Maximal Attractor for the System of a Landau-Ginzburg Theory for Structural Phase Transitions in Shape Memory Alloys

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Abstract

In this paper, a system of partial differential equations modelling the dynamics of martensitic phase transitions in shape memory alloys is further investigated. In this system, the free energy is assumed to be in the Landau-Ginzburg form and nonconvex in the order parameter; the materials are assumed to be viscous. In a previous paper published in SIAM J. Math. Anal. the global existence of a unique solution and the compactness of the orbit have been established; in the present paper the existence of a compact maximal attractor is proved.

1 Introduction

In this paper, we further study a system modelling the thermomechanical developments in a one-dimensional heat-conducting viscous solid of constant density ϱ (assumed to be normalized to unity, i.e. $\varrho = 1$) which is subject to heating and loading. The basic assumptions on the system under investigation are the following ones: we think of metallic solids that not only respond to a change of the strain ε by an elastic (possibly nonlinear) stress $\sigma = \sigma(\varepsilon)$, but also to a change of the curvature of their metallic lattices by a couple stress $\mu = \mu(\varepsilon_x)$. We assume that the Helmholtz free energy density F is a potential of Landau-Ginzburg form, that is,

$$F = F(\varepsilon, \varepsilon_x, \theta)\,, \tag{1.1}$$

where θ denotes the absolute temperature. To cover systems modelling first-order stress-induced and temperature-induced solid-solid phase transitions accompanied by hysteresis phenomena, we do not assume that F is a convex function of the order parameter ε.
A particular class of materials, where both stress-induced and temperature-induced first-order phase transitions occur that lead to a rather spectacular hysteretic behavior, are the so-called shape memory alloys (for details, we refer to the monograph [4] by Brokate and Sprekels). In these materials the metallic lattice is deformed by shear and the assumption of a constant density is justified. The shape memory effect itself is due to martensitic phase transitions between different configurations of the crystal lattice, namely austenite and martensitic twins. For an account of the physical properties of shape memory alloys, we refer the reader to the papers by Falk [8], [9]. In [8], [9] Falk has proposed a Laudau-Ginzburg theory, using the strain ε as order parameter, to explain the occurrence of the martensitic transitions in shape memory alloys. In this connection, we also refer to the works of Müller et al. (cf. [1], [14]).
The simplest form for the free energy density F, which accounts quite well for the experimentally observed behavior and takes couple stresses into account, is given by (see Falk [8], [9])

$$F(\varepsilon, \varepsilon_x, \theta) = F_0(\theta) \,+\, F_1(\varepsilon)\,\theta \,+\, \frac{\delta}{2}\,\varepsilon_x^2\,, \tag{1.2}$$

where

$$F_1(\varepsilon) = \alpha_1\varepsilon^2\,, \quad F_2(\varepsilon) = \alpha\varepsilon^4 - \alpha_1\theta_1\varepsilon^2\,, \tag{1.3}$$

$$F_0(\theta) = -\,C_V\,\theta\,\log\left(\frac{\theta}{\theta_2}\right) \,+\, C_V\,\theta \,+\, \tilde{C}\,, \tag{1.4}$$

with θ_1 , δ , α_i (i=1,2,3), θ_2 , C_V , $\tilde{C}$ being positive constants. The positive constant C_V denotes the specific heat. Observe that in the interesting range of temperature, for θ close

to θ_1, F is not a convex function of the shear strain ε. In fact, $F(\cdot, \varepsilon_x, \theta)$ may have up to three minima which correspond to the austenitic and the two martensitic phases.
We want to forecast the dynamics of the phase transitions in the one-dimensional situation. To this end, let $\Omega = (0,1)$, and, for $t > 0, \Omega_t = \Omega \times (0,t)$. Then the balance laws of momentum and internal energy read

$$u_{tt} - \sigma_x + \mu_{xx} = 0, \quad \text{in } \Omega_\infty\,, \tag{1.5}$$

$$U_t + q_x - \sigma\varepsilon_t - \mu\varepsilon_{xt} = 0, \quad \text{in } \Omega_\infty\,. \tag{1.6}$$

The second law of thermodynamics is expressed by the Clausius-Duhem inequality

$$S_t + \left(\frac{q}{\theta}\right)_x \geq 0, \quad \text{in } \Omega_t\,. \tag{1.7}$$

Here u, σ, μ, U, q, ε, S and θ denote displacement, stress, couple stress, internal energy density, heat flux, shear strain, entropy density and absolute temperature, respectively.
For one-dimensional, homogeneous, thermoviscoelastic materials, we have the following constitutive relations:

$$\varepsilon = u_x\,, \quad \sigma = \frac{\partial F}{\partial \varepsilon} + \gamma\varepsilon_t\,, \quad \mu = \frac{\partial F}{\partial \varepsilon_x}\,, \quad S = -\frac{\partial F}{\partial \theta}\,, \quad U = F + \theta S\,. \tag{1.8}$$

Here, $\gamma > 0$ denotes the viscosity.
For the heat flux q, we assume Fourier's law

$$q = -\,k\,\theta_x\,, \tag{1.9}$$

where $k > 0$ is the heat conductivity (assumed constant). Obviously, this assumption implies the validity of (1.7) so that the second law of thermodynamics is satisfied automatically.
Inserting the constitutive relations into the balance laws (1.5)–(1.6), we obtain the system of partial differential equations

$$u_{tt} - (f_1\theta + f_2)_x - \gamma\varepsilon_{xt} + \delta u_{xxxx} = 0\,, \quad \text{in } \Omega_\infty\,, \tag{1.10}$$

$$C_V\theta_t - k\theta_{xx} - f_1\theta\varepsilon_t - \gamma\varepsilon_t^2 = 0\,, \quad \text{in } \Omega_\infty\,, \tag{1.11}$$

$$\varepsilon = u_x\,, \quad \text{in } \Omega_\infty\,, \tag{1.12}$$

where

$$f_1 = f_1(\varepsilon) = F_1'(\varepsilon)\,, \quad f_2 = f_2(\varepsilon) = F_2'(\varepsilon)\,. \tag{1.13}$$

In addition, we prescribe the initial and boundary conditions

$$u|_{x=0} = \varepsilon_x|_{x=0} = 0\,, \quad \varepsilon|_{x=1} = (\gamma u_{xt} - \delta u_{xxx} + \sigma_1)|_{x=1} = 0\,, \tag{1.14}$$

where

$$\sigma_1 = f_1\theta + f_2\,, \tag{1.15}$$

as well as

$$\theta_x|_{x=0,1} = 0\,, \tag{1.16}$$

and

$$u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)\,, \quad \theta(x,0) = \theta_0(x) > 0\,, \quad x \in \overline{\Omega}\,. \tag{1.17}$$

The mechanical meaning of the boundary conditions is clear. For instance, the second one at $x = 1$ simply means stress-free.
We now employ the idea of Andrews [2] and Pego [17] to simplify the problem by introducing the velocity potential

$$p(x,t) = \int_1^x u_t(y,t)dy\,. \tag{1.18}$$

Then

$$\varepsilon_t = p_{xx}\,, \quad \text{in } \Omega_\infty\,, \tag{1.19}$$

and (1.10)–(1.11) turn out to be

$$p_t - \gamma p_{xx} + \delta\varepsilon_{xx} - \sigma_1 = 0\,, \quad \text{in } \Omega_\infty\,, \tag{1.20}$$

$$C_V\theta_t - k\theta_{xx} - f_1\theta p_{xx} - \gamma p_{xx}^2 = 0\,, \quad \text{in } \Omega_\infty\,. \tag{1.21}$$

Accordingly, the initial and boundary conditions (1.14), (1.16), (1.17) become

$$p_x|_{x=0} = \varepsilon_x|_{x=0} = 0\,, \tag{1.22}$$

$$p|_{x=1} = p_{xx}|_{x=1} = \varepsilon|_{x=1} = 0\,, \tag{1.23}$$

$$\varepsilon(x,0) = \varepsilon_0(x) := u_{0x}(x)\,, \quad p(x,0) = p_0(x) := \int_1^x u_1(y)dy\,, \quad \theta(x,0) = \theta_0(x)\,. \tag{1.24}$$

It is easy to see that if (u, v, θ) is a smooth solution to (1.10)–(1.17), then (ε, p, θ) is a smooth solution to (1.19)–(1.24) and vice versa. Therefore, it suffices to consider the problem (1.19)–(1.24). In the sequel, we assume without loss of generality that $C_V = 1$.
In the previous paper [22] it has been proved that for any given initial data $(\varepsilon_0, p_0, \theta_0) \in H^3 \times H^3 \times H^1$ satisfying the compatibility conditions, that is, $p_{0x}|_{x=0} = \varepsilon_{0x}|_{x=0} = 0$, $p_0|_{x=1} = p_{0xx}|_{x=1} = \varepsilon_{0x}|_{x=1} = 0$, the problem (1.19)–(1.24) admits a unique global solution. Moreover, the orbit defined by the solution is compact in $H^3 \times H^3 \times H^1$. In the present paper, we further investigate the dynamics defined by the system, namely, we want to prove the existence of a compact maximal attractor.
Before stating and proving our results, let us first recall the related results in the literature. When $\delta = 0$, Dafermos [6], Dafermos & Hsiao [7], Chen & Hoffmann [5], Jiang [11], proved the global existence of classical solution to the system (1.10)–(1.12) with various boundary conditions for a class of solid-like materials. However, the asymptotic behavior was not considered in these papers. Recently, on the basis of Dafermos [6] and of Dafermos & Hsiao [7], T. Luo [13] further investigated the asymptotic behavior of smooth solution as time tends to infinity for a special class of solid-like materials in which $e = C_V\theta$, $F_2 = 0$ and $\delta = 0$. Racke and Zheng [18], and Shen, Zheng & Zhu [20], respectively, obtained global existence, uniqueness and asymptotic behavior of weak solutions to (1.10)– (1.12) for $\delta = 0$ in the case when both ends of the rod are thermally insulated and when at least one end is stress-free (or both ends are clamped, respectively).
Concerning the case $\delta > 0$, we refer to Sprekels & Zheng [21] for the case $\delta > 0$, $\gamma = 0$, and to Hoffmann & Zochowski [10] for the case $\delta > 0$, $\gamma > 0$, for global existence and

uniqueness results for the model of shape memory alloys with the Helmholtz free energy density being a potential of Landau-Ginzburg form. However, the a priori estimates of the solutions obtained in these papers depend on T. Consequently, the asymptotic behavior of the solutions as time tends to infinity could not be discussed there.
In this direction, we would also like to refer to Andrews [2], Andrews & Ball [3], and Pego [17], for the isothermal and purely viscoelastic case.
To study the existence of a compact attractor, we first multiply (1.20) by $-p_{xx}$, add the result to (1.21) and integrate with repect to x over Ω to obtain that

$$\frac{d}{dt}\int_0^1 \left(\theta + F_2(\varepsilon) + \frac{1}{2}p_x^2 + \frac{\delta}{2}\varepsilon_x^2\right)(t)\,dx = 0. \tag{1.25}$$

Thus,

$$E_1(t) := \int_0^1 \left(\theta + F_2(\varepsilon) + \frac{1}{2}p_x^2 + \frac{\delta}{2}\varepsilon_x^2\right)(t)\,dx = E_1(0), \tag{1.26}$$

where $E_1(0)$ is a constant depending on the initial data. Obviously, this energy conservation indicates that there can be no absorbing ball for initial data varying *in the whole space*. Instead, we should rather consider the dynamics in a restricted set which is invariant for the orbit. In this regard the situation is quite similar to that encountered for the Cahn–Hilliard equation (cf., for instance, [23]).
Now let us consider the space

$$\begin{aligned} H \;:=\; & \Big\{(\varepsilon,p,\theta)\in H^3[0,1]\times H^3[0,1]\times H^1[0,1] \;:\; p_x|_{x=0}=\varepsilon_x|_{x=0}=0\,, \\ & p|_{x=1}=p_{xx}|_{x=1}=\varepsilon_x|_{x=1}=0\Big\}, \end{aligned} \tag{1.27}$$

which becomes a Hilbert space when equipped with the usual inner product and norm of $H^3\times H^3\times H^1$.
Next, let $\beta_1>0$, $\beta_2\in I\!R$, be arbitrary given constants, and let

$$\begin{aligned} H_{\beta_1,\beta_2} \;:=\; & \Big\{(\varepsilon,p,\theta)\in H \;:\; \theta>0\,,\quad \int_0^1\big(\log\theta - F_1(\varepsilon)\big)\,dx > \beta_2\,, \\ & \int_0^1\left(\theta + F_2(\varepsilon) + \frac{1}{2}p_x^2 + \frac{\delta}{2}\varepsilon_x^2\right)dx < \beta_1\Big\}\,. \end{aligned} \tag{1.28}$$

Apparently H_{β_1,β_2} is an open subset of H.
We are now in the position to state our main theorem.

Theorem 1.1 *For every $\beta_1>0$, $\beta_2<0$, the semigroup $S(t)$ defined by (1.19)–(1.24) maps H_{β_1,β_2} into itself. In addition, it possesses in H_{β_1,β_2} a maximal attractor A_{β_1,β_2} which is compact.*

The notation in this paper will be as follows: L^p, $1\le p\le\infty$, $W^{m,\infty}$, $m\in I\!N$, $H^1\equiv W^{1,2}$, $H_0^1=W_0^{1,2}$, denote the usual (Sobolev) spaces on (0,1). In addition, $(\cdot,\cdot)$ stands for the inner product in L^2, and $\|\cdot\|_B$ denotes the norm in the space B; we also put $\|\cdot\|:=\|\cdot\|_{L_2}$. We denote by $C^k(I,B)$, $k\in I\!N_0$, the space of k-times continuously differentiable functions from $I\subset I\!R$ into a Banach space B, and, likewise, by $L^p(I,B)$, $1\le p\le\infty$, the corresponding Lebesgue-spaces. Finally, ∂_t or $\frac{d}{dt}$ or a subscript t and ∂_x or a subscript x denote the (partial) derivatives with respect to t and x, respectively.

2 Proof of Theorem 1.1

Since we have proved in the previous paper [22] that for any initial data $(\varepsilon_0, p_0, \theta_0) \in H$ with $\theta_0 > 0$ the problem (1.19)–(1.24) admits a unique global solution and that the orbit defined by the problem is compact in $H^3 \times H^3 \times H^1$, by Theorem I.1.1 in the book by R. Temam [23] it suffices to prove that the semigroup $S(t)$ defined by the problem maps H_{β_1,β_2} into itself and that there is a bounded subset B of H_{β_1,β_2} such that B is absorbing in H_{β_1,β_2}. Although for any given initial data the convergence of solution of the problem as time tends to infinity has been considered in the previous paper [22], the proof of the existence of an absorbing set is significantly different from the line of argumentation in that paper. What we need now are uniform estimates of solutions with respect to initial data varying in a bounded subset of H_{β_1,β_2}.

In the sequel, we always assume that the initial data for the problem (1.19)–(1.24) belong to H_{β_1,β_2}. We denote by C a positive constant, which may vary from place to place, that may depend on β_1, β_2, but not on the initial data.

The proof of the existence of an absorbing set consists of the following lemmas.

Lemma 2.1 *For any $t > 0$, the following estimates hold.*

$$\|\varepsilon(t)\| + \|\varepsilon(t)\|_{L^6} + \|p_x(t)\| + \|\varepsilon_x(t)\| + \|\theta(t)\|_{L^1} \le C\,, \tag{2.1}$$

$$\|p(t)\|_{L^\infty} + \|\varepsilon(t)\|_{L^\infty} \le C\,, \tag{2.2}$$

$$\theta(x,t) > 0\,, \quad \forall\ (x,t)\ \in\ [0,1] \times I\!R^+\,, \tag{2.3}$$

$$E_1(t) = E_1(0) < \beta_1\,. \tag{2.4}$$

Proof. First, applying the maximum principle to (1.21), we have

$$\theta(x,t) > 0 \quad \forall\ x, t\,. \tag{2.5}$$

As derived in the previous section, the energy conservation (2.4) holds for all $t \ge 0$. Using Young's inequality, we find that

$$F_2(\varepsilon) \ge C_1 \varepsilon^6 - C_2\,, \tag{2.6}$$

from which (2.1) follows. Inequality (2.2) is then a consequence of the boundary conditions and of Poincare's inequality, which concludes the proof of the assertion. □

Lemma 2.2 *For any $t > 0$, the following estimates hold.*

$$\int_0^t \int_0^1 \left(\frac{\theta_x^2}{\theta^2} + \frac{p_{xx}^2}{\theta} \right) dx d\tau \le C\,, \tag{2.7}$$

$$\int_0^t \|p_x(\tau)\|^2 d\tau \le \int_0^t \|p_x(\tau)\|^2_{L^\infty} d\tau \le C\,, \quad \int_0^t \|p(\tau)\|^2_{L^\infty} d\tau \le C\,, \tag{2.8}$$

$$E_2(t) := \int_0^1 (\log\theta - F_1(\varepsilon))(t)\, dx \ge E_2(0) > \beta_2\,, \tag{2.9}$$

$$\int_0^t \|p_x(\tau)\|^{n+2} d\tau \le C\,, \quad \forall\ \ n \ge 0\,, \tag{2.10}$$

$$\int_0^1 \theta(t)\, dx \ge C > 0\,. \tag{2.11}$$

PROOF. Multiplication of (1.21) by θ^{-1} and integration with respect to x over Ω yields

$$\frac{d}{dt}\int_0^1 (\log\theta - F_1(\varepsilon))(t)\,dx \;-\; \int_0^1 \left(\frac{k\theta_x^2}{\theta^2} + \frac{\gamma p_{xx}^2}{\theta}\right)(t)\,dx = 0\,, \tag{2.12}$$

from which (2.9) follows. In addition, since $\log\theta \le \theta - 1$ for all $\theta > 0$, we have

$$\int_0^t\int_0^1 \left(\frac{k\theta_x^2}{\theta^2} + \frac{\gamma p_{xx}^2}{\theta}\right) dx d\tau \le C\,. \tag{2.13}$$

Next, observe that it follows from $p_x|_{x=0} = 0$ that

$$p_x(x,t) = p_x(0,t) \,+\, \int_0^x p_{xx}(y,t)\,dy = \int_0^x p_{xx}(y,t)\,dy\,. \tag{2.14}$$

Hence,

$$\begin{aligned}
&\int_0^t \|p_x(\tau)\|_{L^\infty}^2 d\tau \le \int_0^t \Big(\int_0^1 |p_{xx}(x,\tau)|\,dx\Big)^2 d\tau \\
\le\ & \int_0^t \Big(\int_0^1 \sqrt{\theta}\,\frac{|p_{xx}|}{\sqrt{\theta}}\,dx\Big)^2 d\tau \le \int_0^t \Big(\int_0^1 \theta\,dx\Big)\int_0^1 \frac{p_{xx}^2}{\theta} dx\,d\tau \\
\le\ & C\int_0^t\int_0^1 \frac{p_{xx}^2}{\theta} dx d\tau \le C\,.
\end{aligned} \tag{2.15}$$

Therefore also

$$\int_0^t \|p_x(\tau)\|^2 d\tau \le \int_0^t \|p_x\|_{L^\infty}^2 d\tau \le C\,. \tag{2.16}$$

Combining (2.8) with (2.1) yields

$$\int_0^t \|p_x(\tau)\|^{n+2} d\tau \le C \quad \forall \;\; n \ge 0\,. \tag{2.17}$$

Finally, we conclude from (2.12) and Jensen's inequality that

$$-\,\log\Big(\int_0^1 \theta(t)\,dx\Big) \le -\int_0^1 \log\,\theta(t)\,dx \le C\,, \tag{2.18}$$

from which (2.11) follows. □

Note that it follows from the Lemmas 2.1 and 2.2 that the semigroup $S(t)$ maps H_{β_1,β_2} into itself. Also, we will see in what follows that (2.17) plays a very important role in deriving uniform a priori estimates.

Lemma 2.3 *For any $t > 0$, the following estimates hold.*

$$\frac{d}{dt}\|\theta_x(t)\|^2 \,+\, \|\theta_t(t)\|^2 \le C\left(\|\theta(t)\|_{L^\infty}\,\|p_{xx}(t)\|_{L^\infty}^2 \,+\, \|p_{xx}(t)\|_{L^4}^4\right)\,, \tag{2.19}$$

$$\frac{d}{dt}(\|p_{xt}(t)\|^2 \,+\, \delta\,\|\varepsilon_{xt}(t)\|^2) \,+\, \|\varepsilon_{tt}(t)\|^2 \le C\int_0^1 (\theta^2\varepsilon_t^2 + \varepsilon_t^2 + \theta_t^2)(t)\,dx\,, \tag{2.20}$$

$$\frac{\gamma}{2}\frac{d}{dt}\|\varepsilon_{xt}(t)\|^2 - \frac{d}{dt}(p_{xt}(t),\varepsilon_{xt}(t)) - \|p_{xxt}(t)\|^2 + \frac{\delta}{2}\|\varepsilon_{xxt}(t)\|^2 \le C\int_0^1 (\theta^2\varepsilon_t^2 + \varepsilon_t^2 + \theta_t^2)(t)\,dx\,. \tag{2.21}$$

Proof. For the sake of brevity, we omit the arguments of functions from now on. Multiplying (1.21) by θ_t and integrating with repect to x over Ω, we obtain that

$$
\begin{aligned}
\frac{k}{2}\frac{d}{dt}\|\theta_x\|^2 + \|\theta_t\|^2 &= \int_0^1 \left(f_1\,\theta\,\theta_t\,p_{xx} + \gamma\,\theta_t\,p_{xx}^2\right)dx \\
&\le C\left(\|\theta\,p_{xx}\|\,\|\theta_t\| + \left(\int_0^1 p_{xx}^4\,dx\right)^{\frac{1}{2}}\|\theta_t\|\right) \\
&\le C\left(\|\theta\|_{L^\infty}^{\frac{1}{2}}\|p_{xx}\|_{L^\infty}\left(\int_0^1 \theta\,dx\right)^{\frac{1}{2}}\|\theta_t\| + \|p_{xx}\|_{L^4}^2\,\|\theta_t\|\right), \qquad (2.22)
\end{aligned}
$$

and (2.19) follows from the Cauchy-Schwarz inequality.
Next, we differentiate (1.20) with respect to t, multiply the result by $-\varepsilon_{tt}$, and integrate with repect to x over Ω to obtain

$$
\begin{aligned}
0 &= (p_{tt}, -p_{xxt}) + \gamma\,\|\varepsilon_{tt}\|^2 + (\delta\,\varepsilon_{xt}, \varepsilon_{xtt}) + \int_0^1 \sigma_{1t}\,\varepsilon_{tt}\,dx \\
&= (p_{xtt}, p_{xt}) + \gamma\,\|\varepsilon_{tt}\|^2 + \delta\,(\varepsilon_{xt}, \varepsilon_{xtt}) + \int_0^1 \left(f_1'\,\varepsilon_t\,\theta + f_2'\,\varepsilon_t + f_1\,\theta_t\right)\varepsilon_{tt}\,dx\,. \qquad (2.23)
\end{aligned}
$$

Combining this with (2.2) yields

$$
\frac{1}{2}\frac{d}{dt}\left(\|p_{xt}\|^2 + \delta\,\|\varepsilon_{xt}\|^2\right) + \gamma\,\|\varepsilon_{tt}\|^2 \le \frac{\gamma}{2}\,\|\varepsilon_{tt}\|^2 + C\int_0^1 \left(\theta^2\,\varepsilon_t^2 + \varepsilon_t^2 + \theta_t^2\right)dx\,, \qquad (2.24)
$$

and (2.20) follows.
Finally, we differentiate (1.20) with respect to t, multiply the result by ε_{xxt}, and integrate the result with respect to x over Ω to arrive at

$$
\begin{aligned}
0 &= (p_{tt}, \varepsilon_{xxt}) - \gamma\,(\varepsilon_{tt}, \varepsilon_{xxt}) + \delta\|\varepsilon_{xxt}\|^2 - \int_0^1 \varepsilon_{xxt}\,\sigma_{1t}\,dx \\
&= (p_{xxtt}, \varepsilon_t) + \gamma\,(\varepsilon_{xtt}, \varepsilon_{xt}) + \delta\|\varepsilon_{xxt}\|^2 - \int_0^1 \varepsilon_{xxt}\,\sigma_{1t}\,dx \\
&= \frac{d}{dt}(p_{xxt}, \varepsilon_t) - \|p_{xxt}\|^2 + \frac{\gamma}{2}\frac{d}{dt}\|\varepsilon_{xt}\|^2 + \delta\|\varepsilon_{xxt}\|^2 - \int_0^1 \varepsilon_{xxt}\,\sigma_{1t}\,dx\,. \qquad (2.25)
\end{aligned}
$$

However, by integration by parts, we have

$$
(p_{xxt}, \varepsilon_t) = -(p_{xt}, \varepsilon_{xt})\,. \qquad (2.26)
$$

Observe also that

$$
\sigma_{1t} = f_1'(\varepsilon)\,\varepsilon_t\,\theta + f_1(\varepsilon)\,\theta_t + f_2'(\varepsilon)\,\varepsilon_t\,. \qquad (2.27)
$$

Hence, combining (2.25)–(2.27) and applying the Cauchy–Schwarz inequality, we see that (2.21) holds. The proof of the assertion is complete. □

Now let $\eta \in (0,1)$ be a small positive constant chosen in such a way that

$$
E(t) := \|p_{xt}(t)\|^2 + \delta\|\varepsilon_{xt}(t)\|^2 + \frac{\gamma\,\eta}{2}\,\|\varepsilon_{xt}(t)\|^2 - \eta\,(p_{xt}(t), \varepsilon_{xt}(t)) \qquad (2.28)
$$

is positive definite, i.e., such that there is a constant $C > 0$ satisfying

$$
E(t) \ge C\left(\|p_{xt}(t)\|^2 + \|\varepsilon_{xt}(t)\|^2\right). \qquad (2.29)
$$

We then have

Lemma 2.4 *For any $t > 0$ it holds that*

$$\frac{d}{dt}E(t) + \frac{\delta\,\eta}{2}\,\|\varepsilon_{xxt}(t)\|^2 + (1-\eta)\,\|\varepsilon_{tt}(t)\|^2 \le C\int_0^1 \Big(\theta^2\,\varepsilon_t^2 + \varepsilon_t^2 + \theta_t^2\Big)(t)\,dx\,. \tag{2.30}$$

Proof. Multiplying (2.21) by η, then adding the result up with (2.20), yields (2.30). □

We now introduce the mean value of temperature,

$$\bar\theta(t) := \int_0^1 \theta(x,t)\,dx\,. \tag{2.31}$$

We then obtain the following result.

Lemma 2.5 *For any $t > 0$ it holds*

$$\frac{d}{dt}\,\|(\theta-\bar\theta)(t)\|^2 + k\,\|\theta_x(t)\|^2 \le C\,\Big(\|p_{xx}(t)\|^2_{L^\infty} + \|p_{xx}(t)\|^4\Big)\,. \tag{2.32}$$

Proof. It follows from the energy equation (1.11) that

$$\bar\theta_t - \int_0^1 f_1\,\theta\,\varepsilon_t\,dx - \gamma\|\varepsilon_t\|^2 = 0\,. \tag{2.33}$$

Thus,

$$(\theta-\bar\theta)_t - k\,\theta_{xx} - \Big(f_1\,\theta\,\varepsilon_t - \int_0^1 f_1\,\theta\,\varepsilon_t\,dx\Big) - \gamma\,(\varepsilon_t^2 - \|\varepsilon_t\|^2) = 0\,. \tag{2.34}$$

Multiplication of (2.34) by $\theta-\bar\theta$ and integration with respect to x yields (again the argument t is omitted)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\,\|\theta-\bar\theta\|^2 + k\,\|\theta_x\|^2\\
\le\;& \int_0^1 |\theta-\bar\theta|\,\Big|f_1\,\theta\,\varepsilon_t - \int_0^1 f_1\,\theta\,\varepsilon_t\,dx\Big|\,dx + \gamma\int_0^1 |\theta-\bar\theta|\,\Big|\varepsilon_t^2 - \|\varepsilon_t\|^2\Big|\,dx\\
\le\;& C\,\|\theta-\bar\theta\|_{L^\infty}\,\Big(\|p_{xx}\|_{L^\infty} + \|p_{xx}\|^2\Big)\\
\le\;& \frac{k}{2}\,\|\theta_x\|^2 + C\,\Big(\|p_{xx}\|^2_{L^\infty} + \|p_{xx}\|^4\Big)\,,
\end{aligned} \tag{2.35}$$

from which (2.32) follows. □

Lemma 2.6 *There exists some $\mu > 0$, depending only on β_1 and β_2, such that for any $t > 0$ it holds*

$$\begin{aligned}
&\frac{d}{dt}\Big(\|(\theta-\bar\theta)(t)\|^2 + \|\theta_x(t)\|^2 + \mu\,E(t)\Big) + C\,\Big(\|(\theta-\bar\theta)(t)\|^2 + \|\theta_x(t)\|^2 + \mu\,E(t)\Big)\\
&\quad\le C\,\Big(\|p_x(t)\|^2 + \|p_x(t)\|^6 + \|p_x(t)\|^8 + \|p_x(t)\|^{14}\Big)\,.
\end{aligned} \tag{2.36}$$

Proof. We multiply (2.30) by a positive constant μ (which is yet to be specified), and then add the result up with (2.19) and (2.32) to find that

$$\begin{aligned} & \frac{d}{dt}\left(\|(\theta-\bar\theta)(t)\|^2+\|\theta_x(t)\|^2+\mu E(t)\right)+\|\theta_t(t)\|^2+k\,\|\theta_x(t)\|^2 \\ & \quad +C\mu\left(\|p_{xxxx}(t)\|^2+\|\varepsilon_{tt}(t)\|^2\right) \\ \leq\; & C\left(\|\theta(t)\|_{L^\infty}\,\|p_{xx}(t)\|^2_{L^\infty}+\|p_{xx}(t)\|^4_{L^4}\right)\;+\;C\left(\|p_{xx}(t)\|^2_{L^\infty}+\|p_{xx}(t)\|^4\right) \\ & \quad +C\mu\int_0^1\left(\theta^2p_{xx}^2+p_{xx}^2+\theta_t^2\right)(t)\,dx. \end{aligned} \tag{2.37}$$

Now, we choose μ small enough to cancel the term $\int_0^1\theta_t^2(t)\,dx$ on the right–hand side of (2.37). Next, we apply Nirenberg's inequality in the following way to the remaining terms on the right–hand side of (2.37):

$$\|p_{xx}(t)\|_{L^4}\;\leq\;C\,\|p_{xxxx}(t)\|^{\frac{5}{12}}\,\|p_x(t)\|^{\frac{7}{12}}\,, \tag{2.38}$$

$$\|p_{xx}(t)\|_{L^\infty}\;\leq\;C\,\|p_{xxxx}(t)\|^{\frac12}\,\|p_x(t)\|^{\frac12}\,, \tag{2.39}$$

$$\|p_{xx}(t)\|\;\leq\;C\,\|p_{xxxx}(t)\|^{\frac13}\,\|p_x(t)\|^{\frac23}\,, \tag{2.40}$$

$$\|\theta(t)\|_{L^\infty}\;\leq\;C_1\,\|\theta_x(t)\|^{\frac23}\,\|\theta(t)\|^{\frac13}_{L^1}\;+\;C_2\,\|\theta\|_{L^1}\,. \tag{2.41}$$

Therefore, we obtain from Young's inequality that

$$\begin{aligned} & \frac{d}{dt}\left(\|(\theta-\bar\theta)(t)\|^2+\|\theta_x(t)\|^2+\mu\,E(t)\right)+\|\theta_t(t)\|^2+k\,\|\theta_x(t)\|^2 \\ & \quad +\frac{C\mu}{2}\left(\|p_{xxxx}(t)\|^2+\|\varepsilon_{tt}(t)\|^2\right) \\ \leq\; & C\left(\|p_x(t)\|^2+\|p_x(t)\|^6+\|p_x(t)\|^8+\|p_x(t)\|^{14}\right)\,. \end{aligned} \tag{2.42}$$

Now observe that (1.19), (1.22) and (2.28) imply that

$$\begin{aligned} E(t) &\leq C\left(\|p_{xt}(t)\|^2+\|\varepsilon_{xt}(t)\|^2\right) \\ &\leq C\left(\|p_{xxt}(t)\|^2+\|p_{xxxx}(t)\|^2\right) \\ &= C\left(\|\varepsilon_{tt}(t)\|^2+\|p_{xxxx}(t)\|^2\right)\,. \end{aligned} \tag{2.43}$$

Also, owing to Poincaré's inequality,

$$\|(\theta-\bar\theta)(t)\|\leq C\,\|\theta_x(t)\|\,. \tag{2.44}$$

The assertion now follows from the inequalities (2.42)–(2.44). □

Conclusion of the proof of Theorem 1.1. It follows from (2.36) and (2.10) that, for any $t>0$,

$$\begin{aligned} & \|(\theta-\bar\theta)(t)\|^2\;+\;\|\theta_x(t)\|^2\;+\;\mu\,E(t) \\ \leq\; & C\left(\|\theta_0\|^2_{H^1}+\|p_0\|^2_{H^3}+\|\varepsilon_0\|^2_{H^3}\right)e^{-Ct}\;+\;C\,. \end{aligned} \tag{2.45}$$

By virtue of the boundary conditions, and owing to (1.18)–(1.20), we have that

$$\|\theta(t)\|_{H^1}^2 + \|p(t)\|_{H^3}^2 + \|\varepsilon(t)\|_{H^3}^2 \le C\left(\|(\theta-\bar{\theta})(t)\|^2 + \|\theta_x(t)\|^2 + \mu\, E(t)\right) + C\,. \quad (2.46)$$

Therefore, the existence of an absorbing ball set in H_{β_1,β_2} is proved. From Theorem I.1.1 in R. Temam [23], we can infer that there is a compact maximal attractor A_{β_1,β_2} in H_{β_1,β_2}. The proof of Theorem 1.1 is finally complete. □

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