Lyapunov functions for cocycle attractors in nonautonomous difference equations

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Abstract

The construction of a Lyapunov function characterizing the pullback attraction of a cocycle attractor of a nonautonomous discrete time dynamical system involving Lipschitz continuous mappings is presented.

1 Introduction

The Kishinev school of dynamical systems founded by K.S. Sibirsky has made many wide ranging and important contributions to theory of dynamical systems, above all in connection with multivalued and nonautonomous systems, for which the references [2, 3, 4, 5, 11, 12, 15] are but a small sample. In this paper we consider a result that falls within this Kishinev tradition, namely the construction of a Lyapunov function that characterizes the pullback attraction of a cocycle attractor of a nonautonomous discrete time dynamical system.

We consider a nonautonomous difference equation

$$x_{n+1} = f_n(x_n) \tag{1}$$

on $\mathbb{R}^d$ where f_n is a Lipschitz continuous mapping from $\mathbb{R}^d$ into $\mathbb{R}^d$ with domain Dom_n which is an open, but not necessarily bounded, subset of $\mathbb{R}^d$ such that $f_n(Dom_{n)}) \subset Dom_{n+1}$ for each $n \in \mathbb{Z}$.

Such a difference equation (1) generates a cocycle mapping $\Phi : Dom_\Phi \to \mathbb{R}^d$, where $Dom_\Phi := \mathbb{N} \times \bigcup_{n_0 \in \mathbb{Z}} \{n_0 \times Dom_{n_0}\}$, through iteration by

$$\Phi(n, n_0, x_0) = f_{n_0+n-1} \circ \cdots \circ f_{n_0}(x_0) \tag{2}$$

for each $x_0 \in Dom_{n_0}$, $n_0 \in \mathbb{Z}$ and $n \in \mathbb{N}$. This cocycle mapping Φ satisfies the *initial condition property*

$$\Phi(0, n_0, x_0) = x_0 \tag{3}$$

for each $x_0 \in Dom_{n_0}$, $n_0 \in \mathbb{Z}$ and the *cocycle property*

$$\Phi(m + n, n_0, x_0) = \Phi(m, n_0 + n, \Phi(n, n_0, x_0)) \tag{4}$$

for each $x_0 \in Dom_{n_0}$, $n_0 \in \mathbb{Z}$ and n, $m \in \mathbb{N} \cup \{0\}$.

The cocycle property (4) is the nonautonomous counterpart of the group or semigroup evolutionary property of an autonomous dynamical system. The cocycle formalism provides a natural generalization to nonautonomous systems which retains the original state space in contrast to the skew–product flow formalism that represents the nonautonomous system as an autonomous system on the cartesian product of the original state space and some function space, see [14]. This is particularly advantageous in numerical dynamics [9, 10] and for random systems [1, 13].

It is too great a restriction of generality to consider as invariant just a single subset A^*, say, of $\mathbb{R}^d$ to be invariant w.r.t. every mapping f_{n_0}, that is, to satisfy $f_{n_0}(A^*) = A^*$ for all $n_0 \in \mathbb{Z}$. Instead we will say that a family $\widehat{A} = \{A_{n_0};\, n_0 \in \mathbb{Z}\}$ of nonempty sets with $A_{n_0} \subset Dom_{n_0}$ for each $n_0 \in \mathbb{Z}$ is *invariant* under Φ or Φ–*invariant* if

$$\Phi(n, n_0, A_{n_0}) = A_{n_0+n}, \qquad n_0 \in \mathbb{Z}, n \in \mathbb{N},$$

or, equivalently, if $f_{n_0}(A_{n_0}) = A_{n_0+1}$ for all $n_0 \in \mathbb{Z}$. Consequently every trajectory of Φ is Φ–invariant, with each of the sets A_{n_0} consisting of a single point. Some of these trajectories could have certain attractive properties w.r.t. the other trajectories, as can invariant families of more complicated, non-singleton sets. The task is how to formulate such attraction, particularly so the limit sets are also invariant. For this the concept of pullback attraction of random dynamical systems [1, 13] (see also [9, 10] is appropriate and leads to the concept of a pullback or cocycle attractor.

Let $H^*(A, B)$ denote the Hausdorff separation or semi–metric between nonempty compact subsets A and B of $\mathbb{R}^d$, and is defined by

$$H^*(A, B) := \max_{a \in A} \operatorname{dist}(a, B)$$

where $\operatorname{dist}(a, B) := \min_{b \in B} \|a - b\|$.

The most obvious way to formulate asymptotic behaviour for a nonautonomous dynamical system is consider the limit set of the forwards trajectory $\{\Phi(n, n_0, x_0\}_{n \geq 0}$ as $n \to \infty$ for each fixed initial value (n_0, x_0), which now depends on both the starting time n_0 and the starting point x_0. This has been extensively investigated in [3, 6, 8, 16], but has the disadvantage that the resulting (omega) limit sets $\omega^+(n_0, x_0)$ are generally not invariant under Φ. On the other hand, if we consider a Φ–invariant family $\widehat{A} = \{A_{n_0};\, n_0 \in \mathbb{Z}\}$ such forwards convergence would take the form

$$H^*(\Phi(n, n_0, x_0), A_{n_0+n}) \to 0 \quad \text{as} \quad n \to \infty.$$

To ensure convergence to a specific compnent set A_{n_0} for a fixed n_0, we would have to start progressively earlier in order to finish at time n_0. This leads to the concept of *pullback convergence*

$$H^*(\Phi(n, n_0 - n, x_0), A_{n_0}) \to 0 \quad \text{as} \quad n \to \infty,$$

that was first considered in connection with random dynamical systems, which are intrinsically nonautonomous [1, 3, 4, 13]. The invariant family $\widehat{A}$ is then called a pullback or cocycle attractor.

In this paper we construct a Lyapunov function which characterizes such pullback attraction and attactors. The main result is stated in the next section, a lemma on the existence of a pullback absorbing neighbourhood family is proved in Section 3, and finally in Section 4 an appropriate Lyapunov function is defined and shown to satisfy the properties asserted in the the theorem.

2 Lyapunov Functions for Pullback Attractors

A Φ–invariant family of compact subsets $\widehat{A} = \{A_{n_0};\ n_0 \in \mathbb{Z}\}$ will be called a *cocycle attractor* if it satisfies the *pullback attraction*

$$\lim_{n\to\infty} H^*\left(\Phi(n, n_0 - n, D_{n_0-n}), A_{n_0}\right) = 0 \tag{5}$$

for all $n_0 \in \mathbb{Z}$ and all $\widehat{D} = \{D_{n_0};\ n_0 \in \mathbb{Z}\}$ belonging to a *basin of attraction system* $\mathcal{D}_{att}$ consisting of families of sets $\widehat{D} = \{D_{n_0};\ n_0 \in \mathbb{Z}\}$ such that D_{n_0} is bounded and $D_{n_0} \subset Dom_{n_0}$ for each $n_0 \in \mathbb{Z}$ with the properties:

i) there exists a $\widehat{D}^{(int)} \in \mathcal{D}_{att}$ such that $A_{n_0} \subset \mathrm{int} D^{(int)}_{n_0}$ for each $n_0 \in \mathbb{Z}$; and

ii) $\widehat{D}^{(1)} = \left\{D^{(1)}_{n_0}\ ;\ n_0 \in \mathbb{Z}\right\} \in \mathcal{D}_{att}$ if $\widehat{D}^{(2)} = \left\{D^{(2)}_{n_0}\ ;\ n_0 \in \mathbb{Z}\right\} \in \mathcal{D}_{att}$ and $D^{(1)}_{n_0} \subseteq D^{(2)}_{n_0}$ for all $n_0 \in \mathbb{Z}$.

Obviously $\widehat{A} \in \mathcal{D}_{att}$. Although somewhat complicated, the use of such a basin of attractionsystem allows us to consider both nonuniform and local attraction regions which are typical in nonautonomous systems.

Our main result is the construction of a Lyapunov function that characterizes this pullback attraction.

Theorem 1 *Let f_{n_0} be uniformly Lipschitz continuous on Dom_{n_0} for each $n_0 \in \mathbb{Z}$ and let $\widehat{A}$ be a family of nonempty compact Φ–invariant sets that is pullback attracting with respect to Φ with a basin of attraction system $\mathcal{D}_{att}$. Then there exists a Lipschitz continous function $V : \mathbb{N} \times \bigcup_{n_0\in\mathbb{Z}} \{\{n_0\} \times \mathcal{D}_{att}(n_0)\} \to \mathbb{R}^+$, where $\mathcal{D}_{att}(n_0) := \bigcup_{\widehat{D}\in\mathcal{D}_{att}} D_{n_0}$ for each $n_0 \in \mathbb{Z}$, such that*

Property 1 (upper bound): *For all $n_0 \in \mathbb{Z}$ and $x_0 \in \mathcal{D}_{att}(n_0)$*

$$V(n_0, x_0) \le \mathrm{dist}(x_0, A_{n_0}); \tag{6}$$

Property 2 (lower bound): *For each $n_0 \in \mathbb{Z}$ there exists a function $a(n_0, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with $a(n_0, 0) = 0$ and $a(n_0, r) > 0$ for all $r > 0$ which is monotonic increasing in r such that*

$$a(n_0, \mathrm{dist}(x_0, A_{n_0})) \le V(n_0, x_0) \tag{7}$$

for all $x_0 \in \mathcal{D}_{att}(n_0)$;

Property 3 (Lipschitz condition): *For all $n_0 \in \mathbb{Z}$ and $x_0,\ y_0 \in \mathcal{D}_{att}(n_0)$*

$$\left|V(n_0, x_0) - V(n_0, y_{)})\right| \le \|x_0 - y_0\|; \tag{8}$$

Property 4 (pullback convergence): *For all $n_0 \in \mathbb{Z}$ and any $\widehat{D} \in \mathcal{D}_{att}$*

$$\operatorname{limsup}_{n\to\infty} \sup_{z_{n_0-n}\in D_{n_0-n}} V(n_0, \Phi(n, n_0 - n, z_{n_0-n})) = 0. \tag{9}$$

In addition,

Property 5 (forwards convergence): *There exists $\widehat{N} \in \mathcal{D}_{att}$ which is positively invariant under Φ and consists of nonempty compact sets N_{n_0} with $A_{n_0} \subset \operatorname{int} N_{n_0}$ for each $n_0 \in \mathbb{Z}$ such that*

$$V(n_0 + 1, \Phi(1, n_0, x_0)) \leq e^{-1} V(n_0, x_0) \tag{10}$$

for all $x_0 \in N_{n_0}$ and hence

$$V(n_0 + j, \Phi(j, n_0, x_0)) \leq e^{-j} V(n_0, x_0) \tag{11}$$

for all $x_0 \in N_{n_0}$ and $j \in \mathbb{N}$.

Note 1: It would be nice to use $\Phi(n, n_0 - n, x_0)$ for a fixed x_0 in the pullback convergence property (9), but this may not always be possible due to nonuniformity of the attraction region, i.e. we may not have a $\widehat{D} \in \mathcal{D}_{att}$ and an $x_0 \in D_{n_0-n}$ for all $n \in \mathbb{N}$.

Note 2: The forwards convergence inequality (11) does not imply forwards Lyapunov stability or asymptotic stability. Athough we then have

$$a(n_0 + j, \operatorname{dist}(\Phi(j, n_0, x_0), A_{n_0+j})) \leq e^{-j} V(n_0, x_0)$$

there is no guarantee (without additional assumptions) that

$$\inf_{j\geq 0} a(n_0 + j, r) > 0$$

for $r > 0$, so $\operatorname{dist}(\Phi(j, n_0, x_0), A_{n_0+j})$ need not become small as $j \to \infty$.

As a counterexample consider the cocycle mapping Φ on $\mathbb{R}$ generated $f_n = f$ for $n \leq 0$ and $f_n = g$ for $n \geq 1$ where the mappings f, $g : \mathbb{R} \to \mathbb{R}$ are given by $f(x) := \frac{1}{2}x$ and $g(x) := \max\{0, 4x(1 - x)\}$ for all $x \in \mathbb{R}$. Then $\widehat{A}$ with $A_{n_0} = \{0\}$ for all $n_0 \in \mathbb{Z}$ is pullback attracting for Φ but is not forwards Lyapunov asymptotically stable. (Note we can restrict f, g to $[-R, R] \to [-R, R]$ for any fixed $R > 1$ to ensure the required uniform Lipschitz continuity of the f_n).

Note 3: We can rewrite the forwards convergence inequality (11) as

$$V(n_0, \Phi(j, n_0 - j, x_{n_0-j})) \leq e^{-j} V(n_0 - j, x_{n_0-j}) \leq e^{-j} \operatorname{dist}(x_{n_0-j}, A_{n_0-j})$$

for all $x_{n_0-j} \in N_{n_0-j}$ and $j \in I\!N$.

We will say that $\widehat{D} \in \mathcal{D}_{att}$ is *past–tempered* with respect to $\widehat{A}$ if

$$\lim_{j\to\infty} \frac{1}{j} \log^+ H^*(D_{n_0-j}, A_{n_0-j}) = 0$$

for each $n_0 \in \mathbb{Z}$, or equivalently if

$$\lim_{j\to\infty} e^{-\gamma j} H^*(D_{n_0-j}, A_{n_0-j}) = 0$$

for each $n_0 \in \mathbb{Z}$ and every real $\gamma > 0$. This says that there is at most subexponential growth backwards in time of the starting sets. It is reasonable to restrict our atention to such sets.

For a past-tempered set $\widehat{D} \subset \widehat{N}$ we thus have

$$V(n_0, \Phi(j, n_0 - j, x_{n_0-j})) \leq e^{-j} H^*(D_{n_0-j}, A_{n_0-j}) \longrightarrow 0$$

as $j \to \infty$, and hence

$$a(n_0, \mathrm{dist}(\Phi(j, n_0 - j, x_{n_0-j}), A_{n_0})) \leq e^{-j} H^*(D_{n_0-j}, A_{n_0-j}) \longrightarrow 0$$

as $j \to \infty$. Since n_0 is fixed in the lower expression, this implies the pullback convergence

$$\lim_{j\to\infty} H^*(\Phi(j, n_0 - j, D_{n_0-j}), A_{n_0}) = 0.$$

A rate of pull–back convergence for more general sets $\widehat{D} \in \mathcal{D}_{att}$ will be considered in the appendix.

3 Pullback Absorbing Neighbourhood Systems

We will say that a family $\widehat{B} = \{B_{n_0}\ ;\ n_0 \in \mathbb{Z}\} \in \mathcal{D}_{att}$ of nonempty compact subsets with nonempty interior is a *pullback absorbing neighbourhood system* for a Φ–pullback attractor $\widehat{A}$ if it is positively invariant w.r.t. Φ in the sense that

$$\Phi(n, n_0, B_{n_0}) \subseteq B_{n_0+n} \qquad \forall n \in I\!N, n_0 \in \mathbb{Z}$$

and if it pullback attracts all $\widehat{D} \in \mathcal{D}_{att}$, that is for each $\widehat{D} \in \mathcal{D}_{att}$ and $n_0 \in \mathbb{Z}$ there exists an $N(\widehat{D}, n_0) \in I\!N$ such that

$$\Phi(n, n_0 - n, D_{n_0-n}) \subseteq \mathrm{int} B_{n_0}, \qquad \forall n \geq N.$$

Obviously we then have $\widehat{A} \subset \widehat{B} \in \mathcal{D}_{att}$. Moreover, by positive invariance and the cocycle property we have

$$\Phi(n + m, n_0 - n - m, B_{n_0-n-m}) \subset \Phi(n, n_0 - n, B_{n_0-n})$$

for all n, $m \in I\!N$ and $n_0 \in \mathbb{Z}$. From this we see that

$$A_{n_0} = \bigcap_{n \in I\!N} \Phi(n, n_0 - n, B_{n_0 - n}), \qquad \forall n_0 \in \mathbb{Z}.$$

The following lemma shows that there always exists such a pullback absorbing neighbourhood system for any given cocycle attractor. This will be required for the construction of the Lyapunov function for the proof of Theorem 1

Lemma 2 *If $\widehat{A}$ is a cocycle attractor with a basin of attraction system $\mathcal{D}_{att}$ for a cocycle Φ which is continuous in its spatial variable, then there exists a pullback absorbing neighbourhood system $\widehat{B} \subset \mathcal{D}_{att}$ of $\widehat{A}$ w.r.t. Φ.*

Proof: For each $n_0 \in \mathbb{Z}$ pick $\delta_{n_0} > 0$ such that $B[A_{n_0}; \delta_{n_0}] := \{x \in I\!R^d : \operatorname{dist}(x, A_{n_0}) \leq \delta_{n_0}\} \subset \mathcal{D}_{att}(n_0)$ and define

$$B_{n_0} := \overline{\bigcup_{j \geq 0} \Phi(j, n_0 - j, B[A_{n_0 - j}; \delta_{n_0 - j}])}.$$

Obviously $A_{n_0} \subset \operatorname{int} B[A_{n_0}; \delta_{n_0}] \subset B_{n_0}$. To show positive invariance we use the cocycle property in what follows.

$$\begin{aligned}
\Phi(1, n_0, B_{n_0}) &= \overline{\bigcup_{j \geq 0} \Phi(1, n_0, \Phi(j, n_0 - j, B[A_{n_0 - j}; \delta_{n_0 - j}]))} \\
&= \overline{\bigcup_{j \geq 0} \Phi(j + 1, n_0 - j, B[A_{n_0 - j}; \delta_{n_0 - j}])} \\
&= \overline{\bigcup_{i \geq 1} \Phi(i, n_0 + 1 - i, B[A_{n_0 + 1 - i}; \delta_{n_0 + 1 - i}])} \\
&\subseteq \overline{\bigcup_{i \geq 0} \Phi(i, n_0 + 1 - i, B[A_{n_0 + 1 - i}; \delta_{n_0 + 1 - i}])} \;=\; B_{n_0 + 1},
\end{aligned}$$

so $\Phi(1, n_0, B_{n_0}) \subseteq B_{n_0 + 1}$. By this and the cocycle property again we obtain

$$\begin{aligned}
\Phi(2, n_0, B_{n_0}) &= \Phi(1, n_0 + 1, \Phi(1, n_0, B_{n_0})) \\
&\subseteq \Phi(1, n_0 + 1, B_{n_0 + 1}) \subseteq B_{n_0 + 2}.
\end{aligned}$$

The general positive invariance assertion then follows by induction.

Now by the continuity of $\Phi(j, n_0 - j, \cdot)$ and the compactness of $B[A_{n_0 - j}; \delta_{n_0 - j}]$, the set $\Phi(j, n_0 - j, B[A_{n_0 - j}; \delta_{n_0 - j}])$ is compact for each $j \geq 0$ and $n_0 \in \mathbb{Z}$. Moreover, by pullback convergence, there exists an $N = N(n_0, \delta_{n_0}) \in I\!N$ such that

$$\Phi(j, n_0 - j, B[A_{n_0 - j}; \delta_{n_0 - j}]) \subseteq B[A_{n_0}; \delta_{n_0}] \subset B_{n_0}$$

for all $j \geq N$. Hence

$$\begin{aligned}
\Phi(1, n_0, B_{n_0}) &= \overline{\bigcup_{j \geq 0} \Phi(j, n_0 - j, B[A_{n_0-j}; \delta_{n_0-j}])} \\
&\subseteq B[A_{n_0}; \delta_{n_0}] \bigcup \overline{\bigcup_{0 \leq j < N} \Phi(j, n_0 - j, B[A_{n_0-j}; \delta_{n_0-j}])} \\
&= \overline{\bigcup_{0 \leq j < N} \Phi(j, n_0 - j, B[A_{n_0-j}; \delta_{n_0-j}])},
\end{aligned}$$

which is compact, so B_{n_0} is compact.

To see that $\widehat{B}$ so constructed is pullback absorbing w.r.t. $\mathcal{D}_{att}$, let $\widehat{D} \in \mathcal{D}_{att}$. Fix $n_0 \in \mathbb{Z}$. Since $\widehat{A}$ is pullback attracting, there exists an $N(\widehat{D}, \delta_{n_0}, n_0) \in \mathbb{N}$ such that

$$H^* \left(\Phi(j, n_0 - j, D_{n_0-j}), A_{n_0}\right) < \delta_{n_0}$$

for all $j \geq N(\widehat{D}, \delta_{n_0}, n_0)$. But $(\Phi(j, n_0 - j, D_{n_0-j}) \subset \mathrm{int} B[A_{n_0}; \delta_{n_0}]$ and $B[A_{n_0}; \delta_{n_0}]$ $\subset B_{n_0}$, so

$$\Phi(j, n_0 - j, D_{n_0-j}) \subset \mathrm{int} B_{n_0}$$

for all $j \geq N(\widehat{D}, \delta_{n_0}, n_0)$. Hence $\widehat{B}$ is pullback absorbing as required. □

4 Proof of Theorem 1

We want to construct a Lyapunov function $V(n_0, x_0)$ that characterizes a pullback attractor $\widehat{A}$ and satisfies properties 1–5 of Theorem 1.

For this we define for all $n_0 \in \mathbb{Z}$ and $x_0 \in \mathcal{D}_{att}(n_0) := \bigcup_{\widehat{D} \in \mathcal{D}_{att}} D_{n_0}$ as

$$V(n_0, x_0) := \sup_{n \in \mathbb{N}} e^{-T_{n_0,n}} \mathrm{dist}\left(x_0, \Phi(n, n_0 - n, B_{n_0-n})\right)$$

where

$$T_{n_0,n} = n + \sum_{j=1}^{n} \alpha^+_{n_0-j}$$

with $T_{n_0,n} = 0$. Here $\alpha_n = \log L_n$, where L_n is the uniform Lipschitz constant of f_n on Dom_n, and $a^+ = (a + |a|)/2$, i.e. the positive part of a real number a.

Note 4: We have $T_{n_0,n} \geq n$ and $T_{n_0,n+m} = T_{n_0,n} + T_{n_0-n,m}$ for all n, $m \in \mathbb{N}$ and $n_0 \in \mathbb{Z}$.

4.1 Proof of property 1

Since $e^{-T_{n_0,n}} \leq 1$ for all $n \in \mathbb{N}$ and $\mathrm{dist}\left(x_0, \Phi(n, n_0 - n, B_{n_0-n})\right)$ is monotonically increasing from $0 \leq \mathrm{dist}\left(x_0, \Phi(0, n_0, B_{n_0})\right)$ at $n = 0$ to $\mathrm{dist}\left(x_0, A_{n_0}\right)$ as $n \to \infty$, we have

$$\begin{aligned} V(n_0, x_0) &= \sup_{n \in I\!N} e^{-T_{n_0,n}} \operatorname{dist}(x_0, \Phi(n, n_0 - n, B_{n_0-n})) \\ &\leq 1 \cdot \operatorname{dist}(x_0, A_{n_0}). \end{aligned}$$

4.2 Proof of property 2

If $x_0 \in A_{n_0}$ we have $V(n_0, x_0) = 0$ by Property 1, so let us assume that we have $x_0 \in \mathcal{D}_{att}(n_0) \setminus A_{n_0}$. Now in

$$V(n_0, x_0) = \sup_{n \geq 0} e^{-T_{n_0,n}} \operatorname{dist}(x_0, \Phi(n, n_0 - n, B_{n_0-n}))$$

the supremum involves the product of an exponential decreasing quantity bounded below by zero and a bounded increasing function, since the $\Phi(n, n_0 - n, B_{n_0-n})$ are a nested family of compact sets decreasing to $A_{n_0}$with increasing n. In particular,

$$\operatorname{dist}(x_0, A_{n_0}) \geq \operatorname{dist}(x_0, \Phi(n, n_0 - n, B_{n_0-n})), \qquad \forall n \in I\!N.$$

Hence there exists an $N^* = N^*(n_0, x_0) \in I\!N$ such that

$$\frac{1}{2}\operatorname{dist}(x_0, A_{n_0}) \leq \operatorname{dist}(x_0, \Phi(n, n_0 - n, B_{n_0-n})) \leq \operatorname{dist}(x_0, A_{n_0})$$

for all $n \geq N^*$ but not for $n = N^* - 1$. Then from above

$$\begin{aligned} V(n_0, x_0) &\geq e^{-T_{n_0,N^*}} \operatorname{dist}(x_0, \Phi(N^*, n_0 - N^*, B_{n_0-N^*})) \\ &\geq \frac{1}{2} e^{-T_{n_0,N^*}} \operatorname{dist}(x_0, A_{n_0}). \end{aligned}$$

Define

$$\hat{N}(n_0, r) := \sup\{N^*(n_0, x_0) : \operatorname{dist}(x_0, A_{n_0}) = r\}$$

We have $\hat{N}(n_0, r) < \infty$ for $x_0 \notin A_{n_0}$ with $\operatorname{dist}(x_0, A_{n_0}) = r$ and $\hat{N}(n_0, r)$ is nondecreasing with $r \to 0$. To see this note that by the triangle rule

$$\operatorname{dist}(x_0, A_{n_0}) \leq \operatorname{dist}(x_0, \Phi(n, n_0 - n, B_{n_0-n})) + H^*(\Phi(n, n_0 - n, B_{n_0-n}), A_{n_0}).$$

Also by pullback convergence there exists an $N(n_0, r/2)$ such that

$$H^*(\Phi(n, n_0 - n, B_{n_0-n}), A_{n_0}) < \frac{1}{2} r$$

for all $n \geq N(n_0, r/2)$. Hence for $\operatorname{dist}(x_0, A_{n_0}) = r$ and $n \geq N(n_0, r/2)$ we have

$$r \leq \operatorname{dist}(x_0, \Phi(n, n_0 - n, B_{n_0-n})) + \frac{1}{2} r,$$

that is

$$\frac{1}{2}r \le \operatorname{dist}(x_0, \Phi(n, n_0 - n, B_{n_0-n})).$$

Obviously we have $\hat{N}(n_0, r) \le N(n_0, r/2)$.

Finally, we define

$$a(n_0, r) := \frac{1}{2}r\ e^{-T_{n_0,\hat{N}(n_0,r)}}. \tag{12}$$

Note that there is no guarantee here (without further assumptions) that $a(n_0, r)$ does not go to 0 for fixed $r \ne 0$ as $n_0 \to \infty$.

4.3 Proof of property 3

We have $|V(n_0, x_0) - V(n_0, y_0)|$

$$\begin{aligned}
&= \left|\sup_{n\in I\!N} e^{-T_{n_0,n}} \operatorname{dist}\left(x_0, \Phi(n, n_0 - n, B_{n_0-n})\right) - \sup_{n\in I\!N} e^{-T_{n_0,n}} \operatorname{dist}\left(y_0, \Phi(n, n_0 - n, B_{n_0-n})\right)\right| \\
&\le \sup_{n\in I\!N} e^{-T_{n_0,n}} \left|\operatorname{dist}\left(x_0, \Phi(n, n_0 - n, B_{n_0-n})\right) - \operatorname{dist}\left(y_0, \Phi(n, n_0 - n, B_{n_0-n})\right)\right| \\
&\le \sup_{n\in I\!N} e^{-T_{n_0,n}} \|x_0 - y_0\| \le \|x_0 - y_0\|.
\end{aligned}$$

4.4 Proof of property 4

Assume the opposite. Then there exists an $\varepsilon_0 > 0$, a sequence $n_j \to \infty$ in $I\!N$ and points $x_j \in \Phi(n_j, n_0 - n_j, D_{n_0-n_j})$ such that $V(n_0, x_j) \ge \varepsilon_0$ for all $j \in I\!N$. Since $\widehat{D} \in \mathcal{D}_{att}$ and $\widehat{B}$ is pullback absorbing, there exists an $N = N(\widehat{D}, n_0) \in I\!N$ such that

$$\Phi(n_j, n_0 - n_j, D_{n_0-n_j}) \subset B_{n_0}, \qquad \forall n_j \ge N.$$

Hence for all j such that $n_j \ge N$ we have $x_j \in B_{n_0}$, which is a compact set, so there exists a convergent subsequence $x_{j'} \to x^* \in B_{n_0}$. But we also have

$$x_{j'} \in \overline{\bigcup_{n\ge n_{j'}} \Phi(n, n_0 - n, D_{n_0-n})}$$

and

$$\bigcap_{n_{j'}} \overline{\bigcup_{n\ge n_{j'}} \Phi(n, n_0 - n, D_{n_0-n})} \subseteq A_{n_0}$$

by the definition of a cocycle attractor. Hence we must have $x^* \in A_{n_0}$ and $V(n_0, x^*) = 0$. But V is Lipschitz continuous in its second variable by property 3, so

$$\varepsilon_0 \le V(n_0, x_{j'}) = \|V(n_0, x_{j'}) - V(n_0, x^*)\| \le \|x_{j'} - x^*\|,$$

which contradicts the convergence $x_{j'} \to x^*$. Hence property 4 must hold.

Proof of property 5

Define

$$N_{n_0} := \{x_0 \in B[B_{n_0};1] \ : \ \Phi(1,n_0,x_0) \in B_{n_0+1}\},$$

where $B[B_{n_0};1] = \{x_0 \ : \ \mathrm{dist}(x_0, B_{n_0} \leq 1\}$ is bounded because B_{n_0} is compact and $I\!R^d$ is locally compact, so N_{n_0} is bounded. It is also closed, hence compact, since $\Phi(1,n_0,\cdot)$ is continuous and B_{n_0+1} is compact. Now $A_{n_0} \subset \mathrm{int}B_{n_0}$ and $B_{n_0} \subset N_{n_0}$, so $A_{n_0} \subset \mathrm{int}N_{n_0}$. In addition,

$$\Phi(1,n_0,N_{n_0}) \subset B_{n_0+1} \subset N_{n_0+1},$$

so $\widehat{N}$ is positive invariant.

It remains to establish the exponential decay inequality (10). For this we will need the following Lipschitz condition

$$\|\Phi(1,n_0,x_0) - \Phi(1,n_0,y_0)\| \leq e^{\alpha_{n_0}}\|x_0 - y_0\|$$

for all x_0, $y_0 \in Dom_{n_0}$ on $\Phi(1,n_0,\cdot) \equiv f_{n_0}(\cdot)$. It follows from this that

$$\mathrm{dist}(\Phi(1,n_0,x_0), \Phi(1,n_0,C_{n_0})) \leq e^{\alpha_{n_0}}\mathrm{dist}(x_0, C_{n_0})$$

for any compact subset $C_{n_0} \subset Dom_{n_0}$.

From the definition of V we have

$$\begin{aligned} V(n_0+1,\Phi(1,n_0,x_0)) &= \sup_{n\geq 0} e^{-T_{n_0+1,n}}\mathrm{dist}(\Phi(1,n_0,x_0), \Phi(n,n_0-n,B_{n_0-n})) \\ &= \sup_{n\geq 1} e^{-T_{n_0+1,n}}\mathrm{dist}(\Phi(1,n_0,x_0), \Phi(n,n_0-n,B_{n_0-n})) \end{aligned}$$

since $\Phi(1,n_0,x_0) \in B_{n_0+1}$ when $x_0 \in N_{n_0}$. Hence re-indexing and then using the cocycle property and the Lipschitz condition on $\Phi(1,n_0,\cdot)$ we have

$$\begin{aligned} V(n_0+1,\Phi(1,n_0,x_0)) &= \sup_{j\geq 0} e^{-T_{n_0+1,j+1}}\mathrm{dist}(\Phi(1,n_0,x_0), \Phi(j+1,n_0-j,B_{n_0-j})) \\ &= \sup_{j\geq 0} e^{-T_{n_0+1,j+1}}\mathrm{dist}(\Phi(1,n_0,x_0), \Phi(1,n_0,\Phi(j,n_0-j,B_{n_0-j}))) \\ &\leq \sup_{j\geq 0} e^{-T_{n_0+1,j+1}}e^{\alpha_{n_0}}\mathrm{dist}(x_0, \Phi(j,n_0-j,B_{n_0-j})) \end{aligned}$$

Now $T_{n_0+1,j+1} = T_{n_0,j} + 1 - \alpha^+_{n_0}$, so

$$\begin{aligned} V(n_0+1,\Phi(1,n_0,x_0)) &\leq \sup_{j\geq 0} e^{-T_{n_0+1,j+1}+\alpha_{n_0}}\mathrm{dist}(x_0, \Phi(j,n_0-j,B_{n_0-j})) \\ &= \sup_{j\geq 0} e^{-T_{n_0,j}-1-\alpha^+_{n_0}+\alpha_{n_0}}\mathrm{dist}(x_0, \Phi(j,n_0-j,B_{n_0-j})) \end{aligned}$$

$$\leq \quad e^{-1} \sup_{j\geq 0} e^{-T_{n_0,j}} \mathrm{dist}(x_0, \Phi(j, n_0 - j, B_{n_0-j}))$$

$$\leq \quad e^{-1} V(n_0, x_0),$$

which is the desired inequality.

Moreover, since $\Phi(1, n_0, x_0) \in B_{n_0+1} \subset N_{n_0+1}$, the proof continues inductively to give

$$V(n_0 + j, \Phi(j, n_0, x_0)) \leq e^{-j} V(n_0, x_0)$$

for all $j \in I\!N$.

This completes the proof of Theorem 1. $\square$

Appendix: Rate of pull-back convergence

Since $\widehat{B}$ is a pullback absorbing neighbourhood system for every $n_0 \in \mathbb{Z}$, $n \in I\!N$ and $\widehat{D} \in \mathcal{D}_{att}$ there exists an $N(\widehat{D}, n_0, n) \in I\!N$ such that

$$\Phi(m, n_0 - n - m, D_{n_0-n-m}) \subseteq B_{n_0-n}, \qquad \forall m \geq N.$$

Hence by the cocycle property we have

$$\begin{aligned}
\Phi(n+m, n_0 - n - m, D_{n_0-n-m}) &= \Phi(n, n_0 - n, \Phi(m, n_0 - n - m, D_{n_0-n-m})) \\
&\subseteq \Phi(n, n_0 - n, B_{n_0-n}), \quad \forall m \geq N, \\
&= \Phi(i, n_0 - i, \Phi(n - i, n_0 - n, B_{n_0-n})), \quad \forall \leq i \leq n, \\
&\subseteq \Phi(i, n_0 - i, B_{n_0-i})
\end{aligned}$$

where we have used the forward positive invariance of $\widehat{B}$ in the last line. Hence we have

$$\Phi(n+m, n_0 - n - m, D_{n_0-n-m}) \subseteq \Phi(i, n_0 - i, B_{n_0-i})$$

for all $m \geq N(\widehat{D}, n_0, n)$ and $0 \leq i \leq n$, or equivalently

$$\Phi(m, n_0 - m, D_{n_0-m}) \subseteq \Phi(i, n_0 - i, B_{n_0-i})$$

for all $m \geq n + N(\widehat{D}, n_0, n)$ and $0 \leq i \leq n$. This means that for any $z_{n_0-m} \in D_{n_0-m}$ the supremum in

$$V(n_0, \Phi(m, n_0 - m, z_{n_0-m})) = \sup_{i\geq 0} e^{-T_{n_0,i}} \mathrm{dist}\,(\Phi(m, n_0 - m, z_{n_0-m}), \Phi(i, n_0 - i, B_{n_0-i}))$$

need only be considered over $i \geq n$. Hence

$$V(n_0, \Phi(m, n_0 - m, z_{n_0-m}))$$

$$
\begin{aligned}
&= \sup_{i\geq n} e^{-T_{n_0,i}} \operatorname{dist}\left(\Phi(m,n_0-m,z_{n_0-m}),\Phi(i,n_0-i,B_{n_0-i})\right)\\
&\leq e^{-T_{n_0,n}} \sup_{j\geq 0} e^{-T_{n_0-n,j}} \operatorname{dist}\left(\Phi(m,n_0-m,z_{n_0-m}),\Phi(n+j,n_0-n-j,B_{n_0-n-j})\right)\\
&\leq e^{-T_{n_0,n}} \operatorname{dist}\left(\Phi(m,n_0-m,z_{n_0-m}),A_{n_0}\right)\\
&\leq e^{-T_{n_0,n}} \operatorname{dist}\left(B_{n_0},A_{n_0}\right)
\end{aligned}
$$

since $A_{n_0} \subseteq \Phi(n+j,n_0-n-j,B_{n_0-n-j})$ and $\Phi(m,n_0-m,z_{n_0-m}) \in B_{n_0}$.

We thus have

$$V(n_0,\Phi(m,n_0-m,z_{n_0-m})) \leq e^{-T_{n_0,n}} \operatorname{dist}\left(B_{n_0},A_{n_0}\right)$$

for all $z_{n_0-m} \in D_{n_0-m}$, $m \geq n + N(\widehat{D},n_0,n)$ and $n \geq 0$.

We can assume that the mapping $n \mapsto n + N(\widehat{D},n_0,n)$ is monotonic increasing in n (by taking a larger $N(\widehat{D},n_0,n)$ if necessary), and is hence invertible. Let the inverse of $m = n + N(\widehat{D},n_0,n)$ be $n = M(m) = M(\widehat{D},n_0,m)$. Then

$$V(n_0,\Phi(m,n_0-m,z_{n_0-m})) \leq e^{-T_{n_0,M(m)}} \operatorname{dist}\left(B_{n_0},A_{n_0}\right)$$

for all $m \geq N(\widehat{D},n_0,0) \geq 0$. Usually we will have $N(\widehat{D},n_0,0) > 0$. We can modify the expression to hold for all $m \geq 0$ by replacing $M(m)$ by $M^*(m)$ defined for all $m \geq 0$ and introducing a constant $K_{\widehat{D},n_0} \geq 1$ to account for the behaviour over the finite time set $0 \leq m < N(\widehat{D},n_0,0)$. This will give us

$$V(n_0,\Phi(m,n_0-m,z_{n_0-m})) \leq K_{\widehat{D},n_0} e^{-T_{n_0,M^*(m)}} \operatorname{dist}\left(B_{n_0},A_{n_0}\right)$$

for all $m \geq 0$.

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