

Approximation and Commutator Properties of Projections onto
Shift-Invariant Subspaces and Applications to
Boundary Integral Equations

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Abstract. The main purpose of the present paper is to prove approximation and commutator properties for projections mapping periodic Sobolev spaces onto shift-invariant spaces generated by a finite number of compactly supported functions. With these prerequisites at hand and using certain localization techniques, we then characterize the stability of generalized Galerkin-Petrov schemes for solving periodic pseudodifferential equations in terms of elliptic type estimates of the numerical symbol. Moreover, we establish optimal convergence rates for the approximate solutions with respect to the Sobolev norms.

1 Introduction

It is well known that one of the central problems of the numerical analysis for pseudodifferential equations is to find conditions ensuring the stability of the numerical scheme in consideration. One possible approach to stability analysis for variable symbols is a reduction to the case of constant symbols by means of certain localization techniques which could be viewed as a numerical counterpart to the well known principle of freezing coefficients in the theory of partial differential equations. The main ingredients for applying such techniques are certain superapproximation results for the projections defining the numerical schemes.

Of course, the basic idea of localizing techniques has a long history in theory as well as in the numerical analysis of partial differential equations. The first papers addressing this particular aspect seem to be [Si1], [Si2], where classical Galerkin schemes with trigonometric trial functions for singular integral equations are investigated. The analysis of piecewise linear spline collocation for Cauchy singular integral equations in [PS] already involved implicitly certain *discrete commutator properties* and localization arguments as well which also played then a crucial role in various subsequent papers treating one- and multidimensional problems (see e.g. [AW2], [P], [PR1], [PR2], [S2], [CM1], [PSr1], [PSr2], [DPS], [MP], [SW], [NS1]). An explicit abstract formulation of these principles was given in [P1], [P]. For an overview of the various univariate results see also [PSi].

The main purpose of the present paper is to prove approximation and commutator properties for projections mapping periodic Sobolev spaces onto shift-invariant spaces generated by a finite number of compactly supported functions. (Note that such spaces are frequently used as trial spaces in numerical procedures for engineering applications, see [MP].) To our knowledge, these results (see Theorems 2.1 through 2.4) are new in the present form and generality and should be of some independent interest. In particular, they cover all results in the univariate case and for uniform grids known in the literature (e.g. for spline or classical wavelet spaces). With these prerequisites at hand we characterize in the last section the stability of generalized Galerkin-Petrov schemes defined by the aforementioned projections in terms of elliptic type estimates of the numerical symbol. Further, we establish optimal convergence rates for the approximate solutions with respect to the Sobolev norms. Note that our results can be extended to the corresponding spaces of functions defined on \mathbb{R}^n or \mathbb{T}^n , respectively. Moreover, the estimates obtained in the present paper are important in the case of function spaces over subdomains of \mathbb{R}^n , since the crucial ingredients of our analysis are local in nature.

2 Notations and the main approximation results

Let us denote by $\|\cdot\|_t$ the norm of the periodic Sobolev space $H^t(\mathbb{T})$ of order $t \in \mathbb{R}$ on the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. The main ingredients for applying localization techniques to pseudodifferential equations with variable symbols are certain superapproximation results for the projections $(P_m)_{m \in \mathbb{N}_0}$ defining the numerical schemes. These results are sometimes referred to as *discrete commutator properties* and have the form

$$\|(1 - P_m) f P_m u\|_s \leq c 2^{-m(t-s+\delta)} \|P_m u\|_t \quad (\text{CP I})$$

as well as

$$\|P_m f (1 - P_m) u\|_s \leq c \delta_m 2^{-m(t-s)} \|u\|_t \quad (\text{CP II})$$

where c and δ are positive constants independent of $u \in H^t(\mathbb{T})$ (but depending on s and t , in general). The orders t and s satisfy $t \geq s$ and are restricted by the choice of the projections. Here f is a smooth periodic function, $(\delta_m)_{m \in \mathbb{N}_0}$ tends to zero as $m \rightarrow \infty$ and P_m is the projection onto a finite dimensional space defined over a uniform mesh with mesh size $h = 2^{-m}$. In view of multiresolution analysis, we have chosen the step size to be a power of two. From now on, we use the letter c to denote a generic constant the value of which varies from instance to instance. In addition, we say that the projections $(P_m)_{m \in \mathbb{N}_0}$ have the *approximation property* for $s \leq t$ if and only if

$$\|(1 - P_m) u\|_s \leq c 2^{-m(t-s)} \|u\|_t \quad (\text{AP})$$

holds for any $u \in H^t(\mathbb{T})$. Another important property is the *inverse property* for $s \leq t$ which means

$$\|P_m u\|_t \leq c 2^{m(t-s)} \|P_m u\|_s. \quad (\text{IP})$$

The main purpose of this paper is to prove (CP I) and (CP II) for appropriate families of projections. For $M \in \mathbb{N}$, let

$$\Lambda_M := \mathbb{Z} \cap \left[-\frac{M}{2}, \frac{M}{2}\right).$$

We choose a sequence $\phi := (\phi^j)_{j \in \Lambda_M}$ of generators for the spaces of approximating functions satisfying

$$\phi^j \in \mathcal{L}_2 := \left\{ f \in L^2(\mathbb{R}) : \sum_{k \in \mathbb{Z}} |f(\cdot + k)| \in L^2([0, 1]) \right\}$$

for $j \in \Lambda_M$. It is easy to see that $\mathcal{L}_2 \subseteq L^1(\mathbb{R})$ and therefore $\hat{f} \in \mathcal{C}(\mathbb{R})$ for $f \in \mathcal{L}_2$, where

$$\hat{f}(x) := \int_{\mathbb{R}} e^{-2\pi i x y} f(y) dy$$

is the Fourier transform on $L^2(\mathbb{R})$. For the step size $h := \frac{1}{N}$, $N := 2^m$ with $m \in \mathbb{N}_0$, we define the approximation spaces

$$S_m(\phi) := \text{Lin} \{ \phi_{k,m}^j := 2^{\frac{m}{2}} [\phi^j(2^m \cdot -k)] : k \in \Lambda_N, j \in \Lambda_M \} . \quad (1)$$

Hereby the periodization operator $[\cdot]$ is given by $[f] := \sum_{l \in \mathbb{Z}} f(\cdot + l)$ for $f \in \mathcal{L}_2$.

Next, we consider a family of distributions $\eta := (\eta^j)_{j \in \Lambda_M} \in (H^{-s'}(\mathbb{R}))^M$, $s' \geq 0$, with compact support to define the functionals

$$\eta_{k,m}^l(f) := 2^{-\frac{m}{2}} \eta^l(f(2^{-m}(\cdot + k))) , \quad l \in \Lambda_M, k \in \Lambda_N , \quad (2)$$

for $f \in H^{s'}(\mathbb{T})$. We require that $S_m(\phi) \subseteq H^{s'}(\mathbb{T})$ and

$$\eta^r(\phi^j(\cdot - k)) = \delta_{r,j} \delta_{0,k} \quad (3)$$

for $r, j \in \Lambda_M$, $k \in \mathbb{Z}$. Therefore, the operators

$$Q_m(f) := \sum_{\substack{k \in \Lambda_N \\ l \in \Lambda_M}} \eta_{k,m}^l(f) \phi_{k,m}^l \quad (4)$$

are projections defined for sufficiently smooth functions f . From now on, we denote by P_m the orthogonal projections onto $S_m(\phi)$ in $H^0(\mathbb{T}) = L^2(\mathbb{T})$. We remark that these projections have a representation like (4), too, provided the functions $\phi := (\phi^j)_{j \in \Lambda_M}$ have compact support and their integer translates form a Riesz bases. Indeed, we have

$$P_m(u) = \sum_{\substack{k \in \Lambda_N \\ l \in \Lambda_M}} \langle u, \phi_{k,m}^l \rangle \psi_{k,m}^l$$

for $u \in L^2(\mathbb{T})$ with $(\psi^j)_{j \in \Lambda_M}$ defined by

$$\begin{aligned} \left(\hat{\psi}^j(pM + l + x) \right)_{l \in \Lambda_M, j \in \Lambda_M} &:= \\ \left(\hat{\phi}^j(pM + l + x) \right)_{l \in \Lambda_M, j \in \Lambda_M} &\left(\sum_{m \in \mathbb{Z}} \overline{\hat{\phi}^r(m + x)} \hat{\phi}^j(m + x) \right)_{r \in \Lambda_M, j \in \Lambda_M}^{-1} \end{aligned} \quad (5)$$

for $p \in \mathbb{Z}$. We will require the linear independence of the generators which is stronger than Riesz stability. A family of compactly supported functions $\phi := (\phi^j)_{j \in \Lambda_M}$ is called *linearly independent* if the mapping $((c_k^j)_{j \in \Lambda_M})_{k \in \mathbb{Z}} \mapsto \sum_{j \in \Lambda_M, k \in \mathbb{Z}} c_k^j \phi^j(\cdot - k)$ is injective on the space of sequences of M -dimensional complex vectors. It is known (cf. [DBDVR]) that in the univariate case there always exist linearly independent generators if the spaces $S_m(\phi)$ are generated by compactly supported functions. By definition the functions $\phi := (\phi^j)_{j \in \Lambda_M}$ are said to satisfy the *Strang-Fix* condition of order $d \in \mathbb{N}$ if there exists a finite

linear combination $\tilde{\phi}$ of integer translates of the $(\phi^j)_{j \in \Lambda_M}$ which fulfils the usual Strang-Fix condition of order d , i.e. $\tilde{\phi}^{(k)}(l) = 0$ for any $l \in \mathbb{Z} \setminus \{0\}$, $k = 0, \dots, d-1$, and $\tilde{\phi}(0) \neq 0$. The integer translates of $\tilde{\phi}$ reproduce algebraic polynomials up to degree $d-1$ if $\tilde{\phi}$ is a compactly supported continuous function of bounded variation satisfying the Strang-Fix condition of order d (cf. [DeVL]).

For the approximating functions, we require

Hypothesis: *There exist compactly supported functions $\gamma := (\gamma^j)_{j \in \Lambda_M}$ and $M \times M$ matrices $\omega := (\omega_l)_{l \in \mathbb{Z}}$ with exponential decay such that both $\Pi(x) := \sum_{l \in \mathbb{Z}} \omega_l e^{2\pi i l x}$ is invertible on $[0, 1]$ and*

$$(\phi^j)_{j \in \Lambda_M} = \omega *'_M (\gamma^j)_{j \in \Lambda_M} := \sum_{l \in \mathbb{Z}} \omega_l (\gamma^j(\cdot - l))_{j \in \Lambda_M}. \quad (6)$$

The integer translates of γ have to be linearly independent. Further, there exist numbers $0 < \rho < 1$, $d, d' \in \mathbb{N}_0$ with $d' \geq d$ satisfying

1. $S_m(\phi) \subseteq H^s(\mathbb{T}) \cap H^{s'}(\mathbb{T})$ for any $s < d + \rho$, $m \in \mathbb{N}_0$ where s' is the Sobolev index of the functionals η ;
2. the integer translates of ϕ reproduce algebraic polynomials up to degree $d' \in \mathbb{N}_0$;
3. $S_m(\phi)$ fulfil the inverse property, i.e. $\|u_m\|_t \leq c 2^{m(t-s)} \|u_m\|_s$ for any $u_m \in S_m(\phi)$, $s \leq t < d + \rho$.

The third condition implies the inverse property of the projections and is fulfilled for generators satisfying Definition 4.4 (cf. also Theorem 3.1 in [PSch]). In particular, splines of order r and defect M , i.e. piecewise polynomials of degree less than or equal to $r-1$ which are $r-M-1$ times continuously differentiable, fulfil the inverse property for $s \leq t < r - M + \frac{1}{2}$ (cf. Example 3.3 in [PSch]). Further, if ϕ in (5) satisfies the Hypothesis then so does ψ .

Now we are in the position to formulate the main theorems.

Theorem 2.1 *Let ϕ fulfil the Hypothesis. Then the projections $(Q_m)_{m \in \mathbb{N}_0}$ defined by (4) satisfy (AP) and (CP II) for $s = 0$ and any $s' \leq t \leq d' + 1$.*

In contrast to [DPS], to prove Theorem 2.1 we do not need any refinability of the generators, i.e. the spaces $S_m(\phi)$ are not required to be nested.

Next, we give a generalization under the additional assumptions that the projections are orthogonal in $H^0(\mathbb{T})$ and the spaces $S_m(\phi)$ are nested, i.e. in the case when ϕ is refinable.

Theorem 2.2 *Let ϕ satisfy the Hypothesis and let $S_m(\phi) \subseteq S_{m+1}(\phi)$ for $m \in \mathbb{N}$. Then the orthogonal projections $(P_m)_{m \in \mathbb{N}_0}$ onto $S_m(\phi)$ fulfil the approximation property (AP) for $-d' - 1 \leq s < d + \rho$, $-d - \rho < t \leq d' + 1$ and $s \leq t$.*

Note that Theorem 2.2 and the subsequent Corollaries 2.3, 2.5, 2.8 and Theorem 2.4 generalize the results obtained in [DPS], Sect. 5, for $M = 1$, at least in the univariate case. Moreover, here the range of Sobolev indices is larger than in the aforementioned paper.

If the generators are splines of order r with knots of multiplicity M , then we are in the situation of [MP]. In this case, Theorem 2.2 is valid with $d' = r - 1$, $d = r - M$ and $\rho = \frac{1}{2}$. Then the results can be obtained from Theorem 3.4 in [MP].

Corollary 2.3 *Let ϕ satisfy the Hypothesis and let $S_m(\phi) \subseteq S_{m+1}(\phi)$ for $m \in \mathbb{N}$. Then the projections $(Q_m)_{m \in \mathbb{N}_0}$ defined by (4) satisfy (AP) for any $s \leq t$ with $s' \leq t \leq d' + 1$, $0 \leq s < d + \rho$.*

Proof. Using the inverse estimate we conclude that

$$\|(1 - Q_m)u\|_s \leq \|(1 - P_m)u\|_s + c 2^{ms} \|(1 - Q_m)u\|_0 \leq c 2^{-m(t-s)} \|u\|_t \quad (7)$$

for $u \in H^t(\mathbb{T})$. ■

Now we turn to the commutator property (CP I) for the orthogonal projections.

Theorem 2.4 *Let ϕ satisfy the Hypothesis and let the orthogonal projections $(P_m)_{m \in \mathbb{N}_0}$ onto $S_m(\phi)$ fulfil the approximation property for $s \leq t$ with $0 \leq t \leq d' + 1$ and $0 \leq s < d + \rho$. Then the orthogonal projections $(P_m)_{m \in \mathbb{N}_0}$ have the following properties:*

- i) (CP I) is valid for $-d' - 1 \leq s \leq t < d + \rho$;*
- ii) (CP II) is valid for $s \leq t \leq d' + 1$ and $-d - \rho < s < d + \rho$.*

Using the boundedness of the projections $(Q_m)_{m \in \mathbb{N}_0}$ in $\|\cdot\|_s$ for $s' \leq s < d + \rho$, we infer from Theorem 2.4

Corollary 2.5 *Let ϕ fulfil the Hypothesis and let $S_m(\phi) \subseteq S_{m+1}(\phi)$ for $m \in \mathbb{N}$. Then the projections $(Q_m)_{m \in \mathbb{N}_0}$ satisfy (CP I) for $s' \leq s \leq t < d + \rho$.*

Proof. The assertion follows directly from the identity

$$1 - Q_m = (1 - P_m) + Q_m(P_m - 1). \quad (8)$$

The proofs of Theorems 2.1, 2.2 and 2.4 are deferred until the next section. ■

The interpolation property of Sobolev spaces allows to reduce the proofs of (CP I) and (CP II) from different orders of Sobolev spaces to the order $s = 0$.

Proposition 2.6 *Let $(R_m)_{m \in \mathbb{N}_0}$ be a family of projections and let $0 \leq a \leq b \leq c$ be real numbers such that for any*

$$0 \leq s < b, \quad a \leq t \leq c \quad \text{with } s \leq t \quad (9)$$

the approximation property (AP) and for $0 \leq s \leq t < b$ the inverse property (IP) are satisfied. Then we infer from (CP I) (resp. (CP II)) for $s = 0$ and $a \leq t \leq c$ that (CP I) (resp. (CP II)) is valid for any s and t restricted by (9).

Proof. Let (CP I) be satisfied with $s = 0$ and $a \leq t \leq c$. Assume $0 < s < b$ and $s \leq t$. For $s < t$ choose $\epsilon > 0$ such that $s + \epsilon < t$ and $s + \epsilon < b$. We obtain

$$\|(1 - R_m) f R_m u\|_{s+\epsilon} \leq c 2^{-m(t-s-\epsilon)} \|R_m u\|_t \quad (10)$$

by the approximation property. In the case $s = t$, we conclude (10) from the uniform boundedness of the projections in $\|\cdot\|_{s+\epsilon}$ with $s + \epsilon < b$ and from (IP). Using the interpolation inequality

$$\|\cdot\|_s \leq c \|\cdot\|_0^{1-\frac{s}{s+\epsilon}} \|\cdot\|_{s+\epsilon}^{\frac{s}{s+\epsilon}} \quad (\text{IE})$$

we conclude from (CP I) for $s = 0$ and (10) that

$$\|(1 - R_m) f R_m u\|_s \leq c 2^{-m(t-s)} 2^{-m\delta \frac{\epsilon}{s+\epsilon}} \|R_m u\|_t.$$

For (CP II) the assertion follows directly from the inverse property. ■

Let $(\tilde{R}_m)_{m \in \mathbb{N}_0}$ be another family of projections with the same ranges as $(R_m)_{m \in \mathbb{N}_0}$, i.e. $\mathcal{R}(\tilde{R}_m) = \mathcal{R}(R_m)$ for any $m \in \mathbb{N}_0$, and such that $\|\tilde{R}_m\|_0 \leq c$. Then all properties excepting (CP II) remain valid with $(R_m)_{m \in \mathbb{N}_0}$ replaced by $(\tilde{R}_m)_{m \in \mathbb{N}_0}$. More precisely, because of (8), the following holds.

Corollary 2.7 *If the $(R_m)_{m \in \mathbb{N}_0}$ fulfil the assumptions of Proposition 2.6 then so do the $(\tilde{R}_m)_{m \in \mathbb{N}_0}$. Further we infer from (CP I) for $(R_m)_{m \in \mathbb{N}_0}$ and $s = 0$, $a \leq t \leq c$ that (CP I) is valid for $(\tilde{R}_m)_{m \in \mathbb{N}_0}$ and any s and t restricted by (9).* ■

Further, we infer from Proposition 2.6, Theorem 2.1 and Corollary 2.3

Corollary 2.8 *Let ϕ satisfy the Hypothesis and let $S_m(\phi) \subseteq S_{m+1}(\phi)$ for $m \in \mathbb{N}$. Then the projections $(Q_m)_{m \in \mathbb{N}_0}$ defined by (4) satisfy (CP II) for any $s \leq t$ with $s' \leq t \leq d' + 1$, $0 \leq s < d + \rho$.* ■

Remark 2.9 Let $(R_m)_{m \in \mathbb{N}_0}$ be a family of projections satisfying (CP I) and (CP II). Let $L : H^t(\mathbb{T}) \rightarrow H^{t-r}(\mathbb{T})$ be a bounded operator with $r \in \mathbb{R}$ given. Using the identity

$$R_m f L - f R_m L = R_m f (1 - R_m) L - (1 - R_m) f R_m L ,$$

we obtain from (CP I) and (CP II) that

$$\|R_m f L u - f R_m L u\|_{s-r} \leq c 2^{-m(t-s)} \delta_m \|u\|_t \quad (11)$$

is valid for the corresponding Sobolev indices $s \leq t$, where the constant does not depend on $u \in H^t(\mathbb{T})$.

Property (11) has been proved in [SW] for the case of quolocation projections R_m , periodic pseudodifferential operators L and functions u belonging to subspaces of smoothest splines. In this case, property (CP I) follows from Theorem 2.1 in [SW], however, Theorem 2.1 can not be applied, since the functionals occurring in the quolocation do not have compact support. Nevertheless, it is our conjecture that (CP II) is satisfied for quolocation projections, too. If so, then, for the quolocation method studied in [SW], results similar to Theorems 4.8 and 4.10 of the present paper can be derived.

3 Proof of the Theorems

The proof of Theorem 2.1 is based on techniques of [DPS], but uses no refinability. It will be divided into two steps. We show first (AP) and then (CP II). To this end we need the l th forward differences of u defined by

$$(\Delta_h^l u)(x) := \sum_{j=0}^l \binom{l}{j} (-1)^{l-j} u(x + jh)$$

for $h \in \mathbb{R}$. As usually $\|\cdot\|_0(\Omega)$ denotes the L^2 -norm relative to some domain $\Omega \subseteq \mathbb{R}$. The corresponding l th order modulus of continuity is given by

$$\omega_l(u, t, \Omega) := \sup_{|h| \leq t} \|\Delta_h^l u\|_0(\Omega_{h,l}) ,$$

where

$$\Omega_{h,l} := \{x \in \Omega : x + jh \in \Omega, j = 0, \dots, l\} .$$

For $\Omega = \mathbb{T}$ we write $\omega_l(u, t)$ instead of $\omega_l(u, t, \mathbb{T})$. Now we are ready to introduce the Besov norms

$$\|u\|_{B_{2,2}^t(\Omega)} := \|u\|_0(\Omega) + |u|_t(\Omega)$$

where for any fixed $l \in \mathbb{N}$, $l > t$

$$|u|_t(\Omega)^2 := \sum_{j=0}^{\infty} 2^{2jt} \omega_l(u, 2^{-j}, \Omega)^2 .$$

In the following we need the norm equivalence (see e.g. [DP], [T])

$$\|\cdot\|_t(\Omega) \sim \|\cdot\|_{B_{2,2}^t(\Omega)} \quad (12)$$

for any $0 < t < l$ where Ω is an interval. We remind that $\eta \in H^{-s'}(\mathbb{R})$ with support in $\Gamma := (-a, a)$ for some fixed $a \in \mathbb{N}$.

Proof of Theorem 2.1. Step 1. For any $t \geq s'$ we obtain from (12)

$$\begin{aligned} |\eta_{k,m}^l(u)|^2 &= |2^{-\frac{m}{2}} \eta^l(u(2^{-m}(\cdot + k)))|^2 \leq c 2^{-m} \|u(2^{-m}(\cdot + k))\|_t^2(\Gamma) \\ &\leq c \left(\|u\|_0(\Gamma_k^m)^2 + \sum_{j=0}^{\infty} 2^{2tj} \omega_{d+1}(u, 2^{-m-j}, \Gamma_k^m)^2 \right) \\ &\leq c \left(\|u\|_0(\Gamma_k^m)^2 + 2^{-2tm} \|u\|_{B_{2,2}^t(\Gamma_k^m)}^2 \right) \end{aligned} \quad (13)$$

with $\Gamma_k^m := 2^{-m}(k + \Gamma)$. By our hypothesis

$$(\phi_{k,m}^j)_{j \in \Lambda_M} = \sum_{i \in \Lambda_N} \omega_{i-k}^m (\gamma_{i,m}^j)_{j \in \Lambda_M} \quad (14)$$

is valid with

$$\omega_k^m := \sum_{p \in \mathbb{Z}} \omega_{pN+k}$$

where $N = 2^{-m}$. Using the exponential decay of the coefficients, one concludes that

$$\sum_{k \in \Lambda_N} \sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\| \leq c , \quad \sum_{l \in \Lambda_N} \sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\| \leq c \quad (15)$$

with $\square_{l,\gamma}^m := \{i \in \Lambda_N : \cup_{j \in \Lambda_M} \text{supp} \gamma_{i,m}^j \cap \square_l^m \neq \emptyset\}$, $\square := [-\frac{1}{2}, \frac{1}{2}]$ and $\square_l^m := 2^{-m}(l + \square)$.

Lemma 3.1 For $0 \leq s \leq t$, $s' \leq t$, $s < d + \rho$, $u \in H^t(\mathbb{T})$ we have

$$\|Q_m u\|_{s(\square_l^m)} \leq c 2^{ms} \sum_{k \in \Lambda_N} \left(\sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\| \right) \left(\|u\|_0(\Gamma_k^m) + 2^{-mt} \|u\|_{B_{2,2}^t(\Gamma_k^m)} \right) .$$

Proof. From our hypothesis and (14) we deduce that

$$\|\phi_{k,m}^j\|_s(\square_l^m) \leq c 2^{ms} \sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\| .$$

Since

$$\|Q_m u\|_s(\square_l^m) \leq \sum_{\substack{k \in \Lambda_N \\ j \in \Lambda_M}} |\eta_{k,m}^j(u)| \|\phi_{k,m}^j\|_s(\square_l^m) ,$$

the assertion follows from (13). ■

Because of the linear independence of the integer translates of γ there exist functionals $(F^j)_{j \in \Lambda_M}$ with compact support $\mathcal{F} := \cup_{j \in \Lambda_M} \text{supp } F^j$ such that

$$|F^j(g)| \leq \int_{\mathcal{F}} |g(x)|^2 dx$$

for $g \in L^2(\mathbb{R})$ and that

$$G_m u := \sum_{\substack{k \in \Lambda_N \\ j \in \Lambda_M}} F_{k,m}^j(u) \gamma_{k,m}^j$$

are projections (cf. [B-AR]).

Lemma 3.2 *Let $\Omega \subseteq \mathbb{R}$ be a fixed bounded interval. Then we have, for $0 \leq s \leq t \leq d' + 1$ with $s < d + \rho$,*

$$\|G_m u - u\|_{B_{2,2}^s(\Omega_k^m)} \leq c 2^{-m(t-s)} \|u\|_{B_{2,2}^t(\tilde{\Omega}_k^m)}$$

for any $u \in H^t(\mathbb{T})$ where $\tilde{\Omega}_k^m := \cup_{l \in \Lambda_N} \{\mathcal{F}_l^m : \cup_{j \in \Lambda_M} \text{supp } \gamma_{l,m}^j \cap \Omega_k^m \neq \emptyset\} \cup \Omega_k^m$.

Proof. Without loss of generality, we may assume that m is sufficiently large such that $|\Omega_k^m| < 1$. Because of ii) in the Hypothesis the G_m reproduce all polynomials up to degree d' on Ω_k^m , and hence,

$$\|G_m u - u\|_{B_{2,2}^s(\Omega_k^m)} \leq \|G_m(u - p)\|_{B_{2,2}^s(\Omega_k^m)} + \|u - p\|_{B_{2,2}^s(\Omega_k^m)} . \quad (16)$$

Thus, when $s = 0$, we conclude that

$$\|G_m u - u\|_0(\Omega_k^m) \leq c \inf_{p \in \Pi_{d'}} \|u - p\|_0(\tilde{\Omega}_k^m) \quad (17)$$

where $\Pi_{d'}$ is the space of all polynomials of degree less than or equal to d' . A Whitney type estimate (cf. [DP]) ensures the existence of a polynomial $p_0 \in \Pi_{d'}$ such that

$$\|u - p_0\|_0(\tilde{\Omega}_k^m) \leq c \omega_{d'+1}(u, 2^{-m}, \tilde{\Omega}_k^m) \leq c 2^{-mt} \left(\sum_{j=0}^{\infty} 2^{2jt} \omega_{d'+1}(u, 2^{-j}, \tilde{\Omega}_k^m)^2 \right)^{\frac{1}{2}} \quad (18)$$

$$\leq c 2^{-mt} \|u\|_{B_{2,2}^t}(\tilde{\Omega}_k^m). \quad (19)$$

Using the inverse property we obtain, by (19),

$$\begin{aligned} \|G_m(u - p_0)\|_{B_{2,2}^s}(\Omega_k^m) &\leq \left\| \sum_{j \in \Lambda_M, l \in \Omega_{k,\gamma}^m} F_{l,m}^j(u - p_0) \gamma_{l,m}^j \right\|_s \\ &\leq c 2^{ms} \|u - p_0\|_0(\tilde{\Omega}_k^m) \leq c 2^{-m(t-s)} \|u\|_{B_{2,2}^t}(\tilde{\Omega}_k^m). \end{aligned} \quad (20)$$

The estimate $\omega_{d'+2}(u, t, \Omega) \leq c \|u\|_0(\Omega)$ reveals

$$\begin{aligned} &\sum_{l=0}^{\infty} 2^{2sl} \omega_{d'+2}(u - p_0, 2^{-l}, \Omega_k^m)^2 \\ &= \sum_{l=0}^m 2^{2sl} \omega_{d'+2}(u - p_0, 2^{-l}, \Omega_k^m)^2 + \sum_{l=m+1}^{\infty} 2^{2sl} \omega_{d'+2}(u - p_0, 2^{-l}, \Omega_k^m)^2 \\ &\leq c \sum_{l=0}^m 2^{2sl} \|u - p_0\|_0(\Omega_k^m)^2 + 2^{-2m(t-s)} \sum_{l=m+1}^{\infty} 2^{2sl} 2^{2l(t-s)} \omega_{d'+2}(u - p_0, 2^{-l}, \Omega_k^m)^2 \\ &\leq c 2^{-2m(t-s)} \|u\|_{B_{2,2}^t}(\Omega_k^m)^2 + 2^{-2m(t-s)} \sum_{l=m+1}^{\infty} 2^{2lt} \omega_{d'+2}(u, 2^{-l}, \Omega_k^m)^2 \\ &\leq c 2^{-2m(t-s)} \|u\|_{B_{2,2}^t}(\Omega_k^m)^2 \end{aligned} \quad (21)$$

where we have used $\Delta_h^{d'+2} p_0 = 0$. Thus, the assertion follows from (16), (20) and (21). ■

Now we are ready to prove the approximation property. Using $Q_m u - u = -Q_m(G_m u - u) + (G_m u - u)$, we infer from the Lemmas 3.1 and 3.2 that

$$\begin{aligned} \|Q_m u - u\|_0^2 &\leq c \sum_{l \in \Lambda_N} \|Q_m u - u\|_0(\square_l^m)^2 \\ &\leq c 2^{-2mt} \sum_{l \in \Lambda_N} \left(\sum_{k \in \Lambda_N} \left(\sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\| \right) \|u\|_{B_{2,2}^t}(\tilde{\Gamma}_k^m) + \|u\|_{B_{2,2}^t}(\square_l^m) \right)^2 \\ &\leq c 2^{-2mt} \sum_{l \in \Lambda_N} \left(\sum_{k \in \Lambda_N} \sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\| \right) \left(\sum_{k \in \Lambda_N} \sum_{i \in \square_{l,\gamma}^m} \|\omega_{i-k}^m\| \|u\|_{B_{2,2}^t}(\tilde{\Gamma}_k^m)^2 \right) \\ &\quad + c 2^{-2mt} \sum_{l \in \Lambda_N} \|u\|_{B_{2,2}^t}(\square_l^m)^2 \\ &\leq c 2^{-2mt} \sum_{k \in \Lambda_N} \|u\|_{B_{2,2}^t}(\tilde{\Gamma}_k^m)^2 \leq c 2^{-2mt} \|u\|_{B_{2,2}^t}^2 \leq c 2^{-2mt} \|u\|_t^2 \end{aligned}$$

where we have used $\text{diam } \tilde{\Gamma}_k^m \doteq 2^{-m}$.

Step 2. The same arguments as in the proof of Theorem 5.4 in [DPS] with $r_{l,k} := \sum_{i \in \square_{k,\gamma}^m} \|\omega_{i-l}^m\|$, $N_m := 2^{\frac{m}{2}}$, $K_{k,m} := \{l \in \Lambda_N : |l - k| < 2^{\frac{m}{2}}\}$, d replaced by d' , imply (CP II) for $s = 0$. \blacksquare

Proof of Theorem 2.2. First we show the assertion for $0 \leq s \leq t$. A crucial point is to prove the equivalence of the norm

$$\|u\|_{\phi,t}^2 := \|u\|_0^2 + \sum_{m \in \mathbb{N}} 2^{2mt} \|(P_m - P_{m-1})u\|_0^2 \quad (22)$$

and the Sobolev norm of order t on $\cup_{m \in \mathbb{N}_0} S_m(\phi)$. By Theorem 2.1 the orthogonal projections P_m onto $S_m(\phi)$ satisfy for $0 \leq t \leq d' + 1$ the estimate

$$\|(1 - P_m)f\|_0 \leq c 2^{-mt} \|f\|_t, \quad f \in H^t(\mathbb{T}). \quad (23)$$

Now we introduce the norm $|||\cdot|||_{d'+1}$ defined by $|||f|||_{d'+1} := \|f^{(d'+1)}\|_0 + |f(0)|$ for $f \in H^{d'+1}(\mathbb{T})$. Using the norm equivalence of $\|\cdot\|_{d'+1}$ and $|||\cdot|||_{d'+1}$ and the fact that the constant functions are contained in $S_m(\phi)$, we obtain from (23) the relation

$$\|(1 - P_m)f\|_0 \leq c 2^{-m(d'+1)} \|f^{(d'+1)}\|_0, \quad f \in H^{d'+1}(\mathbb{T}). \quad (24)$$

From (24) we get similarly to the proof of Proposition 4.1 in [DK] that

$$\|(1 - P_m)f\|_0 \leq c \omega_{d'+1}(f, 2^{-m}) \quad (25)$$

for $f \in H^{d'+1}(\mathbb{T})$.

Lemma 3.3 *For any $u \in H^{d+s}(\mathbb{T})$, $0 \leq s \leq 1$, we have*

$$\omega_{d+1}(u, t) \leq c t^{d+s} \|u\|_{d+s}.$$

Proof. For $u \in H^s(\mathbb{T})$, $0 \leq s \leq 1$, it is known that

$$\begin{aligned} \omega_1(u, t)^2 &\leq c \frac{1}{t} \int_{-t}^t \int_0^{1-h} |f(x+h) - f(x)|^2 dx dh \\ &\leq c t^{2s} \int_0^1 \int_{-1}^2 \frac{|f(x) - f(y)|^2}{|\sin(\pi(x-y))|^{1+2s}} dx dy \leq c t^{2s} \|u\|_s^2. \end{aligned}$$

Therefore, we obtain for any $u \in H^{d+s}(\mathbb{T})$ that

$$\omega_{d+1}(u, t) \leq c t^d \omega_1(u^{(d)}, t) \leq c t^{d+s} \|u^{(d)}\|_s \leq c t^{d+s} \|u\|_{d+s} .$$

■

Let $0 \leq s < \rho$. From the inverse property and Lemma 3.3 we conclude that

$$\omega_{d+1}(u_m, t) \leq c \min\{\|u_m\|_0, t^{d+s} \|u_m\|_{d+s}\} \leq c \min\{1, (2^m t)\}^{d+s} \|u_m\|_0 \quad (26)$$

for $u_m \in S_m(\phi)$. In view of (25) and (26), Theorem 4.1 of [DK] applies and yields

$$\|u\|_t \doteq \|u\|_{\phi, t}, \quad u \in \cup_{m \in \mathbb{N}_0} S_m(\phi)$$

for any fixed $0 \leq t < d + \rho$. By (23) the smooth functions are contained in $\text{clos}_{\|\cdot\|_{\phi, t}} \cup_{m \in \mathbb{N}_0} S_m(\phi)$. Hence, the norm equivalence is valid even on $H^t(\mathbb{T})$. Arguing as in the proof to Theorem 5.1 of [DPS], we obtain for $0 \leq s \leq t \leq d' + 1$, $s < d + \rho$, that

$$\begin{aligned} \|(1 - P_m)u\|_s^2 &\leq c \left(\|(1 - P_m)u\|_0^2 + \sum_{j=1}^{\infty} 2^{2js} \|(P_j - P_{j-1})(1 - P_m)u\|_0^2 \right) \\ &= c \left(\|(1 - P_m)u\|_0^2 + \sum_{j=m+1}^{\infty} 2^{2js} \|(P_j - P_{j-1})(1 - P_m)u\|_0^2 \right) \\ &\leq c \sum_{j=m+1}^{\infty} 2^{2j(s-t)} \|u\|_t^2 \leq c 2^{-2m(t-s)} \|u\|_t^2 \end{aligned}$$

where $u \in H^t(\mathbb{T})$. In the last step we have used (23). It remains to prove the case $s < 0$. If $s \leq t < 0$, then

$$\begin{aligned} \|u - P_m u\|_s &= \sup_{0 \neq v \in H^{-s}(\mathbb{T})} \frac{\langle u - P_m u, v \rangle}{\|v\|_{-s}} = \sup_{0 \neq v \in H^{-s}(\mathbb{T})} \frac{\langle u, v - P_m v \rangle}{\|v\|_{-s}} \\ &\leq \|u\|_t \sup_{0 \neq v \in H^{-s}(\mathbb{T})} \frac{\|v - P_m v\|_{-t}}{\|v\|_{-s}} \leq c 2^{m(s-t)} \|u\|_t . \end{aligned}$$

In the case $s < 0 \leq t \leq d' + 1$ we obtain

$$\begin{aligned} \|u - P_m u\|_s &= \sup_{0 \neq v \in H^{-s}(\mathbb{T})} \frac{\langle u - P_m u, v \rangle}{\|v\|_{-s}} = \sup_{0 \neq v \in H^{-s}(\mathbb{T})} \frac{\langle u - P_m u, v - P_m v \rangle}{\|v\|_{-s}} \\ &\leq \|u - P_m u\|_0 \sup_{0 \neq v \in H^{-s}(\mathbb{T})} \frac{\|v - P_m v\|_0}{\|v\|_{-s}} \leq c 2^{m(s-t)} \|u\|_t . \end{aligned}$$

■

Proof of Theorem 2.4. Step 1. For any $h > 0$, we have

$$\Delta_h^{d+1} f u_m(x) = \sum_{k=0}^{d+1} \binom{d+1}{k} \Delta_h^k f(x) \Delta_h^{d+1-k} u_m(x + kh) .$$

Hence,

$$\begin{aligned} \|\Delta_h^{d+1} f u_m\|_0(\Omega)^2 &\leq \\ &c \left(\sum_{k=1}^{d+1} \|\Delta_h^k f\|_{0,\infty}(\Omega)^2 \|\Delta_h^{d+1-k} u_m\|_0(\Omega_{h,k})^2 + \|f\|_{0,\infty}(\Omega)^2 \|\Delta_h^{d+1} u_m\|_0(\Omega_{h,d+1})^2 \right) \end{aligned}$$

where $\|f\|_{l,\infty}(\Omega) := \sup_{\nu \leq l} \sup_{x \in \Omega} |f^{(\nu)}(x)|$. Now we conclude from (17) and (18), with d' replaced by d , that

$$\begin{aligned} \|G_m f u_m - f u_m\|_0(\square_k^m)^2 &= \|G_m(f - f(kh))u_m - (f - f(kh))u_m\|_0(\square_k^m)^2 \\ &\leq c \omega_{d+1}((f - f(kh))u_m, 2^{-m}, \square_k^m)^2 \\ &\leq c \left[2^{-2m} \|f\|_{1,\infty}^2 \omega_{d+1}(u_m, 2^{-m}, \square_k^m)^2 \right. \\ &\quad \left. + \sum_{q=1}^{d+1} 2^{-2mq} \|f\|_{q,\infty}^2 \omega_{d+1-q}(u_m, 2^{-m}, \square_k^m)^2 \right] . \end{aligned}$$

Since $u_m \in H^d(\mathbb{T})$ we have $\omega_l(u_m, \tau, \Omega) \leq c \tau^l \|u_m\|_d(\Omega)$ for $0 \leq l \leq d$. So, we infer

$$\begin{aligned} \|G_m f u_m - f u_m\|_0(\square_k^m)^2 &\leq c \|f\|_{d+1,\infty}^2 \left\{ 2^{-2m} \omega_{d+1}(u_m, 2^{-m}, \square_k^m)^2 + 2^{-2m(d+1)} \|u_m\|_d(\square_k^m)^2 \right\} \end{aligned}$$

and therefore

$$\|G_m f u_m - f u_m\|_0^2 \leq c \|f\|_{d+1,\infty}^2 \left\{ 2^{-2m} \omega_{d+1}(u_m, 2^{-m})^2 + 2^{-2m(d+1)} \|u_m\|_d^2 \right\} .$$

For fixed s'' with $d < s'' < d + \rho$, we obtain

$$\begin{aligned} \|G_m f u_m - f u_m\|_0^2 &\leq c \|f\|_{d+1,\infty}^2 \left\{ 2^{-2m(s''+1)} 2^{-2ms''} \omega_{d+1}(u_m, 2^{-m})^2 + 2^{-2m(d+1)} \|u_m\|_d^2 \right\} \\ &\leq c \|f\|_{d+1,\infty}^2 2^{-2m(d+1)} \left(\sum_{l=0}^{\infty} 2^{-2ls''} \omega_{d+1}(u_m, 2^{-l})^2 + \|u_m\|_d^2 \right) \\ &\leq c \|f\|_{d+1,\infty}^2 2^{-2m(d+1)} \|u_m\|_{s''}^2 \end{aligned}$$

where we have used (12). For $\delta := 1 - \rho < d + 1 - s''$, we conclude from the inverse property that

$$\begin{aligned} \|P_m f u_m - f u_m\|_0 &\leq \|G_m f u_m - f u_m\|_0 \\ &\leq c \|f\|_{d+1, \infty} 2^{-m\delta} 2^{-mt} \|u_m\|_t \end{aligned}$$

for any $t \leq s''$. Hence, by Proposition 2.6, the assertion i) is shown for $0 \leq s \leq t < d + \rho$. Thus, it remains to consider the case $-d' - 1 \leq s < 0$. Applying the approximation property, we obtain

$$\|(1 - P_m) f P_m u\|_s \leq c 2^{ms} \|(1 - P_m) f P_m u\|_0.$$

If $t \geq 0$ the assertion follows from (CP I) for $s = 0$. Otherwise we apply (CP I) with $\tilde{t} := 0$ and employ the inverse property.

Step 2. Next, we prove the second part of Theorem 2.4. Because of part i) there exists $\delta > 0$ such that

$$\begin{aligned} \|P_m f (1 - P_m) u\|_0^2 &\leq \|(1 - P_m) \bar{f} P_m f (1 - P_m) u\|_0 \|u\|_0 \\ &\leq c 2^{-m\delta} \|P_m f (1 - P_m) u\|_0 \|u\|_0 \leq c 2^{-m\delta} \|u\|_0^2 \end{aligned}$$

for any $u \in L^2(\mathbb{T})$. Let $0 \leq s \leq t \leq d' + 1$ with $s < d + \rho$ be fixed. Then we conclude from (IP) and (AP) that

$$\begin{aligned} \|P_m f (1 - P_m) u\|_s &\leq c 2^{ms} \|P_m f (1 - P_m) u\|_0 \\ &\leq c 2^{-m(\delta/2-s)} \|(1 - P_m) u\|_0 \leq c 2^{-m(\delta/2+t-s)} \|u\|_t \end{aligned}$$

provided $u \in H^t(\mathbb{T})$.

In the case $s < 0$ we proceed as in the proof of Corollary 5.2 in [DPS] to obtain

$$\begin{aligned} \|P_m f (1 - P_m) u\|_s &= \sup_{\|v\|_{-s}=1} \langle (1 - P_m) u, (1 - P_m) \bar{f} P_m v \rangle \\ &\leq \|(1 - P_m) u\|_s \sup_{\|v\|_{-s}=1} \|(1 - P_m) \bar{f} P_m v\|_{-s} \\ &\leq c 2^{-m(t-s+\delta)} \|u\|_t \sup_{\|v\|_{-s}=1} \|P_m v\|_{-s} \leq c 2^{-m(t-s+\delta)} \|u\|_t \end{aligned}$$

where we have applied (AP) to the first factor and Theorem 2.4, i) to the second factor on the right hand side of the second inequality. ■

4 Applications to pseudodifferential equations

In the present section, we establish some convergence and stability results for numerical methods to solving periodic pseudodifferential equations. First we introduce the class of pseudodifferential operators which will be studied throughout the remainder of this paper. For $r \in \mathbb{R}$ we denote by $S^r(\mathbb{T})$ the class of symbols which consists of functions $\sigma \in C^\infty(\mathbb{T} \times \mathbb{Z})$ satisfying

$$|\partial_x^\beta \Delta_1^\alpha \sigma(x, \xi)| \leq c_{\alpha, \beta} (1 + |\xi|)^{r-\alpha} \quad \text{for all } x \in \mathbb{T}, \xi \in \mathbb{Z},$$

where $\Delta_1 \xi$ is the forward difference operator with respect to ξ with step size 1 defined in Section 3 and α, β are non-negative integers. The corresponding pseudodifferential operator with the symbol σ is given by

$$\begin{aligned} \sigma(x, D)u(x) &:= \sum_{\xi \in \mathbb{Z}} e^{2\pi i x \xi} \sigma(x, \xi) \tilde{u}(\xi), \quad u \in C^\infty(\mathbb{T}), \\ \tilde{u}(\xi) &:= \int_0^1 e^{-2\pi i \xi x} u(x) dx. \end{aligned}$$

It is well known that the symbol of a pseudodifferential operator is uniquely determined up to a function belonging to $\cap_{r \in \mathbb{R}} S^r(\mathbb{T})$.

In what follows we restrict ourselves to the subclass $\Sigma^\mu(\mathbb{T}) \subset S^r(\mathbb{T})$ of all symbols $\sigma \in S^r(\mathbb{T})$ which admit a decomposition $\sigma = \sigma_0 + \sigma_1$, where $\sigma_1 \in S^{r_1}(\mathbb{T})$ with $r_1 < r := \operatorname{Re} \mu$, $\mu \in \mathbb{C}$, and $\sigma_0 \in C^\infty(\mathbb{T} \times \mathbb{R} \setminus \{0\})$. The function σ_0 is required to be positively homogeneous of degree μ , i.e.,

$$\sigma_0(x, \lambda \xi) = \lambda^\mu \sigma_0(x, \xi)$$

for $\lambda > 0$ and $\xi \neq 0$. Without loss of generality, we assume $\sigma_0(x, 0) = 1$. We will denote by $\Psi^r(\mathbb{T})$, and $\Phi^\mu(\mathbb{T})$, the class of pseudodifferential operators (Ψ DO's) which admit a decomposition $L = \sigma(x, D) + K$ where $\sigma \in S^r(\mathbb{T})$, and $\Sigma^\mu(\mathbb{T})$ respectively, and K is a smoothing operator given by $K u(x) = \int u(y) k(x, y) dy$ with $k \in C^\infty(\mathbb{T} \times \mathbb{T})$. Note that if $L \in \Phi^\mu(\mathbb{T})$, then $L : H^s(\mathbb{T}) \rightarrow H^{s-r}(\mathbb{T})$ is a bounded operator. We remark that the class $\Phi^\mu(\mathbb{T})$ contains all classical operators occurring in boundary element methods (cf. [AW1], [DPS]).

Our central objective is to solve the pseudodifferential equation

$$L u = f \tag{27}$$

where $L \in \Phi^\mu(\mathbb{T})$ and $f \in H^{s-r}(\mathbb{T})$ are given. For the solution of (27) we examine the numerical methods considered in [PSch]. To this end we have to introduce finite-dimensional trial spaces of approximating functions and sets of test functionals. We choose a sequence $\phi := (\phi^j)_{j \in \Lambda_M} \in \mathcal{L}_2^M$ of generators for the spaces of approximating functions. Then the trial spaces are defined by (1).

Now we turn to the test functionals. Choose a family of distributions $\eta := (\eta^j)_{j \in \Lambda_M} \in (H^{-s'}(\mathbb{R}))^M$, $s' \geq 0$, with compact support to define the test functionals

$$\eta_{k,m}^l(f) := 2^{-\frac{m}{2}} \eta^l(f(2^{-m}(\cdot + k))) \quad , \quad l \in \Lambda_M, k \in \Lambda_N \quad , \quad (28)$$

for $f \in H^{s'}(\mathbb{T})$. The numerical method which we are going to investigate is the Galerkin-Petrov method corresponding to the aforementioned trial spaces and test functionals. This method reads as follows:

Find an approximate solution $u_m \in S_m(\phi)$ such that

$$\eta_{k,m}^l(Lu_m) = \eta_{k,m}^l(f) \quad , \quad l \in \Lambda_M, k \in \Lambda_N \quad (29)$$

for a fixed and all sufficiently large $m \in \mathbb{N}_0$. The scheme (29) corresponding to the trial and test spaces generated by ϕ and η , respectively, is called *numerical method* $\{\eta, \phi\}$ for the operator L . The following two examples are particular realizations of the scheme (29).

Example 4.1 *Collocation method:* Choose a strictly increasing sequence $(\epsilon_j)_{j \in \Lambda_M} \in [0, 1]^M$ and define the test functionals by

$$\eta^j(f) := f(\epsilon_j)$$

for $j \in \Lambda_M$. So, we have to find a solution $u_m \in S_m(\phi)$ satisfying

$$L u_m(2^{-m}(\epsilon_j + k)) = f(2^{-m}(\epsilon_j + k)) \quad , \quad j \in \Lambda_M, k \in \Lambda_N \quad .$$

Example 4.2 *Galerkin method:* Let $(\phi^j)_{j \in \Lambda_M} \in \mathcal{L}_2^M$ be a family of compactly supported functions. Then the test functionals in (29) are defined by (28) and

$$\eta^j(f) := \langle f, \phi^j \rangle_{L^2(\mathbb{R})} \quad , \quad j \in \Lambda_M,$$

for $f \in L^2(\mathbb{T})$.

Example 4.3 *Biorthogonal Galerkin method:* Let $(\tilde{\eta}^j)_{j \in \Lambda_M} \in \mathcal{L}_2^M$ be a family of compactly supported functions biorthogonal to $(\phi^j)_{j \in \Lambda_M}$, i.e.

$$\langle \phi^r, \tilde{\eta}^s(\cdot - k) \rangle_{L^2(\mathbb{R})} = \delta_{r,s} \delta_{0,k}$$

for $r, s \in \Lambda_M$ and $k \in \mathbb{Z}$. Then the test functionals in (29) are defined by (28) and

$$\eta^j(f) := \langle f, \tilde{\eta}^j \rangle_{L^2(\mathbb{R})} \quad , \quad j \in \Lambda_M,$$

for $f \in L^2(\mathbb{T})$.

It turns out that the convergence analysis of the numerical method (29) essentially depends on the behavior of the matrix valued function $[\eta\sigma_0\phi]$ defined by

$$[\eta\sigma_0\phi](y, x) := \sum_{l \in \mathbb{Z}} \left(\eta^r (e^{2\pi i(l+x)\cdot}) \sigma_0(y, l+x) \hat{\phi}^s(l+x) \right)_{(r,s) \in \Lambda_M^2}, \quad x \in [-\frac{1}{2}, \frac{1}{2}].$$

This function $[\eta\sigma_0\phi]$ will be called *numerical symbol* of the numerical method $\{\eta, \phi\}$ for the operator L with principal symbol σ_0 . Using the notation

$$\begin{aligned} \hat{\eta}^r(x) &:= \overline{\eta^r(e^{2\pi i x \cdot})}, \\ \hat{\eta}_p(x) &:= \left(\hat{\eta}^r(pM + l + x) \right)_{(l,r) \in \Lambda_M^2}, \\ \hat{\phi}_p(x) &:= \left(\hat{\phi}^r(pM + l + x) \right)_{(l,r) \in \Lambda_M^2}, \\ f_p(x) &:= \text{diag}(f(pM + l + x))_{l \in \Lambda_M}, \end{aligned} \quad (30)$$

for $f : \mathbb{R} \rightarrow \mathbb{C}$, the numerical symbol takes the simple form

$$[\eta\sigma_0\phi](y, x) = \sum_{p \in \mathbb{Z}} \hat{\eta}_p(x)^* (\sigma_0(y, \cdot))_p(x) \hat{\phi}_p(x). \quad (31)$$

To define a class of admissible numerical methods (cf. [PSch]) we need the notation

$$\langle x \rangle := \begin{cases} |x| & \text{if } x \neq 0 \\ 1 & \text{else} \end{cases}.$$

We recall the following definition from [PSch].

Definition 4.4 *The numerical method $\{\eta, \phi\}$ is called s -admissible for ΨDO 's in $\Phi^\mu(\mathbb{T})$, $s \in \mathbb{R}$, if the following is satisfied:*

- i) *the matrices $\hat{\phi}_0$ and $\hat{\eta}_0$ are invertible on $[-\frac{1}{2}, \frac{1}{2}]$;*
- ii) *$\sum_{p \neq 0} \|\langle x \rangle_p^s \hat{\phi}_p(x) \hat{\phi}_0(x)^{-1} \langle x \rangle_0^{-s}\|^2$ is uniformly bounded on $[-\frac{1}{2}, \frac{1}{2}]$;*
- iii) *$\sum_{p \neq 0} \|\hat{\eta}_p(x)^* |x|_p^\mu \hat{\phi}_p(x)\|$ is convergent on $[-\frac{1}{2}, \frac{1}{2}]$.*

Here the matrices $\langle \cdot \rangle_p$, $|\cdot|_p^\mu$ arising in ii) and iii) are defined by (30) and $\|\cdot\|$ means any matrix norm. The letter s denotes the Sobolev index of the space $H^s(\mathbb{T})$.

Remark 4.5 *Properties i) and ii) are sufficient conditions for a certain discrete Sobolev norm to be equivalent to the continuous Sobolev norm (see Section 3 in [PSch]). Condition i) is stronger than the Riesz stability. Property ii) is a uniform Strang-Fix condition combined with a growth condition for the $(\hat{\phi}^j)_{j \in \Lambda_M}$ (see Section 4 in [PSch]). The last condition ensures that the numerical symbol is well defined.*

To interpret the numerical scheme (29) as a projection method we have to assume

Hypothesis H: There exist functions $\psi := (\psi^j)_{j \in \Lambda_M} \in \mathcal{L}_2^M$ such that

- $\hat{\psi}_0$ is invertible on $[-\frac{1}{2}, \frac{1}{2}]$ and condition ii) of Definition 4.4 is fulfilled with s and ϕ replaced by $s - r$ and ψ , respectively;
- ψ satisfies the duality conditions $\eta^l(\psi^j(\cdot - k)) = \delta_{l,j}\delta_{0,k}$ for $l, j \in \Lambda_M, k \in \mathbb{Z}$;
- $\left\| |x|_0^{s-r} \hat{\psi}_0(x) \hat{\eta}_0(x)^* |x|_0^{r-s} \right\| \leq c$ and $\left\| |x|_0^{s-r} (\hat{\psi}_0(x) \hat{\eta}_0(x)^*)^{-1} |x|_0^{r-s} \right\| \leq c$
for $x \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$.

Sufficient conditions for the above hypothesis are formulated in Section 3 of [PSch]. There are also given some hints how to construct such functions ψ in a general situation. Note that the last condition of Hypothesis H is a uniform Strang-Fix condition. Moreover, the second property implies that the operators

$$Q_m f := \sum_{\substack{k \in \Lambda_N \\ l \in \Lambda_M}} \eta_{k,m}^l(f) \psi_{k,m}^l \quad (32)$$

are projections defined for sufficiently smooth functions f . In view of the notation (32) and the representation of $u_m \in S_m(\phi)$ as

$$u_m = u *' \phi := \sum_{j \in \Lambda_M} \sum_{k \in \Lambda_N} u_k^j \phi_{k,m}^j \quad (33)$$

with the coefficient vector $u := ((u_k^j)_{j \in \Lambda_M})_{k \in \Lambda_N} \in \mathbb{C}^{MN}$, the numerical scheme (29) is equivalent to the projection equation

$$Q_m L(u *' \phi) = Q_m f .$$

Definition 4.6 *The numerical method $\{\eta, \phi\}$ is called stable for $L : H^s(\mathbb{T}) \rightarrow H^{s-r}(\mathbb{T})$ if*

$$\|Q_m L u_m\|_{s-r} \geq c \|u_m\|_s$$

for any $u_m \in S_m(\phi)$ and sufficiently large $m \in \mathbb{N}_0$.

Our strategy is to reduce the problem of stability of the numerical method for L to that of the Ψ DO of convolution type defined by the principal symbol $\sigma_0(y, \cdot)$ for fixed y . The stability and convergence analysis of such operators has been developed in [PSch]. The aforementioned reduction is based on localization techniques introduced in [P] and [DPS] and essentially uses the properties of the projections proved in Section 2. To this end we need the concept of *local stability* (cf. [P], [DPS]). Following [P], we denote for fixed $y \in \mathbb{T}$ by $\mathcal{M}_y \subset C^\infty(\mathbb{T})$ the *localizing classes* consisting of functions which are equal to 1 in a neighborhood of y . From now on P_m are the orthogonal projections onto $S_m(\phi)$ relative to the scalar product in $H^0(\mathbb{T})$.

Definition 4.7 *The numerical method $\{\eta, \phi\}$ is locally stable for $L : H^s(\mathbb{T}) \rightarrow H^{s-r}(\mathbb{T})$ if for each $y \in \mathbb{T}$ there exist $g_y \in \mathcal{M}_y$ and operators $T_y, T'_y \in \Psi^{r'}(\mathbb{T})$, $r' < r$, and linear operators $C_{y,m}, D_{y,m} : S_m(\phi) \rightarrow S_m(\phi)$ with $\sup_{m \in \mathbb{N}} \|C_{y,m}\|_{H^{s-r}(\mathbb{T}), H^s(\mathbb{T})} < \infty$, $\sup_{m \in \mathbb{N}} \|D_{y,m}\|_{H^{s-r}(\mathbb{T}), H^s(\mathbb{T})} < \infty$ such that*

$$\begin{aligned} Q_m g_y (\sigma_y(D) + T_y) C_{y,m} &\stackrel{\dot{=}_{s-r}}{=} Q_m g_y P_m, \\ D_{y,m} Q_m (\sigma_y(D) + T'_y) g_y P_m &\stackrel{\dot{=}_s}{=} P_m g_y P_m. \end{aligned}$$

Here for any two sequences of operators B_m, C_m the notation $B_m \stackrel{\dot{=}_s}{=} C_m$ stands for $\lim_{m \rightarrow \infty} \|B_m - C_m\|_s = 0$, and $\sigma_y(D)$ is the operator with the symbol $\sigma_0(y, \cdot)$.

Applying the localization principle proved in [P] and using the results of Section 2, we can now deduce the equivalence of local and global stability. To this end, we assume

$$L : H^s(\mathbb{T}) \rightarrow H^{s-r}(\mathbb{T}) \quad (34)$$

to be an invertible Ψ DO belonging to $\Phi^\mu(\mathbb{T})$, $r = \operatorname{Re} \mu$, and the functions ϕ to fulfil the Hypothesis of Section 2. Further suppose ψ satisfies the Hypothesis of Section 2 with ϕ, d, ρ, s replaced by $\psi, \tilde{d}, \tilde{\rho}, s - r$, respectively. We assume that ψ and ϕ are refinable. The number s' is determined by the choice of the functionals in (28). The following restrictions on the parameter s in (34) are necessary to ensure the approximation and commutator properties by using the results of Section 2. We require that the Sobolev index s fulfills the inequalities

$$\begin{aligned} -d' - 1 &\leq s < d + \rho \quad \text{and} \\ 0 &\leq s - r < \tilde{d} + \tilde{\rho} \quad \text{or} \quad -d' - 1 \leq s - r < d + \rho \quad \text{if } P_m = Q_m. \end{aligned} \quad (35)$$

Using the results of Section 2 and proceeding as in the proofs of Propositions 6.4 and 6.5 of [DPS], we obtain

Theorem 4.8 *Let $\{\eta, \phi\}$ be s -admissible for Ψ DO's in $\Phi^\mu(\mathbb{T})$ with s restricted by (35) and let η fulfil the Hypothesis H. Assume that $L : H^s(\mathbb{T}) \rightarrow H^{s-r}(\mathbb{T})$ is invertible and $L \in \Phi^\mu(\mathbb{T})$, $r = \operatorname{Re} \mu$. Then the numerical method $\{\eta, \phi\}$ is stable for L if and only if one of the following conditions is satisfied:*

- i) *The method is local stable.*
- ii) *The method is stable for the operator $\sigma_y(D)$ with symbol $\sigma_0(y, \cdot)$ for all $y \in \mathbb{T}$.*

■

Combining now Theorem 4.8 and Theorem 2.6 of [PSch] establishes the main result of this section.

Theorem 4.9 *Suppose that the assumption of Theorem 4.8 is fulfilled. Then the numerical method $\{\eta, \phi\}$ is stable for the operator L if and only if*

$$\left\| \left(\langle x \rangle_0^{s-r} (\hat{\eta}_0(x)^{-1})^* [\eta \sigma_0 \phi](y, x) \hat{\phi}_0(x)^{-1} \langle x \rangle_0^{-s} \right)^{-1} \right\| \leq c \quad (36)$$

for any $x, y \in [-\frac{1}{2}, \frac{1}{2}]$.

■

The stability combined with approximation properties of the projections guarantees error estimates via standard arguments. (For details, we refer the reader to the proof of Theorem 6.3 in [DPS] or to the proof of Theorem 13.14 in [PSi].)

Theorem 4.10 *Let the assumption of Theorem 4.8 be satisfied. Further suppose that the method $\{\eta, \phi\}$ is stable for $L : H^s(\mathbb{T}) \rightarrow H^{s-r}(\mathbb{T})$ and, in addition, s is restricted by $-d' - 1 \leq s < \tilde{d} + \tilde{\rho}$. Suppose that $f \in H^{t-r}(\mathbb{T})$ for some $t \geq s$ with $\tilde{d}' + 1 \geq t - r \geq s'$ where s' is defined by the test functionals (cf. (28)). Let u denote the exact solution of (27) and let u_m denote the unique solution of (29) whose existence is guaranteed by Theorem 4.9. Then*

$$\|u - u_m\|_s \leq c 2^{-m(t-s)} \|u\|_t \quad (37)$$

is valid. If, in addition $s' \leq s - r$, then

$$\|u - u_m\|_{t'} \leq c 2^{-m(t-t')} \|u\|_t, \quad \max\{-d' - 1, r\} \leq t' \leq s. \quad (38)$$

In the case of the Galerkin method, (38) holds for $\max\{-d' - 1, -d' - 1 - r\} \leq t' \leq s$.

■

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References

- [AW1] Arnold, D.N., Wendland, W.L.: On the asymptotic convergence of collocation methods. *Math. Comp.* **41**, 383-397 (1983).
- [AW2] Arnold, D.N., Wendland, W.L.: The convergence of spline collocation for strongly elliptic equations on curves. *Numer. Math.* **47**, 317-341 (1985).
- [B-AR] Ben-Artzi, A., Ron, A.: On the integer translates of a compactly supported function: dual bases and linear projectors. *SIAM J. Math. Anal.* **21**, 1550-1562 (1990).
- [CDM] Cavaretta, A.S., Dahmen, W., Micchelli, C.A.: Stationary Subdivision. *Mem. Am. Math. Soc.* **93** no. 453 (1991).

- [CM1] Costabel, M., McLean, W.: Spline collocation for strongly elliptic equations on the torus. *Num. Math.* **62**, 511-538 (1992).
- [DK] Dahmen, W., Kunoth, A.: Multilevel preconditioning. *Numer. Math.* **63**, 315-344 (1992).
- [DPS] Dahmen, W., Prössdorf, S., Schneider, R.: Wavelet approximation methods for pseudodifferential equations I: Stability and convergence. *Math. Z.* **215**, 583-620 (1994).
- [DBDVR] De Boor, C., DeVore, R.A., Ron, A.: The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$. *J. Func. Anal.* **119**, 37-78 (1994).
- [DBDVR1] De Boor, C., DeVore, R.A., Ron, A.: Approximation from shift-invariant subspaces of $L^2(\mathbb{R})^d$. *Trans. Am. Math. Soc.* **341** no. 2, 787-806 (1994).
- [DeVL] DeVore, R.A., Lorentz, G.G.: *Constructive Approximation* (Grundlehren Math. Wiss.). Berlin Heidelberg New York: Springer 1993.
- [DP] DeVore, R., Popov, V.: Interpolation of Besov spaces. *Trans. Am. Math. Soc.* **305**, 397-414 (1988).
- [JL] Jia, R.Q., Lei, J.: Approximation by multiinteger translates of functions having global support. *J. Approx. Theory* **72**, 2-23 (1993).
- [JM] Jia, R.Q., Micchelli, C.A.: Using the refinement equation for the construction of prewavelets II: Powers of two. In: Laurent, P., Le Méhanté, A., Schumaker, L.L. (eds.) pp. 209-246. *Curves and Surfaces*. New York: Academic Press 1991.
- [JS] Jia, R.Q., Sivakumar, N.: On the linear independence of integer translates of box splines with rational directions. *Lin. Alg. and Appl.* **135**, 19-31 (1990).
- [MP] McLean, W., Prössdorf, S.: Boundary element collocation methods using splines with multiple knots. *Num. Math.* **74**, 419-451 (1996).
- [NS1] Nitsche, J., Schatz, A.: On local approximation properties of L_2 -projection on spline subspaces. *Appl. Anal.* **2**, 161-168 (1972).
- [P1] Prössdorf, S.: A localization principle in the theory of finite element methods. In: Friedrich, V. et al. (eds.) *Probleme und Methoden der Mathematischen Physik. 8. Tagung in Karl-Marx-Stadt, 1983.* (Teubner Texte Math., Bd. 63, pp. 169-177) Leipzig: Teubner 1984.
- [P] Prössdorf, S.: Ein Lokalisierungsprinzip in der Theorie der Spline-Approximationen und einige Anwendungen. *Math. Nachr.* **119**, 239-255 (1984).
- [PR1] Prössdorf, S., Rathsfeld, A.: A spline collocation method for singular integral equations with piecewise continuous coefficients. *Integral Equations Oper. Theory* **7**, 536-560 (1984).

- [PR2] Prössdorf, S., Rathsfeld, A.: Mellin techniques in the numerical analysis for one-dimensional singular integral equations. Report R-Math-06/88. Berlin: Karl-Weierstraß-Institut 1988.
- [PS] Prössdorf, S., Schmidt, G.: A finite element collocation method for singular integral equations. *Math. Nachr.* **100**, 33-60 (1981).
- [PSr1] Prössdorf, S., Schneider, R.: A spline collocation method for multi-dimensional strongly elliptic pseudodifferential operators of order zero. *Integral Equations Oper. Theory* **4**, 399-435 (1991).
- [PSr2] Prössdorf, S., Schneider, R.: Spline approximation methods for multidimensional periodic pseudodifferential equations. *Integral Equations Oper. Theory* **15**, 626-672 (1992).
- [PSch] Prössdorf, S., Schult, J.: Multiwavelet approximation methods for pseudodifferential equations on curves. Stability and convergence analysis. Preprint 325 (1997), Weierstraß-Institut für Angewandte Analysis und Stochastik. In: *The Numerical Treatment of Boundary Integral Equations* (eds. K.E. Atkinson, Y. Xu), *Adv. Comput. Math.* (submitted).
- [PSi] Prössdorf, S., Silbermann, B.: *Numerical Analysis for Integral and Related Operator Equations*. Berlin: Akademie Verlag 1991, and Boston Basel Stuttgart: Birkhäuser 1991.
- [S1] Schmidt, G.: The convergence of Galerkin and collocation methods with splines for pseudodifferential equations on closed curves. *Z. Anal. Anw.* **3**, 4, 371-384 (1984).
- [S2] Schmidt, G.: Splines und die näherungsweise Lösung von Pseudodifferentialgleichungen auf geschlossenen Kurven. Report R-Math-09/86. Berlin: Karl-Weierstraß-Institut 1986.
- [Si1] Silbermann, B.: Lokale Theorie des Reduktionsverfahrens für Toeplitzoperatoren. *Math. Nachr.* **104**, 137-146 (1981).
- [Si2] Silbermann, B.: Lokale Theorie des Reduktionsverfahrens für singuläre Integraloperatoren. *Z. Anal. Anwen.* **1**, 45-56 (1982).
- [SW] Sloan, I.H., Wendland, W.: Commutator properties for periodic splines. Preprint 96/16, Universität Stuttgart, SFB 404, Mehrfeldprobleme in der Kontinuumsmechanik.
- [SS1] Strela, S., Strang, G.: Orthogonal multiwavelets with vanishing moments. Preprint.
- [SS2] Strela, S., Strang, G.: Biorthogonal multiwavelets and finite elements. Preprint.
- [T] Triebel, H.: *Interpolation Theory, Function spaces, Differential Operators*. Amsterdam New York Oxford: North-Holland 1978.