Edge Asymptotics for the Radiosity Equation over Polyhedral Boundaries

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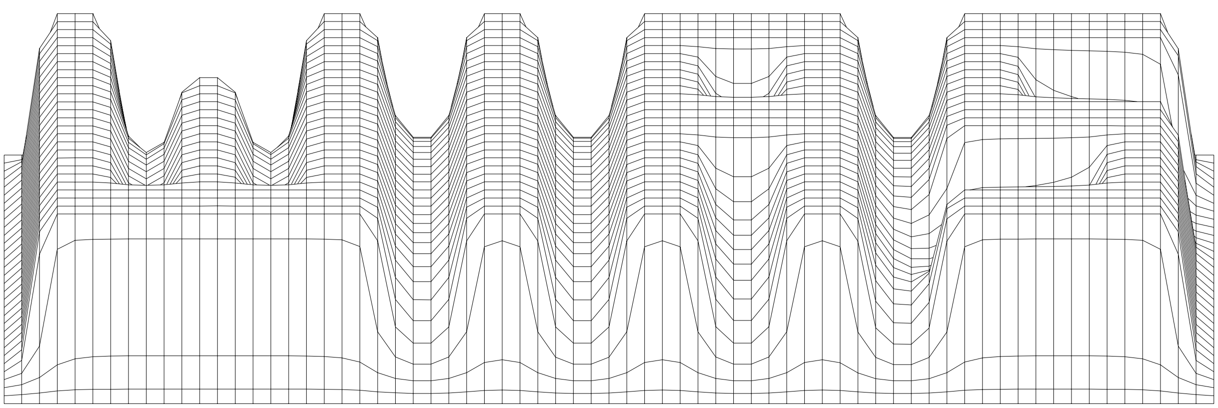
October 7, 1997



1991 Mathematics Subject Classification. 45A05.

Keywords. radiosity equation, edge asymptotics.

Abstract

In the present paper we consider the radiosity equation over the boundary of a polyhedral domain. Similarly to corresponding results on the double layer potential equation, the solution of the second kind integral equation with non-compact integral operator is piecewise continuous. The partial derivatives, however, are not bounded. In the present paper we derive the first term in the asymptotic expansion of the solution in the vicinity of an edge. Note that, knowing this term, optimal mesh gradings can be designed for the numerical solution of this equation.

1 Introduction

The radiosity equation for a Lambertian diffuse reflector takes the form (cf. e.g. [5, 2] and for the numerical treatment cf. also [8, 3])

$$\begin{aligned} u(P) - \frac{\varrho(P)}{\pi}\int_S u(Q)G(P,Q)\mathrm{d}_Q S &= E(P), \quad P \in S, \qquad (1.1)\\ G(P,Q) &:= \frac{[n_P\cdot(Q-P)][n_Q\cdot(P-Q)]}{|P-Q|^4}V(P,Q), \end{aligned}$$

where S is the boundary $\partial\Omega$ of a bounded domain $\Omega \subseteq \mathbb{R}^3$ and n_P is the unit normal to S pointing into Ω at $P \in S$. The right-hand side E is the known emissivity function, and the coefficient ϱ is the reflectivity satisfying $0 \le \varrho(P) < 1$. The visibility function $V(P,Q)$ is 1 if the straight line segment $\{P + \lambda\ \overrightarrow{PQ}\colon\ 0 < \lambda < 1\}$ is contained in the interior of Ω and $V(P,Q) = 0$ otherwise. The unknown function u is the radiosity. We write (1.1) shortly as $(I - K_S)u = E$.

For the edge asymptotics, we need the following assumptions:

(A_1) The surface S is the boundary of a polyhedron Ω and O the point at which we seek the asymptotic expansion of u is an edge point. The tangent cone of S at O is the union of two half planes H_1 and H_2. The coordinate system with coordinates (x,y,z) is chosen such that $O = (0,0,0)$ and

$$\begin{aligned} H_1 &:= \{(x,y,0):\ -\infty < y < \infty,\ 0 \le x < \infty\},\\ H_2 &:= \{(x\cos\varphi, y, x\sin\varphi):\ -\infty < y < \infty,\ 0 \le x < \infty\} \end{aligned}$$

where the angle φ between H_1 and H_2 satisfies $0 < \varphi < 2\pi$.

(A_2) Let us denote the union of the visibility set $\{W \in S:\ V(P,W) = 1\}$ with the two faces containing O by Ω_P and its boundary $\partial\Omega_P \subseteq S$ by Γ_P. For P in a small neighbourhood of O, we assume that Γ_P is the union of two parts Γ and $\tilde{\Gamma}_P$ with Γ independent of P and $\tilde{\Gamma}_P$ varying with P. Usually Γ is the union of edges such that one of the adjacent faces is visible from O and the other not. The polygonal curve $\tilde{\Gamma}_P$ is the boundary line of the shadows on S thrown by Γ if a light source is placed at P (cf. Figure 1). We suppose that $\tilde{\Gamma}_O$ contains no vertex and intersects each edge of S in at most one point. Moreover, we assume that, for any $Q \in \tilde{\Gamma}_O$, there exists exactly one point between O and Q which belongs to S.

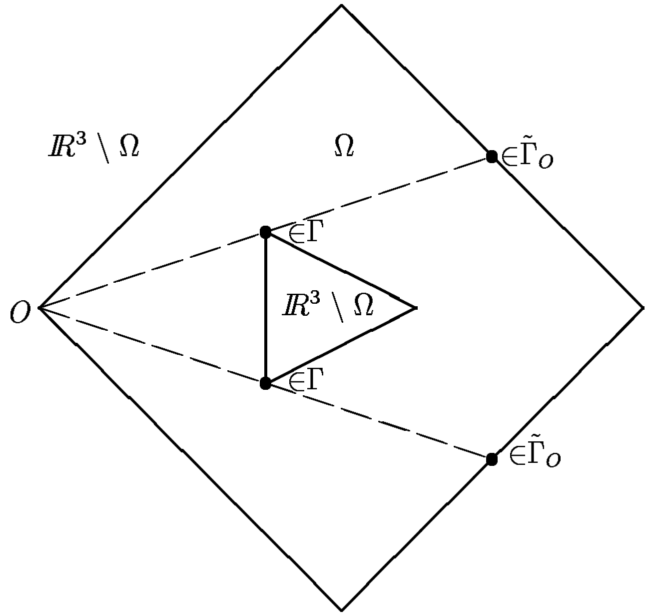


Figure 1: Cross section of Ω through O.

(A_3) Suppose E is bounded and twice continuously differentiable over each face of S, i.e. the derivatives exist in the interior of the face and extend continuously to the boundary.

(A_4) Suppose ϱ is twice continuously differentiable over each face of S and equal to the constant values ϱ_1 and ϱ_2 over $H_1 \cap S$ and $H_2 \cap S$, respectively.

In the case that O is an interior point of a face of S, i.e., in the case $\varphi = \pi$, the solution u of (1.1) is twice continuously differentiable (cf. Corollary 3.1 and Remark 3.1). For the case $\varphi \neq \pi$, we get

Theorem 1.1 *If the Assumptions (A_1)-(A_4) are fulfilled, then in a neighbourhood of O the solution u of (1.1) can be represented as*

$$\begin{aligned} u(P) &= u(x_P, y_P, 0) = \psi_1^{H_1}(y_P) + \psi_2^{H_1}(y_P)x_P^\gamma + \psi_3^{H_1}(x_P, y_P), \\ u(Q) &= u(x_Q \cos\varphi, y_Q, x_Q \sin\varphi) = \psi_1^{H_2}(y_Q) + \psi_2^{H_2}(y_Q)x_Q^\gamma + \psi_3^{H_2}(x_Q, y_Q). \end{aligned} \tag{1.2}$$

Here $P = (x_P, y_P, 0) \in H_1$, $Q = (x_Q \cos\varphi, y_Q, x_Q \sin\varphi) \in H_2$ and x_P as well as x_Q are just the distances of P and Q to the edge $H_1 \cap H_2$. The singularity exponent γ, $0 < \gamma < 1$ is a constant depending only on $\sqrt{\varrho_1 \varrho_2}$, and φ (cf. Figure 2, Table 1, (2.11), and the end of the next section). The functions $\psi_1^{H_1}$ and $\psi_1^{H_2}$ are twice continuously differentiable, the singularity coefficients $\psi_2^{H_1}$ and $\psi_2^{H_2}$ once continuously differentiable and, for a sufficiently small $\delta > 0$, the remainder functions $\psi_3^{H_1}$ and $\psi_3^{H_2}$ are once continuously differentiable if $y \in [-\delta, \delta]$ and $x \in (0, \delta]$. Moreover, for a suitably small $\varepsilon > 0$, there holds

$$\left| \frac{\partial^k}{\partial y^k} \frac{\partial^l}{\partial x^l} \psi_3^{H_i}(x, y) \right| \leq Cx^{\gamma - l + \varepsilon}, \ k, l = 0, 1.$$

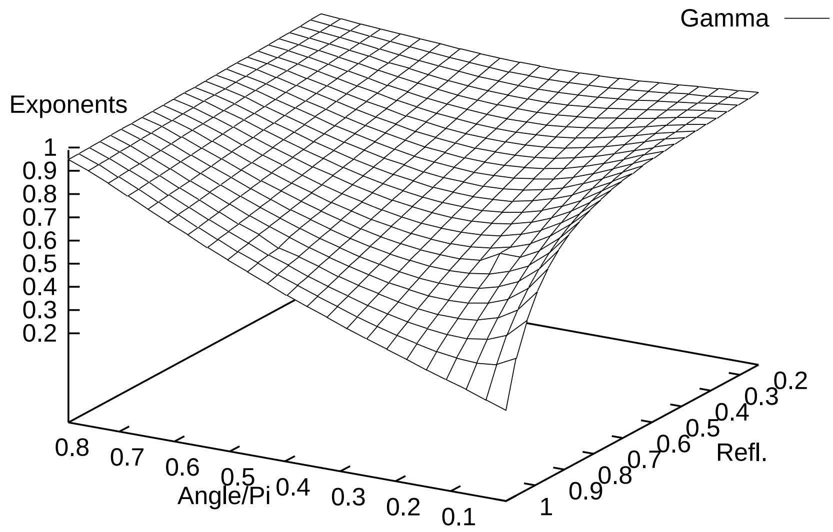


Figure 2: Exponent γ depending on $[\varrho_1^{1/2}\varrho_2^{1/2}]$ and φ/π.

Note that, the first term in the asymptotics may be helpful to design optimal mesh gradings for numerical methods to solve (1.1). We believe that, analogously to the case of the double layer integral operator, in many situations the edge singularities will be stronger than those of the vertices. In this case, only the edge singularities are important for the mesh gradings.

The remainder of the present paper is devoted to the verification of Theorem 1.1. Using localization techniques, we reduce the derivation of the asymptotics to the analysis of a one-dimensional Mellin convolution equation. A sketch of this localization and the computation of the exponent γ from the zeros of the Mellin symbol is provided in Section 2. Details follow in Section 3. In Section 4 we present the results of a numerical test in which we have tried to compute the exponents γ of the edge asymptotics approximately. Finally, we remark that the Mellin techniques applied in this paper are well known. A complete overview over the historical development, however, would be longer than the present article. Therefore we only mention the two quite recent works [6, 7], where the asymptotic behaviour of solutions to partial differential equations and to one-dimensional integral equations is analyzed.

2 The Exponents of the Asymptotics

Now we reduce the computation of the asymptotics to the solution of a Mellin convolution equation over a one-dimensional curve. As we shall see in the next section, the asymptotics depends on local properties only. Thus, without loss of generality, we may suppose:

Angle φ/π	Reflectivity $\sqrt{\varrho_1, \varrho_2}$	Exponent γ
0.100	0.200	0.983
0.100	0.467	0.954
0.100	0.733	0.859
0.100	1.000	0.208
0.367	0.200	0.910
0.367	0.467	0.771
0.367	0.733	0.610
0.367	1.000	0.419
0.633	0.200	0.921
0.633	0.467	0.830
0.633	0.733	0.747
0.633	1.000	0.670
0.900	0.200	0.985
0.900	0.467	0.974
0.900	0.733	0.964
0.900	1.000	0.954

Table 1: Some values of exponent γ.

- The influence of remote boundary parts of S can be neglected. We suppose that S coincides with the tangent cone $T := H_1 \cup H_2$.
- We observe that $V(P, Q)$ is different from 1 only if at least one argument P or Q is not close to O. Hence, for the localized situation, we may suppose that the visibility function V is identically equal to 1.
- The right-hand side function E and the solution V depend smoothly on the variable y in edge direction. Therefore we can freeze the dependence on y and suppose $E(x, y, z) = E(x, 0, z)$ and $u(x, y, z) = u(x, 0, z)$. We introduce

$$\begin{aligned} E_1(x) &:= E(x, 0, 0), \\ E_2(x) &:= E(x \cos \varphi, 0, x \sin \varphi), \\ u_1(x) &:= u(x, 0, 0), \\ u_2(x) &:= u(x \cos \varphi, 0, x \sin \varphi). \end{aligned} \tag{2.1}$$

For $P = (x_P, y_P, 0) \in H_1$ and $Q = (x_Q \cos \varphi, y_Q, x_Q \sin \varphi) \in H_2$, we get $n_P = (0, 0, 1)$ and $n_Q = (\sin \varphi, 0, -\cos \varphi)$. The kernel k_T of the integral operator K_T (cf. (1.1) and replace S by T) over T takes the form

$$k_T(U, W) = \begin{cases} 0 & \begin{array}{l} \text{if } U, W \in H_1 \\ \text{or if } U, W \in H_2 \end{array} \\ \dfrac{\varrho_1 \sin^2 \varphi \, x_P x_Q}{\pi \left[x_P^2 + x_Q^2 - 2 \cos \varphi \, x_P x_Q + (y_P - y_Q)^2\right]^2} & \text{if } U = P, W = Q. \\ \dfrac{\varrho_2 \sin^2 \varphi \, x_P x_Q}{\pi \left[x_P^2 + x_Q^2 - 2 \cos \varphi \, x_P x_Q + (y_P - y_Q)^2\right]^2} & \text{if } W = P, U = Q \end{cases} \tag{2.2}$$

Using (2.1) and the fact that u is independent of y, we arrive at

$$\begin{aligned} K_T u(P) &= \int_0^\infty \int_{-\infty}^\infty u(Q) k_T(P,Q) \mathrm{d}y_Q \mathrm{d}x_Q & (2.3) \\ &= \frac{\varrho_2 \sin^2\varphi}{\pi} \int_0^\infty u_2(x_Q) x_P x_Q \int_{-\infty}^\infty \frac{\mathrm{d}y_Q}{\left[x_P^2 + x_Q^2 - 2\cos\varphi\, x_P x_Q + (y_P - y_Q)^2\right]^2} \mathrm{d}x_Q \\ &= \frac{\varrho_2 \sin^2\varphi}{2} \int_0^\infty u_2(x_Q) \frac{x_P x_Q}{\sqrt{x_P^2 + x_Q^2 - 2\cos\varphi\, x_P x_Q}^3} \mathrm{d}x_Q \\ &= \int_0^\infty k_2\left(\frac{x_P}{x_Q}\right) \frac{1}{x_Q} u_2(x_Q) \mathrm{d}x_Q \ , \\ k_2(\xi) &:= \frac{\varrho_2 \sin^2\varphi}{2} \frac{\xi}{\sqrt{\xi^2 + 1 - 2\xi\cos\varphi}^3}. & (2.4) \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} K_T u(Q) &= \int_0^\infty \int_{-\infty}^\infty u(P) k_T(Q,P) \mathrm{d}y_P \mathrm{d}x_P = \int_0^\infty k_1\left(\frac{x_Q}{x_P}\right) \frac{1}{x_P} u_1(x_P) \mathrm{d}x_P \ , & (2.5) \\ k_1(\xi) &:= \frac{\varrho_1 \sin^2\varphi}{2} \frac{\xi}{\sqrt{\xi^2 + 1 - 2\xi\cos\varphi}^3}. & (2.6) \end{aligned}$$

Consequently, $K_T u$ is independent of the edge variable y. Equation (1.1) over T is equivalent to the one-dimensional system of equations

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 0 & K_2 \\ K_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} , & (2.7) \\ K_i u_i(x) &:= \int_0^\infty k_i\left(\frac{x}{\xi}\right) \frac{1}{\xi} u_i(\xi) \mathrm{d}\xi, \end{aligned}$$

where K_i is a Mellin convolution operator. To analyze a Mellin convolution operator we need the Mellin transform. Using this, we shall derive a representation for the Mellin transform of the solution functions u_1 and u_2. From this representation we shall obtain the asymptotics.

For a function f over the half axis, we introduce its Mellin transform $\mathcal{M}f = \hat{f}$ by

$$\hat{f}(z) := \int_0^\infty f(x) x^{z-1} \mathrm{d}x.$$

If $|f(x)| \leq Cx^{-\beta}$ for $x \longrightarrow \infty$, if $|f(x)| \leq Cx^{-\alpha}$ for $x \longrightarrow 0$, and if $\alpha < \beta$, then $\hat{f}(z)$ exists and is analytic for $\alpha < Re\, z < \beta$. Knowing $\hat{f}$ and the decay property $|\hat{f}(z)| \leq C(1+|z|)^{-2}$, the function f can be reconstructed by the inverse $\mathcal{M}^{-1}$ of $\mathcal{M}$.

$$f(x) = \frac{1}{2\pi i} \int_{\{z:\, Re\, z = z_0\}} \hat{f}(z) x^{-z} \mathrm{d}z, \ \alpha < z_0 < \beta.$$

In particular, the estimate $|\hat{f}(z)| \leq C(1+|z|)^{-2}$, $\alpha < z < \beta$ implies that, for sufficiently small positive ε,

$$|f(x)| \leq C \begin{cases} x^{-\alpha-\varepsilon} & \text{for } x \longrightarrow 0 \\ x^{-\beta+\varepsilon} & \text{for } x \longrightarrow \infty \end{cases} . \tag{2.8}$$

Now the Mellin convolution operator is transformed into multiplication by

$$(K_i u_i)^\wedge(z) \;=\; \hat{k}_i(z)\hat{u}_i(z).$$

In particular, for the identity I, we get $(Iu_i)^\wedge(z) = 1 \cdot \hat{u}_i(z)$. Consequently, (2.7) leads to

$$\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} \;=\; \begin{pmatrix} 1 & -\hat{k}_2 \\ -\hat{k}_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \end{pmatrix}. \tag{2.9}$$

We observe $\hat{k}_i = \varrho_i\, m(z)$ and (cf. [4], p. 310, formula (22) and change the exponent of the sine function from the wrong value $\nu - 0.5$ to the correct value $0.5 - \nu$)

$$\begin{aligned} m(z) \;&:=\; \int_0^\infty \frac{\sin^2\varphi\, x^z}{2\sqrt{x^2+1-2x\cos\varphi}^{\,3}} \mathrm{d}x \\ &=\; \sin\varphi\; B(z+1, 2-z)\sqrt{\frac{1+\cos\varphi}{1-\cos\varphi}}\; {}_2F_1\left(1-z, z; 2; \frac{1+\cos\varphi}{2}\right), \end{aligned}$$

where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function and ${}_2F_1$ is the hypergeometric series

$${}_2F_1(a_1, a_2; b; z) \;:=\; \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\Gamma(a_2+k)\Gamma(b)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b+k)} \frac{z^k}{k!}.$$

Note that $m(z)$ is analytic for $-1 < Re\, z < 2$.

The asymptotics of a function f can be determined from its Mellin transform. For a fixed complex number z_0, the inverse Mellin transform of the function $z \mapsto 1/(z - z_0)$ is $x \mapsto x^{-z_0}$. Consequently, if $\hat{f}(z)$ is meromorphic for $-1 < Re\, z < 2\varepsilon$ and has simple poles only, then we get

$$f(x) \;\sim\; \sum_{z_0:\,\text{pole of } \hat{f} \text{ in } \{z:\, -1 < Re\, z < \varepsilon\}} c_{z_0} x^{-z_0} \;+\; O(x^{1-\varepsilon}), \quad x \longrightarrow 0, \tag{2.10}$$

for any small $\varepsilon > 0$. The numbers c_{z_0} are constants. Hence, it suffices to determine the poles of $\hat{u}_1$ and $\hat{u}_2$. The functions E_1 and E_2 are smooth by assumption. They satisfy $E_i \sim e_i + O(x^{1-\varepsilon})$ with constants e_i. This means that the functions $\hat{E}_i$, $i = 1, 2$ are meromorphic with a simple pole at 0. By (2.9) the functions $\hat{u}_i$, $i = 1, 2$ have poles at 0 too. This corresponds to a constant (i.e. not depending on y) term $\psi_1 = \psi_1^{H_i}$ in the asymptotic expansion (1.2). To get further terms, we need the poles of the matrix function on the right hand side of (2.9), i.e., the zeros of the determinant

$$det \begin{pmatrix} 1 & -\hat{k}_2 \\ -\hat{k}_1 & 1 \end{pmatrix} \;=\; \Big(1 - \sqrt{\varrho_1\varrho_2} m(z)\Big)\Big(1 + \sqrt{\varrho_1\varrho_2} m(z)\Big).$$

We seek the zero $z = z_0$ with the smallest absolute real part $|Re\, z|$. In view of (2.10), this leads to $u_1 = \psi_1 + \psi_2 x^\gamma + O(x^{\gamma+\varepsilon})$ with $\gamma = -z_0$. To determine γ, the following lemma is useful.

Lemma 2.1 *i) For $-1 < Re\, z < 2$, we get $m(Re\, z) \geq |Re\, m(z)|$. Moreover, if $Im\, z \neq 0$, then there holds even $m(Re\, z) > |Re\, m(z)|$.*

ii) The function $(-1,0.5] \ni y \mapsto m(y)$ is strictly monotone decreasing. It takes the values $m(-1+0)=+\infty$ and $m(0)=(1+\cos\varphi)/2$.

Proof. i) The first assertion follows from

$$\begin{aligned} Re\, m(z) &= \int_0^\infty \frac{\sin^2\varphi\, x^{Re\, z}}{2\sqrt{x^2+1-2x\cos\varphi}^3}\cos(Im\, z\log x)\mathrm{d}x \\ |Re\, m(z)| &\le \int_0^\infty \frac{\sin^2\varphi\, x^{Re\, z}}{2\sqrt{x^2+1-2x\cos\varphi}^3}\mathrm{d}x = m(Re\, z). \end{aligned}$$

ii) We conclude

$$\int_0^\infty \frac{x^y}{\sqrt{x^2+1-2x\cos\varphi}^3}\mathrm{d}x = \int_0^1 \ldots + \int_1^\infty \ldots .$$

Substituting $x=\xi^{-1}$ for the variable of integration, we arrive at

$$\begin{aligned} \int_0^1 \ldots &= \int_\infty^1 \frac{\xi^{-y}}{\sqrt{\xi^{-2}+1-2\xi^{-1}\cos\varphi}^3}\frac{-\mathrm{d}\xi}{\xi^2}, = \int_1^\infty \frac{\xi^{1-y}}{\sqrt{\xi^2+1-2\xi\cos\varphi}^3}\mathrm{d}\xi, \\ \int_0^\infty \frac{x^y}{\sqrt{x^2+1-2x\cos\varphi}^3}\mathrm{d}x &= \int_1^\infty \frac{1}{\sqrt{x^2+1-2x\cos\varphi}^3}\left\{x^y+x^{1-y}\right\}\mathrm{d}x. \end{aligned}$$

The last expression is monotone decreasing since $y\mapsto x^y+x^{1-y}$ is decreasing for $y<0.5$. ∎

Corollary 2.1 *The function $(1-\sqrt{\varrho_1\varrho_2}m(z))(1+\sqrt{\varrho_1\varrho_2}m(z))$ has exactly one real zero in the strip $-1 < Re\, z < 0.5$. This zero is negative and it is just that zero in the strip $-1 < Re\, z < 0.5$ with the smallest absolute real part $|Re\, z|$.*

Proof. From Lemma 2.1 ii) we derive that there is exactly one simple real zero of $(1-\sqrt{\varrho_1\varrho_2}m(z))$ in the interval $(-1,0)$ and no further real zero in $[0,0.5)$. The function $z\mapsto(1+\sqrt{\varrho_1\varrho_2}m(z))$ is positive on $(-1,0.5)$. Let the negative real zero be $-\gamma$ and suppose z is another zero of $(1\pm\sqrt{\varrho_1\varrho_2}m(z))$ with $Im\, z\neq 0$ and $-1<Re\, z<0.5$. Then we get

$$\begin{aligned} Re\, m(z) &= \pm\frac{1}{\sqrt{\varrho_1\varrho_2}}, \\ m(Re\, z) &> \frac{1}{\sqrt{\varrho_1\varrho_2}}, \\ (1-\sqrt{\varrho_1\varrho_2}m(Re\, z)) &< 0. \end{aligned}$$

Since $(1-\sqrt{\varrho_1\varrho_2}m(0))>0$ and since $y\mapsto(1-\sqrt{\varrho_1\varrho_2}m(y))$ is strictly monotone increasing over (-1,0.5), there is a zero of $y\mapsto(1-\sqrt{\varrho_1\varrho_2}m(y))$ between $Re\, z$ and 0. Thus $Re\, z < -\gamma < 0$. ∎

The actual value of γ depending on $\sqrt{\varrho_1\varrho_2}$ and on φ can be computed numerically. Using Maple, this can be done for an angle of e.g. $\varphi=1.2345$ and a reflectivity of e.g. $\sqrt{\varrho_1\varrho_2}=$

0.12345 with the following program:

```
with(inttrans);
readlib(hypergeom);
Sy := (a,r,y) -> 1 - r * sin(a) * Beta(y+1, 2-y)*
      sqrt( (1 + cos(a))/(1 - cos(a)) ) * hypergeom([1-y, y], [2], (1 + cos(a))/2);
Ex := (a,r) -> -fsolve( Sy(a,r,y) = 0 , y, -1..0.5);
evalf(Ex(1.2345, 0.12345));
```

Results of such computations are presented in Figure 2 and Table 1. Finally, we note that $m(z)$ depends only on $\sin^2\varphi$ and $\cos\varphi$ but not on φ. Hence, the exponent $\gamma = \gamma(\sqrt{\varrho_1\varrho_2}, \varphi)$ satisfies

$$\gamma\left(\sqrt{\varrho_1\varrho_2}, \varphi\right) \quad = \quad \gamma\left(\sqrt{\varrho_1\varrho_2}, 2\pi - \varphi\right). \tag{2.11}$$

3 Details of the Localization

Now we turn to the details of the localization arguments mentioned in the beginning of Section 2. We introduce cut off functions χ and χ' which are smooth and concentrated in a neighbourhood of O. For these and their supports, we assume $\chi(U) = 1$ and $\chi'(U) = 1$ in a small vicinity of O and

$$\operatorname{supp}\chi' \subseteq \{P \in S : \ \chi(P) = 1\} \subseteq \operatorname{supp}\chi \subseteq T \cap S.$$

Moreover, we assume that $V(U,W) = 1$ for any $U, W \in \operatorname{supp}\chi$. To get an equation over T, we write

$$\begin{aligned}
(I - K_S)\,u &= E, \\
\chi'\,(I - K_S)\,\chi u &= \chi' E + \chi' K_S\,(1-\chi)\,u, \\
\chi'\,(I - K_T)\,\chi u &= \chi' E + \chi' K_S\,(1-\chi)\,u, \\
(I - K_T)\,[\chi' u] &= R \;:=\; -R_1 + R_2 + R_3, \\
R_1 &:= [K_T\chi' - \chi' K_T]\,\chi u, \\
R_2 &:= \chi' E, \\
R_3 &:= \chi' K_S\,(1-\chi)\,u.
\end{aligned} \tag{3.1}$$

Clearly, R_2 is piecewise twice continuously differentiable by Assumption (A_3).

Lemma 3.1 *i) The function R_3 is continuously differentiable over each face of T. Suppose additionally that, for any directional derivative ∂u of the solution u and for any point $U \in S$ not belonging to a small neighbourhood of the vertices, the estimate*

$$|\partial u(U)| \quad \le \quad C\,dist^q, \tag{3.2}$$

is valid, where the exponent q satisfies $-1 < q < 0$ and where $dist$ stands for the distance of U to the set of edge points of S. Then R_3 is twice continuously differentiable over each face of T.

ii) The derivatives $\partial^k_{y_P} R_1(P)$ of R_1, taken in edge direction y_P over the half plane H_1, are continuous for $k = 0, 1, 2$. The additional derivatives $\partial_{x_P}\partial^k_{y_P} R_1(P)$, $k = 0, 1$ in the direction perpendicular to the edge are continuous at the points $P \in H_1$ which do not belong to the edge. If C stands for a general positive constant independent of P, then

$$\left|\partial^k_{y_P} R_1(P)\right| \leq C, \quad k = 0, 1, 2, \tag{3.3}$$

$$\left|\partial_{x_P}\partial^k_{y_P} R_1(P)\right| \leq C \begin{cases} \log|x_P|^{-1} & \text{if } x_P \leq 0.5 \\ 1 & \text{if } x_P > 0.5 \end{cases}, \quad k = 0, 1. \tag{3.4}$$

Similar estimates hold over H_2.

iii) The function $[\chi' u]$ is continuously differentiable with respect to the edge variable over each face of T. For $P \in H_1$, we get

$$\left|\partial^k_{y_P} [\chi' u](P)\right| \leq C, \quad k = 0, 1. \tag{3.5}$$

The function is twice continuously differentiable and (3.5) holds with $k = 2$ if (3.2) is valid. Similar estimates hold over H_2.

Proof. i) In the case $V \equiv 1$ the kernel function of the integral operator $\chi' K_S(1-\chi)$ and all its derivatives are continuous over each face of S and T (cf. Assumption (A_4)). Hence, assertion i) follows even without the assumption (3.2). In the case that $V \not\equiv 1$ the computation of the derivatives is more sophisticated. To get a formula, we fix a unit vector $\vec{d}$ and consider the directional derivative

$$\partial_P f(P) := \lim_{h \longrightarrow 0} \frac{f(P + h\vec{d}) - f(P)}{h}$$

at the point $P = O$. For a $Q \in \tilde{\Gamma}_O$, we know from Assumption (A_2) that there is exactly one $W \in S$ such that OWQ are collinear. To each point $O' = O + h\vec{d}$ with sufficiently small h there is exactly one point $Q' \in \tilde{\Gamma}_{O'}$ such that $O'WQ'$ are collinear. The shift vector $\overrightarrow{QQ'}$ is the image of the shift vector $\overrightarrow{OO'} \subseteq T$ via "reflection" at point W. The length of this vector satisfies (cf. Figure 3)

$$\left|\overrightarrow{QQ'}\right| =: \tilde{\mu}(h) = \mu \cdot h + o(h), \quad \mu = \frac{|\overrightarrow{WQ}| \sin(\overrightarrow{OW}, \overrightarrow{OO'})}{|\overrightarrow{OW}| \sin(\overrightarrow{QW}, \overrightarrow{QQ'})}.$$

By $\varphi(Q)$, $0 < \varphi(Q) < \pi$ we denote the angle at Q between $\tilde{\Gamma}_O$ and $\overrightarrow{QQ'}$. We introduce the factor

$$\mu(Q) := \begin{cases} \mu \sin\varphi(Q) & \text{if } V(O', Q) = 1 \\ -\mu \sin\varphi(Q) & \text{if } V(O', Q) = 0 \end{cases}$$

and the kernel k_S

$$k_S(U, W) := \frac{\varrho(U)}{\pi} \frac{[n_W \cdot (U - W)][n_U \cdot (U - W)]}{|U - W|^4},$$

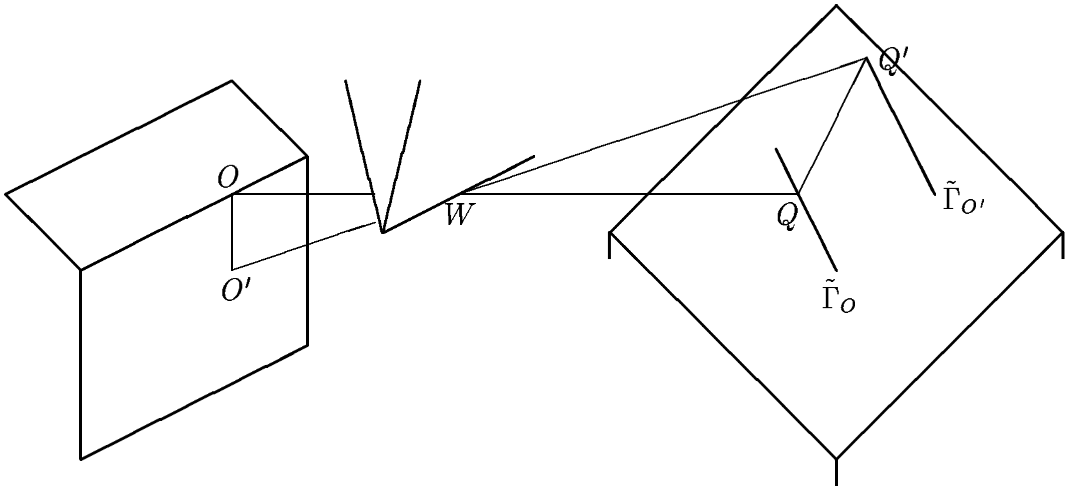


Figure 3: Neighbourhoods of O and Q.

which is roughly speaking the kernel G of (1.1) without multiplication by the visibility function V. With this notation and that from the introduction we get

$$\begin{aligned}\partial_O\left[\chi' K_S(1-\chi)u\right](O) &= \int_{\Omega_O}\partial_O\left[\chi'(O)k_S(O,Q)\right](1-\chi(Q))u(Q)\mathrm{d}_Q S + \\ &\quad \int_{\tilde{\Gamma}_O}\left[\chi'(O)k_S(O,Q)(1-\chi(Q))\right]\mu(Q)u(Q)\mathrm{d}_Q\tilde{\Gamma}_0.\end{aligned} \tag{3.6}$$

Indeed, without loss of generality, we assume that $\tilde{\Gamma}_O$ is a straight line and that $V(O',Q)$ is equal to 1. Retaining the notation $O' = O + h\vec{d}$ and setting $\vec{d}(Q) := \overrightarrow{QQ'} / \mid \overrightarrow{QQ'} \mid$, we observe

$$\Omega_{O'} \setminus \Omega_O = \{Q + \lambda\vec{d}(Q) : \ Q \in \tilde{\Gamma}_O, \ 0 \le \lambda \le \tilde{\mu}(h)\}.$$

Consequently, we obtain

$$\begin{aligned}&\frac{1}{h}\left\{[\chi' K_S(1-\chi)u](O') - [\chi' K_S(1-\chi)u](O)\right\} \\ = &\int_{\Omega_O}\frac{1}{h}\left[\chi'(O')k_S(O',Q) - \chi'(O)k_S(O,Q)\right](1-\chi(Q))u(Q)\mathrm{d}_Q S + \\ &\frac{1}{h}\int_{\Omega_{O'}\setminus\Omega_O}\chi'(O')k_S(O',Q)(1-\chi(Q))u(Q)\mathrm{d}_Q S.\end{aligned}$$

Clearly, the first term tends to the first term on the right-hand side of (3.6). The second can be written as

$$\begin{aligned}\int_{\tilde{\Gamma}_O}\frac{1}{h}\int_0^{\tilde{\mu}(h)}&\left\{\chi'(O')k_S\Big(O',[Q+\lambda\vec{d}(Q)]\Big)\Big(1-\chi(Q+\lambda\vec{d}(Q))\Big)\right. \\ &\left. u(Q+\lambda\vec{d}(Q))\right\}\sin\,\varphi(Q)\mathrm{d}\lambda\mathrm{d}_Q\tilde{\Gamma}_O.\end{aligned}$$

Taking into account that the functions $Q \mapsto k_S(O',Q)$, $Q \mapsto \chi(Q)$, and $Q \mapsto u(Q)$ are continuous at non edge points and that $\tilde{\mu}(h)\sin\,\varphi(Q) = \mu(Q)h + o(h)$, we conclude that the last expression tends to the second term on the right-hand side of (3.6). Thus (3.6) is proved.

Now the continuity of $\partial_P R_3$ follows easily from (3.6), the continuity of $\tilde{\Gamma}_O$, and the smoothness of kernel $[\chi'(O)k_S(O,Q)(1-\chi(Q))]$ with respect to O and Q. To get the second derivative of R_3, we have to differentiate (3.6) once again. The first term on the right-hand side can be treated analogously to the first derivative of R_3. For the derivative of the second term, we observe that $\tilde{\Gamma}_O$ is a polygonal curve each side of which depends differentiably on O. The kernel function $Q \mapsto [\chi'(O)k_S(O,Q)(1-\chi(Q))]$ and μ are continuously differentiable and the derivatives of u remain integrable by assumption (3.2). Hence, the second integral on the right-hand side of (3.6) is continuously differentiable too. The function R_3 is twice continuously differentiable if (3.2) holds.

ii) Let $k_C(U,W) = [\chi'(U) - \chi'(W)]k_T(U,W)\chi(W)$ stand for the kernel of the integral operator $[K_T\chi' - \chi' K_T]\chi$. If $\operatorname{diam}\operatorname{supp}\chi$ denotes the diameter of the support of χ, then it is not hard to see that

$$|k_C(U,W)| \;\le\; C\begin{cases} |U-W|^{-1} & \text{if } |U| \le 2\operatorname{diam}\operatorname{supp}\chi \\ 0 & \text{if } U,W \text{ both on } H_1 \text{ or } H_2 \\ |U|^{-2} & \text{if } |U| \ge 2\operatorname{diam}\operatorname{supp}\chi \end{cases}, \tag{3.7}$$

$$|\partial_U k_C(U,W)| \;\le\; C\begin{cases} |U-W|^{-2} & \text{if } |U| \le 2\operatorname{diam}\operatorname{supp}\chi \\ 0 & \text{if } U,W \text{ both on } H_1 \text{ or } H_2 \\ |U|^{-3} & \text{if } |U| \ge 2\operatorname{diam}\operatorname{supp}\chi \end{cases}. \tag{3.8}$$

Hence, for $U \in H_1$ and $|U| \le 2\operatorname{diam}\operatorname{supp}\chi$, we get

$$|\partial_U R_1(U)| \;\le\; C\int_{\operatorname{supp}\chi\cap H_2} |U-W|^{-2}\mathrm{d}_W T \;\le\; C\log\operatorname{dist}(U,H_2)^{-1}$$

and, for $|U| \ge 2\operatorname{diam}\operatorname{supp}\chi$,

$$|\partial_U R_1(U)| \;\le\; C\int_{\operatorname{supp}\chi\cap H_2} |U|^{-3}\mathrm{d}_W T \;\le\; C|U|^{-3}.$$

This and the corresponding results for $U \in H_2$ prove the assertions for the first derivative taken in arbitrary direction.

Now suppose that $U = P \in H_1$ and that the direction is parallel to the edge, i.e., $\partial_P = \partial_{y_P}$. We can replace (3.7) by

$$|k_C(P,Q)| \;\le\; C\begin{cases} |n_P\cdot(P-Q)|\,|P-Q|^{-2} & \text{if } |P| \le 2\operatorname{diam}\operatorname{supp}\chi \\ 0 & \text{if } Q \in H_1 \\ |P|^{-2} & \text{if } |P| \ge 2\operatorname{diam}\operatorname{supp}\chi \end{cases}. \tag{3.9}$$

The factor $n_P\cdot(P-Q)$ equals $-x_Q\sin\varphi$ and is independent of the variable y_P in edge direction. Consequently, (3.8) can be improved to

$$|\partial_{y_P} k_C(P,Q)| \;\le\; C\begin{cases} |n_P\cdot(P-Q)|\,|P-Q|^{-3} & \text{if } |P| \le 2\operatorname{diam}\operatorname{supp}\chi \\ 0 & \text{if } Q \in H_1 \\ |P|^{-3} & \text{if } |P| \ge 2\operatorname{diam}\operatorname{supp}\chi \end{cases}. \tag{3.10}$$

We end up with

$$|\partial_{y_P} R_1(P)| \le C\int_{\operatorname{supp}\chi\cap H_2} \frac{|n_P\cdot(P-Q)|}{|P-Q|^3}\mathrm{d}_Q T = C\left|\int_{\operatorname{supp}\chi\cap H_2} \frac{n_P\cdot(P-Q)}{|P-Q|^3}\mathrm{d}_Q T\right| \le C.$$

The last estimate is a well-known fact for double layer kernels $n_P \cdot (P-Q)|P-Q|^{-3}$. Namely, the integral over this kernel is the solid angle under which the surface $\operatorname{supp}\chi \cap H_2$ is seen from the point P. This angle is smaller than the full solid angle 4π. Thus all the assertions for the first order derivatives are proved.

Using only the results on first order derivatives, we shall prove in part iii) that $\partial_{y_P}[\chi' u]$ is continuous. If we shift O and the cut off functions a little bit, we get the piecewise continuous differentiability at all points close to the edge. In other words, even $\partial_{y_P}[\chi u]$ is continuous. On the other hand, the operator K_T has a kernel $k_T(P,Q)$ which depends in edge direction only on the difference $y_P - y_Q$. Consequently, K_T commutes with differentiation in edge direction and we arrive at

$$\partial_{y_P} R_1 \;=\; \left[K_T(\partial_{y_P}\chi') - (\partial_{y_P}\chi')K_T\right]\chi u + \left[K_T\chi' - \chi' K_T\right]\partial_{y_Q}[\chi u].$$

Repeating the arguments from above, we conclude

$$\begin{aligned} |\partial_{x_P}\partial_{y_P} R_1(P)| \;&\le\; C\begin{cases} C\log \operatorname{dist}(P,H_2)^{-1} & \text{if } |P| \le 2\operatorname{diam}\operatorname{supp}\chi \\ C|P|^{-3} & \text{else}\end{cases}, \\ |\partial^2_{y_P} R_1(P)| \;&\le\; \begin{cases} C & \text{if } |P| \le 2\operatorname{diam}\operatorname{supp}\chi \\ C|P|^{-3} & \text{else}\end{cases}. \end{aligned}$$

iii) In view of the fact that $K_T\partial_y = \partial_y K_T$ we get

$$(I - K_T)\,\partial_y^k[\chi' u] \;=\; \partial_y^k R, \quad k = 0,1,2. \tag{3.11}$$

Note that $I - K_T$ is a bounded and invertible operator in L^∞ (cf. e.g. [2]) mapping L^∞ into the space of piecewise continuous functions which are continuous over each face of T. Hence, $I - K_T$ is bounded and invertible in the space of piecewise continuous functions. Since the right-hand side in (3.11) is piecewise continuous and continuous over each face of T for $k = 0,1$, we conclude that $\partial_y^k[\chi' u]$ is continuous and bounded for $k = 0,1$. Moreover, if (3.2) is valid, then the right-hand side is piecewise continuous for $k = 2$. Thus under this assumption $\partial_y^2[\chi' u]$ is continuous and bounded over each face of T. ∎

Corollary 3.1 *If O is a point in the interior of a face, then $\varphi = \pi$, T is a plane, $K_T = 0$, and $R_1 \equiv 0$. In this case Lemma 3.1 i) and (3.1) imply that the solution u is continuously differentiable at O. Moreover, if (3.2) holds, then u is even twice continuously differentiable at O.*

Now we set

$$\begin{aligned} u_1(x_P) \;&:=\; [\chi' u](x_P, 0, 0), \\ u_2(x_Q) \;&:=\; [\chi' u](x_Q\cos\varphi, 0, x_Q\sin\varphi). \end{aligned}$$

We freeze $[\chi' u]$ over $T \cap \{(x,0,z):\ x,z \in \mathbb{R}\}$ by defining

$$v(x_W, y_W, z_W) \;:=\; [\chi' u](x_W, 0, z_W)$$

and set $R_4 := K_T([\chi' u] - v)$. Thus (3.1) changes into

$$[\chi' u] - K_T v \;=\; R_4 + R. \tag{3.12}$$

Introducing

$$\begin{aligned} r_1(x_P) &= [R_4+R](x_P,0,0),\\ r_2(x_Q) &= [R_4+R](x_Q\cos\varphi,0,x_Q\sin\varphi) \end{aligned}$$

and restricting (3.12) to $T\cap\{(x,0,z):\ x,z\in I\!R\}$ yields (compare (2.7))

$$\begin{pmatrix} u_1\\ u_2\end{pmatrix} - \begin{pmatrix} 0 & K_2\\ K_1 & 0\end{pmatrix}\begin{pmatrix} u_1\\ u_2\end{pmatrix} = \begin{pmatrix} r_1\\ r_2\end{pmatrix}. \tag{3.13}$$

Lemma 3.2 *i) For the matrix operator on the right-hand side of (3.13), we get the inverse*

$$\begin{pmatrix} I & -K_2\\ -K_1 & I\end{pmatrix}^{-1} = \begin{pmatrix} I+K_{1,1} & K_{1,2}\\ K_{2,1} & I+K_{2,2}\end{pmatrix}, \tag{3.14}$$

where the $K_{j,l}$ are Mellin convolution operators

$$K_{j,l}f(x) := \int_0^\infty k_{j,l}\left(\frac{x}{\xi}\right)\frac{1}{\xi}f(\xi)\mathrm{d}\xi$$

with kernels $k_{j,l}$ such that

$$k_{j,l}(\zeta) = c_{j,l}\theta(\zeta)\zeta^\gamma + \tilde{k}_{j,l}(\zeta), \tag{3.15}$$

$$\left|\left(\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^k \tilde{k}_{j,l}(\zeta)\right| \le C\left\{\begin{array}{ll} \zeta^{\gamma+\varepsilon} & \text{if } \zeta<1\\ \zeta^{-\varepsilon} & \text{if } \zeta\ge 1\end{array}\right., \quad k=0,1. \tag{3.16}$$

Here the $c_{j,l}$ are constants, γ is the exponent mentioned in Sections 1 and 2, and ε is a sufficiently small positive number. The function $\zeta\mapsto\theta(\zeta)$ is a smooth cut off function which is equal to 1 for $\zeta\le 0.5$ and equal to 0 for $\zeta\ge 1$.

ii) The functions r_i, $i=1,2$ are continuously differentiable over $(0,\infty)$ and satisfy

$$|r_i(x)| \le C, \quad \left|\frac{\mathrm{d}}{\mathrm{d}x}r_i(x)\right| \le C\log|x|^{-1} \text{ if } x\le 0.5. \tag{3.17}$$

Proof. i) Recall that $\hat{k}_j(z)$ is analytic for $-1<Re\,z<2$. Applying the definition of the Mellin transform we conclude

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^k k_j\right]^\wedge(z) = (-z)^k\hat{k}_j(z), \quad k=0,1,\dots \tag{3.18}$$

and we obtain that even the $(-z)^k\hat{k}_j(z)$ are analytic and bounded in the strip $-1+\varepsilon< Re\,z<2-\varepsilon$. Hence, $|\hat{k}_j(z)|\le C(1+|z|)^{-k}$ holds for any positive integer k and any z in this strip. From

$$\begin{pmatrix} 1+\hat{k}_{1,1} & \hat{k}_{1,2}\\ \hat{k}_{2,1} & 1+\hat{k}_{2,2}\end{pmatrix} = \begin{pmatrix} 1 & -\hat{k}_2\\ -\hat{k}_1 & 1\end{pmatrix}^{-1}$$

we obtain that the $\hat{k}_{j,l}$ have a simple pole at $z = -\gamma$ and no further poles for $-\gamma - \varepsilon < Re\, z < \varepsilon$ (cf. Corollary 2.1). On the other hand, the support of $(x\frac{\mathrm{d}}{\mathrm{d}x} - \gamma)[\theta(x)x^\gamma] = \theta'(x)x^{\gamma+1}$ is contained in $[0.5, 1]$. Consequently, the Mellin transform of $(x\frac{\mathrm{d}}{\mathrm{d}x})^k(x\frac{\mathrm{d}}{\mathrm{d}x} - \gamma)[\theta(x)x^\gamma]$, $k = 0, 1, \ldots$ is an entire function which is uniformly bounded over the strip $-1 < Re\ z < 2$. In view of (3.18), we conclude that $(-z)^k(-1)(z+\gamma)[\theta(\cdot)(\cdot)^\gamma]^\wedge(z)$ is entire and uniformly bounded over the strip $-1 < Re\ z < 2$. Hence, the Mellin transform $[\theta(\cdot)(\cdot)^\gamma]^\wedge(z)$ is meromorphic in $-1 < Re\ z < 2$. The only pole is the simple pole at $z = -\gamma$ and

$$\left|[\theta(\cdot)(\cdot)^\gamma]^\wedge(z)\right| \;\leq\; C(1+|z|)^k, \quad k = 0, 1, \ldots$$

holds for $|z+\gamma| > \varepsilon$ and a constant C depending on k and the small ε. For a suitably chosen $c_{j,l}$ the function $\tilde{k}^\wedge_{j,l}(z) = \hat{k}_{j,l}(z) - c_{l,j}[\theta(\cdot)(\cdot)^\gamma]^\wedge(z)$ is analytic over the strip $-\gamma-\varepsilon < Re\ z < \varepsilon$ and satisfies

$$\left|\tilde{k}^\wedge(z)\right| \;\leq\; C(1+|z|)^k, \quad k = 0, 1, \ldots .$$

Applying the inverse Mellin transform and using (2.8), we arrive at (3.16).

ii) In view of Lemma 3.1 i) and ii) it is sufficient to consider the restriction of function R_4 to $T \cap \{(x, 0, z) :\ x, z \in \mathbb{R}\}$. The proof of the continuous differentiability for this function, however, is completely analogous to the proof of Lemma 3.1 ii). The only difference is that, for $U = P = (x_P, 0, 0)$ and $W = Q = (x_Q \cos\varphi, y_Q, x_Q \sin\varphi)$, the estimate $|\chi'(U) - \chi'(W)| \leq C|U - W|$ for the factor $[\chi'(U) - \chi'(W)]$ is to be replaced by the estimate (cf. Lemma 3.1 iii))

$$\begin{aligned} \left|[\chi' u](Q) - v(Q)\right| &= \left|[\chi' u]\left(x_Q \cos\varphi, y_Q, x_Q \sin\varphi\right) - [\chi' u]\left(x_Q \cos\varphi, 0, x_Q \sin\varphi\right)\right| \\ &\leq Cy_Q \;\leq\; C|P - Q|. \end{aligned}$$

for the factor $[[\chi' u](Q) - v(Q)]$. ■

Corollary 3.2 *There exist constant numbers c_0 and c_γ and a differentiable function g such that, for $x \leq 0.5$ and for sufficiently small $\varepsilon > 0$,*

$$u_i(x) \;=\; c_0 + c_\gamma x^\gamma + g(x), \tag{3.19}$$

$$|g(x)| \;\leq\; Cx^{\gamma+\varepsilon}, \quad \left|\frac{\mathrm{d}}{\mathrm{d}x} g(x)\right| \;\leq\; Cx^{\gamma+\varepsilon-1}. \tag{3.20}$$

Of course, c_0, c_γ, and g depend on i.

Proof. Applying (3.14) to (3.13), we observe that it is sufficient to prove the desired representation (3.19) for $K_{j,l} r_l$. We get

$$\begin{aligned} (K_{j,l} r_l)(x) &= r_l(0)\left(K_{j,l} 1\right)(x) + \left[K_{j,l}\left(r_l - r_l(0)\right)\right](x) \\ &= \hat{k}_{j,l}(0) r_l(0) + c_{j,l} \int_0^\infty \theta\left(\frac{x}{\xi}\right)\left(\frac{x}{\xi}\right)^\gamma \frac{1}{\xi}\left[r_l(\xi) - r_l(0)\right] \mathrm{d}\xi \\ &\quad + \int_0^\infty \tilde{k}_{j,l}\left(\frac{x}{\xi}\right)\frac{1}{\xi}\left[r_l(\xi) - r_l(0)\right] \mathrm{d}\xi \\ &= c_0 + c_\gamma x^\gamma + g(x), \end{aligned} \tag{3.21}$$

where $g = g_1 + g_2$ and

$$
\begin{aligned}
g_1(x) &:= c_{j,l} \int_0^\infty \left\{1 - \theta\left(\frac{x}{\xi}\right)\right\} \left(\frac{x}{\xi}\right)^\gamma \frac{1}{\xi} \left[r_l(\xi) - r_l(0)\right] \mathrm{d}\xi, \\
g_2(x) &:= \int_0^\infty \tilde{k}_{j,l}\left(\frac{x}{\xi}\right) \frac{1}{\xi} \left[r_l(\xi) - r_l(0)\right] \mathrm{d}\xi, \\
c_\gamma &:= c_{j,l} \int_0^\infty \xi^{-1-\gamma} \left[r_l(\xi) - r_l(0)\right] \mathrm{d}\xi.
\end{aligned}
$$

Note that the integral defining c_γ is finite by Lemma 3.2 ii). It remains to estimate g. Now suppose $x < 0.5$. Then Lemma 3.2 yields

$$
\begin{aligned}
|g_2(x)| &\leq C \int_0^x \left(\frac{x}{\xi}\right)^{-\varepsilon} \frac{1}{\xi} \xi \log|\xi|^{-1} \mathrm{d}\xi + C \int_x^{0.5} \left(\frac{x}{\xi}\right)^{\gamma+\varepsilon} \frac{1}{\xi} \xi \log|\xi|^{-1} \mathrm{d}\xi \\
&\quad + C \int_{0.5}^\infty \left(\frac{x}{\xi}\right)^{\gamma+\varepsilon} \frac{1}{\xi} \mathrm{d}\xi \leq C x^{\gamma+\varepsilon}.
\end{aligned} \tag{3.22}
$$

The derivative takes the form

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} g_2(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\infty \tilde{k}_{j,l}\left(\frac{x}{\xi}\right) \frac{1}{\xi} \left[r_l(\xi) - r_l(0))\right] \mathrm{d}\xi \\
&= \int_0^\infty \tilde{k}'_{j,l}\left(\frac{x}{\xi}\right) \left(-\frac{1}{\xi}\right) \frac{1}{\xi} \left[r_l(\xi) - r_l(0))\right] \mathrm{d}\xi \\
&= -x^{-1} \int_0^\infty \left[\left(\frac{x}{\xi}\right) \tilde{k}'_{j,l}\left(\frac{x}{\xi}\right)\right] \frac{1}{\xi} \left[r_l(\xi) - r_l(0))\right] \mathrm{d}\xi.
\end{aligned}
$$

This can be estimated as in (3.22).

For g_1 we conclude that, if x is small,

$$
\begin{aligned}
|g_1(x)| &\leq C x^\gamma \int_0^\infty \left|1 - \theta\left(\frac{x}{\xi}\right)\right| \xi^{-1-\gamma} \left|r_l(\xi) - r_l(0)\right| \mathrm{d}\xi \\
&\leq C x^\gamma \int_0^{2x} \xi^{-\gamma} \log \xi \mathrm{d}\xi \leq C x^{1-\varepsilon}, \\
\left|\frac{\mathrm{d}}{\mathrm{d}x} g_1(x)\right| &= \Bigg| c_{j,l} x^\gamma \int_0^\infty (-1) \theta'\left(\frac{x}{\xi}\right) \xi^{-2-\gamma} \left[r_l(\xi) - r_l(0)\right] \mathrm{d}\xi + \\
&\quad c_{j,l} \gamma x^{\gamma-1} \int_0^\infty \left\{1 - \theta\left(\frac{x}{\xi}\right)\right\} \xi^{-1-\gamma} \left[r_l(\xi) - r_l(0)\right] \mathrm{d}\xi \Bigg| \\
&\leq C x^\gamma \int_{0.5}^1 \xi^{-2-\gamma} \left|r_l(\xi) - r_l(0)\right| \mathrm{d}\xi + C x^{\gamma-1} \int_0^{2x} \xi^{-\gamma} \log \xi \mathrm{d}\xi \\
&\leq C x^{-\varepsilon}.
\end{aligned}
$$

This completes the proof of (3.20). ∎

Remark 3.1 *All the constants in Corollary 3.2 are independent of the point O at which we consider the asymptotic expansion. From this, Lemma 3.1 iii), and Corollary 3.1, we conclude (3.2) for any point U not close to a vertex. Hence, the assertions on the second order derivatives in Lemma 3.1 i) and iii) and in Corollary 3.1 are true.*

Corollary 3.3 *There exist constant numbers d_0 and d_γ and a differentiable function h such that, for $x \le 0.5$ and for sufficiently small $\varepsilon > 0$,*

$$\partial_y[\chi' u](x,0,0) = d_0 + d_\gamma x^\gamma + h(x), \tag{3.23}$$

$$|h(x)| \le C x^{\gamma+\varepsilon}, \quad \left| \frac{\mathrm{d}}{\mathrm{d}x} h(x) \right| \le C x^{\gamma+\varepsilon-1}. \tag{3.24}$$

A similar representation holds for the function $x \mapsto \partial_y[\chi' u](x \cos\varphi, 0, x \sin\varphi)$.

Proof. The starting point is

$$(I - K_T)\left[\partial_y[\chi' u]\right] = \partial_y R.$$

From this we can proceed analogously to the derivation of Corollary 3.2 from (3.1). Instead of the function u_i we have

$$v_1(x) := \partial_y[\chi' u](x,0,0),$$
$$v_2(x) := \partial_y[\chi' u](x\cos\varphi, 0, x\sin\varphi).$$

Setting $w(x,y,z) := \partial_y[\chi' u](x,0,z)$, we get $R_5 := K_T(\partial_y[\chi' u] - w)$ instead of R_4. The right-hand sides r_i in (3.13) are to be replaced by $s_1(x) := [R_5 + \partial_y R](x,0,0)$ and $s_2(x) := [R_5 + \partial_y R](x\cos\varphi, 0, x\sin\varphi)$. Using Lemma 3.1 and the arguments of the proof to Lemma 3.2 ii), we get that, analogously to (3.17),

$$|s_i(x)| \le C, \quad \left| \frac{\mathrm{d}}{\mathrm{d}x} s_i(x) \right| \le C \log|x|^{-1} \text{ if } x \le 0.5. \tag{3.25}$$

Instead of the entities in (3.21) we arrive at the corresponding entities

$$d_0 := \hat{k}_{j,l}(0) s_l(0), \tag{3.26}$$

$$d_\gamma := c_{j,l} \int_0^\infty \xi^{-\gamma-1} \left[s_l(\xi) - s_l(0)\right] \mathrm{d}\xi, \tag{3.27}$$

$$h := h_1 + h_2, \tag{3.28}$$

$$h_1(x) := c_{j,l} \int_0^\infty \left\{1 - \theta\left(\frac{x}{\xi}\right)\right\} \left(\frac{x}{\xi}\right)^\gamma \frac{1}{\xi} \left[s_l(\xi) - s_l(0)\right] \mathrm{d}\xi, \tag{3.29}$$

$$h_2(x) := \int_0^\infty \tilde{k}_{j,l}\left(\frac{x}{\xi}\right) \frac{1}{\xi} \left[s_l(\xi) - s_l(0)\right] \mathrm{d}\xi. \tag{3.30}$$

With this notation the estimate for h is analogous to that for g. ∎

Now let us consider the y dependence of the coefficients c_0, d_0, c_γ, d_γ, and of the function h and g in Corollaries 3.2 and 3.3. To this end let the point O and the coordinate system with coordinates (x,y,z) be fixed. Let $\tilde{O} := (0, y_{\tilde{O}}, 0)$ be another edge point and denote the coordinates corresponding to this point by $(\tilde{x}, \tilde{y}, \tilde{z})$. Obviously, $(\tilde{x}, \tilde{y}, \tilde{z}) = (x, y - y_{\tilde{O}}, z)$. If we apply Corollary 3.3 to $\tilde{O}$, we get

$$\partial_y[\chi' u](x, y_{\tilde{O}}, 0) = \tilde{d}_0 + \tilde{d}_\gamma + \tilde{h}(x) \tag{3.31}$$

together with the corresponding estimates for $\tilde{h}$. We write $d_0(y_{\tilde{O}}) := \tilde{d}_0$, $d_\gamma(y_{\tilde{O}}) := \tilde{d}_\gamma$, and $h(x, y_{\tilde{O}}) := \tilde{h}(x)$ and note that these entities are defined by (3.26)-(3.30) with s_l

replaced by $s_l(x, y_{\tilde{O}}) := \tilde{s}_l(x)$. The function $\tilde{s}_l(x)$ is the restriction of $[\tilde{R}_5 + \partial_y R]$ to $\{(x, y_{\tilde{O}}, z) :\ x, y \in I\!R\}$ and $\tilde{R}_5 := K_T(\partial_y[\chi' u] - \partial_y[\chi' u](\cdot, y_{\tilde{O}}, \cdot))$. All these functions depend continuously on $y_{\tilde{O}}$ and even the estimates (3.25) for s_l replaced by $\tilde{s}_l$ are uniform with respect to $y_{\tilde{O}}$. Consequently, the functions $d_0(y_{\tilde{O}})$, $d_\gamma(y_{\tilde{O}})$, and $h(x, y_{\tilde{O}})$ are continuous with respect to $y_{\tilde{O}}$.

Integrating (3.31) with respect to $y = y_{\tilde{O}}$, we obtain

$$\begin{aligned} u(x,y,0) &= u(x,0,0) + \int_0^y \partial_y u(x,\eta,0)\mathrm{d}\eta \\ &= c_0 + c_\gamma + g(x) + \int_0^y d_0(\eta)\mathrm{d}\eta + x^\gamma \int_0^y d_\gamma(\eta)\mathrm{d}\eta + \int_0^y h(x,\eta)\mathrm{d}\eta \\ &= \left[c_0 + \int_0^y d_0(\eta)\mathrm{d}\eta\right] + \left[c_\gamma + \int_0^y d_\gamma(\eta)\mathrm{d}\eta\right] x^\gamma + \left[g(x) + \int_0^y h(x,\eta)\mathrm{d}\eta\right]. \end{aligned} \tag{3.32}$$

Clearly, the function $y \mapsto u(+0, y, 0) = [c_0 + \int d_0(\eta)\mathrm{d}\eta]$ is twice continuously differentiable by Lemma 3.1 iii) and Remark 3.1. Equation (3.32) and the estimates (3.20) and (3.24) imply Theorem 1.1.

4 Numerical Test

In order to verify the first term of the edge asymptotics numerically, we consider the two-piece wedge boundary of two triangles meeting along the x-axis at an angle of φ, i.e., $S = T_1 \cup T_2$ with

$$\begin{aligned} T_1 &:= \{(x,y,0) :\ 0 < x < 1,\ 0 < y < 1 - x\}, \\ T_2 &:= \{(x, y\cos\varphi, y\sin\varphi) :\ 0 < x < 1,\ 0 < y < 1 - x\}. \end{aligned}$$

Note that this S can be considered as a part of a polyhedral boundary $\tilde{S}$. If ϱ and E vanish over $\tilde{S} \setminus S$, then the equation (1.1) over $\tilde{S}$ reduces to an equation (1.1) over the open surface S. We choose

$$\begin{aligned} E(x,y,0) &:= \frac{\sin y}{0.61}, \\ E(x, y\cos\varphi, y\sin\varphi) &:= \frac{\sin\{y\sin\varphi\}}{0.61}, \\ \varrho(P) &:= \begin{cases} \varrho_1 & \text{if } P \in T_1 \\ \varrho_2 & \text{if } P \in T_2 \end{cases}. \end{aligned}$$

Note that the special choice of the right-hand side E ensures that the "constant" terms $\psi_1^{H_i}(y)$ in (1.2) are close to zero.

Using a program package of Atkinson which is an extended version of the package [1], we solve (1.1) numerically by piecewise linear collocation. The surface is divided into 2048 uniform triangles of diameter $h = 0.0442$. The approximate solution is linear over each of the triangles but not necessarily continuous. In the interior of each triangle three collocation points are chosen. Thus the number of degrees of freedom is 6144. The integrals in the coefficients of the arising linear system of equations are computed by a change of variable followed by Gaussian quadrature if the integral is singular. They are computed by a suitable subdivision combined with a seven point scheme if the integral

ϱ_1	ϱ_2	φ	γ	γ_h
0.6	0.6	0.5π	0.718	0.710
0.3	0.7	0.5π	0.782	0.814
0.6	0.6	0.9π	0.972	0.983

Table 2: Some values of approximate exponent γ_h.

is non-singular. The linear system is solved directly by Gaussian elimination. We denote the approximate solution for u by u_h.

To find an approximation for the exponent γ, we consider the edge point $(x_0, 0, 0)$, $x_0 = 23/64$ which is relatively far from the boundary points of the open surface S. We restrict the solution u_h to $\{(x_0, y, 0) : \; 0 < y < 1 - x_0\}$ and try to verify the asymptotic expansion $u_h(x_0, y, 0) \sim C + Cy^\gamma + \dots$. If this is the real asymptotic behaviour, then

$$\gamma \approx \gamma_h := \frac{\log[u_h(x_0, 4y_0, 0) - u_h(x_0, 2y_0, 0)] - \log[u_h(x_0, 2y_0, 0) - u_h(x_0, y_0, 0)]}{\log 2}, \quad (4.1)$$

where the error $|\gamma - \gamma_h|$ is small for sufficiently small y_0 and sufficiently small discretization errors $\|u - u_h\|_{L^\infty(S)}$. For our numerical tests, we have chosen $y_0 = 0.01$. Note that this choice guarantees that the points $(x_0, y_0, 0)$, $(x_0, 2y_0, 0)$, and $(x_0, 4y_0, 0)$ (cf. (4.1)) belong to three different subdivision triangles. Much smaller values y_0 would lead to the situation that all the three points used for (4.1) are contained in one triangle. Then, due to the linearity of the approximation u_h, we would get $\gamma_h = 1$. Much larger values y_h lead to larger errors $|\gamma - \gamma_h|$ due to the influence of higher order terms in the asymptotics.

In Table 2 we present some approximate values which seem to be in relatively good agreement with the values predicted by Theorem 1.1. Note, however, that in general the approximation of γ by γ_h is not very accurate. For the case $\varrho_1 = \varrho_2 = 0.6$ and $\varphi = \pi/2$, we compare the solution u_h with a lower level solution u_h and get the estimate 0.008 for the discretization error. With this error tolerance applied to (4.1), we can only conclude that the true γ is contained in the interval $[0.41, 1.01]$.

Acknowledgements. This work was greatly inspired by the author's visit to Professor K.E. Atkinson at Iowa City. It was supported by the University of Iowa and by a grant of Deutsche Forschungsgemeinschaft under grant numbers Pr 336/5-1 and Pr 336/5-2. The author is especially grateful to Professor K.E. Atkinson for providing him with the codes for the numerical tests.

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