the log-likelihood statistics l_n in (2.14) are asymptotically Gaussian $N(-u_n^2/2, u_n^2/2)$ under the null hypothesis H_0 and $N(u_n^2/2, u_n^2/2)$ under the Bayesian alternative H_{n,π^n}.

In considerable problem we have

Theorem 1 *Let $\tau \geq 2$. Then*

1. Lower bounds:

$$\beta_n(\alpha, V_n) \geq \Phi(t_\alpha - u_n) + o(1).$$

2. Upper bounds. Let us consider the sequences of the tests

$$\psi_{n,\alpha} = \mathbf{1}_{\{\tilde{l}_n > T_{n,\alpha}\} \cup \{\max_i |x_i| > H_n\}}. \tag{2.18}$$

Here H_n is some sequence such that $n\Phi(H_n) \to 0$, $T_{n,\alpha} = t_\alpha u_n - u_n^2/2$. If $\tau > 3$, then $\tilde{l}_n = l_n$ and if $\tau \leq 3$, then $\tilde{l}_n = l_n(\tilde{b}_n, \tilde{h}_n)$ where the values $\tilde{b}_n, \tilde{h}_n$ correspond to $p, q, R_{n,2}$ and changed $\tilde{R}_{n,1} = R_{n,1}(1 - b_n^{-3})$. Then $\alpha(\psi_{n,\alpha}) = \alpha + o(1)$,

$$\beta_n(\psi_{n,\alpha}, V_n) \leq \Phi(t_\alpha - u_n) + o(1).$$

It means that the sequence of the tests $\psi_{n,\alpha}$ is asymptotically minimax.

Corollary 2.1

1. Let $p \leq q$. Then

$$\beta_n(\alpha, V_n) = \Phi(t_\alpha - u_n) + o(1).$$

2. Let $p \leq q$, $p \leq 2$. Then

$$\beta_n(\alpha, V_n) = \beta_n(\psi_{n,\alpha}, V_n) + o(1) = \Phi(t_\alpha - \tilde{u}_n) + o(1).$$

Here $\tilde{u}_n^2 = nb_n^4/2$ and $\psi_{n,\alpha} = \mathbf{1}_{\{\tilde{l}_n > t_\alpha\}}$ where $\tilde{l}_n = (2n)^{-1/2} \sum_{i=1}^n (x_i^2 - 1)$ is the sequence of xi-square tests.

It means that there is the unit family of tests which is asymptotically minimax for any $p \leq q$, $p \leq 2$, $R_{n,1}$, $R_{n,2}$.

Proof of the Corollary. By making $R_{n,1}$ smaller, we can assume that $u_n = O(1)$. The statement n.1 follows from the boundness b_n in the case $p \le q$ by the Lemma 1.1. If $p \le 2$ also, then $h_n = 1$. Note that $u_n \sim \tilde{u}_n$ in this case and the statement n.2 follows from Ingster, 1993, n. 5.2.

2.2 Infinitely divisible case: $\tau \in (1, 2)$

Put $c_n = 2n\Phi(-T_n)$ and assume without loss of generality that $c_n \to c \in [0, \infty]$.

For $\tau \in (1, 2)$ and $c \in (0, \infty)$ let us define two independent infinitely divisible random variables $\zeta^0 = \zeta^0_{c,\tau}$ and $\zeta^\Delta = \zeta^\Delta_{c,\tau}$ with the characteristic functions

$$\log \varphi^0(z) = iz\gamma^0 + \int_{+0}^{\infty} (e^{izt} - 1 - \frac{izt}{1+t^2}) dL^0(t), \tag{2.19}$$

$$\log \varphi^\Delta(z) = \int_{+0}^{\infty} (e^{izt} - 1) dL^\Delta(t). \tag{2.20}$$

Here $L^0 = L^0_{c,\tau}$ and $L^\Delta = L^\Delta_{c,\tau}$ are the Levi spectral functions (see Petrov, 1981) which are zero for $t < 0$ and for $t > 0$

$$L^0(t) = -c(e^t - 1)^{-\tau}, \tag{2.21}$$

$$L^\Delta(t) = -\frac{d}{dt} L^0(t) = -\frac{c}{\tau - 1}(e^t - 1)^{1-\tau}. \tag{2.22}$$

The constant γ^0 in (2.19) is defined by the relation

$$E\zeta^0 = \gamma^0 + \int_0^\infty \frac{t^3}{1+t^2} dL^0(t) = cI^0(\tau)$$

where

$$I^0(\tau) = \int_0^\infty (\log(1 + u^{-1/\tau}) - u^{-1/\tau}) du.$$

This is equivalent to

$$E \exp \zeta^0 = 1.$$

Note that for the Levi spectral functions $L = L^0$ and $L = L^\Delta$ one has from (2.21) and (2.22) that

$$\int_{|x|>1} |x|^p dL(x) < \infty$$

for any $p > 0$ which implies that the random variables ζ^0 and ζ^Δ have finite moments of any order. The distributions of ζ^0 and ζ^Δ are absolutely continuous. The support of ζ^0 is R^1 but the support of ζ^Δ is the positive halfline $R^1_+ = \{t \ge 0\}$ (see Petrov (1981) for general theorems which imply these properties).

Let $F^0 = F^0_{c,\tau}$ and $F^1 = F^1_{c,\tau}$ be the distribution functions of ζ^0 and of $\zeta^1 = \zeta^1_{c,\tau} = \zeta^0 + \zeta^\Delta$ and let $t^0(\alpha) = t^0_{c,\tau}(\alpha)$ be the $(1-\alpha)$-quantile of ζ^0: $F^0(t^0(\alpha)) = 1 - \alpha$.

Then (Ingster, 1997, Theorem 2) the following relations hold.

1. If $c = 0$, then $\beta_n(\alpha, P_{\pi_n}) \to 1 - \alpha$.
2. If $c = \infty$, then $\beta_n(\alpha, P_{\pi_n}) \to 0$.

3. Let $c \in (0, \infty)$. Then $l_n \to \zeta^0$ under $P_{n,0}$ - probability, $l_n \to \zeta^1$ under P_{n,π^n} - probability and

$$\beta_n(\alpha, P_{\pi_n}) \to F^1(t^0(\alpha)).$$

In considerable problem we have

Theorem 2 *Let $\tau \in (1,2)$. Then*

1. Lower bounds:

$$\beta_n(\alpha, V_n) \geq \beta_n(\alpha, P_{\pi_n}) + o(1).$$

2. Upper Bounds. Let us consider the sequences of the tests (2.17) where H_n is some sequence such that $n\Phi(H_n) \to 0$, $T_{n,\alpha} = t^0_{c,\tau}(\alpha)$ and $\tilde{l}_n = l_n(\tilde{b}_n, \tilde{h}_n)$; here the values $\tilde{b}_n, \tilde{h}_n$ correspond to $p, q, R_{n,2}$ and to changed $\tilde{R}_{n,1} = R_{n,1}(1 - b_n^{-3})$. Then $\alpha(\psi_{n,\alpha}) = \alpha + o(1)$,

$$\beta_n(\psi_{n,\alpha}, V_n) \leq \beta_n(\alpha, P_{\pi_n}) + o(1).$$

It means that the sequence of the tests $\psi_{n,\alpha}$ is asymptotically minimax.

2.3 Degenerate case: $\tau \in [1/2, 1]$

Let $nh_n \to \infty$. Put

$$\lambda_n = nh_n\Phi(b_n - T_n)$$

and assume without loss of generality that $\lambda_n \to \lambda \in [0, \infty]$.

Then (Ingster, 1997, Theorem 3)

1. If $\lambda = 0$, then $\beta_n(\alpha, P_{\pi_n}) \to 1 - \alpha$.

2. If $\lambda = \infty$, then $\beta_n(\alpha, P_{\pi_n}) \to 0$.

3. Let $\lambda \in (0, \infty)$. Then $l_n \to -\lambda$ under null hypothesis H_0 and

$$\beta_n(\alpha, P_{\pi_n}) \to (1 - \alpha)\exp(-\lambda).$$

For considerable problem we have

Theorem 3

1. Lower bounds. Let $\tau = 1$.

a) Let $nh_n \to \infty$; also $nh_n/\log n \to \infty$ or nh_n is an integer. Then

$$\beta_n(\alpha, V_n) \geq \beta_n(\alpha, P_{\pi_n}) + o(1) = (1 - \alpha)\exp(-\lambda_n) + o(1).$$

b) Let $nh_n = O(1)$ and nh_n is an integer. Then

$$\beta_n(\alpha, V_n) \geq (1 - \alpha)(\Phi(\sqrt{2\log n} - b_n))^{nh_n} + o(1).$$

2. Upper bounds. Let us consider the sequences of the tests

$$\psi_{n,\alpha} = \mathbf{1}_{\{\max_i |x_i| > H_{n,\alpha}\}}. \tag{2.23}$$

where $H_{n,\alpha} = \sqrt{2\log n} + o(1)$ is such sequence that $(1 - 2\Phi(-H_{n,\alpha}))^n = 1 - \alpha$ (this implies $\alpha(\psi_{n,\alpha}) = \alpha$).

a) Let $\tau = 1$, $nh_n \to \infty$. Then

$$\beta_n(\psi_{n,\alpha}, V_n) \le \beta_n(\alpha, P_{\pi_n}) + o(1) = (1-\alpha)\exp(-\lambda_n) + o(1).$$

b) Let $\tau = 1$, $nh_n = O(1)$. Then

$$\beta_n(\psi_{n,\alpha}, V_n) \le (1-\alpha)(\Phi(\sqrt{2\log n} - b_n))^{nh_n} + o(1).$$

3. Consistent properties. Let one of the following assumptions hold:
(i): $\tau < 1$; (ii): $\tau = 1$ and $\lambda_n \to \infty$; (iii): $\tau \in (1, \infty)$ and $c_n \to \infty$.
Then for the tests (2.23) one has: $\beta_n(\psi_{n,\alpha}, V_n) \to 0$ for any $\alpha \in (0,1)$.

Remark 2.1 *1. The nn. 1,2 of the Theorem mean that for $\tau = 1$ the tests (2.23) are asymptotically minimax, if $nh_n/\log n \to \infty$ or nh_n is an integer.*

2. The n. 3 of the Theorem means that for $\tau < 2$ the tests (2.23) are asymptotically consistent in minimax sense, if it is possible to construct asymptotically consistent sequence of tests.

3 Proof of the Theorems : lower bounds

To obtain the lower bounds we can assume: if $\tau \ge 2$, then $u_n \asymp 1$; if $\tau \in (1,2)$, then $c_n \asymp 1$. Note that these relations imply $b_n^2 = O(\log n)$ and $nh_n/\log n \to \infty$ (see Ingster, 1997). For $\tau = 1$ assume $\lambda_n \asymp 1$ and $nh_n/\log n \to \infty$ or $k_n = nh_n$ is an integer. Note that $b_n^2 = O(\log n)$ also in this case.

For an integer $k = k_n$ let us consider the sets $\tilde{V}_n = V_{n,k_n,b_n}$:

$$\tilde{V}_n = \{v \in R^n : \ v = b_n(t_1, \dots, t_n), t_i \in \{-1, 0, 1\}, i = 1, \dots, n, \quad \sum_{i=1}^n t_i = k_n\}.$$

It is clear that $\tilde{V}_n \subset V_n$ for $k_n = nh_n$ which implies the inequality

$$\beta_n(\alpha, V_n) \ge \beta_n(\alpha, \tilde{V}_n). \tag{3.24}$$

It was shown in Ingster, 1997, Theorem 4, that if $k_n \to \infty$, then

$$\beta_n(\alpha, \tilde{V}_n) = \beta_n(\alpha, P_{\pi_n}) + o(1) \tag{3.25}$$

and if $k_n = O(1)$, then

$$\beta_n(\alpha, \tilde{V}_n) = (1-\alpha)(\Phi(\sqrt{2\log n} - b_n))^{k_n} + o(1)$$

which imply the lower bounds of the Theorems 1-3 for an integer $k_n = nh_n$.

Let nh_n is not integer, $nh_n/\log n \to \infty$. By making $R_{2,n}$ smaller: $R_{2,n}^* = R_{2,n}(1 - \delta_{n,0})$, we can get an integer $k_n = nh_n^*$; $nh_n > k_n > nh_n - 1$ with

$$h_n^* = h_n(1 - \delta_{n,1}), \ b_n^* = b_n(1 + \delta_{n,2}), \quad \delta_{n,0} \asymp \delta_{n,1} \asymp \delta_{n,2} = O((nh_n)^{-1}) \to 0$$

which implies the decrease $\beta_n(\alpha, \tilde{V}_n)$ only.

Analogous to the proof of the Theorem 4 in Ingster, 1997, let us observe the following *asymptotical continuity property* of the values $\beta_n(\alpha, P_{\pi_n})$ which follows from Theorems 1 - 3 in this paper.

Let

$$h_n^* = h_n((1-\delta_{n,1}),\ b_n^* = b_n((1-\delta_{n,2}),\ \delta_{n,1} = o(1),\ \delta_{n,2} = o(b_n^{-2}),\ b_n \to 0.$$

Then for any $\alpha \in (0,1)$

$$\beta_n(\alpha, P_{\pi^n(b_n^*, h_n^*)}) = \beta_n(\alpha, P_{\pi^n(b_n, h_n)}) + o(1).$$

Using this continuity property and (3.24), (3.25) one has the lower bounds from the Theorems, if $b_n^2 = O(nh_n)$. The last relation holds by $nh_n/\log n \to \infty$. The lower bounds are proved.

4 General remarks to the proofs of the upper bounds

First, note that to prove the upper bounds of the Theorems 1-3 we can assume $b_n \to \infty$ by the results for $b_n = O(1)$ follow from Ingster, 1993. Also we can assume that $u_n \asymp 1,\ c_n \asymp 1,\ \lambda_n \asymp 1$ respectively (see the proof of the Theorem 1 in Ingster, 1997). These imply the relation (1.13).

Also remind some technical relations from Ingster, 1997. In the cases $b_n \to \infty$ and $u_n = O(1)$ or $c_n = O(1)$ or $\lambda_n = O(1)$ one has

$$T_n \to \infty, \quad h_n \to 0, \quad h_n \sim 2\exp(\frac{b_n^2}{2} - T_n b_n). \tag{4.26}$$

Let us put

$$z_n(x) = h_n \xi(x, b_n), \quad w_n(x) = \log(1 + z_n(x))$$

and note that for any $\delta > 0$ and $y = |x| - T_n > \delta$ the following representations hold

$$z_n(x) = e^{b_n y}(1 + o(1)), \quad w_n(x) = \log(1 + e^{b_n y}(1 + o(1))), \quad \sup_{|x| \le T_n - \delta} |w_n(x)| \to 0. \tag{4.27}$$

Here and later we use the well know relations: as $x \to \infty,\ \delta = o(x)$

$$\Phi(-x) \sim \frac{1}{x\sqrt{2\pi}} e^{-x^2/2},\ \Phi(-x+\delta) \sim \Phi(-x)e^{x\delta} \tag{4.28}$$

Define the function $T_n(t)$ by

$$w_n(T_n(t)) = t, \quad T_n(t) \ge 0. \tag{4.29}$$

By $w_n(|x|)$ is increasing in $|x|$, it follows from (4.27), (2.15) that for any $t > 0$

$$T_n(t) = T_n + \frac{\log(e^t - 1)}{b_n} + o\left(\frac{1}{b_n}\right),\ \{x : w_n(x) < t\} = (-T_n(t), T_n(t)). \tag{4.30}$$

Remind (Ingster, 1997) that under the assumptions $u_n \asymp 1$ and $\tau \geq 2$ one has

$$n\Phi(-T_n) \to 0;\ b_n^2 \sim \frac{\log n}{2(\tau-1)},\ \log nh_n \sim \frac{2\tau-3}{4(\tau-1)}\log n; \tag{4.31}$$

under the assumptions $c_n \asymp 1$ and $\tau \in (1,2)$ one has

$$c_n = 2n\Phi(-T_n) \asymp 1;\ b_n^2 \sim \frac{2\log n}{\tau^2},\ \log nh_n \sim (1-\tau^{-1})^2 \log n; \tag{4.32}$$

and under the assumptions $\lambda_n \asymp 1$ and $\tau = 1$ one has

$$c_n = 2n\Phi(-T_n) = o(\lambda_n) \to 0;\ b_n^2 \sim 2\log n,\ \log nh_n = o(1). \tag{4.33}$$

Note that by the asymptotical normality of $\tilde{l}_n$ (Ingster, 1997, Theorem 1) and continuity property we have the relations:

$$\alpha + o(1) \leq \alpha(\psi_{n,\alpha}) \leq \begin{cases} \alpha + o(1), & \text{if } n\Phi(H_n) \to 0, \\ \alpha + 2\delta + o(1), & \text{if } n\Phi(H_n) \leq \delta. \end{cases}$$

By these relations to proof the upper bounds in the Theorems 1, 2 it is enough for any $\delta > 0$ to estimate the second kind error probabilities $\beta_n(\psi_{n,\alpha}, V_n)$ for such tests $\psi_{n,\alpha} = \psi_{n,\alpha,\delta}$ that $n\Phi(H_n) \leq \delta$.

To estimate $\beta_n(\psi_{n,\alpha}, V_n)$ we use the inequality:

$$\begin{aligned}\beta_n(\psi_{n,\alpha}, v) = P_v(\tilde{l}_n \leq T_{n,\alpha}, \max_i |x_i| \leq H_n) \leq \\ \min\{P_v(\max_i |x_i| \leq H_n),\ P_v(\tilde{l}_n \leq T_{n,\alpha})\}.\end{aligned} \tag{4.34}$$

Let us consider the probabilities $P_v^1 = P_v(\max_i |x_i| \leq H_n)$. Note that for any $\delta > 0$ there is such $B > 0$ that $P_v^1 < \delta$ if any of the following two relations hold:

$$\max_i |v_i| > H_n + B, \quad \sum_i \Phi(|v_i| - H_n) > B. \tag{4.35}$$

In fact,

$$P_v^1 = \prod_i (\Phi(H_n - |v_i|) - \Phi(-H_n - |v_i|)) < \Phi(H_n - \max_i |v_i|) \leq \Phi(-B), \tag{4.36}$$

$$\begin{aligned} P_v^1 &= \prod_i (1 - \Phi(-H_n + |v_i|) - \Phi(-H_n - |v_i|)) < \\ &< \prod_i (1 - \Phi(-H_n + |v_i|)) < \exp(-\sum_i \Phi(-H_n + |v_i|)) \leq \exp(-B). \end{aligned} \tag{4.37}$$

By the relations (4.34) - (4.35) it is enough to estimate the probabilities $P_v(\tilde{l}_n \leq T_{n,\alpha})$ (or the probabilities $P_v(\hat{l}_n \leq T_{n,\alpha})$ where $\hat{l}_n = \tilde{l}_n \mathbf{1}_{\{\max_i |x_i| \leq \hat{H}_n\}}$ is $\hat{H}_n$-truncated $\tilde{l}_n$) for $v \in V_n \cap V_B^n$ where

$$V_B^n = \{v \in R^n :\ \max_i |v_i| \leq B + H_n,\ \sum_i \Phi(|v_i| - H_n) \leq B\} \tag{4.38}$$

for large enough $B>0$ and $\hat{H}_n \geq H_n > 0$.

Introduce the values Q_n. If $\tau > 2$, then put $Q_n = T_n - b_n + d_n$, if $\tau > 2$, then put$Q_n = T_n/2$ (this corresponds to $d_n = b_n - T_n/2$). Here $d_n \to \infty$ is such sequence that $d_n = o(b_n)$ (using (4.26), (4.31) one can check that these relations hold for $\tau = 2$ also). If $\tau \in (1,2)$, then put $Q_n = \eta b_n,\ \eta \in (1,\tau)$.

Let us consider the sets

$$
\begin{aligned}
V_{n,1} &= \{v \in V_n \cap V_B^n : \ \max_i |v_i| \leq Q_n\}, \\
V_{n,2} &= \{v \in V_n \cap V_B^n : \ \max_i |v_i| > Q_n\}.
\end{aligned}
$$

Choose the values H_n. Let $\tau > 3$. Put $H_n = \sqrt{2\log n}$. By (4.31) one has $Q_n - H_n \to \infty$ which implies $V_{n,2} = \emptyset$ in this case.

Let$\tau \in [2,3]$. Using (4.31), (4.28) put $H_n = T_n - t_n/b_n$ with such $t_n \to \infty$ that $n\Phi(H_n) \to 0$ and if $\tau > 2$, then $t_n = o(b_n)$ and if $\tau = 2$, then $t_n = o(b_n/d_n),\ 2d_n = 2b_n - T_n$.

Let $\tau \in (1,2)$. Using (4.32) , (4.28) put $H_n = T_n + t(\delta)/b_n$ for such $t(\delta)$ that $n\Phi(H_n) < \delta$.

Denote $m_n(v)$ the number of coordinates $|v_i| > Q_n$.

Lemma 4.1 *For any $\tau \in (1,3]$ one has*

$$\sup_{v \in V_{n,2}} m_n(v) = o(nh_n/b_n^3).$$

Proof of the Lemma. Assume that there is such sequence $v = v^{(n)} \in V_{n,2}$ and $A > 0$ that $m_n(v^{(n)}) > Anh_n b_n^{-3}$.

First, let $\tau \geq 2$. By $H_n < T_n$ using (4.28) one has for large enough n and some constant $B > 0$:

$$
\begin{aligned}
\sum_i \Phi(|v_i| - H_n) &\geq m_n(v)\Phi(d_n - b_n) \geq Bb_n^{-4} nh_n \exp(-(b_n - d_n)^2/2) \\
&\gg nh_n^2 e^{b_n^2} \Phi(T_n - 2b_n) = 2u_n^2 \asymp 1
\end{aligned}
$$

by if $T_n - 2b_n > -B$ for any $B < \infty$ (it corresponds to $\tau > 2$), then using (4.26) one has

$$
\begin{aligned}
& b_n^4 h_n e^{b_n^2} \Phi(T_n - 2b_n) \asymp b_n^4 h_n \exp(b_n^2 + (b_n - d_n)^2/2) \asymp \\
& \asymp b_n^4 \exp(-b_n(d_n - B) - d_n^2/2) \to 0
\end{aligned}
$$

and if $T_n - 2b_n = -d_n/2 \to \infty$, (it corresponds to $\tau = 2$), then

$$b_n^4 h_n e^{b_n^2} \Phi(-d_n/2) \asymp b_n^4 d_n^{-1} \exp(-(b_n - d_n)d_n/2) \to 0.$$

Let $\tau \in (1,2)$. By $n\Phi(-H_n) \leq \delta$ one has for some $b > 0$:

$$
\begin{aligned}
& \sum_i \Phi(|v_i| - H_n) \geq bm_n(v)\Phi(\eta b_n - T_n) \asymp \\
& \asymp b_n^4 nh_n \exp(-(T_n - \eta b_n)^2/2) \gg nT_n^{-1} e^{-T_n^2/2} \asymp n\Phi(-T_n) \asymp 1
\end{aligned}
$$

by for $\eta \in (1,\tau)$

$$h_n^{-1}\exp(-T_n^2/2) \asymp \exp(-(T_n-b_n)^2/2) = o(\exp(-(T_n-\eta b_n)^2/2)).$$

These relations contradict to the assumption $v = v^{(n)} \in V_B^n$ which proves the Lemma.

Put $\tilde v = (\tilde v_1, ..., \tilde v_n)$ where $\tilde v_i = v_i \mathbf{1}_{\{|v_i| \le Q_n\}}$. It is clear that $\tilde v \in V_B^n$. Also if $v \in V_{n,2}$, then using (1.13) and the Lemma 4.1 one has for large enough n :

$$\tilde v \in \tilde V_n = V_n^{p,q}(\tilde R_{1,n}, R_{2,n}), \quad \tilde R_{1,n} = R_{1,n}(1-b_n^{-3})$$

by

$$\sum_i |\tilde v_i|^p \ge \sum_i |v_i|^p - m_n(v)(H_n+B)^p \ge R_{1,n}^p - o(nh_n b_n^{p-3}) \ge \tilde R_{1,n}.$$

Note that the set $X_{n,\alpha} = \{x \in R^n : \tilde l_n(x) \le T_{n,\alpha}\}$ is convex and symmetrical on any coordinate x_i. Appling the Anderson's lemma (see Ibragimov and Hasminskii, 1981) to coordinate cross-sections of the set $X_{n,\alpha}$ we get the inequality $P_{n,v}(X_{n,\alpha} \le P_{n,\tilde v}(X_{n,\alpha}$ which implies for large enough n

$$\beta_n(\psi_{n,\alpha}, V_n) \le \min\{\delta, \sup_{v \in \tilde V_{n,Q_n}} P_{n,v}(\tilde l_n \le T_{n,\alpha}\} \tag{4.39}$$

where

$$\tilde V_{n,Q_n} = \{v \in \tilde V_n \cap V_B^n : \max_i |v_i| \le Q_n\}.$$

By the (4.39) later we deal with the estimation of the probabilities $P_{n,v}(\tilde l_n \le T_{n,\alpha})$ (or the values $P_{n,v}(\hat l_n \le T_{n,\alpha})$ for $\hat H_n$-truncated statistics l_n with $\hat H_n \ge H_n$) for $v \in \tilde V_{n,Q_n}$. To simplicity we will omit often later the wave in the notations $\tilde l_n$, $\tilde V_n$ and so on.

5 Upper bounds for $\tau \ge 2$

For $\hat H_n$-truncated statistics l_n with $\hat H_n = T_n - t_n/b_n \ge H_n$ ($\hat H_n = H_n$, if $\tau < 3$) and for $v \in V_{n,Q_n}$ using (4.27), (4.30) one can assume

$$\hat w_n = \sup_x |\hat w_n(x)| = w_n(\hat H_n) \to 0 \tag{5.40}$$

where $\hat w_n(x) = w_n(x)\mathbf{1}_{\{|x| \le \hat H_n\}}$ are the $\hat H_n$-truncated items in the sum for $\hat l_n = \sum_i \hat w_n(x_i)$. This implies the asymptotical (0,1)-normality of the statistics $(\hat l_n - \hat m_n)/(\hat D_n)^{1/2}$ under $P_{n,v^{(n)}}$-probability for any sequence $v^{(n)} \in R^n$. Here $\hat m_n = \hat m_n(v^{(n)})$ and $\hat D_n = \hat D_n(v^{(n)})$ are the $P_{n,v^{(n)}}$-mean and $P_{n,v^{(n)}}$-variance of $\hat l_n$. It is to obtain the relations:

$$\Delta D_n = \hat D_n(v^{(n)}) - \hat D_n(0) = o(\Delta m_n), \tag{5.41}$$

$$\Delta m_n = \hat m_n(v^{(n)}) - \hat m_n(0) \ge 2nh_n^2 \sinh^2(b_n^2/2)\Phi(\hat H_n - 2b_n). \tag{5.42}$$

In fact, the relations (5.40), (5.41) and the Central Limit Theorem (Petrov, 1981) imply that if $\Delta m_n \to \infty$, then $\hat{l}_n \to \infty$ and if $\Delta m_n = O(1)$, then $\hat{l}_n$ is $(\Delta m_n - u_n^2/2,\ u_n^2)$-asymptotically Gaussian under $P_{n,v^{(n)}}$-probability; $\Phi(T_n - 2b_n) \sim \Phi(\hat{H}_n - b_n)$ by the choosing $\hat{H}_n$ and the right-hand side in (5.42) is $u_n^2(1 + o(1)$. We have:

$$P_{n,v^{(n)}}(\hat{l}_n \le T_{n,\alpha}) = \Phi\left(t_\alpha - \frac{\Delta m_n}{u_n}\right) + o(1) \le \Phi\left(t_\alpha - u_n\right) + o(1).$$

To prove (5.41), (5.42) put

$$\begin{aligned} \mu_n(v) &= \int_{|x|<\hat{H}_n} z_n(x)\left(d\Phi(x+v) - d\Phi(x)\right) = h_n \int_{|x|<\hat{H}_n} \xi(x,b_n)\xi(x,v)d\Phi(x), \\ \sigma_n^2(v) &= \int_{|x|<\hat{H}_n} z_n^2(x)\left(d\Phi(x+v) - d\Phi(x)\right) = h_n \int_{|x|<\hat{H}_n} \xi^2(x,b_n)\xi(x,v)d\Phi(x). \end{aligned}$$

By (5.40) one has

$$\begin{aligned} \Delta m_n &= \sum_i \mu_n(v_i^{(n)}) + O\left(\sum_i \sigma_n^2(v_i^{(n)})\right), \qquad (5.43) \\ \Delta D_n &= O\left(\sum_i \sigma_n^2(v_i^{(n)})\right) + o(1). \end{aligned}$$

Using the relations: if $b \to \infty,\ H \to \infty,\ H - |v| \to \infty$, then

$$\int_{|x|<H} \xi(x,b)\xi(x,v)d\Phi(x) \sim 2\sinh^2(bv/2)\Phi(H - b - |v|), \qquad (5.44)$$

$$\int_{|x|<H} \xi^2(x,b_n)\xi(x,v)d\Phi(x) \le e^{b^2}\sinh^2(bv)\Phi(H - 2b - |v|)(1 + o(1)) \qquad (5.45)$$

one can check that

$$\sum_i \sigma_n^2(v_i^{(n)}) = o\left(\sum_i \mu_n(v_i^{(n)})\right)$$

and by (5.43) this implies (5.41). Also these imply that (5.40) follows from the inequality

$$\inf_{v \in V_{n,Q_n}} \sim_i 2\sinh^2(b_n v_i/2)\Phi(\hat{H}_n - b_n - |v_i|) \ge 2nh_n^2 \sinh^2(b_n^2/2)\Phi(\hat{H}_n - 2b_n) \qquad (5.46)$$

which follows from the

Lemma 5.1 *Denote* $\Xi_n = \Xi_n^{p,q}(R_{1,n},\ R_{2,n},\ Q_n)$ *the set of the collections* $\bar{r} = (r_i, ..., r_n)$ *of the probability measures on* $(R^1, \mathcal{B})$*:*

$$\Xi_n = \{\bar{r}:\ \sum_i \int |u|^p r_i(du) \ge R_{1,n}^p,\ \sum_i \int |u|^q r_i(du) \le R_{2,n}^q;\ r_i([-Q_n, Q_n]) = 1\}.$$

Let the values $h_n,\ b_n$ *are defined by (1.13),* $p > q,\ R_{1,n} \le R_{2,n},\ b_n \to \infty$ *and for some* $d_n,\ \delta \in (0,1)$ *the following constraints hold:*

$$b_n < Q_n \le (1-\delta)H_n + \sqrt{2H_n b_n - 3b_n^2},\ H_n > 2b_n - d_n;\ d_n \to \infty,\ d_n = o(b_n). \qquad (5.47)$$

Put

$$\Phi_n(r) = \int \phi_n(v) r(dv),\ \ F_n(\bar r) = \sum_i \Phi_n(r_i)$$

where

$$\phi(v) = \phi_n(v) = 2\sinh^2(b_n v/2)\Phi(H_n - b_n - |v|). \tag{5.48}$$

Then for large enough n the following equality holds:

$$\inf_{\bar r \in \Xi_n} F_n(\bar r) = F_n(\bar r_n^*) = n\Phi(\pi_n) = nh_n\phi(b_n) \tag{5.49}$$

where

$$\bar r_n^* = (\pi_n, ..., \pi_n),\ \ \pi_n = \pi(h_n, b_n) = (1-h_n)\delta_0 + \frac{h_n}{2}(\delta_{b_n} + \delta_{-b_n}).$$

In particularly, for large enough n one has

$$\inf_{v \in V_{n,Q_n}} \sum_i \phi(v_i) \geq nh_n\phi(b_n). \tag{5.50}$$

Proof of the Lemma is given in sec. 8.

Note that the analogous to (5.49) equality for $\tau > 2$ and for $\phi(v) = 2\sinh^2(b_n v/2)$ (which formally corresponds to (5.47), (5.48) with $H_n = Q_n = \infty$) follows from Ingster, 1990, 1993.

The Theorem 1 is proved.

6 Upper bounds for $\tau \in (1,2)$

6.1 The outline of the proof

By (4.39) it is enough to show that for any sequence $v^{(n)} \in V_{n,Q_n}$ one has

$$\limsup(P_{n,v^{(n)}}\left(l_n \leq T^0_{c,\tau}(\alpha) - \beta_n(\alpha, P_{\pi^n})\right)) \leq 0. \tag{6.51}$$

Put

$$L_n^0(t) = -2\Phi(-T_n(t)),\ \ L_n(t) = L_n^0(t) + \Delta L_n(t)$$

where the values $T_n(t)$ are defined by (4.29) and

$$\begin{aligned} \Delta L_n(t) &= -\sum_i \left(P_{1,v_i^{(n)}}(x:\ w_n(x) > t) - P_{1,0}(x:\ w_n(x) > t)\right) = \\ &= -\sum_i \left(\Phi(-T_n(t) + |v_i^{(n)}|) + \Phi(-T_n(t) - |v_i^{(n)}|) - 2\Phi(-T_n(t))\right). \end{aligned} \tag{6.52}$$

It is clear that $\Delta L_n(t) \leq 0$. Also one can easily check that $\Delta L_n(t) \to 0$ as $t \to \infty$ and for large enough n and any $t_0 > 0$ the functions $\Delta L_n(t)$ are bounded from below on $t \geq t_0$ and their derivatives are positive and bounded on $t \geq t_0$. Thus the set of the functions $\{\Delta L_n(t)\}$ are equicontinuous in $C[t_0,\infty)$ for any $t_0>0$ and we can assume that for every $t>0$

$$\Delta L_n(t)\to\Delta L(t) \tag{6.53}$$

where $\Delta L(t)$ is continuous on $(0,\infty)$ and the properties of Levi spectrum hold:

$$\Delta L(t)\le 0,\ \ \Delta L(t)\to 0 \text{ as } t\to\infty,\ \ \Delta L(t) \text{ nonincrease on } t>0$$

and (we will show below) for any $\delta>0$

$$\int_{+0}^{\delta} t^2 d\Delta L(t)<\infty. \tag{6.54}$$

Put

$$\mu_n=\sum_i \mu_{n,i}=2h_n\sum_i \sinh^2(b_n v^{(n)}/2)\Phi(T_n-b_n-|v_i^{(n)}|). \tag{6.55}$$

The outline of the proof is following. We show that if $\mu_n\to\infty$, then $l_n\to\infty$ under $P_{n,v^{(n)}}$-probability. Let $\mu_n=O(1)$. Then, by pass to subsequence, we can find such constant $b\ge 0$ that the limit $P_{n,v^{(n)}}$-distribution of the statistics l_n is the distribution of the sum $\zeta=\zeta^0+\Delta\zeta$. Here ζ^0 and $\Delta\zeta$ are an independent infinite divisible random variables. The variable ζ^0 corresponds to the limit P_0-distribution of the statistics l_n and is described in sec. 2.2. The variable $\Delta\zeta$ has not Gaussian component also and is supported on the half-line (b,∞) (it means that the characteristic function $\phi_{\Delta\zeta}(z)$ of the $\Delta\zeta$ is of the form:

$$\log\phi_{\Delta\zeta}(z)=ibz+\int_0^\infty(\exp(izt)-1)d\Delta L(t). \tag{6.56}$$

Then we obtain the relation

$$\Delta L(t)=L^{\Delta}(t)+\Lambda(t),\ \ t>0 \tag{6.57}$$

where $L^{\Delta}(t)$ is the Levi spectrum which has described by (2.22) and $\Lambda(t),\ t>0$ is the Levi spectrum also:

$$\Lambda(t)\le 0,\ \ d\Lambda(t)/dt\ge 0,\ \ \Lambda(t)\to 0 \text{ as } t\to\infty$$

and the relation (6.54) implies the analogous relation for $\Lambda(t)$: for any $\delta>0$

$$\int_{+0}^{\delta} t^2 d\Lambda(t)<\infty. \tag{6.58}$$

These relations imply the equality

$$\zeta=\zeta_0+\zeta^\delta+\eta=\zeta^1+\eta \tag{6.59}$$

where the random variables $\zeta_0,\ \zeta^\delta,\ \eta$ are independent, $\zeta^1=\zeta_0+\zeta^\delta$ corresponds to the limit distribution of l_n under the Bayesian alternatives P_{π^n} and $\eta\ge b\ge 0$ with the probability 1. By (6.59) and by the Theorem 2 in Ingster, 1997 one has

$$P_{n,v^{(n)}}(l_n\le T^0_{c,\tau}(\alpha))\le P(\zeta^1\le T^0_{c,\tau}(\alpha))+o(1)=\beta_n(\alpha,P_{\pi^n})+o(1)$$

which imply (6.51) and the upper bounds of the Theorem 2.

6.2 The study of the limit distribution of l_n

To realize the outline above introduce the functions

$$\mu_n(t) = \sum_i \mu_{n,i}(t) = 2h_n \sum_i \int |x| < T_n(t)\xi(x, b_n)\xi(x, v_i^{(n)})d\Phi(x),$$
$$\sigma_n^2(t) = \sum_i \sigma_{n,i}^2(t) = 2h_n^2 \sum_i \int |x| < T_n(t)\xi^2(x, b_n)\xi(x, v_i^{(n)})d\Phi(x)$$

and put

$$\sigma_n^2 = \sum_i \sigma_{n,i}^2 = 2h_n^2 e^{b_n^2} \sum_i \sinh^2(b_n v^{(n)})\Phi(T_n - 2b_n - |v_i^{(n)}|).$$

Using the relations (5.44), (5.45) and (4.28) one can check that

$$\mu_{n,i}(t) = \begin{cases} \mu_{n,i}, & \text{if } |v_i^{(n)}| \le T_n - b_n, \\ \mu_{n,i}(e^t - 1)^{|v_i^{(n)}|/b_n + 1 - \tau_n}, & \text{if } |v_i^{(n)}| > T_n - b_n \end{cases} \tag{6.60}$$

and for large enough n

$$\sigma_{n,i}^2(t) \le B\sigma_{n,i}^2(e^t - 1)^{|v_i^{(n)}|/b_n + 2 - \tau_n}$$

(here and later we denote B some positive constants, may be, different, which do not depend on n). Also one can check that

$$\sigma_{n,i}^2 = \begin{cases} o(\mu_{n,i}(t)), & \text{if } |v_i^{(n)}| \le T_n - b_n, \\ O(\mu_{n,i}(t)), & \text{if } |v_i^{(n)}| > T_n - b_n \end{cases}$$

which imply for large enough n and small enough $t > 0$ the relation

$$\sigma_n^2(t) \le B\mu_n(t)(t + o(1)). \tag{6.61}$$

Let us estimate the differences of the means and of the variances

$$\Delta E_n = E_{n,v^{(n)}} l_n - E_{n,0} l_n, \ \Delta D_n = D_{n,v^{(n)}} l_n - D_{n,0} l_n.$$

It is clear that

$$\Delta E_n = \sum_i \int w_n(x)\xi(x, v_i^{(n)})d\Phi(x). \tag{6.62}$$

Denote

$$\tilde{D}_{n,v^{(n)}} = \sum_i \int w_n^2(x)d\Phi(x + v_i^{(n)}), \ \tilde{D}_{n,0} = \sum_i \int w_n^2(x)d\Phi(x).$$

By $E_{n,0} l_n = nE_{1,0} w_n(x) = O(1)$ it is clear that $\Delta D_n \le \Delta\tilde{D}_n + o(1)$ and

$$\Delta\tilde{D}_n = \tilde{D}_{n,v^{(n)}} - \tilde{D}_{n,0} = \sum_i \int w_n^2(x)\xi(x,)d\Phi(x). \tag{6.63}$$

For any $t > 0$ put

$$\Delta E_n = \Delta E_n^+(t) + \Delta E_n^-(t),\ \ \Delta\tilde{D}_n = \Delta\tilde{D}_n^+(t) + \Delta\tilde{D}_n^-(t)$$

where the items with the index + and − correspond to the sums of the integrals in (6.62) and (6.63) over the sets $|x| > T_n(t)$ and $|x| < T_n(t)$.

Let us obtain the relations

$$\begin{aligned}\Delta E_n^+(t) &= O(1), \quad \Delta\tilde{D}_n^+(t) = O(1), &(6.64)\\ \Delta E_n^-(t) &= \mu_n(t)(1 + O(t + o(1))), &(6.65)\\ \Delta\tilde{D}_n^-(t) &= O(\mu_n(t)). &(6.66)\end{aligned}$$

To obtain (6.64), note the equality:

$$2\xi(x,v)d\Phi(x) = d\Phi(x + |v|) + d\Phi(x - |v|) - 2d\Phi(x) \tag{6.67}$$

which implies the equality analogous to (6.52):

$$\Delta E_n^+(t) = A_n(t) + B_n(t) - 2C_n(t) \tag{6.68}$$

Here by (4.27), (4.30), (4.28) one has

$$\begin{aligned}A_n(t) &\sim \sum_i \left(\frac{T_n - |v_i^{(n)}|}{b_n}\right) \Phi(|v_i^{(n)}| - T_n) \int_{u>e^t-1} u^{-(\tau+1-|v_i^{(n)}|/b_n)} \log(1+u)du,\\ B_n(t) &\sim \sum_i \left(\frac{T_n + |v_i^{(n)}|}{b_n}\right) \Phi(-|v_i^{(n)}| - T_n) \int_{u>e^t-1} u^{-(\tau+1+|v_i^{(n)}|/b_n)} \log(1+u)du,\\ C_n(t) &\sim c\tau \int_{u>e^t-1} u^{-(\tau+1)} \log(1+u)du\end{aligned}$$

where the boundedness of the values $A_n(t),\ B_n(t),\ C_n(t)$ follows from the boundedness of the integrals above (for $|v_i^{(n)}|/b_n \le \eta < \tau$ by the definition of the set V_{n,Q_n}) and from the boundedness of the sums $\sum_i \Phi(-T_n + |v_i^{(n)}|)$ by (6.52), (6.53). The boundedness of the values $\Delta\tilde{D}_n^+(t)$ follows from the analogous estimations.

The relations (6.65), (6.66) follow from (6.61) and the relation $\log(1+z) = z + O(z^2)$ for $-1 + B^{-1} < z < B$.

Assume $\mu_n \to \infty$. Then by the boundedness of the values $E_{n,0}l_n,\ D_{n,0}l_n$ and from the relations (6.60), (6.62) - (6.66) one has

$$E_{n,v^{(n)}}l_n \asymp \mu_n \to \infty,\ D_{n,v^{(n)}}l_n \le BE_{n,v^{(n)}}l_n$$

and using the Chebyshev inequality one has $l_n \to \infty$ under $P_{n,v^{(n)}}$-distribution.

Assume $\mu_n = O(1)$. Let us obtain the relations:

$$\lim_{n\to\infty} \sum_i P_{1,v_i^{(n)}}\{x:\ w_n(x) > t\} = --(L_0(t) + \Delta L(t)) \text{ for any } t > 0; \tag{6.69}$$

$$\lim_{t\to 0}\lim_{n\to\infty} \Delta\tilde{D}_n^-(t) = 0, \tag{6.70}$$

$$\liminf_{t\to 0}\liminf_{n\to\infty} \Delta E_n^-(t) \ge 0, \tag{6.71}$$

$$\liminf_{t_2\to 0}\liminf_{t_1\to 0}\liminf_{n\to\infty} (\Delta E_n^-(t_2) - \Delta E_n^-(t_1)) = 0. \tag{6.72}$$

By pass to subsequence in (6.71) one can assume that there exist the limit $b \geq 0$ of the left-hand side (6.71). Then the relations (6.69) - (6.72) imply that the limit $P_{n,v^{(n)}}$-distribution of the statistics l_n is the distribution of the sum $\zeta = \zeta^0 + \Delta\zeta$. In fact (see Petrov, 1981), (6.69) implies the necessary form of the Levi spectrums, (6.70) and analogous relation (4.7) in Ingster, 1997 imply that the limit $P_{n,v^{(n)}}$-distribution of l_n has no Gaussian component and (6.54) holds.

Let us consider the component γ. One has the equality: $\gamma_\zeta = \gamma^0 + \Delta\gamma$ where for any $t > 0$

$$\Delta\gamma = \lim_{n\to\infty} \sum_i \int_{|w_n|<t} w_n(x)\xi(x, v_i^{(n)})d\Phi(x) - \int_{+0}^{t} \frac{x^3}{1+x^2} d\Delta L(x) + \int_t^\infty \frac{x}{1+x^2} d\Delta L(x). \tag{6.73}$$

Let $t \to 0$ in (6.73). Then the relations (6.71), (6.54) imply that the limit of the first item in right-hand side is nonnegative and the limit of the second item is 0. Also by (6.69), (6.72) one can pass to the limit in the third item by :

$$\lim_{t\to 0} \int_0^t \frac{x}{1+x^2} d\Delta L(x) = \lim_{t_2\to 0} \lim_{t_1\to 0} \int_{t_1}^{t_2} x d\Delta L(x) =$$
$$\lim_{t_2\to 0} \lim_{t_1\to 0} \lim_{n\to\infty} (\Delta E_n^-(t_2) - \Delta E_n^-(t_1)) = 0.$$

These imply the equality

$$\Delta\gamma = b + \int_{+0}^{\infty} \frac{x}{1+x^2} d\Delta L(x)$$

which is equivalent to (6.56).

6.3 Proof of the (6.69) - (6.72)

The relation (6.69) follows from (6.53) and from the analogous relation (4.2) in Ingster, 1997 for the limit $P_{n,0}$-spectrum L_0. The relations (6.60), (6.61) imply the estimators

$$\Delta\tilde{D}^-(t) \leq B\sigma_n^2(t) \leq B\mu_n(t + o(1))$$

which imply (6.70). The relation (6.71) follows from (6.65), (6.66), (6.60).

To prove (6.72) let us consider the equality

$$\Delta E_n^-(t_2) - \Delta E_n^-(t_1) = \Sigma_{n,1}(t_1, t_2) + \Sigma_{n,2}(t_1, t_2).$$

Here $\Sigma_{n,l}(t_1, t_2)$, $l = 1, 2$ correspond to the sums of the items in (6.62) with such i that $|v_i^{(n)}| \leq A_n$ (for $l = 1$) and $|v_i^{(n)}| \in (A_n, Q_n]$ (for $l = 2$) where $A_n = \delta_n/T_n$, $\delta_n \to 0$. By $|\xi(x, v)| \leq \delta_n^2(1 + o(1))/2$ for $|v| \leq A_n$, $|x| \leq T_n(t_2)$, using (4.27), (4.30) one has for any $t_2 > t_1 > 0$:

$$\Sigma_{n,1}(t_1, t_2) \leq n\delta_n^2 \int_{T_n(t_1)}^{T_n(t_2)} |w_n(x)| d\Phi(x)(1 + o(1)) \sim$$
$$\delta_n^2 c\tau \int_{e^{t_1}-1}^{e^{t_1}-1} |\log(1 + u)| u^{-(\tau+1)} du \to 0 \text{ as } n \to \infty. \tag{6.74}$$

To estimate $\Sigma_{n,2}(t_1,t_2)$ denote $m_n(A_n)$ the number of the items in this sum. By (1.13) one has

$$m_n(A_n) \le A_n^{-q} \sum_i |v_i^{(n)}|^q \le (b_n/A_n)^q nh_n \asymp b_n^{2q}\delta_n^{-q} nh_h$$

and by (4.32) for any $\epsilon_1 > 0$ one can choose such $\delta_n = n^{-\epsilon_0}$ and $\epsilon_0 > \epsilon_1/q$ that

$$m_n(A_n) \le Bn^{1+\epsilon_1}h_n = o(n). \tag{6.75}$$

By analogy with (6.52), (6.68) put

$$\Sigma_{n,2}(t_1,t_2) = A_n(t_1,t_2) + B_n(t_1,t_2) - 2C_n(t_1,t_2)$$

where A, B, C are the sums which correspond to the items in the right-hand side of (6.67). By analogy with the estimation of the values $\Delta E_n^+(t)$ one can obtain the estimators: for any $t_2 > t_1 > 0$

$$C_n(t_1,t_2) \le (1+o(1))c\tau m_n(A_n)n^{-1}\phi_1(t_1,t_2) \to 0 \ \text{ as } n \to \infty$$

where

$$\phi_1(t_1,t_2) = \int_{e^{t_1}-1}^{e^{t_1}-1} |\log(1+u)|u^{-(\tau+1)}du$$

and analogously $B_n(t_1,t_2) \to 0$. To estimate $A_n(t_1,t_2)$ put

$$A_n(t_1,t_2) = A_{n,1}(t_1,t_2) + A_{n,2}(t_1,t_2).$$

Here the values $A_{n,l}(t_1,t_2)$, $l = 1,2$ correspond to the sums of the items in with such i that $|v_i^{(n)}| \in (A_n, b_n(1-\epsilon)]$ (for $l = 1$) and $|v_i^{(n)}| \in (b_n(1-\epsilon), Q_n]$ (for $l = 2$) where $0 < \epsilon < \min(2-\tau,\ 2\tau-2)$. Using (4.27), (4.30) (6.75) one has for small enough $t_2 > t_1 > 0$:

$$A_{n,1}(t_1,t_2) \le Bm_n(A_n)b_n^{-1}\exp(-(T_n - b_n(1-\epsilon))^2/2)\phi_1(t_1,t_2) \to 0 \ \text{ as } n \to \infty$$

by for small enough $\epsilon_1 > 0$

$$\frac{m_n(A_n)}{b_n}e^{-(T_n-b_n(1-\epsilon))^2/2} \le \frac{m_n(A_n)}{nh_nb_n}e^{-\epsilon b_n^2(\tau_n-1-\epsilon/2)} \to 0.$$

Also one has for small enough $t_2 > t_1 > 0$:

$$A_{n,2}(t_1,t_2) \le B_n\phi_\epsilon(t_1,t_2);\ B_n \le B\sum_i \Phi(T_n - |v_i^{(n)}|) = O(1) \ \text{ as } n \to \infty$$

where

$$\phi_\epsilon(t_1,t_2) = \int_{e^{t_1}-1}^{e^{t_1}-1} |\log(1+u)|u^{-(\tau+\epsilon)}du \asymp \int_{t_1}^{t_2} u^{1-\tau-\epsilon}du$$

and by the choosing ϵ

$$\lim_{t_2\to 0}\lim_{t_1\to 0} \phi_\epsilon(t_1,t_2) \to 0.$$

The estimators above imply (6.72).

Now it is necessary to obtain the relation (6.57). By pass $n \to \infty$ this relation follows from the inequalities

$$\Delta L_n \leq L^{\Delta}_{\pi_n}, \ \frac{d}{dt}\Delta L_n \geq \frac{d}{dt} L^{\Delta}_{\pi_n} \tag{6.76}$$

where ΔL_n is defined by (6.52) and (see Ingster, 1997, the relation (4.11))

$$L^{\Delta}_{\pi_n}(t) = n\left(P_{1,0}(w_n(x) > t) - P_{1,\pi_n}(w_n(x) > t)\right) \to L^{\Delta}(t).$$

The following Lemmas imply the inequalities (6.76).

Lemma 6.1 *Let us change the constraints (5.47) of the Lemma 5.1 onto following: for some $B \geq 1$*

$$H_n - b_n \geq d_n \to \infty, \quad BH_n \geq Q_n > b_n; \tag{6.77}$$

and change the functionals (5.48) onto

$$\phi(v) = \Phi(-H_n + v) + \Phi(-H_n - v) - 2\Phi(-H_n). \tag{6.78}$$

Then for large enough n the equality (5.49) and the inequality (5.50) hold.

Lemma 6.2 *Let us change the constraints (5.47) of the Lemma 5.1 onto following:*

$$H_n - b_n \geq d_n \to \infty, \ (2-\delta)H_n - b_n \geq Q_n > b_n; \tag{6.79}$$

and change the functionals (5.48) onto

$$\phi(v) = \exp(-(H_n - v)^2/2) + \exp(-(H_n + v)^2/2) - 2\exp(-H_n^2/2). \tag{6.80}$$

Then for large enough n the equality (5.49) and the inequality (5.50) hold.

Proof of the Lemmas is given in sec 8.

The upper bounds of the Theorem 2 are proved.

7 The study of the tests (2.23): upper bounds and consistent properties

Put $\lambda_{n,\alpha} = \lambda_{n,\alpha}(b_n, h_n) = nh_n\Phi(b_n - H_{n,\alpha})$.

Lemma 7.1

1. Assume (i): $b_n - H_{n.\alpha} \to \infty$; or (ii): $nh_n \to \infty$ and $\lambda_{n,\alpha} \to \infty$. Then $\beta_n(\psi_{n,\alpha}, V_n) \to 0$.

2. Assume $nh_n \to \infty$ and $\lambda_{n,\alpha} = O(1)$. Then

$$\beta_n(\psi_{n,\alpha}, V_n) \leq (1-\alpha)\exp(-\lambda_{n,\alpha}) + o(1). \tag{7.81}$$

3. Assume $nh_n = O(1)$. Then

$$\beta_n(\psi_{n,\alpha}, V_n) \leq (1-\alpha)\Phi(\sqrt{2\log n} - b_n)^{nh_n} + o(1). \tag{7.82}$$

Proof of the Lemma. By $p > q$ one has for all $v \in V_n$

$$R_{n,1}^p \le \sum_i |v_i|^p \le \max_i |v_i|^{p-q} \sum_i |v_i|^q \le \max_i |v_i|^{p-q} R_{n,2}^q.$$

This relations and (1.13) imply

$$\inf_{v \in V_n} \max_i |v_i| \ge b_n = (R_{n,1}^p / R_{n,2}^q)^{1/(p-q)}. \tag{7.83}$$

In analogy with (4.36) using (7.83) one has

$$\beta_n(\psi_{n,\alpha}, V_n) \le \Phi(H_{n,\alpha} - \inf_{v \in V_n} \max_i |v_i|) \le \Phi(H_{n,\alpha} - b_n) \to 0$$

as $b_n - H_{n.\alpha} \to \infty$.

Let $nh_n \to \infty$. One can assume that

$$H_{n\alpha} \ge b_n + d_n \text{ for some sequence } d_n \to \infty,\ d_n = o(b_n). \tag{7.84}$$

In fact, put $V_n(t) = V_n^{p,q}(tR_{n,1}, tR_{n,2}),\ t \in (0,1)$ and note that

$$\beta_n(\psi_{n,\alpha}, V_n) \le \beta_n(\psi_{n,\alpha}, V_n(t)).$$

One can get this relation from the proof of Anderson's lemma (see Ibragimov and Hasminskii, 1981). The direct proof follows from the equality (in analogy with (4.37))

$$\beta_n(\psi_{n,\alpha}, v) = \prod_i (1 - \Phi(H_{n,\alpha} - |v_i|) - \Phi(-H_{n,\alpha} - |v_i|)) \tag{7.85}$$

and from the decrease on $|v_i|$ of the functions under the product. The set $V_n(t)$ is corresponding to the values $h_n(t) = h_n,\ b_n(t) = tb_n$. Thus, if $\limsup H_{n,\alpha} - b_n < \infty$, then one can choose such $t = t_n < 1$ that (7.84) holds and $\lambda_{n,\alpha}(b_n(t_n), h_n) \to \infty$.

Put $Q_n = H_{n\alpha} + d_n/2$. Assume $v \in V_n,\ \max_i |v_i| \ge Q_n$. Then $\beta_n(\psi_{n,\alpha}, V_n) \to 0$ by (4.36).

Assume $v \in V_n,\ \max_i |v_i| < Q_n$. Then using (7.85) one has

$$\begin{aligned} \beta_n(\psi_{n,\alpha}, v) &= (1-\alpha) \prod_i \left(1 - \frac{\phi(|v_i|, H_{n,\alpha})}{1 - 2\Phi(-H_{n,\alpha})}\right) \\ &\le (1-\alpha) \exp\left(-\frac{F(v, H_{n,\alpha})}{1 - 2\Phi(-H_{n,\alpha})}\right) \end{aligned} \tag{7.86}$$

where

$$F(v, H) = \sum_i \phi(v_i, H),\ \phi(t, H) = \Phi(-H + t) + \Phi(-H - t) - 2\Phi(-H).$$

Using the Lemma 6.1 for $H_n = H_{n,\alpha}$ one has $F(v, , H_{n,\alpha}) \ge \lambda_{n,\alpha}$ which imply

$$\beta_n(\psi_{n,\alpha}, v) \le (1-\alpha) \exp(-\lambda_{n,\alpha}(1 + o(1)).$$

Let $nh_n = O(1)$. One can assume that $H_{n,\alpha} - b_n = O(1)$. Then analogously to (7.86) we have:

$$\beta_n(\psi_{n,\alpha}, v) = (1-\alpha)\exp(-F(v, H_{n,\alpha})), \ \ F(v,H) = \sum_i \phi(v_i, H)$$

where

$$\phi(v,H) = \log\left(\frac{P_{1,0}([-H,H])}{P_{1,v}([-H,H])}\right) = \log(\Phi(H)-\Phi(-H)) - \log(\Phi(H-v)-\Phi(-H-v)). \tag{7.87}$$

Then the inequality (7.82) follows from the

Lemma 7.2 *Let us change the constraints (5.47) of the Lemma 5.1 onto following:*

$$H_n - b_n = O(1), \ \ b_n + O(1) \geq Q_n > b_n; \tag{7.88}$$

and change the functionals (5.48) onto (7.87). Then for large enough n the equality (5.49) and the inequality (5.50) hold.

Proof of the Lemma 7.2 is given in sec 8.

The Lemma 7.1 is proved.

The following Lemma which has been proved in Ingster, 1997.

Lemma 7.3 *Let $k_n = nh_n \to \infty$.*

1. Let $\tau_n < \infty, \quad \lambda_n \to \infty$. Then $\lambda_{n,\alpha} \to \infty$.

2. Let $\tau = 1, \quad \lambda \asymp 1$. Then $\lambda_{n,\alpha} = \lambda_n + o(1)$.

We get the upper bounds and consistent properties of the Theorem 3 from the Lemmas 7.1, 7.3 and from the simple remark: if $\tau \in (1, \infty)$, then using (4.26) , (4.28) one has

$$c_n = 2\Phi(-T_n) \sim \lambda_n \frac{\tau - 1}{\tau} \asymp \lambda_n.$$

The Theorem 3 is proved.

8 Proof of the Lemmas 5.1-7.2

We give the outline of the proofs and omit some simple calculations.

The linear convex problems of minimization are considered in the Lemmas: to minimized the functional

$$\inf_{\bar{r} \in \Xi_n} F(\bar{r}); \quad F(\bar{r}) = \sum_{i=1}^n \int \phi(v) r_i(dv)$$

where the convex set $\Xi_n = \{\bar{r} = (r_1, ..., r_n)\}$ of collections of the probability measures r_i on the real line is defined by the constraints: $r_i([-Q_n, Q_n] = 1, \ i = 1, ..., n$ and

$$\sum_{i=1}^n \int \phi_1(v) r_i(dv) \geq H_1, \ \sum_{i=1}^n \int \phi_2(v) r_i(dv) \leq H_2.$$

Here the functions $\phi = \phi_n$ are defined by (5.48), (6.78), (6.80), (7.87), $\phi(t) = \phi(-t) \geq 0$, $\phi(0) = 0$ and $\phi_1(v) = |v|^p$, $\phi_2(v) = |v|^q$, $H_1 = H_{1,n} = R^p_{1,n}$, $H_2 = H_{2,n} = R^q_{2,n}$.

By the symmetry of the problems one can find the infimum on the collections $\bar{r}^* = (r^*, ..., r^*) \in \Xi_n$ of the equal symmetrical measures r^*. Also using the method of subdifferentials and the Theorem by Kuhn and Tucker (see, for example, Ioffe and Tikhomirov (1976), pp. 76-77) one can get the sufficient conditions of infimum: there exist such $\lambda = \lambda_n > 0$, $\mu = \mu_n > 0$, $\eta = \eta_n$ that the following relations hold:

$$\phi(v) - \lambda\phi_1(v) + \mu\phi_2(v) \geq \eta \text{ for all } v \in [-Q_n, Q_n] \tag{8.89}$$

and

$$r^*(\{v : \ \phi(v) - \lambda\phi_1(v) + \mu\phi_2(v) = \eta\}) = 1. \tag{8.90}$$

It is enough to check that $r^* = \pi(b_n, h_n)$ satisfies to (8.90) and (8.89) holds for some $\lambda > 0$, $\mu > 0$, η. The relation (8.90) implies the equality in (8.89) for $v = 0$, $v = b$ and $v = -b$; $b = b_n \to \infty$ as $n \to \infty$.

Put $\eta = 0$ and

$$\lambda = \frac{b\phi'(b) - q\phi(b)}{b^p(p-q)}, \ \mu = \frac{b\phi'(b) - p\phi(b)}{b^q(p-q)} \tag{8.91}$$

which imply the necessary equalities in (8.89). Thus we need to check the inequalities: $\lambda > 0$, $\mu > 0$ and

$$\psi(v) = \phi(v) + (v/b)^q \frac{b\phi'(b) - p\phi(b)}{p-q} - (v/b)^p \frac{b\phi'(b) - q\phi(b)}{p-q} \geq 0, \ 0 \leq v \leq Q_n. \tag{8.92}$$

For large enough $b = b_n$ the inequalities $\lambda > 0$, $\mu > 0$ follow from the relations:

$$\phi'(b) > 0, \ \eta(b) = \phi(b)/b\phi'(b) \to 0 \ \text{ as } \ b \to \infty. \tag{8.93}$$

The relations (8.93) hold under assumptions of the Lemmas.

By $f(t) = (t^q - t^p)/(p-q) > 0$, $t \in (0,1)$, $f(1-z) \sim z$ as $z \to 0$ and $\phi(v) \geq 0$, one can easily check that (8.92) holds, if $v \in \Delta_n^- = (0, b_n^-)$ where $b_n^- = b_n(1 - B\eta(b_n))$ for large enough $B = B(p,q)$.

Fix small enough $\delta > 0$. Assume $Q_n \geq b_n^+ = b_n + \delta d_n$ (this is possible under assumptions of the Lemmas 5.1-6.2) and $v \in \Delta_n^+ = [b_n^+, Q_n]$. To satisfy (8.92) for $v \in \Delta_n^+$ it is enough the following : for some $B > 0$

$$Bb_n \geq Q_n, \ \ \phi(v) \geq \phi(b_n^+), \ v \in \Delta_n; \ \ b_n\phi'(b_n)/\phi(b_n^+) \to 0 \ \text{ as } b_n \to \infty. \tag{8.94}$$

Under assumptions of the Lemmas 5.1-6.2 using (4.28) one can easily check that (8.94) hold. Note that the relation $\phi(v) \geq \phi(b_n^+)$, $v \in \Delta_n$ in (8.94) follows from the constraints on Q_n in the Lemmas 5.1, 6.2 and from the increase of $\phi(v)$ on v in the Lemma 6.1.

Let $v \in \Delta_n^0 = (b_n^-, b_n^*)$ where $b_n^* = \min(Q_n, b_n^+)$ (note that $b_n^* = Q_n$ under the assumptions of the Lemma 7.2). The relations (8.91) imply the tangency of the functions $\psi(v)$ and $v = 0$ at the point $v = b_n$. The inequality (8.92) follows from the convexity of $\psi(v),\ v \in \Delta_n^0$. To convexity it is enough the following:

$$\phi_n'' = \inf_{v \in \Delta_n^0} \phi''(v) > 0,\ \phi'(b_n)/b_n\phi_n'' \to 0 \ \text{ as } b_n \to \infty\ . \tag{8.95}$$

One can easily check (8.95) under the assumption of the Lemmas. In fact, $\phi_n'' = \phi''(b_n^-)$ under assumptions of the Lemmas 5.1-6.2; $\phi_n'' \asymp \phi'(b_n) \asymp 1$ under assumptions of the Lemma 7.2.

The Lemmas are proved.

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