A hysteresis approach to phase-field models

P. Krejčí [1] J. Sprekels [1]

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[1] Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, D–10117 Berlin, Germany

Abstract

Phase-field systems as mathematical models to forecast the evolution of processes involving phase transitions have drawn a considerable interest in recent years. However, while they are capable of capturing many of the experimentally observed phenomena, they are only of restricted value in modelling hysteresis effects occuring during phase transition processes. To overcome this shortcoming, a new approach to phase-field models is proposed in this paper which is based on the mathematical theory of hysteresis operators developed in the past fifteen years. The approach taken here leads to highly nonlinearly coupled systems of differential equations containing hysteretic nonlinearities at different places. For such a system, well-posedness and thermodynamic consistency are proved. Due to the lack of smoothness (hysteresis operators are, as a rule, non-differentiable) in the system, the method of proof has to be different from those usually employed for classical phase-field systems.

1 Introduction

The theory of hysteresis operators developed in the past fifteen years (let us at least refer to the monographs [13], [18], [24], [4], [14] devoted to this subject) has proved to be a powerful tool for solving mathematical problems in various branches of applications such as solid mechanics, material fatigue, ferromagnetism, phase transitions, and many others. In this paper we propose an approach using hysteresis operators to classical *phase-field models* for phase transitions and their generalizations.

The motivation for such an approach is quite obvious: in nature, many phase transitions are accompanied by hysteresis effects (rather they are driving mechanisms behind their occurence). On the other hand, the nonconvex free energy functionals (typically, double-well potentials) usually considered in phase-field models may induce hysteresis effects by themselves (cf., for instance, Chapter 4 in [4]); however, they are by far too simplistic to give a correct account of the complicated loopings due to the storage and deletion of internal memory that are observed in thermoplastic materials or ferromagnets. Therefore, there is certainly a *deficiency* in present phase-field theories and a *need* for a theory involving hysteresis operators (incidentally, the ancient Greek word "hysteresis" just means "deficiency" or "need"!). An additional motivation comes from the fact that hysteresis operators also arise quite naturally already in simple classical phase-field models. To demonstrate this, let us consider the well-known model for melting and solidification which is usually referred to as the *relaxed Stefan problem with undercooling and overheating* (see [9], [22], [23], for instance).

To fix things, suppose that the phase transition takes place in some open and bounded container $\Omega \subset \mathbb{R}^N$ during the time period $[0,T]$, where $T > 0$ is some final time. Then the mathematical problem consists in finding real-valued functions $\theta = \theta(x,t)$ (absolute temperature) and $\chi = \chi(x,t)$ (phase fraction, the *order parameter* of the phase transition) in $\Omega \times]0,T[$. The function χ is allowed to take values only in the interval $[0,1]$, where $\chi = 1$ corresponds to the liquid phase, $\chi = 0$ to the solid phase and $\chi \in]0,1[$ to the mushy region. The evolution of the system is governed by the balance of internal energy

$$U_t \;=\; -\mathrm{div}\; q \;+\; \psi\,, \tag{1.1}$$

where $U = U(\theta,\chi)$ is the internal energy, q is the heat flux which we assume here to obey Fourier's law

$$q \;=\; -\,\kappa\nabla\theta \tag{1.2}$$

with a constant heat conduction coefficient $\kappa > 0$, and ψ is the heat source density, and by the melting/solidification law

$$\mu\,\chi_t \;\in\; -\,\partial_\chi\,F(\theta,\chi)\,, \tag{1.3}$$

where $F = F(\theta,\chi)$ is the free energy, ∂_χ is the partial subdifferential with respect to χ and $\mu > 0$ is a fixed relaxation coefficient. In order to ensure the thermodynamical consistency of the model, we have to require that

$$\theta(x,t) > 0 \quad \text{a.e. in } \Omega\times\,]0,T[\,, \tag{1.4}$$

and that the Clausius-Duhem inequality $S_t \geq -\mathrm{div}\left(\dfrac{q}{\theta}\right) + \dfrac{\psi}{\theta}$ holds, which in view of (1.1), (1.2) and (1.4) is certainly the case if only

$$U_t \;\leq\; \theta\,S_t \quad \text{a.e.}\,, \tag{1.5}$$

where $S := \dfrac{1}{\theta}\,(U - F)$ denotes the entropy.

A standard choice [9] for F is given by

$$F \;:=\; F_0(\theta) \;+\; \lambda(\chi) \;+\; \theta\,I(\chi) \;-\; \frac{L}{\theta_c}(\theta \;-\; \theta_c)\chi\,, \tag{1.6}$$

$$U \;:=\; c_V\,\theta \;+\; \lambda(\chi) \;+\; L\,\chi\,, \tag{1.7}$$

where

$$F_0(\theta) \;:=\; c_V\,\theta(1 \;-\; \log\theta)\,, \tag{1.8}$$

$$\lambda(\chi) \;:=\; \alpha\,\chi(1 \;-\; \chi)\,. \tag{1.9}$$

Here I is the indicator function of the interval $[0,1]$ and L (latent heat), θ_c (melting temperature), c_V (specific heat) and $\alpha < L$ (limit of undercooling/overheating) are positive constants (see Fig. 1). Note that the graph $\Gamma_\theta(\chi) \;:=\; \theta\,\big(I(\chi) - L\chi/\theta_c\big) + L\,\chi \,+\, \alpha\,\chi(1 \,-\, \chi)$ is just the "double-obstacle potential" considered in a number of recent papers. We refer the reader to [2], [3], [8], [12].

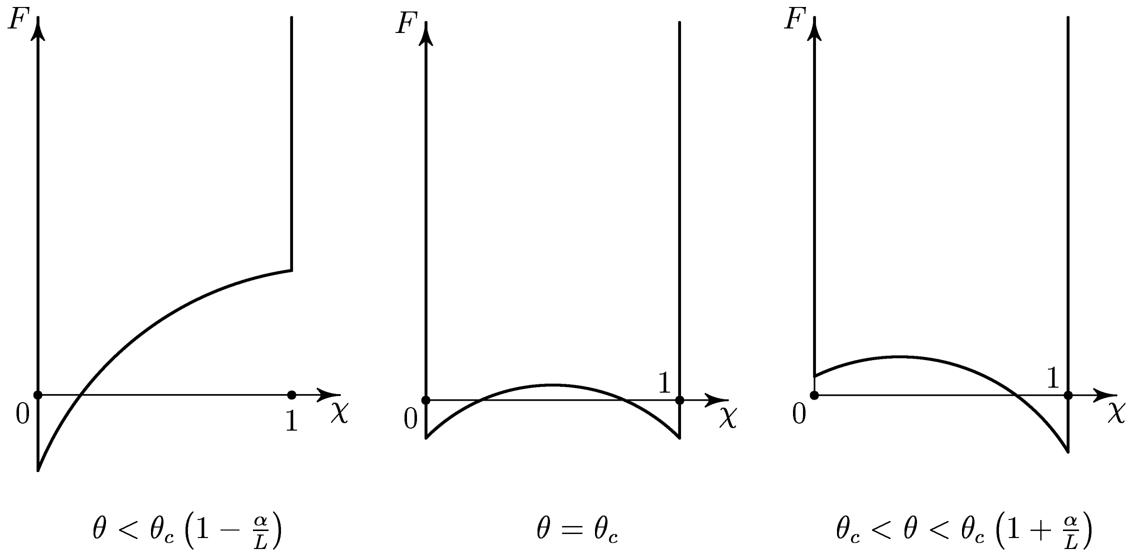

Figure 1: *Free energy F at different temperatures θ.*

The differential inclusion (1.3) then reads

$$\mu \chi_t + \lambda'(\chi) - \frac{L}{\theta_c}(\theta - \theta_c) \in -\partial_\chi I(\chi), \tag{1.10}$$

or, equivalently (see Fig. 2),

$$\chi \in [0,1], \quad \Big(\mu \chi_t + \lambda'(\chi) - \frac{L}{\theta_c}(\theta - \theta_c)\Big)(z - \chi) \geq 0 \quad \forall z \in [0,1]. \tag{1.11}$$

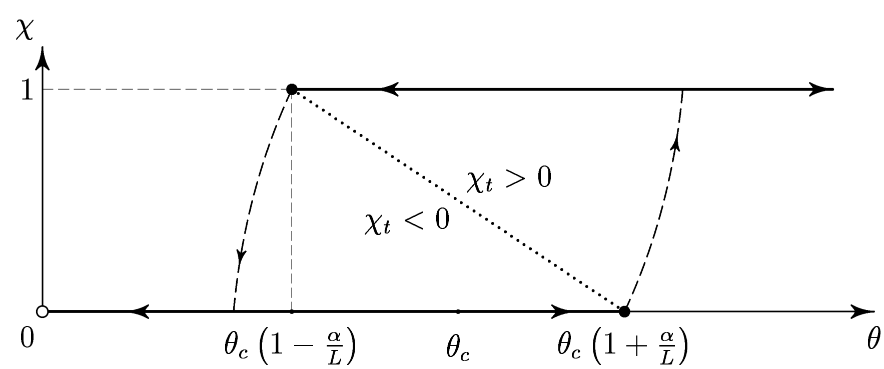

Figure 2: *A $\theta - \chi$ diagram corresponding to (1.11).*

It is easy to see that every solution (θ, χ) of (1.1), (1.2), (1.6)–(1.9), (1.11) for which (1.4) holds, satisfies formally the Clausius-Duhem inequality. Indeed, we have for $\chi \in [0,1]$, $I(\chi) \equiv 0$ and $S = c_V \log\theta + \frac{L}{\theta_c}\chi$, hence

$$U_t - \theta S_t = \Big(\lambda'(\chi) - \frac{L}{\theta_c}(\theta - \theta_c)\Big)\chi_t \leq 0, \tag{1.12}$$

according to (1.11).

We now introduce an auxiliary variable

$$w(x,t) := \frac{1}{\mu}\int_0^t \left(\frac{L}{\theta_c}(\theta - \theta_c) - \lambda'(\chi)\right)(x,\tau)\,d\tau\,. \tag{1.13}$$

Then inequality (1.11) takes the form

$$\chi \in [0,1]\,, \quad (\chi_t - w_t)(z - \chi) \geq 0 \quad \forall\, z \in [0,1]\,. \tag{1.14}$$

At this point, the notion of hysteresis operators comes into play. Variational inequality (1.14) is known to have a unique solution $\chi \in W^{1,1}(0,T)$ for every $w \in W^{1,1}(0,T)$ and initial condition $\chi(0) = \chi^0 \in [0,1]$. According to [13], [24], [4], [14], it is convenient to introduce the solution operator s_Z of (1.14) called *stop*, where the subscript Z stands for the convex constraint $Z = [0,1]$, that is,

$$\chi = s_Z[\chi^0, w]\,. \tag{1.15}$$

The hysteretic input-output behaviour of the stop operator is illustrated in Fig. 3. Along the upper (lower) threshold line $\chi = 1$, ($\chi = 0$), the process is irreversible and can only move to the right (to the left, respectively), while in between, motions in both directions are admissible. This is similar to Prandtl's model of perfect elastoplasticity, where the horizontal parts of the diagram correspond to plastic yielding and the intermediate lines can be interpreted as linearly elastic trajectories.

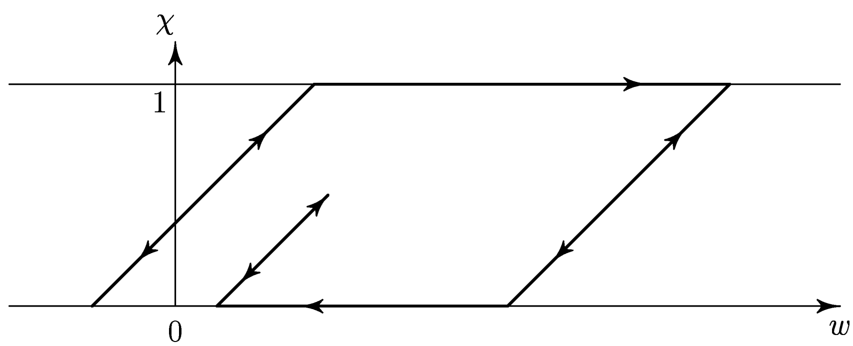

Figure 3: *A diagram of the stop operator (1.15).*

Identity (1.15) enables us to eliminate χ from (1.13) and rewrite the system (1.1) – (1.3) in the form

$$\mu\, w_t = \frac{L}{\theta_c}(\theta - \theta_c) - \lambda'\big(s_Z[\chi^0, w]\big)\,, \tag{1.16}$$

$$\Big(c_V\,\theta + \lambda\,(s_Z[\chi^0, w]) + L\,s_Z[\chi^0, w]\Big)_t - \kappa\,\Delta\theta = \psi\,. \tag{1.17}$$

We thus obtain in a natural way a system of equations for an order parameter w and the absolute temperature θ involving hysteresis operators. In the next section we state precisely the problem for a more general class of systems.

2 Statement of the problem

We consider the system of equations in $\Omega \times]0, T[$

$$\mu\, w_t \,+\, f_1[w] \,+\, f_2[w]\,\theta \,=\, 0\,, \tag{2.1}$$

$$(\theta \,+\, F_1[w])_t \,-\, \Delta\theta \,=\, \psi(x,t,\theta)\,, \tag{2.2}$$

coupled with the initial conditions

$$w(x,0) \,=\, w^0(x)\,, \quad \theta(x,0) \,=\, \theta^0(x)\,, \quad \text{for } x \,\in\, \Omega\,, \tag{2.3}$$

and with the Neumann boundary condition

$$\big\langle \nabla\theta(x,t)\,,\, n(x)\big\rangle \,=\, 0 \quad \text{for } (x,t) \,\in\, \partial\Omega \times]0,T[\,, \tag{2.4}$$

where $n(x)$ is the unit outward normal to $\partial\Omega$ at the point $x \in \partial\Omega$. This simple boundary condition has been chosen in order to make the method of hysteresis operators more transparent, which is our main goal here. We assume that $T > 0$, $\mu > 0$ are given numbers and that $\Omega \subset \mathbb{R}^N$ is a given bounded domain with a lipschitzian boundary.

At the first glance, the system (2.1) – (2.4) does not seem to be very difficult from the mathematical point of view. In fact, if f_1, f_2, F_1 were real-valued functions having suitable properties (smoothness, monotonicity, and the like), then this would be true. However, in our case, f_1, f_2, F_1 will be hysteresis operators and thus, in particular, non-smooth. Also, when dealing with these operators, we will always have to account for the full history of the inputs which makes the theory less obvious.

We now formulate precisely the assumptions on the mappings f_1, f_2, F_1, ψ.

Hypothesis 2.1. *We assume that $f_1, f_2 : C[0,T] \to C[0,T]$ are causal and Lipschitz continuous operators, and that f_2 is bounded. In other words, there exists a constant K_1 such that the inequalities*

$$\big|f_i[w_1](t) \,-\, f_i[w_2](t)\big| \,\le\, K_1 \max_{0\le s\le t} |w_1(s) \,-\, w_2(s)|\,, \quad i \,=\, 1,2\,, \tag{2.5}$$

$$\big|f_2[w](t)\big| \,\le\, K_1\,, \tag{2.6}$$

hold for every $w_1, w_2, w \in C[0,T]$ and $t \in [0,T]$.

Hypothesis 2.2. *The mapping $F_1 : W^{1,2}(0,T) \to W^{1,2}(0,T)$ is causal, and there exist a constant $K_2 > 0$ and a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\left|\frac{d}{dt} F_1[w](t)\right| \,\le\, K_2 |\dot w(t)| \quad \textit{a.e. in }]0,T[\,, \quad \forall\, w \,\in\, W^{1,2}(0,T)\,, \tag{2.7}$$

$$\begin{gathered}\big|F_1[w_1](t) \,-\, F_1[w_2](t)\big| \,\le\, \varphi(R)\, \|w_1 \,-\, w_2\|_{W^{1,2}(0,t)} \\ \forall R > 0,\ \forall\, w_1, w_2 \,\in\, W^{1,2}(0,T) :\ \max\big\{\|w_i\|_{W^{1,2}(0,T)};\ i \,=\, 1,2\big\} \le R,\end{gathered} \tag{2.8}$$

where we denote

$$\|w\|_{W^{1,p}(0,t)} := |w(0)| + \Big(\int_0^t |\dot w(s)|^p\, ds\Big)^{1/p} \quad \forall\, t \in]0,T]\,, \quad 1 \le p < \infty\,. \tag{2.9}$$

We moreover assume that the function ψ *satisfies the condition*

$$\psi_0 := \psi(\,\cdot\,,\,\cdot\,,0) \in L^q(\Omega\times]0,T[)\,, \quad |\psi_\theta(x,t,\theta)| \le K_2 \ a.e., \tag{2.10}$$

for some $q > \dfrac{r_N^2}{r_N - 1}$, *where* $r_N := \max\Big\{2, 1 + \frac{N}{2}\Big\}$.

Hypothesis 2.3. *It holds*

$$\psi_0(x,t) \ge 0 \quad a.e. \ in\ \Omega\times]0,T[\,, \tag{2.11}$$

$$F_1[w](t) \ge 0 \quad \forall\ w \in W^{1,2}(0,T)\,, \quad \forall t \in [0,T]\,, \tag{2.12}$$

and there exist operators $F_2, g : W^{1,2}(0,T) \to W^{1,2}(0,T)$ *and a constant* $K_3 > 0$ *such that the inequalities*

$$0 \le g[w]_t\, w_t \le K_3\, w_t^2\,, \tag{2.13}$$

$$F_i[w]_t - f_i[w]\, g[w]_t \le 0\,, \tag{2.14}$$

hold for each $w \in W^{1,2}(0,T)$ *and a.e.* $t \in]0,T[\,, i = 1,2$.

Let us mention that property (2.13) is called *piecewise* ([24]) or *local* ([14]) *monotonicity* .

Remark 2.4. The domains of definition of the operators f_i, F_i, g can be extended in a natural way to functions which depend on both x and t and appear in (2.1), (2.2). It suffices to keep the same symbols and to put

$$f_i[w](x,t) := f_i[w(x,\,\cdot\,)](t) \quad \text{for } x \in \Omega\,,\ t \in]0,T[\,, \tag{2.15}$$

and similarly for F_i and g, for every function w such that $w(x\,,\,\cdot\,)$ belongs to the original domain of definition for a.e. $x \in \Omega$.

Remark 2.5. System (1.16), (1.17) is a special case of (2.1), (2.2) (up to the constants c_V, κ, indeed); we simply have to put $g[w] := s_Z[\chi^0, w]$, $f_1[w] := \lambda'(g[w]) + L$, $F_1[w] := \lambda(g[w]) + Lg[w]$, $f_2[w] := -L/\theta_c$, $F_2[w] := -\,Lg[w]/\theta_c$. Obviously, Hypotheses 2.1 – 2.3 are fulfilled with these choices.

Remark 2.6. Equations (2.1), (2.2) may be regarded as a phase-field system for the free energy functional $F = F[w,\theta] := \theta(1-\log\,\theta) + F_1[w] + F_2[w]\,\theta$. In the classical case, the relaxation law (1.3) with χ replaced by w is combined with identities of the form $f_i[w] = \delta_w\, F_i[w]$, $i = 1,2$, where δ_w denotes the variation with respect to w, in order to make the model comply with the Second Principle of Thermodynamics. However, since hysteresis operators are, as a rule, *non-differentiable*, we cannot hope to have these identities, as the variation $\delta_w F_i[w]$ of F_i with respect to w does not exist. In this regard, the situation is entirely different from classical phase-field models. On the other hand, inequality (2.14) is a typical condition which guarantees the thermodynamical consistency of hysteresis operators also in other areas of applications. It is fulfilled, in particular, for operators of the form

$$f_i[w] := \mathcal{P}_i\big[g[w]\big], \quad F_i[w] := \mathcal{U}_i\big[g[w]\big], \tag{2.16}$$

where $\mathcal{P}_i$ is a hysteresis operator with a *clockwise admissible hysteresis potential* $\mathcal{U}_i$ in the sense of Section 2.5 in [4]. Note that in this case the "dissipation" over a closed cycle (i.e. $u(t_1) = u(t_2)$, $\mathcal{P}_i[u](t_1) = \mathcal{P}_i[u](t_2)$, $\mathcal{U}_i[u](t_1) = \mathcal{U}_i[u](t_2)$) is positive and equal to the integral

$$\int_{t_1}^{t_2} \mathcal{P}_i[u](t)\frac{du(t)}{dt}\,dt$$

or, in geometrical terms, to the area of the corresponding hysteresis loop, see Fig. 4. A classical example is the Prandtl-Ishlinskii operator

$$\mathcal{P}_i[u] := \int_0^\infty h_i(r)\, s_{Z_r}[u]\,dr, \quad \mathcal{U}_i[u] := \frac{1}{2}\int_0^\infty h_i(r)\, s_{Z_r}^2[u]\,dr, \tag{2.17}$$

where s_{Z_r} is the stop operator with characteristic $Z_r = [-r, r]$ and h_i are given nonnegative density functions.

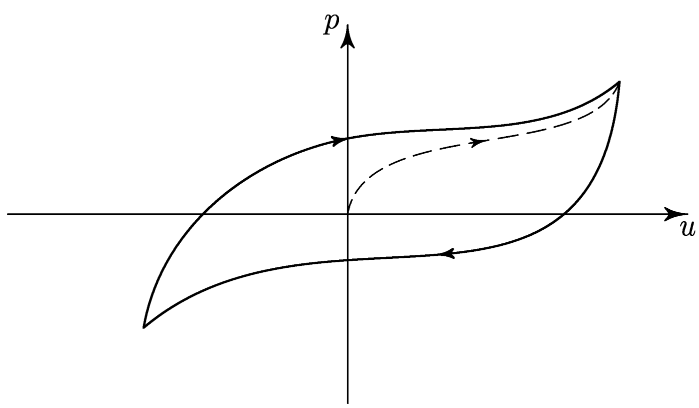

Figure 4: *Clockwise admissibility for* $p = \mathcal{P}_i[u]$

Also here, the condition (1.5) follows from (2.14) provided θ is positive. Indeed, if we define the internal energy $U = U[w,\theta] := \theta + F_1[w]$ and the entropy $S = S[w,\theta] := \log\theta - F_2[w]$, then we obtain formally

$$U_t - \theta\, S_t = F_1[w]_t + \theta\, F_2[w]_t \leq -\mu\, w_t\, g[w]_t \leq 0, \tag{2.18}$$

so that (1.5) is satisfied. We shall see later (cf. Theorem 2.10) that Hypothesis 2.3 ensures also the positivity of θ. In conclusion, inequality (2.14), which reflects the fundamental energy dissipation properties of hysteresis operators f_i, takes over the role of the identity $f_i[w] = \delta_w F_i[w]$ which is meaningless here. We should recall that for constant temperature, (2.18) just means that F decreases in time.

Remark 2.7. Thinking in terms of classical models, the system (2.1), (2.2) can be regarded as a phase-field model of *Caginalp type*, see [5], [4] and the references cited there. One can also consider a hysteresis counterpart of the *Penrose-Fife model* of phase transitions (cf. [19], [4], [6], [7], [11], [10], [16], [17], [21]), in which (2.1) has to be replaced by

$$\mu\, w_t \,+\, f_1[w] \,+\, f_2[w]/\theta \,=\, 0\,. \tag{2.1'}$$

We shall study the Penrose-Fife-type model with (2.1)' in a forthcoming paper.

The next three sections are devoted to the proof of the following theorems.

Theorem 2.8 (Existence). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a lipschitzian boundary, and let Hypotheses* 2.1, 2.2 *hold. Then for every $\theta^0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $w^0 \in L^\infty(\Omega)$ problem* (2.1)–(2.4) *has a solution $(w,\theta) \in \left(L^\infty(\Omega\times]0,T[)\right)^2$ such that $\theta_t\,,\ \Delta\theta \in L^2(\Omega\times]0,T[)\,,\ w_t \in L^\infty(\Omega\times]0,T[)$ and such that* (2.1), (2.2) *are satisfied almost everywhere.*

Theorem 2.9 (Uniqueness and continuous dependence). *Let the hypotheses of Theorem* 2.8 *hold. Let $w_i^0 \in L^\infty(\Omega)\,,\ \theta_i^0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $\psi_i : \Omega\times]0,T[\times\mathbb{R} \to \mathbb{R}\,,\ i = 1,2\,,$ be given functions. Let each of the functions $\psi = \psi_1\,,\ \psi = \psi_2$ satisfy* (2.10), *and let there exist a function $d_\psi \in L^2(\Omega\times]0,T[)$ such that for a.e. $(x,t,\vartheta^i) \in \Omega\times]0,T[\times\mathbb{R}\,,\ i = 1,2\,,$ we have*

$$\left|\psi_1(x,t,\vartheta^1) \,-\, \psi_2(x,t,\vartheta^2)\right| \,\le d_\psi(x,t) \,+\, K_2|\vartheta^1 \,-\, \vartheta^2|\,. \tag{2.19}$$

Let $(w_1,\theta_1)\,,\ (w_2,\theta_2)$ be solutions to (2.1)–(2.4) *corresponding to the data $w_1^0\,,\ \theta_1^0\,,\ \psi_1$ and $w_2^0\,,\ \theta_2^0\,,\ \psi_2\,,$ respectively. Then there exists a constant $C > 0$ such that, for all $t \in [0,T]\,,$*

$$\begin{aligned}\int_0^t\int_\Omega \left|\theta_1 \,-\, \theta_2\right|^2(x,\tau)\,dx\,d\tau \ &\le\ C\Big[t\big(\|w_1^0 \,-\, w_2^0\|^2_{L^2(\Omega)} \\ &\quad+\ \|\theta_1^0 \,-\, \theta_2^0\|^2_{L^2(\Omega)}\big) \,+\, \int_0^t\int_\Omega d_\psi^2(x,\tau)\,dx\,d\tau\Big]\,,\end{aligned} \tag{2.20}$$

$$\begin{aligned}\int_\Omega \|w_1 \,-\, w_2\|^2_{W^{1,2}(0,T)}(x)\,dx \ &\le\ C\Big[\|w_1^0 \,-\, w_2^0\|^2_{L^2(\Omega)} \\ &\quad+\ \|\theta_1^0 \,-\, \theta_2^0\|^2_{L^2(\Omega)} \,+\, \int_0^T\int_\Omega d_\psi^2(x,t)\,dx\,dt\Big]\,.\end{aligned} \tag{2.21}$$

Theorem 2.10 (Thermodynamic consistency). *Let Hypothesis* 2.3 *and the assumptions of Theorem* 2.8 *be fulfilled. Assume that $\theta^0(x) \ge \delta$ a.e. in Ω for some constant $\delta > 0$. Then there exists $\tilde{K} > 0$ such that the solution (w,θ) to* (2.1)–(2.4) *satisfies $\theta(x,t) \ge \delta e^{-\tilde{K}t}$ a.e. , hence* (1.4) *and* (2.18) *hold almost everywhere.*

3 Solution operator of the order parameter equation

We first consider the equation (2.1) with given θ and unknown w separately. Neglecting for the moment the space dependence, we write it in the form

$$\mu \dot{w} + f_1[w] + f_2[w]\,\theta = 0\,, \quad w(0) = w^0\,. \tag{3.1}$$

We have the following result.

Lemma 3.1 (Existence). *Let Hypothesis 2.1 hold, and let $\theta \in L^1(0,T)$ and $w^0 \in \mathbb{R}$ be given. Then there exists a solution $w \in W^{1,1}(0,T)$ of (3.1) such that (3.1) holds a.e., together with the estimate*

$$|\dot{w}(t)| \le C_1\Big(1 + |w^0| + \|\theta\|_{L^1(0,t)} + |\theta(t)|\Big)\,, \tag{3.2}$$

where $C_1 > 0$ is a constant independent of w^0 and θ.

Proof. For each $t \in]0,T]$ put $C_0[0,t] := \{w \in C[0,t];\ w(0) = w^0\}$, and

$$H_0[w](t) := w^0 - \frac{1}{\mu}\int_0^t \big(f_1[w] + f_2[w]\,\theta\big)(s)\,ds\,. \tag{3.3}$$

Then H_0 maps $C_0[0,t]$ into $C_0[0,t]$ for every $t \in]0,T]$, and whenever $w_1, w_2 \in C_0[0,t]$ and $0 \le \tau \le t$, then

$$\Big|H_0[w_1](\tau) - H_0[w_2](\tau)\Big| \le \frac{K_1}{\mu}\int_0^\tau \Big(1 + |\theta(s)|\Big) \max_{0 \le r \le s} |w_1(r) - w_2(r)|\,ds\,. \tag{3.4}$$

To simplify the notation, put for $t \in [0,T]$

$$\alpha(t) := \frac{K_1}{\mu}\big(1 + |\theta(t)|\big)\,, \quad A(t) := \int_0^t \alpha(s)\,ds\,. \tag{3.5}$$

We now define the sequence of *successive approximations* by

$$w_0(t) := w^0\,, \quad w_k(t) := H_0[w_{k-1}](t) \quad \text{for } k \ge 1\,, t \in [0,T]\,. \tag{3.6}$$

It is easily proved by induction that there exists a constant $C > 0$, independent of k, such that, for every $k \ge 1$ and $t \in [0,T]$,

$$\big|w_k(t) - w_{k-1}(t)\big| \le \frac{C}{(k-1)!}\big(A(t)\big)^{k-1}\,. \tag{3.7}$$

Indeed, it suffices to put

$$C := \frac{1}{\mu}\int_0^T \Big|f_1[w^0] + f_2[w^0]\,\theta\Big|(t)\,dt\,. \tag{3.8}$$

Then (3.7) holds for $k = 1$, and assuming (3.7) for some $k \in \mathbb{N}$, we obtain from (3.4) that

$$\begin{aligned} \big|w_{k+1}(t) - w_k(t)\big| &\le \frac{C}{(k-1)!}\int_0^t \alpha(s)\big(A(s)\big)^{k-1}\,ds \\ &\le \frac{C}{k!}\big(A(t)\big)^k\,, \end{aligned} \tag{3.9}$$

and the induction step is complete.

Since the series $\sum_{k=0}^{\infty} \frac{C}{k!}\big(A(t)\big)^k$ converges uniformly, we easily conclude that $\{w_k\}_{k=0}^{\infty}$ is a fundamental sequence in $C_0[0,T]$. Passing to the limit in (3.6) as $k \to \infty$, we conclude from (2.5) that $w = \lim_{k\to\infty} w_k$ is a solution to (3.1).

It remains to derive the estimate (3.2). To this end, let

$$f^0(t) \;:=\; f_1[0](t) \tag{3.10}$$

be the image of the constant function $w \equiv 0$ under f_1. From the identity

$$w(t) \,-\, w^0 \;=\; -\frac{1}{\mu}\int_0^t \Big(f_1[w] \,-\, f^0 \,+\, f^0 \,+\, f_2[w]\,\theta\Big)(s)\,ds\,, \tag{3.11}$$

and from Hypothesis 2.1, which entails, in particular, that

$$\big|f_1[w] \,-\, f^0\big|(t) \;\le\; K_1\Big(|w^0| \,+\, \max_{0\le s\le t}\big\{|w(s) \,-\, w^0|\big\}\Big)\,, \tag{3.12}$$

we infer that

$$|w(t) \,-\, w^0| \;\le\; \frac{K_1}{\mu}\int_0^t \Big(\max_{0\le s\le \tau}\big\{|w(s) \,-\, w^0|\big\} \,+\, |w^0| \,+\, \frac{1}{K_1}\big|f^0(\tau)\big| \,+\, \big|\theta(\tau)\big|\Big)\,d\tau\,. \tag{3.13}$$

Thus,

$$|w(t) \,-\, w^0| \;\le\; \frac{K_1}{\mu}\int_0^t e^{\frac{K_1}{\mu}(t-\tau)}\Big(|w^0| \,+\, \frac{1}{K_1}\big|f^0(\tau)\big| \,+\, |\theta(\tau)|\Big)\,d\tau\,, \tag{3.14}$$

and (3.2) follows from (3.1), (3.12) and (3.14). □

Lemma 3.2 (Uniqueness and continuous dependence). *Let Hypothesis* 2.1 *hold. Then to every* $M > 0$ *there exists a constant* $C_M > 0$ *such that for every* $\theta_1, \theta_2 \in L^1(0,T)$, $\|\theta_i\|_{L^1(0,T)} \le M$, $i = 1,2$, *the corresponding solutions* w_1, w_2 *of* (3.1) *with initial conditions* w_1^0, w_2^0, *respectively, satisfy the estimates*

$$|w_1(t) \,-\, w_2(t)| \;\le C_M\Big(|w_1^0 \,-\, w_2^0| \,+\, \int_0^t |\theta_1 \,-\, \theta_2|(s)\,ds\Big)\,, \tag{3.15}$$

$$|\dot w_1(t) \,-\, \dot w_2(t)| \;\;\le\;\; C_M\Big(|w_1^0 \,-\, w_2^0| \,+\, \int_0^t |\theta_1 \,-\, \theta_2|(s)\,ds\Big) \cdot \big(1 \,+\, |\theta_1(t)|\big) \,+\, \frac{K_1}{\mu}|\theta_1(t) \,-\, \theta_2(t)| \tag{3.16}$$

for a.e. $t \in\,]0,T[$.

Proof. For $t \in [0,T]$ we have, by virtue of (2.5) and (2.6),

$$|w_1(t) \,-\, w_2(t)| \;\le\; |w_1^0 \,-\, w_2^0| \,+\, \frac{K_1}{\mu}\int_0^t \Big[\big(1 \,+\, |\theta_1(\tau)|\big)\max_{0\le s\le\tau}|w_1(s) \,-\, w_2(s)| \,+\, |\theta_1(\tau) \,-\, \theta_2(\tau)|\Big]\,d\tau\,, \tag{3.17}$$

whence, using Gronwall's lemma,

$$\begin{aligned} |w_1(t) - w_2(t)| &\le e^{\frac{K_1}{\mu}\int_0^t (1+|\theta_1(\tau)|)\,d\tau} \cdot |w_1^0 - w_2^0| \\ &\quad + \frac{K_1}{\mu}\int_0^t e^{\frac{K_1}{\mu}\int_\tau^t (1+|\theta_1(s)|)\,ds} |\theta_1(\tau) - \theta_2(\tau)|\,d\tau \\ &\le e^{\frac{K_1}{\mu}(T+M)}\Big(|w_1^0 - w_2^0| + \frac{K_1}{\mu}\int_0^t |\theta_1(\tau) - \theta_2(\tau)|\,d\tau\Big), \end{aligned} \tag{3.18}$$

i.e. (3.15) holds. Inequality (3.16) then follows immediately from (3.1), (3.15) and Hypothesis 2.1. □

Lemmas 3.1 and 3.2 enable us to introduce the solution operator $\mathcal{P}_p : \mathbb{R} \times L^p(0,T) \to W^{1,p}(0,T)$ of equation (3.1) for every $1 \le p \le \infty$ through the formula

$$w = \mathcal{P}_p[w^0, \theta]. \tag{3.19}$$

$\mathcal{P}_p$ is obviously causal, and it satisfies according to Lemmas 3.1, 3.2 for every $t \in [0,T]$ the following estimates.

Proposition 3.3 *Let Hypothesis* 2.1 *hold. Then there exist a constant* $C_2 > 0$ *and a function* $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for every* $R > 0$ *and every* (w^0, θ), (w_1^0, θ_1), $(w_2^0, \theta_2) \in \mathbb{R} \times L^p(0,T)$ *and* $t \in [0,T]$ *satisfying* $\max\big\{\|\theta_i\|_{L^p(0,t)};\ i = 1,2\big\} \le R$, *we have*

$$\big\|\mathcal{P}_p[w^0, \theta]\big\|_{W^{1,p}(0,t)} \le C_2\big(1 + |w^0| + \|\theta\|_{L^p(0,t)}\big), \tag{3.20}$$

$$\|\mathcal{P}_p[w_1^0, \theta_1] - \mathcal{P}_p[w_2^0, \theta_2]\|_{W^{1,p}(0,t)} \le \gamma(R)\big(|w_1^0 - w_2^0| + \|\theta_1 - \theta_2\|_{L^p(0,t)}\big). \tag{3.21}$$

4 Existence, uniqueness and stability

This section is devoted to the proof of Theorems 2.8 and 2.9. Using the operator $\mathcal{P}_p$ introduced in (3.19), we can formally rewrite the problem (2.1)–(2.4) as a single equation

$$\big(\theta + \mathcal{V}_p[w^0\,,\,\theta]\big)_t - \Delta\theta = \psi(x,t,\theta), \tag{4.1}$$

coupled with initial and boundary conditions (2.3), (2.4), where we have put

$$\mathcal{V}_p[w^0\,,\,\theta](x,t) := F_1\Big[\mathcal{P}_p[w^0(x), \theta(x,\,\cdot)]\Big](t), \tag{4.2}$$

for $x \in \Omega$, $t \in [0,T]$ and $p \in [1,\infty]$. The natural domains of definition of $\mathcal{V}_p$ are the spaces $\mathcal{D}_p^t := L^p(\Omega) \times L^p(\Omega \times]0,t[)$ for $p \in [1,\infty]$ and $t \in]0,T]$. From Proposition 3.3 and Hypothesis 2.2 we see that $\mathcal{V}_p$ maps $\mathcal{D}_p^t$ into $L^p(\Omega; W^{1,p}(0,t))$. Moreover, since for every $p > r$ it holds $\mathcal{V}_r\big|_{\mathcal{D}_p^t} = \mathcal{V}_p$, we may simply write $\mathcal{V}$ in place of $\mathcal{V}_p$, with an implicitly given domain of definition. The operator $\mathcal{V}$ has the following properties.

Proposition 4.1 *Let Hypotheses 2.1, 2.2 hold. Then there exist a constant* $C_3 > 0$ *and a function* $\tilde{\psi} : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for every* $R > 0$, $p \in [1, \infty]$, $(w^0, \theta) \in \mathcal{D}_p^T$, $(w_i^0, \theta_i) \in \mathcal{D}_\infty^T$, $i = 1, 2$ *satisfying* $\max \left\{ \|w_i^0\|_{L^\infty(\Omega)}, \|\theta_i\|_{L^\infty(\Omega \times]0,T[)} : i = 1, 2 \right\} \le R$*, and every* $t \in]0, T]$*, it holds*

$$\|\mathcal{V}[w^0, \theta]_t\|_{L^p(\Omega \times]0,t[)} \le C_3 \Big(1 + \|w^0\|_{L^p(\Omega)} + \|\theta\|_{L^p(\Omega \times]0,t[)} \Big), \tag{4.3}$$

$$\begin{aligned} \Big\| \mathcal{V}[w_1^0, \theta_1] &- \mathcal{V}[w_2^0, \theta_2] \Big\|_{L^2(\Omega; L^\infty(0,t))} \\ &\le \tilde{\psi}(R) \Big(\|w_1^0 - w_2^0\|_{L^2(\Omega)} + \|\theta_1 - \theta_2\|_{L^2(\Omega \times]0,t[)} \Big). \end{aligned} \tag{4.4}$$

Proof. It suffices to use Lemma 3.1, Proposition 3.3, Hypothesis 2.2 and to integrate over Ω. □

According to the above considerations, we reformulate Theorem 2.8 in the following way.

Theorem 4.2 *Let the hypotheses of Theorem 2.8 hold. Then, for every* $w^0 \in L^\infty(\Omega)$ *and* $\theta^0 \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$*, there exists* $\theta \in L^\infty(\Omega \times]0, T[)$ *such that* $\theta_t, \Delta\theta \in L^2(\Omega \times]0, T[)$ *and such that the equation*

$$\big(\theta + \mathcal{V}[w^0, \theta]\big)_t - \Delta\theta = \psi(x, t, \theta) \tag{4.5}$$

is satisfied almost everywhere, together with the initial and boundary conditions (2.3), (2.4).

Note that equation (4.5) does not have the general form considered by Visintin [24], since the operator $\mathcal{V}$ is not piecewise monotone (cf. Remark 2.6). We present here a simple and direct proof of Theorem 4.2 which is based on well-known properties of linear parabolic equations of the following type

$$u_t - \Delta u + u = g, \quad (x,t) \in \Omega \times]0, T[\tag{4.6}$$

$$u(x, 0) = u^0(x), \quad x \in \Omega \tag{4.7}$$

$$\big\langle \nabla u(x,t), n(x) \big\rangle = 0, \quad (x,t) \in \partial\Omega \times]0, T[, \tag{4.8}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a lipschitzian boundary, g, u^0 are given functions and $n(x)$ is the outward normal to $\partial\Omega$ at the point x.

Lemma 4.3

(i) *For every* $p \in [1, \infty[$, $g \in L^p(\Omega \times]0, T[)$ *and* $u^0 \in L^p(\Omega)$ *the solution* u *of* (4.6) – (4.8) *satisfies for every* $t \in [0, T]$ *the estimate*

$$\Big| u(\cdot, t) \Big|_p^p \le \Big| u^0 \Big|_p^p + \int_0^t \Big| g(\cdot, \tau) \Big|_p^p \, d\tau, \tag{4.9}$$

where $|\cdot|_p$ *denotes the norm in* $L^p(\Omega)$.

(ii) *Let r_N and q be as in Hypothesis 2.2. Then there exists a constant $K_\infty > 0$ such that for every $u^0 \in L^\infty(\Omega)$ and $g \in L^q(\Omega\times]0,T[)$ the solution u of (4.6)–(4.8) satisfies the estimate*

$$\|u\|_\infty \le K_\infty \max\left\{|u^0|_\infty\,,\ \|g\|_q\right\}, \tag{4.10}$$

where $\|\cdot\|_q$ denotes the norm of $L^q(\Omega\times]0,T[)$.

Remark on the proof of Lemma 4.3. We do not repeat the detailed proof which can be found in [15], §7 of Chapter III even in much more general cases of variable discontinuous coefficients and anisotropic norms. We just point out that **(i)** is obtained by testing equation (4.6) with $u|u|^{p-2}$ for $p>1$ and $\operatorname{sign}(u)$ for $p=1$. For $p>1$ this yields for instance

$$\begin{aligned} &\frac{1}{p}\int_\Omega |u|^p(x,t)\,dx + \int_0^t\int_\Omega \left(2\left(1-\frac{1}{p}\right)\left|\nabla|u|^{p/2}\right|^2 + |u|^p\right)(x,\tau)\,dx\,d\tau \\ &\le \frac{1}{p}\left|u^0\right|_p^p + \int_0^t\int_\Omega |g|\,|u|^{p-1}(x,\tau)\,dx\,d\tau \end{aligned} \tag{4.11}$$

and it suffices to use Hölder's and Young's inequalities. The proof of **(ii)** also relies on inequality (4.11). We fix some r,s satisfying the inequalities

$$r_N < r < q\frac{r_N-1}{r_N}\,, \qquad \frac{r}{r-1} < s < \frac{r_N}{r_N-1}\,. \tag{4.12}$$

For $p\ge r$ and $w := |u|^{p/2}$ Hölder's inequality applied to (4.11) entails

$$\begin{aligned} &\frac{1}{p}\operatorname*{supess}_{t\in[0,T]}\left|w(\cdot,t)\right|_2^2 + 2\left(1-\frac{1}{p}\right)\|\nabla w\|_2^2 + \|w\|_2^2 \\ &\le \frac{1}{p}E^p + \left\|g\right\|_r \left\|w\right\|_{2\left(\frac{1-1/p}{1-1/r}\right)}^{2(1-1/p)}, \end{aligned} \tag{4.13}$$

where $E := \sup\{|u^0|_{p^*}\,;\ p^*\ge 1\}$. The embedding

$$\|w\|_{2s}^2 \le C\left(\operatorname*{supess}_{t\in[0,T]}\left|w(\cdot,t)\right|^2 + \|\nabla w\|_2^2 + \|w\|_2^2\right) \tag{4.14}$$

enables us to obtain

$$\|w\|_{2s}^2 \le C\left(E^p + \|g\|_r^p + p\|w\|_{\frac{2r}{r-1}}^2\right) \tag{4.15}$$

with a constant C independent of p and the assertion follows from the Moser iteration for $p=(1+\kappa)^n\,r$, $n=0,1,2,\dots$, $\kappa := \dfrac{s(r-1)}{r} - 1 > 0$, and Lemma 5.6 of Chapter II of [15].

Proof of Theorem 4.2. We construct the solution by an easy successive approximation scheme. We define the sequence $\{\theta^k\}_{k=1}^\infty$ recursively as solutions of linear equations

$$\theta_t^k - \Delta\theta^k + \theta^k = \Psi_k(x,t)\,, \quad (x,t)\in\Omega\times]0,T[\,, \tag{4.16}$$

$$\big\langle \nabla\theta^k(x,t)\,,\, n(x)\big\rangle \;=\; 0\,, \quad (x,t)\,\in\,\partial\Omega\times\,]0,T[\,, \tag{4.17}$$

$$\theta^k(x,0) \;=\; \theta^0(x)\,, \quad x\,\in\,\Omega\,, \tag{4.18}$$

where

$$\begin{aligned}\Psi_k(x,t) \;&:=\; \theta^{k-1}(x,t) \,+\, \psi(x,t,\theta^{k-1}(x,t)) \\ &\quad - \mathcal{V}[w^0\,,\,,\theta^{k-1}]_t(x,t)\,, \quad k \,=\, 1,2,\ldots\,,\end{aligned} \tag{4.19}$$

with $\theta^0(x,t) \,\equiv\, \theta^0(x)$.

It follows directly from Hypothesis 2.2, Lemma 4.3 **(i)** and Proposition 4.1 by induction that $\theta^k \,\in\, L^\infty(0,T;\, L^q(\Omega))$. There exists moreover a constant $C_4 \,>\, 0$ independent of k such that

$$\Big|\theta^k(\,\cdot\,,t)\Big|_q^q \,\le\, C_4\Big(1 \,+\, \int_0^t \big|\theta^{k-1}(\,\cdot\,,\tau)\big|_q^q\, d\tau\Big) \tag{4.20}$$

is satisfied for all $k = 1,2,\ldots$ and $t\in[0,T]$.

By induction we therefore have

$$\big|\theta^k(\,\cdot\,,t)\big|_q^q \,\le\, C_4\, e^{C_4\, t}\,. \tag{4.21}$$

In particular, the sequence $\{\theta^k\}_{k=1}^\infty$ is bounded in $L^q(\Omega\times\,[0,T[)$. >From Lemma 4.3 **(ii)** we conclude that $\{\theta^k\}_{k=1}^\infty$ is bounded in $L^\infty(\Omega\times\,]0,T[)$, say

$$\|\theta^k\|_\infty \,\le\, C_5\,. \tag{4.22}$$

Consequently,

$$\|\theta^k_t\|_2^2 \,+\, \|\Delta\theta^k\|_2^2 \,\le\, C_6 \tag{4.23}$$

for some constant $C_6 > 0$ independent of k.

In order to prove the convergence of $\{\theta^k\}$ as $k\to\infty$, we integrate (4.16) with respect to t and subtract the resulting identities corresponding to $k+1$ and k. This yields

$$\begin{aligned}\big(\theta^{k+1} - \theta^k\big)(x,t) \,&-\Delta\int_0^t \big(\theta^{k+1} - \theta^k\big)(x,\tau)\,d\tau + \int_0^t \big(\theta^{k+1} - \theta^k\big)(x,\tau)\,d\tau \\ =\quad & -\Big(\mathcal{V}[w^0\,,\,\theta^k] \,-\, \mathcal{V}[w^0\,,\,\theta^{k-1}]\Big)(x,t) \,+\, \int_0^t \Big(\psi\big(x,\tau,\theta^k(x,\tau)\big) \\ & -\psi\big(x,\tau,\theta^{k-1}(x,\tau)\big)\Big)\,d\tau \,+\, \int_0^t \big(\theta^k \,-\, \theta^{k-1}\big)(x,\tau)\,d\tau\,.\end{aligned} \tag{4.24}$$

Multiplying (4.24) by $(\theta^{k+1} - \theta^k)(x,t)$ and integrating over Ω, we conclude using (2.10), (4.4) and (4.22) that there exists a constant $C_7 > 0$ such that for all $k = 1,2,\ldots$ and $t\in[0,T]$

$$\begin{aligned}&\big|\theta^{k+1} \,-\, \theta^k\big|^2(x,t)\,dx \,+\, \frac{d}{dt}\int_\Omega \Big(\Big|\nabla\int_0^t (\theta^{k+1} \,-\, \theta^k)(x,\tau)\,d\tau\Big|^2 \\ &+\Big|\int_0^t (\theta^{k+1} \,-\, \theta^k)(x,\tau)\,d\tau\Big|^2\Big)\,dx \\ \le\;\; & C_7\int_0^t\int_\Omega \big|\theta^k \,-\, \theta^{k-1}\big|^2(x,\tau)\,dx\,d\tau\,.\end{aligned} \tag{4.25}$$

Integrating (4.25) from 0 to $\bar{t}$, we obtain that

$$\left\|\theta^{k+1} - \theta^k\right\|^2_{L^2(\Omega \times]0,\bar{t}[)} \leq C_7 \int_0^{\bar{t}} \left\|\theta^k - \theta^{k-1}\right\|^2_{L^2(\Omega \times]0,t[)} dt\,, \tag{4.26}$$

holds for every $\bar{t} \in [0,T]$, hence

$$\left\|\theta^{k+1} - \theta^k\right\|^2_{L^2(\Omega \times]0,t[)} \leq \left\|\theta^1 - \theta^0\right\|^2_2 \frac{(C_7\, t)^k}{k!} \quad \forall\ k \in \mathbb{N} \quad \forall\ t \in [0,T]\,. \tag{4.27}$$

The series $\displaystyle\sum_{k=0}^{\infty} \left(\frac{(C_7\, T)^k}{k!}\right)^{1/2}$ is convergent. This means that $\{\theta^k\}_{k=1}^{\infty}$ is a fundamental sequence in $L^2(\Omega\times]0,T[)$ which, by (4.22), is bounded in $L^\infty(\Omega\times]0,T[)$. There exists therefore $\theta \in L^\infty(\Omega\times]0,T[)$ such that $\theta^k \to \theta$ strongly in $L^2(\Omega\times]0,T[)$ and weakly-star in $L^\infty(\Omega\times]0,T[)$. From (4.23) it follows that θ_t, $\Delta\theta$ belong to $L^2(\Omega\times]0,T[)$ and that $\theta^k_t \to \theta_t$, $\Delta\theta^k \to \Delta\theta$, both weakly in $L^2(\Omega\times]0,T[)$. Moreover, by (4.3), (4.4) $\mathcal{V}[w^0\,,\,\theta^k]_t \to \mathcal{V}[w^0\,,\,\theta]_t$, also weakly in $L^2(\Omega\times]0,T[)$. Passing to the limit in (4.16)–(4.19) we see that θ is a solution of (4.5), (2.3), (2.4). This completes the proof of Theorem 4.2 (and, consequently, of Theorem 2.8). □

Proof of Theorem 2.9. We use the same trick as in (4.24)–(4.25). Subtracting equations (4.5) for θ_1, θ_2 and integrating with respect to t, we obtain that

$$\begin{aligned} &(\theta_1 - \theta_2)(x,t) - \Delta \int_0^t (\theta_1 - \theta_2)(x,\tau)\, d\tau \\ = \;& (\theta_1^0 - \theta_2^0)(x) + \big(F_1[w_1^0] - F_1[w_2^0]\big)(x,0) - \big(\mathcal{V}[w_1^0\,,\,\theta_1] \\ & - \mathcal{V}[w_2^0\,,\,\theta_2]\big)(x,t) + \int_0^t \Big(\psi_1\big(x,\tau,\theta_1(x,\tau)\big) - \psi_2\big(x,\tau,\theta_2(x,\tau)\big)\Big)\, d\tau\,. \end{aligned} \tag{4.28}$$

Multiplication by $(\theta_1 - \theta_2)(x,t)$ and integration over Ω yields

$$\begin{aligned} &\int_\Omega \left|\theta_1 - \theta_2\right|^2(x,t)\, dx + \frac{d}{dt}\int_\Omega \left|\nabla \int_0^t (\theta_1 - \theta_2)(x,\tau)\, d\tau\right|^2 dx \\ \leq\; & \int_\Omega \left|\theta_1^0 - \theta_2^0\right|^2(x)\, dx + 4\,T \int_0^t \int_\Omega \Big(d_\psi^2 + K_2\left|\theta_1 - \theta_2\right|^2\Big)\, dx\, d\tau \\ & + C_8\Big(\left|w_1^0 - w_2^0\right|^2 dx + \int_0^t \int_\Omega \left|\theta_1 - \theta_2\right|^2(x,\tau)\, dx\, d\tau\Big)\,, \end{aligned} \tag{4.29}$$

where we used the estimates (2.8), (2.19), (4.4) and (4.22).

To obtain the assertion, it remains to integrate (4.29) from 0 to $\bar{t}$ and to apply a standard Gronwall-type argument. □

5 Thermodynamic consistency

Proof of Theorem 2.10. By Theorem 2.9, and Hypotheses 2.2, 2.3, we have, for a.e. $(x,t) \in \Omega \times]0,T[$,

$$\begin{aligned}
\theta_t - \Delta\theta &= \psi(x,t,\theta) - F_1[w]_t \\
&\geq \psi_0(x,t) - K_2|\theta| - f_1[w]\, g[w]_t \\
&\geq -K_2|\theta| + \frac{g[w]_t}{\mu\, w_t} \cdot f_1[w]\big(f_1[w] + f_2[w]\,\theta\big) \\
&\geq \theta \cdot \Big(- K \operatorname{sign}(\theta) + \frac{g[w]_t}{\mu\, w_t} f_1[w]\, f_2[w]\Big) .
\end{aligned} \tag{5.1}$$

>From hypotheses 2.1, 2.3, we find that

$$0 \leq \frac{g[w]_t}{\mu\, w_t} \leq \frac{K_3}{\mu} \quad \text{a.e.} \;, \quad \big|f_2[w]\big| \leq K_1 \,, \tag{5.2}$$

and from (3.12), (3.14) it follows that

$$\big|f_1[w](x,t)\big| \leq C_9\Big(1 + \int_0^t \big|\theta(x,\tau)\big|\, d\tau\Big) \quad \text{a.e.}\,, \tag{5.3}$$

with some constant C_9 which is independent of x.

Since θ belongs to $L^\infty(\Omega \times]0,T[)$, we see that (5.1) is an inequality of the form

$$\theta_t - \Delta\theta + a(x,t)\,\theta \geq 0 \quad \text{in } \Omega \times]0,T[\tag{5.4}$$

with some function $a \in L^\infty(\Omega \times]0,T[)$. For the sake of definiteness, put

$$\tilde{K} := \|a\|_\infty \,. \tag{5.5}$$

Let us test (5.4) with $\vartheta(x,t) := \big(\delta\, e^{-\tilde{K}t} - \theta(x,t)\big)^+$. This yields

$$\int_0^t \int_\Omega \Big(\big(\delta\, e^{-\tilde{K}\tau} - \vartheta\big)_t \vartheta - \big|\nabla\vartheta\big|^2 + a\,\big(\delta\, e^{-\tilde{K}\tau} - \vartheta\big)\,\vartheta\Big)\, dx\, d\tau \geq 0 \;, \tag{5.6}$$

hence

$$\begin{aligned}
&\int_\Omega \big|\vartheta(x,t)\big|^2 dx + \int_0^t \int_\Omega \Big(|\nabla\vartheta|^2 + \delta(\tilde{K} - a(x,\tau))\, e^{-\tilde{K}\tau}\,\vartheta\Big)\, dx\, d\tau \\
\leq\; &\int_\Omega \big|\vartheta(x,0)\big|^2 dx + \tilde{K} \int_0^t \int_\Omega \big|\vartheta(x,\tau)\big|^2 dx\, d\tau .
\end{aligned} \tag{5.7}$$

By hypothesis, we have $\vartheta(x,0) \equiv 0$. Gronwall's lemma then implies $\vartheta \equiv 0$ and the proof is complete. □

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