# A discretization of Volterra integral equations of the third kind with weakly singular kernels

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**Abstract**

In this paper we propose a method of piecewise constant approximation for the solution of ill-posed third kind Volterra equations

$$p(t)z(t) + \int_0^t \frac{h(t,\tau)}{(t-\tau)^{1-\alpha}} z(\tau)d\tau = f(t), \quad t \in [0,1], \;\; 0 < \alpha < 1.$$

Here $p(t)$ vanishes on some subset of $[t_1, t_2] \subset [0,1]$ and $|p(t)| < \delta$ for $t \in [t_1, t_2]$, where $\delta$ is a sufficiently small positive number. The proposed method gives the accuracy $O(\delta^{2\nu/(2\nu+1)})$ with respect to the $L_2$-norm, where $\nu$ is the parameter of sourcewise representation of the exact solution on $[t_1, t_2]$, and uses no more than $O(\delta^{-(2-\lambda)/\alpha} \log^{2+1/\alpha} \frac{1}{\delta})$ values of Galerkin functionals, where $\lambda \in (0, 1/2)$ is determined in the act of choosing the regularization parameter within the framework of Morozov's discrepancy principle.

## 1 Introduction

We are interested in linear integral equations of the form

$$(pI + H_\alpha)z(t) = f(t), \quad t \in [0,1], \tag{1}$$

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where $I$ is the identity operator,

$$(pI + H_\alpha)z(t) \equiv p(t)z(t) + \int_0^t \frac{h(t,\tau)}{(t-\tau)^{1-\alpha}} z(\tau)d\tau, \quad 0 < \alpha < 1, \tag{2}$$

and the given function $p(t)$ vanishes at least in one point of the interval $[0, 1]$.

In his fundamental papers on integral equations D. Hilbert [2] introduced the notion of integral equations of the first, second and of the third kind. A linear integral equation (1), (2) is said to be of first kind if $p(t) \equiv 0$, of the second kind if $p(t)$ is a non-zero constant, and of the third kind if $p(t)$ is a function with zeros in its domain (otherwise the equation is equivalent to an equation of the second kind). If the function $p(t)$ is continuous and has a finite number of zeros, then the equation (1), (2) is a special case of non-elliptic singular integral equations investigated by S.Prössdorf [11].

Note that Hilbert himself considered the case where $p(t)$ is piecewise constant with values 1 and $-1$ and with jumps at a finite number of points $t = t_i$,

$$0 = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} = 1. \tag{3}$$

He showed that these equations, with some slight modifications, have the same properties as the equations of the second kind. But if $p(t)$ vanishes between two of the points $t_i$, for example, $p(t) \equiv 0, \; t \in [t_1, t_2]$, then, as has been shown by E.Schock [14], the problem of solving the equation (1), (2) is not well posed in the sense of J.Hadamard and regularization techniques are required for solving (1), (2). In our opinion it makes sense to apply the regularization methods even in the case when the function $p(t)$ takes small values at all points on $[t_1, t_2]$, i.e.

$$|p(t)| < \delta, \quad t \in [t_1, t_2], \tag{4}$$

where $\delta$ is a sufficiently small positive constant. Such equations occur, for example, within the framework of the Newton- Kantorovich scheme

$$z_{m+1} = z_m - [\Phi'(z_m)]^{-1}\Phi(z_m) \tag{5}$$

for nonlinear integral equations

$$\Phi(z) \equiv F\left(t, z(t), \int_0^t \frac{k(t,\tau,z(\tau))}{(t-\tau)^{1-\alpha}} d\tau\right) = 0. \tag{6}$$

Here $\Phi'(z_m)$ is a Frechet derivative of $\Phi(z)$ calculated for $z = z_m$, and the singularities of the Frechet derivative give rise to the third-kind integral equation. Namely, $\Phi'(z_m)$ is a related linear integral operator of the form (2), where

$$p(t) = F_v \left( t, z_m(t), \int_0^t \frac{k(t, \tau, z_m(\tau))}{(t-\tau)^{1-\alpha}} d\tau \right),$$

$$h(t, \tau) = F_w \left( t, z_m(t), \int_0^t \frac{k(t, \tau, z_m(\tau))}{(t-\tau)^{1-\alpha}} d\tau \right) k_w(t, \tau, z_m(\tau))$$

(for $G = G(u, v, w)$ we use the following notations: $G_u = \frac{\partial G}{\partial u}$, $G_v = \frac{\partial G}{\partial v}$, $G_w = \frac{\partial G}{\partial w}$). Then the element $[\Phi'(z_m)]^{-1}\Phi(z_m)$ may be obtained by solving an integral equation of the form (1), (2), where

$$f(t) = F \left( t, z_m(t), \int_0^t \frac{k(t, \tau, z_m(\tau))}{(t-\tau)^{1-\alpha}} d\tau \right).$$

If the values of $F_v$ become small then the Newton-Kantorovich scheme (5) leads to the integral equation (1), (2) with an additional peculiarity (4).

For the sake of simplicity, in the sequel we shall assume that in (3) $k = 2$ and the datas $p(t), h(t, \tau), f(t)$ are continuously differentiable functions for $t \in [t_i, t_{i+1}]$, $\tau \in [t_j, t_{j+1}]$, $i, j = 0, 1, 2$. Moreover,

$$|p(t)| \geq d_1, \quad |p(t)| + |p'(t)| \leq d_2, \quad t \in [0, t_1] \cup [t_2, 1], \tag{7}$$

$$|h(t, \tau)| + |h_t(t, \tau)| + |h_\tau(t, \tau)| \leq d_3, \quad |f(t)| + |f'(t)| \leq d_4, \tag{8}$$

$$t \in [t_i, t_{i+1}], \ \tau \in [t_j, t_{j+1}], \ i, j = 0, 1, 2.$$

In this paper we consider some method of discretization for the problems (1), (2) with coefficients satisfying the conditions (4), (7), (8).

# 2 The discretization on the interval of well-posedness

In the sequel we need some results of optimization of the Galerkin scheme for solving operator equations

$$z + Hz = \varphi \tag{9}$$

in the Hilbert space $X$.

Let $\{e_i\}_{i=1}^{\infty}$ be some orthonormal basis of $X$ and let $P_n$ be the orthogonal projector on $span\{e_1, e_2, \ldots, e_n\}$, that is

$$P_n\varphi = \sum_{i=1}^{n}(\varphi, e_i)e_i,$$

where $(\cdot, \cdot)$ is the inner product in the Hilbert space $X$.

We denote by $X^\alpha$, $0 < \alpha < \infty$, a normed subspace of $X$, which is imbedded in $X$ with imbedding constant not exceeding one and such that for any $n = 1, 2, \ldots$

$$\|I - P_n\|_{X^\alpha \to X} \leq cn^{-\alpha}, \tag{10}$$

where the constant $c$ is independent of $n$.

Let us consider the following class of linear operators

$$\mathcal{H}_\beta^\alpha = \left\{H : \|H\|_{X \to X^\alpha} \leq \beta_1, \;\; \|H^*\|_{X \to X^\alpha} \leq \beta_2, \;\; \|(I - H)^{-1}\|_{X \to X} \leq \beta_3\right\},$$

$$\beta = (\beta_1, \beta_2, \beta_3).$$

The Galerkin method applied to equation (9) consists in solving a uniquely solvable equation

$$z_G + P_n H P_n z_G = P_n\varphi$$

and $z_G$ is taken as an approximate solution of (9). It is clear that to construct the approximate solution $z_G$ it is necessary to have the following collection of inner products as an information regarding equation (9):

$$(e_i, He_j), \;\; (e_i, \varphi), \quad i, j = 1, 2, \ldots, n. \tag{11}$$

Information of such type is called the Galerkin information.

Keeping in mind (10) and the well-known error estimate of the Galerkin method (see, for example, [12], p.33) for equation (9) with $H \in \mathcal{H}_\beta^\alpha$ and $\varphi \in X^\alpha$, we have

$$\|z - z_G\|_X \leq c_\beta\|\varphi\|_{X^\alpha}\|I - P_n\|_{X^\alpha \to X} \leq c_{1,\beta}n^{-\alpha}\|\varphi\|_{X^\alpha}, \tag{12}$$

where the constants $c_\beta$ and $c_{1,\beta}$ depend only on $\beta$. In the sequel we shall often use the same symbol $c$ for possibly different constants.

Denote by $Card(IP, \varepsilon)$ the number of inner products of the form (11) required to construct an approximate solution $z_G$ realizing the accuracy $\varepsilon$

with respect to the norm $\|\cdot\|_X$. Then by virtue of (12) for $H \in \mathcal{H}^\alpha_\beta$ and $\varphi \in X^\alpha$, we have

$$Card(IP, \varepsilon) = n^2 + n = O(\varepsilon^{-2/\alpha}). \tag{13}$$

In the sequel a point $(i,j)$ on the coordinate plane will be called the number of the Galerkin functional (inner product) $(e_i, He_j)$.

Let us associate to each operator $H \in \mathcal{H}^\alpha_\beta$ the finite-dimensional operator

$$H_{\Gamma_m} = \sum_{k=1}^{2m} (P_{2^k} - P_{2^{k-1}}) H P_{2^{2m-k}} + P_1 H P_{2^{2m}}.$$

We note that the operator $H_{\Gamma_m}$ acts into the subspace $span\{e_1, e_2, \ldots, e_{2^{2m}}\}$. To construct this operator it is necessary to have the values of the Galerkin functionals $(e_i, He_j)$ with numbers from the following plane set

$$\Gamma_m = \{1\} \times [1, 2^{2m}] \bigcup_{k=1}^{2m} (2^{k-1}, 2^k] \times [1, 2^{2m-k}].$$

If we denote by $Card(\Omega)$ the number of points $(i,j)$ with integer coordinates belonging to $\Omega$ then it is easy to calculate that

$$Card(\Gamma_m) \sim m 2^{2m}. \tag{14}$$

For each equation (9) we determine the sequence of elements

$$z^0 = 0, \;\; z^k = z^{k-1} + (I + H_{\Gamma_m} P_{2^n})^{-1} (P_{2^{2m}} \varphi - z^{k-1} - H_{\Gamma_m} z^{k-1}), \tag{15}$$

$$k = 1, 2, 3, 4, \quad n = [2m/3].$$

All these elements belong to $span\{e_1, e_2, \ldots, e_{2^{2m}}\}$ and to construct $z^1, \ldots, z^4$ we need $Card(\Gamma_m) + 2^{2m}$ values of Galerkin functionals

$$(e_i, He_j), \;\; (i,j) \in \Gamma_m, \quad (e_k, \varphi), \;\; k = 1, 2, \ldots, 2^{2m}. \tag{16}$$

**Theorem 1 (see [9],p.296)** *Let $z$ be the solution of equation (9) with $H \in \mathcal{H}^{\alpha}_{\beta}$, $\varphi \in X^{\alpha}$. Then*

$$\|z - z^4\|_X \leq c_\beta 2^{-2m\alpha} \|\varphi\|_{X^\alpha}.$$

*To represent an approximate solution $z^4$ in the form*

$$z^4 = \sum_{k=1}^{2^{2m}} a_k e_k$$

*it suffices to perform $O(m2^{2m})$ arithmetic operations on the values of Galerkin functionals (16).*

**Corollary 1** *Under the conditions of Theorem 1 the number $Card(IP, \varepsilon)$ of inner products (16) required to construct an approximate solution $z^4$ realizing the accuracy $\varepsilon$ with respect to the norm $\|\cdot\|_X$ has the order*

$$Card(IP, \varepsilon) = O(\varepsilon^{-1/\alpha} \log^{1+1/\alpha} \varepsilon^{-1}). \tag{17}$$

When (17) is compared with (13) it is apparent that for equations (9) with $\mathcal{H}^{\alpha}_{\beta}$, $\varphi \in X^{\alpha}$ the modified Galerkin scheme (15) is more economical than the standard Galerkin method.

Now we apply the modified scheme (15) to the equation (1), (2) considered on the interval $[0, t_1]$.

First of all we rewrite (1), (2) in the form (9), where

$$\varphi(t) = f(t)/p(t), \tag{18}$$

$$Hz(t) = \int_0^t \frac{H(t,\tau)}{(t-\tau)^{1-\alpha}} z(\tau) d\tau, \quad H(t,\tau) = h(t,\tau)/p(t). \tag{19}$$

Thereby as Hilbert space $X$ we take the space $L_2(0, t_1)$ of square-summable functions on $(0, t_1)$ with the usual norm and inner product. Moreover, as $X^\alpha$ we introduce the space $W_2^\alpha(0, t_1)$ of functions $g \in L_2(0, t_1)$ for which

$$||g||_{W_2^\alpha(0,t_1)} := ||g||_{L_2(0,t_1)} + \sup_{0<h<t_1} \frac{\omega_2(g,h)}{h^\alpha} < \infty,$$

where

$$\omega_2(g,h) = \omega_2(g,h;a,b) = \left\{ \sup_{0<\xi<h<b-a} \int_a^{b-\xi} |g(t+\xi) - g(t)|^2 dt \right\}^{1/2}$$

is the integral modulus of continuity of the function $g \in L_2(a,b)$.

If $\{\chi_i(t) = \chi_i(t;a,b)\}_{i=1}^{\infty}$ is the Haar orthonormal basis of piecewise constant functions on the interval $[a,b]$ and $S_m^{(a,b)}$ is the orthogonal projector onto $span\{\chi_1(t;a,b), \chi_2(t;a,b), \ldots, \chi_m(t;a,b)\}$ then it is known [3],p.82, that

$$\|I - S_m^{(a,b)}\|_{W_2^\alpha(a,b)\to L_2(a,b)} \le cm^{-\alpha}. \tag{20}$$

This means that for $X = L_2(0,t_1)$, $X^\alpha = W_2^\alpha(0,t_1)$ and $P_m = S_m^{(0,t_1)}$ the condition (10) holds.

Let $C^1(a,b;c,d)$ be the space of functions $G(t,\tau)$ which are continuously differentiable on $[a,b]\times[c,d]$ with the norm

$$\|G\|_{C^1(a,b;c,d)} = \max_{\substack{a\le t\le b \\ c\le\tau\le d}} \{|G(t,\tau)| + |G_t(t,\tau)| + |G_\tau(t,\tau)|\}.$$

**Lemma 1** *For $H \in C^1(a,b;c,d)$, $0<\alpha<1$, and $t\in[a,b]$ the operators*

$$Hz(t) = \int_a^t \frac{H(t,\tau)}{(t-\tau)^{1-\alpha}} z(\tau)d\tau,$$

$$H^*z(t) = \int_t^b \frac{H(\tau,t)}{(\tau-t)^{1-\alpha}} z(\tau)d\tau$$

*act boundedly from $L_2(a,b)$ into $W_2^\alpha(a,b)$. Moreover,*

$$\max\left\{ \|H\|_{L_2(a,b)\to W_2^\alpha(a,b)}, \ \ \|H^*\|_{L_2(a,b)\to W_2^\alpha(a,b)} \right\} \le c\|H\|_{C^1(a,b;a,b)}.$$

The assertion of the lemma follows immediately from Lemma 31.4 and Theorem 14.2 of [13].

From (7), (8) one can see that the kernel $H(t,\tau) = h(t,\tau)/p(t)$ of the integral operator (19) belongs to the space $C^1(0,t_1;0,t_1)$ and $\varphi(t) = f(t)/p(t)$ belongs to $W_2^\alpha(0,t_1)$. Then by virtue of Lemma 1 the Volterra integral operator (19) belongs to $\mathcal{H}_\beta^\alpha$ for $X = L_2(0,t_1)$, $X^\alpha = W_2^\alpha(0,t_1)$ and for some $\beta$ depending on $d_1, d_2, \ldots, d_4$ (see (7), (8)).

Thus, to the equation (1), (2) considered on the interval $[0, t_1]$ and represented in the form (9), (18), (19), Theorem 1 is applicable. This means that to construct a piecewise constant approximation

$$z_m(t; 0, t_1) = \sum_{k=1}^{2^{2m}} a_k \chi_k(t; 0, t_1)$$

for the solution $z(t) = z(t; 0, t_1)$ of (1), (2) on the interval $[0, t_1]$ realizing the accuracy $\delta$ with respect to the norm $\|\cdot\|_{L_2(0,t_1)}$ it suffices to have no more than

$$Card(IP, \delta) = O(\delta^{-1/\alpha} \log^{1+1/\alpha} \delta^{-1}) \tag{21}$$

values of Galerkin functionals (16), where $(\cdot, \cdot)$ is the inner product in $L_2(0, t_1)$, $e_i = \chi_i(t; 0, t_1)$, and $H, \varphi$ are determined by (18), (19).

Using an argument like that in the proof of Lemma 17.1 [9], we can show that the estimate (21) is order-optimal in the power scale for the class of equations (1), (2) considered on the interval $[0, t_1]$ and having coefficients satisfying the conditions (7), (8).

# 3 The discretization on the interval of ill-posedness

Now we consider the integral equation (1), (2) on the interval $[t_1, t_2]$, where the condition (4) is fulfilled. Moreover, we also admit that the coefficient $p(t)$ vanishes on some subset of $[t_1, t_2]$ having positive Lebesgue measure. As has been shown by E.Schock, [14] in this case the problem of solving this equation on the interval $[t_1, t_2]$ is ill-posed and regularization techniques are required to construct an approximate solution of (1), (2) on $[t_1, t_2]$. In this section we propose one possible approach to such regularization connected with Morozov's discrepancy principle for the method of Tikhonov.

First of all, we assume that

$$h(t, t) \neq 0, \quad t \in [t_1, t_2] \tag{22}$$

and represent (1), (2) in the form

$$p(t)z(t) + \int_{t_1}^{t} \frac{h(t, \tau)}{(t - \tau)^{1-\alpha}} z(\tau) d\tau = f_1(t), \quad t \in [t_1, t_2], \tag{23}$$

where

$$f_1(t) = f(t) - \int_0^{t_1} \frac{h(t,\tau)}{(t-\tau)^{1-\alpha}} z(\tau;0,t_1)d\tau,$$

and $z(t;0,t_1)$ is the solution of (1), (2) on the interval $[0,t_1]$. If $\tilde{z}(t) = z(t;t_1,t_2)$ is the solution of (23) then the first-kind integral equation

$$Az(t) \equiv \int_{t_1}^{t} \frac{h(t,\tau)}{(t-\tau)^{1-\alpha}} z(\tau)d\tau = f_1(t) - p(t)\tilde{z}(t) \tag{24}$$

has the solution $z(t) = z(t;t_1,t_2)$ too. Moreover, from Theorem 31.13 [13] and (22) it follows that $z(t;t_1,t_2)$ is the unique solution of (24). Therefore we can seek an approximate solution of (23) from the first-kind integral equation

$$Az(t) = f_{1,\delta}(t) \tag{25}$$

considered as a perturbed equation for (24). Here

$$f_{1,\delta}(t) = f(t) - \int_0^{t_1} \frac{h(t,\tau)}{(t-\tau)^{1-\alpha}} z_m(\tau;0,t_1)d\tau,$$

and $z_m(t;0,t_1)$ is the piecewise constant approximation for the solution $z(t;0,t_1)$ constructed in the previous section in such a way that

$$\|z(t;0,t_1) - z_m(t;0,t_1)\|_{L_2(0,t_1)} \leq \delta. \tag{26}$$

If, as it is usually in the theory of ill-posed problems, we assume that the solution of (24) can be sourcewise represented, that is, for some $\nu > 0$ and $\rho > 0$

$$z(t;t_1,t_2) = (A^*A)^{\nu} v(t), \quad \|v\|_{L_2(t_1,t_2)} \leq \rho, \tag{27}$$

then the level of perturbation of the right-hand side of (24) is estimated as

$$\|f_1(t) - p(t)z(t;t_1,t_2) - f_{1,\delta}(t)\|_{L_2(t_1,t_2)} \leq$$

$$\leq \| \int_0^{t_1} \tfrac{h(t,\tau)}{(t-\tau)^{1-\alpha}} [z(\tau;0,t_1) - z_m(\tau;0,t_1)]d\tau\|_{L_2(t_1,t_2)} +$$

$$+\delta \|(A^*A)^{\nu} v\|_{L_2(t_1,t_2)} \leq c\delta. \tag{28}$$

Here we used (4), (26), (27) and the fact that the Fredholm integral operator with weakly singular kernel acts boundedly from $L_2(0,t_1)$ into $L_2(t_1,t_2)$.

Now we modify an adaption strategy [5] for discretizing the ill-posed integral equation (25). This strategy, in essence, is as follows: within the framework of a posteriori parameter choice for Tikhonov's regularization, an appropriate discretization in dependence of the regularization parameter has to be chosen.

The same steps as in the previous section lead to the finite-dimensional operator

$$A_{\Gamma_m} = \sum_{k=1}^{2m} (S_{2^k}^{(t_1,t_2)} - S_{2^{k-1}}^{(t_1,t_2)}) A S_{2^{2m-k}}^{(t_1,t_2)} + S_1^{(t_1,t_2)} A S_{2^{2m}}^{(t_1,t_2)}$$

considered as a discretization of the operator $A$ from the left-hand side of (24), (25).

The Tikhonov algorithm with a parameter selection according to the discrepancy principle for solving (24) has the following form:

1. Initialization: $\mu_0, \;\; 0 < q < 1$;

2. Iteration

   (a) $\mu = \mu_k = q^k \mu_0$,

   (b) determine a discretization level $m$ such that

   $$m 2^{-2m\alpha} = c\delta\sqrt{\mu_k}, \tag{29}$$

   (c) compute the inner products

   $$(\chi_i(\cdot; t_1, t_2), f_{1,\delta}(\cdot)), \quad i = 1, 2, \ldots, 2^{2m}, \tag{30}$$

   in $L_2(t_1, t_2)$;

   (d) compute the inner products

   $$(\chi_i(\cdot; t_1, t_2), A\chi_j(\cdot; t_1, t_2)), \quad (i,j) \in \Gamma_m, \tag{31}$$

   required to construct $A_{\Gamma_m}$,

   (e) compute $z^\delta_{\mu_k,m}$ by solving a system of linear algebraic equations corresponding to the equation of Tikhonov's regularization method

   $$\mu_k z + A^*_{\Gamma_m} A_{\Gamma_m} z = A^*_{\Gamma_m} f_{1,\delta} \tag{32}$$

until

$$\|A_{\Gamma_m} z^{\delta}_{\mu_k,m} - f_{1,\delta}\|_{L_2(t_1,t_2)} \le d\delta,$$

where $d > c/q$.

**Theorem 2** *Assume that the solution of (23), (24) can be sourcewise represented in the form (27) for $\nu \in (0, 1/2]$. If $\mu = \mu_N$ satisfies the discrepancy principle*

$$\|A_{\Gamma_m} z^{\delta}_{\mu_k,m} - f_{1,\delta}\|_{L_2(t_1,t_2)} \ge d\delta \quad for \quad k < N, \tag{33}$$

$$\|A_{\Gamma_m} z^{\delta}_{\mu_N,m} - f_{1,\delta}\|_{L_2(t_1,t_2)} \le d\delta \tag{34}$$

*and $m$ is chosen according to (29) for $\mu_k = \mu_N$ then*

$$\|z(\cdot; t_1, t_2) - z^{\delta}_{\mu_N,m}\|_{L_2(t_1,t_2)} \le c\delta^{2\nu/(2\nu+1)}.$$

**Remark.** Without assumption (27) it is possible to prove the convergence $z^{\delta}_{\mu,m} \longrightarrow z(\cdot; t_1, t_2)$ provided $\mu$ is determined by (33), (34) and $m$ is chosen according to (29).

The proof of Theorem 2 is based on the following lemmas.

**Lemma 2** *Let $A$ denote the operator defined by (24). Then*

$$\|A^*A - A^*_{\Gamma_m} A_{\Gamma_m}\|_{L_2(t_1,t_2)\to L_2(t_1,t_2)} \le cm2^{-2m\alpha},$$

$$\|AA^* - A_{\Gamma_m} A^*_{\Gamma_m}\|_{L_2(t_1,t_2)\to L_2(t_1,t_2)} \le cm2^{-2m\alpha},$$

$$\|(A^* - A^*_{\Gamma_m})A\|_{L_2(t_1,t_2)\to L_2(t_1,t_2)} \le cm2^{-2m\alpha}.$$

**Proof.** Taking into account Lemma 1, we find that for $X = L_2(t_1, t_2)$, $X^{\alpha} = W_2^{\alpha}(t_1, t_2)$ the operator $A$ belongs to $\mathcal{H}^{\alpha}_{\beta}$. Then using an argument like that in the proof of Lemma 1 of [10], we get the first estimate of our lemma. The other estimates are established in a similar manner.

**Lemma 3** *Assume that $z(t;t_1,t_2)$ obeys (27). Then for $\nu \in (0,1/2]$*

$$\|z(\cdot;t_1,t_2) - z^{\delta}_{\mu,m}\|_{L_2(t_1,t_2)} \le \frac{\delta}{2\sqrt{\mu}} + \mu^{\nu} c_{\nu,\mu}(v) + \frac{cm2^{-2m\alpha}}{\mu},$$

*where*

$$c_{\nu,\mu}(v) = \mu^{1-\nu}\|(\mu I + A^*A)^{-1} z(\cdot;t_1,t_2)\|_{L_2(t_1,t_2)}.$$

**Proof.** The same steps as in the proof of Lemma 2.5 of [7] lead to the inequality

$$\begin{gathered}\|z(\cdot;t_1,t_2) - z^{\delta}_{\mu,m}\|_{L_2(t_1,t_2)} \le \tfrac{\delta}{2\sqrt{\mu}} + \mu^{\nu} c_{\nu,\mu}(v) + \\ + \|(\mu I + A^*A)^{-1}A^*y - (\mu I + A^*_{\Gamma_m}A_{\Gamma_m})^{-1}A^*_{\Gamma_m} y\|_{L_2(t_1,t_2)},\end{gathered} \tag{35}$$

where

$$y(t) = f_1(t) - p(t) z(t;t_1,t_2).$$

Moreover, from standard estimates using the singular value decomposition of a compact operator $T$ one knows that

$$\begin{gathered}\|(\mu I + T^*T)^{-1}\|_{X\to X} \le \mu^{-1}, \quad \|(\mu I + T^*T)^{-1}T^*\|_{X\to X} \le \tfrac{1}{2\sqrt{\mu}}, \\ \|(\mu I + T^*T)^{-1}T^*T\|_{X\to X} \le 1.\end{gathered} \tag{36}$$

On the other hand, from (27), Lemma 2 and (36) we find that

$$\|(\mu I + A^*A)^{-1}A^*y - (\mu I + A^*_{\Gamma_m}A_{\Gamma_m})^{-1}A^*_{\Gamma_m} y\|_{L_2(t_1,t_2)} \le$$

$$\le \mu^{-1}\|A^*A - A^*_{\Gamma_m}A_{\Gamma_m}\|_{L_2(t_1,t_2)\to L_2(t_1,t_2)} \|(\mu I + A^*A)^{-1}A^*Az(\cdot;t_1,t_2)\|_{L_2(t_1,t_2)} +$$

$$+\mu^{-1}\|(A^* - A^*_{\Gamma_m})Az(\cdot;t_1,t_2)\|_{L_2(t_1,t_2)} \le c\mu^{-1} m 2^{-2m\alpha}\|z(\cdot;t_1,t_2)\|_{L_2(t_1,t_2)} \le$$

$$\le c\frac{m2^{-2m\alpha}}{\mu}. \tag{37}$$

The assertion of the lemma follows from (35)–(37).

**Lemma 4** *If the conditions of Theorem 2 are fulfilled then there exist $d_1, d_2 > 0$ such that*

$$d_1\delta \leq \|Az_{\mu_N} - y\|_{L_2(t_1,t_2)} \leq d_2\delta,$$

*where* $z_{\mu_N} = (\mu_N I + A^*A)^{-1}A^*y, \quad y(t) = f_1(t) - p(t)z(t;t_1,t_2).$

**Proof.** We follow the proof of Lemma 7 and Lemma 10 in [4], [6].
We put $R_\mu(T) = (\mu I + T^*T)^{-1}T^*$. Then

$$Az_{\mu_N} - y = (A_{\Gamma_m} z^\delta_{\mu_N,m} - f_{1,\delta}) - (I - A_{\Gamma_m}R_{\mu_N}(A_{\Gamma_m}))(y - f_{1,\delta}) + \\ +(AR_{\mu_N}(A) - A_{\Gamma_m}R_{\mu_N}(A_{\Gamma_m}))y. \tag{38}$$

Keeping in mind that $T(\mu I + T^*T)^{-1} = (\mu I + TT^*)^{-1}T$ we have

$$(AR_{\mu_N}(A) - A_{\Gamma_m}R_{\mu_N}(A_{\Gamma_m}))y =$$

$$= \mu_N(\mu_N I + A_{\Gamma_m}A^*_{\Gamma_m})^{-1}(AA^* - A_{\Gamma_m}A^*_{\Gamma_m})(\mu_N I + AA^*)^{-1}Az(\cdot;t_1,t_2).$$

Using this formula, (27), (36) and Lemma 2, we obtain the estimate

$$\|(AR_{\mu_N}(A) - A_{\Gamma_m}R_{\mu_N}(A_{\Gamma_m}))y\|_{L_2(t_1,t_2)} \leq$$

$$\leq cm2^{-2m\alpha}\|(\mu_N I + AA^*)^{-1}A\|_{L_2(t_1,t_2)\to L_2(t_1,t_2)}\|z(\cdot;t_1,t_2)\|_{L_2(t_1,t_2)} \leq$$

$$\leq c\frac{m2^{-2m\alpha}}{\sqrt{\mu_N}} \leq c\delta. \tag{39}$$

Moreover, from (28) one sees that

$$\|(I - A_{\Gamma_m}R_{\mu_N}(A_{\Gamma_m}))(y - f_{1,\delta})\|_{L_2(t_1,t_2)} \leq \\ \leq c\mu_N\delta\|(\mu_N I + A_{\Gamma_m}A^*_{\Gamma_m})^{-1}\|_{L_2(t_1,t_2)\to L_2(t_1,t_2)} \leq c\delta. \tag{40}$$

If $\mu = \mu_N$ satisfies (34) then combining (38)–(40), we have

$$\|Az_{\mu_N} - y\|_{L_2(t_1,t_2)} \leq d\delta + c\delta \leq d_2\delta.$$

On the other hand, the same steps as in the proof of Lemma 10 [4] lead to the inequality

$$\|A_{\Gamma_m}z^\delta_{\mu_N,m} - f_{1,\delta}\|_{L_2(t_1,t_2)} \geq qd\delta,$$

where $q$ is the denominator of the geometric progression $\mu_k = q^k \mu_0, k = 1, 2, \ldots, N$. Then combining similarly (38)–(40), by the inverse triangle inequality for $d > c/q$ and for sufficiently large $c$ we have

$$\|Az_{\mu_N} - y\|_{L_2(t_1,t_2)} \geq \|A_{\Gamma_m} z^\delta_{\mu_N,m} - f_{1,\delta}\|_{L_2(t_1,t_2)} - c\delta \geq (qd - c)\delta \geq d_1\delta.$$

The lemma is proved.

**Proof of Theorem 2.** From Lemma 3 it follows that for any $\mu_k$ and $m$ satisfying (29)

$$\|z(\cdot; t_1, t_2) - z^\delta_{\mu_k,m}\|_{L_2(t_1,t_2)} \leq \frac{c\delta}{\sqrt{\mu_k}} + \mu_k^\nu c_{\nu,\mu_k}(v). \quad (41)$$

Moreover, we note that inserting the singular value decomposition shows (see,e.g., [8]) that

$$\|Az_{\mu_k} - y\|^2_{L_2(t_1,t_2)} = \mu_k^{2\nu+1} d^2_{\mu_k,\nu}(v), \quad (42)$$

where $d_{\mu_k,\nu}(v)$ itself is bounded for $0 < \nu \leq 1/2$ and

$$c_{\nu,\mu_k}(v)\left\{d_{\mu_k,\nu}(v)\right\}^{-2\nu/(2\nu+1)} \leq c. \quad (43)$$

Now if $\mu_N$ satisfies (34) and $m$ is chosen according to (29) for $\mu_k = \mu_N$ then from Lemma 4 and (42), (43) we find that

$$\frac{\delta}{\sqrt{\mu_N}} = \delta \left\{ \frac{d_{\mu_N,\nu}(v)}{\|Az_{\mu_N} - y\|_{L_2(t_1,t_2)}} \right\}^{1/(2\nu+1)} \leq$$

$$\leq \delta \left\{ \frac{d_{\mu_N,\nu}(v)}{d_1\delta} \right\}^{1/(2\nu+1)} \leq c\delta^{2\nu/(2\nu+1)}, \quad (44)$$

$$\mu_N^\nu c_{\nu,\mu_N}(v) = c_{\nu,\mu_N}(v) \left( \frac{\|Az_{\mu_N} - y\|_{L_2(t_1,t_2)}}{d_{\mu_N,\nu}(v)} \right)^{2\nu/(2\nu+1)} \leq$$

$$\leq c_{\nu,\mu_N}(v)\left\{d_{\mu_N,\nu}(v)\right\}^{-2\nu/(2\nu+1)} (d_2\delta)^{2\nu/(2\nu+1)} \leq c\delta^{2\nu/(2\nu+1)}. \quad (45)$$

The assertion of the theorem follows from (41), (44), (45) .

To estimate the number $Card(IP, \delta^{2\nu/(2\nu+1)})$ of inner products of the form (30), (31) required to construct an approximate solution $z^\delta_{\mu_N,m}$ realizing the accuracy $\delta^{2\nu/(2\nu+1)}$ with respect to the norm $\|\cdot\|_{L_2(t_1,t_2)}$ we assume that $\mu_N$ satisfying (34) has the order $\delta^{2-2\lambda}$ for some $\lambda \in (0, 1/2)$. This is a sufficiently natural assumption because (see, e.g.[1]) the regularization parameter $\mu$ is normally chosen in dependence of $\delta$ such that

$$\lim_{\delta\to 0} \delta^2 \mu^{-1} = 0, \quad \lim_{\delta\to 0} \mu = 0.$$

Keeping in mind (29) for $\mu_N = q^N \mu_0 = O(\delta^{2-2\lambda})$, we have $N = O(\log \frac{1}{\delta})$, $m2^{2m} = O(\delta^{-(2-\lambda)/\alpha} \log^{1+1/\alpha} \frac{1}{\delta})$. Then the total number $Card(IP, \delta^{2\nu/(2\nu+1)})$ of inner products of the form (30), (31) required to construct an approximate solution with accuracy $\delta^{2\nu/(2\nu+1)}$ within the framework of the algorithm (29)–(32) is no more than

$$Card(IP, \delta^{2\nu/(2\nu+1)}) \le Nm2^{2m} = O\left(\delta^{-(2-\lambda)/\alpha} \log^{2+1/\alpha} \frac{1}{\delta}\right). \qquad (46)$$

To illustrate some advantages of considering (23) as an ill-posed problem we assume for the moment that $p(t) \equiv \delta^q$, $t \in [t_1, t_2]$ and apply to (23) the modified scheme (15) which is order-optimal in the sense of amount of used Galerkin information for Volterra integral equations of the second kind with weakly singular kernels. Then by virtue of Theorem 1 and Corollary 1, for $\varphi = f_1/\delta^q$, $\varepsilon = \delta^{2\nu/(2\nu+1)}$, we have

$$Card(IP, \delta^{2\nu/(2\nu+1)}) = O\left(\delta^{-q-\frac{2\nu}{(2\nu+1)\alpha}} \log^{1+1/\alpha} \frac{1}{\delta}\right). \qquad (47)$$

When (47) is compared with (46) it is apparent that, for example, for $q \ge 2/\alpha$ the discretization scheme (29)–(32) is more efficient than (15) even if $p(t) \ne 0$, $t \in [t_1, t_2]$.

# 4 The discretization on the next interval of well-posedness

In line with our assumptions (7), (8) $|p(t)| \ge d_1$ on the next interval $[t_2, 1]$ and we can rewrite (1), (2) in the form (9) again, where

$$\varphi(t) = \frac{1}{p(t)} \left[ f(t) - \int_0^{t_1} \frac{H(t,\tau) z(\tau; 0, t_1)}{(t-\tau)^{1-\alpha}} d\tau - \int_{t_1}^{t_2} \frac{H(t,\tau) z(\tau; t_1, t_2)}{(t-\tau)^{1-\alpha}} d\tau \right], \qquad (48)$$

$$Hz(t) = \int_{t_2}^{t} \frac{H(t,\tau)}{(t-\tau)^{1-\alpha}} z(\tau) d\tau, \quad H(t,\tau) = h(t,\tau)/p(t). \tag{49}$$

As the perturbed equation for (9), (48), (49) we take the equation

$$z(t) + \int_{t_2}^{t} \frac{H(t,\tau)}{(t-\tau)^{1-\alpha}} z(\tau) d\tau = \varphi_\delta(t), \tag{50}$$

where

$$\varphi_\delta(t) = \frac{1}{p(t)} \left[ f(t) - \int_0^{t_1} \frac{H(t,\tau) z_m(\tau;0,t_1)}{(t-\tau)^{1-\alpha}} d\tau - \int_{t_1}^{t_2} \frac{H(t,\tau) z^\delta_{\mu_N,m}(\tau)}{(t-\tau)^{1-\alpha}} d\tau \right].$$

Then by virtue of (26) and Theorem 2 we find

$$\|\varphi - \varphi_\delta\|_{L_2(t_2,1)} \le c\delta^{2\nu/(2\nu+1)}.$$

Here $\nu$ is the parameter of sourcewise representation (27). Keeping in mind that the operator from left-side of equations (9), (48), (49) and (50) has the inverse operator which acts boundedly from $L_2(t_1,1)$ into $L_2(t_1,1)$, for solutions $z(t;t_2,1)$ and $z_\delta(t;t_2,1)$ of these equations we have

$$\|z(\cdot;t_2,1) - z_\delta(\cdot;t_2,1)\|_{L_2(t_2,1)} \le c\|\varphi - \varphi_\delta\|_{L_2(t_2,1)} \le c\delta^{2\nu/(2\nu+1)}.$$

By virtue of the same reasons as in the section 2, we find that the Volterra integral operator from (50) belongs to $\mathcal{H}^\alpha_\beta$ for $X = L_2(t_2,1)$, $X^\alpha = W^\alpha_2(t_2,1)$ and $\varphi_\delta$ belongs to $W^\alpha_2(t_2,1)$. Thus, from Theorem 1 it follows that to construct a piecewise constant approximation $z_{\delta,m}(t;t_2,1)$ for the solution $z_\delta(t;t_2,1)$ of (50) realizing the accuracy $\delta^{2\nu/(2\nu+1)}$ with respect to the norm $\|\cdot\|_{L_2(t_2,1)}$ it suffices to have no more than

$$Card(IP, \delta^{2\nu/(2\nu+1)}) = O\left(\delta^{-\frac{2\nu}{(2\nu+1)\alpha}} \log^{1+1/\alpha} \frac{1}{\delta}\right) \tag{51}$$

values of Galerkin functionals (16), where $(\cdot,\cdot)$ is the inner product in $L_2(t_2,1)$, $e_i = \chi_i(t;t_2,1)$ and $H$, $\varphi = \varphi_\delta$ are determined by (50).

In such a way, the piecewise constant function

$$z_m(t) = \begin{cases} z_m(t;0,t_1), & t \in [0,t_1) \\ z^\delta_{\mu_N,m}(t), & t \in [t_1,t_2) \\ z_{\delta,m}(t;t_2,1), & t \in [t_2,1] \end{cases}$$

gives the approximate solution of (1), (2) with accuracy $O(\delta^{2\nu/(2\nu+1)})$ in regard to the norm $\|\cdot\|_{L_2(0,1)}$. From (21), (46) and (51) it follows that to construct this approximate solution it suffices to have no more than $O(\delta^{-(2-\lambda)/\alpha}\log^{2+1/\alpha}\frac{1}{\delta})$ values of Galerkin functionals of the form (16), where $\lambda \in (0,1/2)$ is determined in the act of choosing the regularization parameter within the framework of Morozov's discrepancy principle.

# References


[1] Engl, H. W. (1981), Necessary and sufficient conditions for convergence of regularization methods for solving linear operator equations of the first kind, *Numer.Funct.Anal.Optim.* **3**, 201–222.

[2] Hilbert, D. (1912), *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig.

[3] Kashin, B. S., and Saakian, A. A. (1984), *Orthogonal Seriese*, Nauka, Moscow.

[4] Maaß, P.,and Rieder, A. (1996), Wavelet-Accelerated Tikhonov-Phillips regularization with applications, Preprint, Universität Potsdam.

[5] Maaß, P., Pereverzev, S. V., and Solodky, S. G. (1997), An adaptive discretization for Tikhonov-Phillips regularization with discrepancy principle of parameter selection, Preprint, Universität Potsdam (in preparation).

[6] Maaß, P.,and Ramlau, R. (1997), Wavelet accelerated regularization methods for hyperthermia treatment planning, Preprint, Universität Potsdam.

[7] Neubauer, A. (1988), An a posteriori parameter choice for Tikhonov regularization in the presence of modeling error, *Apl.Num.Math.* **4**, 507–519.

[8] Neubauer, A. (1988), An a posteriori parameter selection choice for Tikhonov regularization in Hilbert scales leading to optimal convergence rates, *SIAM J.Numer.Anal.* **25**, 1313–1326.

[9] Pereverzev, S. V. (1996), *Optimization of Methods for Approximate Solution of Operator Equations*, Nova Science Publishers, New York.

[10] Pereverzev, S. V. (1995), Optimization of Projection Methods for Solving ill-posed problems, *Computing* **55**, 113–124.

[11] Prössdorf, S. (1978), *Some classes of singular equations*. North–Holland Publ.Comp., Amsterdam, New York, Oxford.

[12] Prössdorf, S., and Silbermann, B. (1991), Numerical Analysis for Integral and Related Operator Equations, Akademie Verlag Berlin, and Birkhäuser Verlag Basel, Boston, Stuttgart.

[13] Samko, S. G., Kilbas, A. A., and Marichev, O. I. (1987), *Integrals and derivatives of fractional order and some of their applications*, Nauka i Technika, Minsk.

[14] Schock, E. (1985), Integral equations of the third kind, *Studia Mathematica* **81**, 1–11.