STABILITY INDEX FOR ORBITS WITH NONVANISHING DIFFUSION

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bits with nonvanishing diffusion. The obtained general results are applied to investigating stochastic stability and stabilization of orbits on the plane.

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1. Introduction

Consider an autonomous system of stochastic differential equations in the sense of Stratonovich

$$dX = a_0(X)dt + \sum_{r=1}^{q}(\alpha_r(X)a_0(X) + a_r(X)) \circ dw_r(t). \tag{1.1}$$

Here X and $a_r,\ r = 0, ..., q$, are d-dimensional vectors, $\alpha_r,\ r = 1, ..., q$, are scalars, and $w_r(t),\ r = 1, ..., q$, are independent standard Wiener processes on a probability space $(\Omega,\ \mathcal{F},\ \mathbf{P})$.

Let the orbit $\mathbf{O}$ be an invariant manifold for the system (1.1), $a_0(x) \neq 0$ for every $x \in \mathbf{O}$, and $a_r(x) = 0$ for $x \in \mathbf{O}$. For $x \in \mathbf{O}$, $t \geq 0$ introduce the set $\mathbf{S}(x;t) \subset \mathbf{R}^d$:

$$\mathbf{S}(x;t) = \Big\{X :\ X = X(t) = x + \int_0^t a_0(X(s))ds$$

$$+ \sum_{r=1}^{q}\int_0^t (\alpha_r(X(s))a_0(X(s)) + a_r(X(s)))W_r'(s)ds\Big\}\ , \tag{1.2}$$

where $W_r(s),\ r = 1, ..., q$, are arbitrary smooth functions.

Due to the Stroock-Varadhan support theorem (see, for instance, [7]) $\mathbf{S}(x;t) \in \mathbf{O}$. Putting in (1.2) $W_r(s) \equiv 0$, we obtain that

$$\Big\{X :\ X = X(t) = x + \int_0^t a_0(X(s))ds,\ x \in \mathbf{O}\Big\} \in \mathbf{O}.$$

Since $a_0(x) \neq 0$, $x \in \mathbf{O}$, we get from here that the deterministic system of differential equations

$$dX = a_0(X)dt \tag{1.3}$$

has a T-periodic solution $X = \xi(t),\ 0 \leq t < T$, the phase trajectory of which coincides with the orbit $\mathbf{O}$.

The noise in the system (1.1) is subdivided in two parts: the first one acts lengthwise to the field of vectors $a_0(X)$, and the second one vanishes on the orbit $\mathbf{O}$. Let us show that under a highly general hypothesis any stochastic system

$$dX = a_0(X)dt + \sum_{r=1}^{q} b_r(X) \circ dw_r(t), \tag{1.4}$$

which has the orbit $\mathbf{O}$ as an invariant manifold, is of form (1.1). Of course, it is supposed in addition that $a_0(x) \neq 0$, $x \in \mathbf{O}$. As earlier the system (1.3) has a T-periodic solution $X = \xi(t),\ 0 \leq t < T$, the phase trajectory of which coincides with the orbit $\mathbf{O}$. Because

$$\mathbf{S}(x;t) = \Big\{X :\ X = X(t) = x + \int_0^t a_0(X(s))ds + \sum_{r=1}^{q}\int_0^t b_r(X(s))W_r'(s)ds\Big\}$$

$\alpha_r(x)$, $x \in \mathbf{O}$, such that $b_r(x) = \alpha_r(x) a_0(x)$, $r = 1, ..., q$, $x \in \mathbf{O}$. Let us extend the functions $\alpha_r(x)$ from $\mathbf{O}$ in some neighborhood of the orbit $\mathbf{O}$. Introducing the vector functions $a_r(x) = b_r(x) - \alpha_r(x) a_0(x)$ for x belonging to this neighborhood, we arrive at the system (1.1).

The concepts of Lyapunov exponent, moment Lyapunov exponents, and stability index for stationary points (see [8], [9], [1]-[4], [6] and references therein) are carried over for invariant manifolds of non-linear stochastic systems in [12]. But the main attention in [12] is given to the case of orbit with vanishing diffusion ($\alpha_r(x) \equiv 0$), and the case of orbit with nonvanishing diffusion is considered only in a general way for systems of the form (1.4). Introducing systems of the form (1.1) makes possible to study this complicated case more in detail. The obtained general results are applied to investigating stochastic stability and stabilization of orbits on the plane.

2. The linearized system for orthogonal displacement

Let $\mathbf{U}$ be a tubular neighborhood (a toroidal tube) of the orbit $\mathbf{O}$ such that for any point $x \in \mathbf{U}$ one can uniquely find a quantity $\vartheta(x)$, $0 \le \vartheta(x) < T$, for which $\xi(\vartheta(x))$ is the point on the trajectory $\mathbf{O}$ which is the nearest one to x. It is clear that the vector

$$\delta(x) = x - \xi(\vartheta(x))$$

is a displacement from the orbit which is normal to the vector $\xi'(\vartheta(x)) = a_0(\xi(\vartheta(x)))$, i.e.,

$$\sum_{j=1}^{d} (x^j - \xi^j(\vartheta(x))) \cdot a_0^j(\xi(\vartheta(x))) = 0. \tag{2.1}$$

We suppose that all the functions $a_r(x)$, $\alpha_r(x)$, $x \in \mathbf{U}$, are sufficiently smooth.

In what follows it is convenient to consider $\vartheta(x)$ as a multifunction which may take at x any value of $\vartheta(x) + kT$, $k = 0, \pm 1, \pm 2, ...$. Due to the T-periodicity of $\xi(t)$, it does not lead to any misunderstanding.

Let r be sufficiently small such that $\{x : \ |\delta(x)| \le r\} \subset \mathbf{U}$. Denote $\mathbf{U}_r = \{x : |\delta(x)| < r\}$.

Let $X(t)$ be a solution of (1.1) with $X(0) \subset \mathbf{U}_r$. We shall consider it on the random interval $[0, \tau)$ where τ is the first passage time of $X(t)$ to the boundary $\partial \mathbf{U}_r$. We note in connection with this fact that the more rigorous writing of the system (1.1) must include the multiplier $\chi_{\tau > t}$ on the right. For brevity we omit such a multiplier both in the system (1.1) and in the next nonlinear stochastic systems.

Introduce matrices $A_r(x)$ with the elements $a_r^{ij}(x) = \dfrac{\partial a_r^i}{\partial x^j}(x)$, $r = 0, 1, ..., q$; $i, j = 1, ..., d$.

Theorem 2.1. *The displacement $\delta(X(t))$ of the solution $X(t)$ from the orbit $\mathbf{O}$ satisfies the following system*

$$d\delta(X) = (A_0 - \frac{a_0 a_0^\top (A_0 + A_0^\top)}{|a_0|^2}) \delta(X) dt$$

$$+ \sum_{r=1}^{q} \alpha_r (A_0 - \frac{a_0 a_0^\top (A_0 + A_0^\top)}{|a_0|^2}) \delta(X) \circ dw_r(t)$$

where a_0, α_r, and A_r have the quantity $\xi(\vartheta(X(t)))$ as their argument (i.e., they are defined on the orbit $\mathbf{O}$*) and all the $O(\cdot)$ in (2.2) are uniform with respect to x belonging to the closure of* $\mathbf{U}_r$.

Proof. Differentiating (2.1) with respect to x^i and taking into account the equality

$$\xi'(\vartheta(x)) = a_0(\xi(\vartheta(x))),$$

we obtain

$$a_0^i(\xi(\vartheta(x))) - |a_0(\xi(\vartheta(x)))|^2 \cdot \frac{\partial\vartheta}{\partial x^i}(x)$$

$$+\sum_{j=1}^{d}(x^j - \xi^j(\vartheta(x))) \cdot (A_0(\xi(\vartheta(x)))a_0(\xi(\vartheta(x))))^j \cdot \frac{\partial\vartheta}{\partial x^i}(x) = 0.$$

From here

$$\frac{\partial\vartheta}{\partial x^i}(x) = \frac{a_0^i(\xi(\vartheta(x)))}{\varphi(x)}\ , \tag{2.3}$$

where

$$\varphi(x) = |a_0(\xi(\vartheta(x)))|^2 - (A_0(\xi(\vartheta(x)))a_0(\xi(\vartheta(x))), x - \xi(\vartheta(x))).$$

Applying the Stratonovich rule of differentiation to the k-th component of $\delta(X)$, we find

$$d\delta^k(X) = dX^k - d\xi^k(\vartheta(X)) = a_0^k(X)dt + \sum_{r=1}^{q}(\alpha_r(X)a_0(X) + a_r(X))^k \circ dw_r(t)$$

$$-\frac{a_0^k(\xi(\vartheta(X)))}{\varphi(X)} \cdot \sum_{i=1}^{d} a_0^i(\xi(\vartheta(X))) \cdot (a_0^i(X)dt + \sum_{r=1}^{q}(\alpha_r(X)a_0(X) + a_r(X))^i \circ dw_r(t))$$

$$= \frac{1}{\varphi(X)}(a_0^k(X)\varphi(X) - a_0^k(\xi(\vartheta(X)))\sum_{i=1}^{d} a_0^i(\xi(\vartheta(X)))a_0^i(X))dt$$

$$+\sum_{r=1}^{q}\alpha_r(X) \cdot \frac{1}{\varphi(X)}(a_0^k(X)\varphi(X) - a_0^k(\xi(\vartheta(X)))\sum_{i=1}^{d} a_0^i(\xi(\vartheta(X)))a_0^i(X)) \circ dw_r(t)$$

$$+\sum_{r=1}^{q}\frac{1}{\varphi(X)}(a_r^k(X)\varphi(X) - a_0^k(\xi(\vartheta(X)))\sum_{i=1}^{d} a_0^i(\xi(\vartheta(X)))a_r^i(X)) \circ dw_r(t). \tag{2.4}$$

We have (see the above expression for $\varphi(x)$)

$$a_0^k(X)\varphi(X) - a_0^k(\xi(\vartheta(X)))\sum_{i=1}^{d} a_0^i(\xi(\vartheta(X)))a_0^i(X)$$

$$= (a_0^k(X) - a_0^k(\xi(\vartheta(X)))) \cdot |a_0(\xi(\vartheta(X)))|^2$$

$$-a_0^k(\xi(\vartheta(X))) \cdot \sum_{i=1}^{d} a_0^i(\xi(\vartheta(X))) \cdot (a_0^i(X) - a_0^i(\xi(\vartheta(X))))$$

$$-a_0^k(X) \cdot (A_0(\xi(\vartheta(X)))a_0(\xi(\vartheta(X))), \delta(X)). \tag{2.5}$$

and

$$\varphi(X) = |a_0(\xi(\vartheta(X)))|^2 + O(|\delta(X)|). \tag{2.7}$$

Clearly, all the $O(\cdot)$ in (2.6) and (2.7) are uniform with respect to x belonging to the closure of $\mathbf{U}_r$.

From (2.5)–(2.7) we obtain

$$\frac{1}{\varphi(X)}(a_0^k(X)\varphi(X) - a_0^k(\xi(\vartheta(X)))\cdot\sum_{i=1}^{d} a_0^i(\xi(\vartheta(X)))a_0^i(X))$$

$$= (A_0(\xi(\vartheta(X)))\delta(X))^k - \frac{(A_0(\xi(\vartheta(X)))\delta(X), a_0(\xi(\vartheta(X))))}{|a_0(\xi(\vartheta(X)))|^2}a_0^k(\xi(\vartheta(X)))$$

$$-\frac{(A_0(\xi(\vartheta(X)))a_0(\xi(\vartheta(X))), \delta(X))}{|a_0(\xi(\vartheta(X)))|^2}a_0^k(\xi(\vartheta(X))) + O(|\delta(X)|^2). \tag{2.8}$$

Because of

$$a_r(\xi(\vartheta(X))) = 0,$$

we have

$$a_r^k(X) = (A_r(\xi(\vartheta(X)))\delta(X))^k + O(|\delta(X)|^2)$$

and consequently

$$\frac{1}{\varphi(X)}(a_r^k(X)\varphi(X) - a_0^k(\xi(\vartheta(X)))\sum_{i=1}^{d} a_0^i(\xi(\vartheta(X)))a_r^i(X))$$

$$= (A_r(\xi(\vartheta(X)))\delta(X))^k - \frac{(A_r(\xi(\vartheta(X)))\delta(X), a_0(\xi(\vartheta(X))))}{|a_0(\xi(\vartheta(X)))|^2}a_0^k(\xi(\vartheta(X))) + O(|\delta(X)|^2). \tag{2.9}$$

The relations (2.4), (2.8), and (2.9) imply the system (2.2). Theorem 2.1 is proved.

It is not difficult to prove the following theorem.

Theorem 2.2. *The magnitude $\vartheta(X)$ satisfies the following equation*

$$d\vartheta(X) = dt + \sum_{r=1}^{q} \alpha_r \circ dw_r(t) + O(|\delta(X)|)dt + \sum_{r=1}^{q} O(|\delta(X)|)\circ dw_r(t), \tag{2.10}$$

where $\alpha_r,\ r = 1, ..., q,$ have the quantity $\xi(\vartheta(X(t)))$ as their argument (i.e., they are defined on the orbit $\mathbf{O}$) and all the $O(\cdot)$ in (2.10) are uniform with respect to x belonging to the closure of $\mathbf{U}_r$.

Remark 2.1. The relations (2.2), (2.10) can be considered as stochastic differential equations for the process $(\vartheta(X), \delta(X))$ in view of the replacement $X = \xi(\vartheta(X)) + \delta(X)$. The process $(\vartheta(X), \delta(X))$ belongs to a d-dimensional manifold since $a_0^\top(\xi(\vartheta(X)))\delta(X) = 0$.

orthogonal system)

$$d\Delta = B_0(\Theta)\Delta dt + \sum_{r=1}^{q} \beta_r(\Theta) B_0(\Theta)\Delta \circ dw_r(t) + \sum_{r=1}^{q} B_r(\Theta)\Delta \circ dw_r(t) \tag{2.11}$$

$$d\Theta = dt + \sum_{r=1}^{q} \beta_r(\Theta) \circ dw_r(t), \tag{2.12}$$

where

$$B_0(\theta) = A_0(\xi(\theta)) - \frac{a_0(\xi(\theta))a_0^\top(\xi(\theta))(A_0(\xi(\theta)) + A_0^\top(\xi(\theta)))}{|a_0(\xi(\theta))|^2}, \; A_0(\xi(\theta)) = \{\frac{\partial a_0^i}{\partial x^j}(\xi(\theta))\}, \tag{2.13}$$

$$B_r(\theta) = A_r(\xi(\theta)) - \frac{a_0(\xi(\theta))a_0^\top(\xi(\theta))A_r(\xi(\theta))}{|a_0(\xi(\theta))|^2}, \; A_r(\xi(\theta)) = \{\frac{\partial a_r^i}{\partial x^j}(\xi(\theta))\}, \; r = 1, ..., q, \tag{2.14}$$

$$\beta_r(\theta) = \alpha_r(\xi(\theta)). \tag{2.15}$$

Let us note that $\xi(\theta)$ is defined for all θ as a T-periodic function.

Remark 2.2. The matrix $B_r(\theta)$ can be written similar to $B_0(\theta)$:

$$B_r(\theta) = A_r(\xi(\theta)) - \frac{a_0(\xi(\theta))a_0^\top(\xi(\theta))(A_r(\xi(\theta)) + A_r^\top(\xi(\theta)))}{|a_0(\xi(\theta))|^2}.$$

Indeed, due to $a_r(\xi(t)) \equiv 0$ we have for every $k = 1, ..., d$:

$$\sum_{i=1}^{d} \frac{\partial a_r^k}{\partial x^i}(\xi(t))\frac{d\xi^i(t)}{dt} = (A_r(\xi(t))a_0(\xi(t)))^k = 0, \tag{2.16}$$

i.e., $A_r(\xi(t))a_0(\xi(t)) \equiv 0$, and consequently $a_0^\top(\xi(\theta))A_r^\top(\xi(\theta)) \equiv 0$. The formula (2.16) is proved.

Theorem 2.3. *Let $\Delta(t_0) = \delta$, $\Theta(t_0) = \theta$ and let δ be orthogonal to $a_0(\xi(\theta)) = \xi'(\theta)$, i.e., $a_0^\top(\xi(\theta))\delta = 0$. Then $\Delta(t)$ is orthogonal to $a_0(\xi(\Theta(t)))$ for all $t \geq t_0$, i.e.,*

$$a_0^\top(\xi(\Theta(t)))\Delta(t) = \sum_{i=1}^{d} a_0^i(\xi(\Theta(t))) \cdot \Delta^i(t) \equiv 0, \; t \geq t_0 \tag{2.17}$$

Proof. The proof consists in direct checking the identity

$$d(\sum_{i=1}^{d} a_0^i(\xi(\Theta(t))) \cdot \Delta^i(t)) \equiv 0, \; t \geq t_0. \tag{2.18}$$

Theorem 2.3 is proved.

Let $\Delta(0) \neq 0$. Introduce

$$\Lambda = \frac{\Delta}{|\Delta|} \tag{3.1}$$

and consider the process (Θ, Λ). This process satisfies the Khasminskii-type system of stochastic differential equations (see [9], [12]) in the Stratonovich form

$$d\Lambda = b_0(\Theta, \Lambda)dt + \sum_{r=1}^{q} \beta_r(\Theta) b_0(\Theta, \Lambda) \circ dw_r(t) + \sum_{r=1}^{q} b_r(\Theta, \Lambda) \circ dw_r(t) \tag{3.2}$$

$$d\Theta = dt + \sum_{r=1}^{q} \beta_r(\Theta) \circ dw_r(t) \ , \tag{3.3}$$

where the vectors $b_r(\theta, \lambda)$ are equal to

$$b_r(\theta, \lambda) = B_r(\theta)\lambda - (B_r(\theta)\lambda, \lambda)\lambda, \ \ r = 0, 1, ..., q. \tag{3.4}$$

Below the notation $\dfrac{\partial b}{\partial \lambda}$ for the d-dimensional column vector $b = (b^1, ..., b^d)^\top$ means the matrix $\dfrac{\partial b}{\partial \lambda} = \{\dfrac{\partial b^i}{\partial \lambda^j}\}$, $i, j = 1, ..., d$, the notation $\dfrac{\partial c}{\partial \lambda}$ for the scalar c means the d-dimensional vector with the components $\dfrac{\partial c}{\partial \lambda^1}$, ..., $\dfrac{\partial c}{\partial \lambda^d}$, and $\dfrac{\partial^2 c}{\partial \lambda^2}$ means the matrix $\{\dfrac{\partial^2 c}{\partial \lambda^i \partial \lambda^j}\}$, $i, j = 1, ..., d$.

Let us consider the system (3.2)–(3.3) in $\mathbf{R}^{d+1}$, i.e., not only for Λ such that $|\Lambda| = 1$. The infinitesimal operator L of the $(d+1)$-dimensional process defined by the system (3.2)–(3.3) has the following form

$$Lf(\theta, \lambda) = (\frac{\partial f}{\partial \lambda}, b_0 + \frac{1}{2}\sum_{r=1}^{q}(\beta_r \frac{\partial b_0}{\partial \lambda} + \frac{\partial b_r}{\partial \lambda})(\beta_r b_0 + b_r) + \frac{1}{2}\sum_{r=1}^{q} \beta_r \frac{\partial}{\partial \theta}(\beta_r b_0 + b_r))$$

$$+\frac{\partial f}{\partial \theta}(1 + \frac{1}{2}\sum_{r=1}^{q} \beta_r \beta_r') + \frac{1}{2}\sum_{r=1}^{q}(\frac{\partial^2 f}{\partial \lambda^2}(\beta_r b_0 + b_r), (\beta_r b_0 + b_r))$$

$$(\frac{\partial^2 f}{\partial \lambda \partial \theta}, \sum_{r=1}^{q} \beta_r(\beta_r b_0 + b_r)) + \frac{1}{2}\sum_{r=1}^{q} \frac{\partial^2 f}{\partial \theta^2}\beta_r^2, \ (\theta, \lambda) \in \mathbf{R}^{d+1}. \tag{3.5}$$

Let $\Theta(0) = \theta, \ \Lambda(0) = \lambda$ be such that $a_0^\top(\xi(\theta))\lambda = 0, \ |\lambda| = 1$. Then due to (3.1) $a_0^\top(\xi(\theta))\Delta(0) = 0$. Using Theorem 1.3 and again (3.1), we obtain

$$a_0^\top(\xi(\Theta(t)))\Lambda(t) = 0, \ \Lambda^\top(t)\Lambda(t) = 1, \tag{3.6}$$

i.e., (Θ, Λ) is a Markov process on the $(d-1)$-dimensional compact manifold $\mathbf{D}$ defined by the following equations

$$\mathbf{D} = \{(\theta, \lambda) : \ a_0^\top(\xi(\theta))\lambda = 0, \ \lambda^\top \lambda = 1\}$$

in the space of $d+1$ variables θ, $\lambda^1, ..., \lambda^d$.

Under each fixed θ the manifold $\mathbf{D}$ gives a unit sphere $\mathbf{S}^{d-2}$ of the dimension $d-2$ and, consequently, $\mathbf{D}$ is a torus which is equal to the product $\mathbf{O}\times$ $\mathbf{S}^{d-2}$.

this system.

For $|\Delta(t)|^p$, $-\infty < p < \infty$, we obtain the following linear equation

$$d|\Delta(t)|^p = p \cdot (B_0(\Theta)\Lambda, \Lambda) \cdot |\Delta(t)|^p dt$$

$$+p\sum_{r=1}^{q} \beta_r(\Theta)(B_0(\Theta)\Lambda, \Lambda) \cdot |\Delta(t)|^p \circ dw_r(t) + p\sum_{r=1}^{q}(B_r(\Theta)\Lambda, \Lambda) \cdot |\Delta(t)|^p \circ dw_r(t). \tag{3.7}$$

Let $\Delta(0) = \lambda$, $\lambda^\top\lambda = 1$. The next formula defines a strongly continuous semigroup of positive operators on $\mathbf{C}(\mathbf{D})$:

$$T_t(p)f(\theta, \lambda) = Ef(\Theta_\theta(t), \Lambda_{\theta,\lambda}(t))|\Delta_{\theta,\lambda}(t)|^p, \ (\theta, \lambda) \in \mathbf{D}, \ f \in \mathbf{C}(\mathbf{D}). \tag{3.8}$$

This fact can be proved by direct checking the definition of strongly continuous semigroup.

Our urgent aim is to find the generator $A(p)$ of the semigroup $T_t(p)$.

Let $f \in \mathbf{C}^2(\mathbf{D})$ where $f = f(\theta, \lambda)$, $(\theta, \lambda) \in \mathbf{D}$. Let $\mathbf{D} \subset \check{\mathbf{D}} \subset \mathbf{R}^{d+1}$, where $\check{\mathbf{D}}$ is an open set, and let $\check{f} = \check{f}(\theta, \lambda)$, $(\theta, \lambda) \in \check{\mathbf{D}}$, be a twice continuously differentiable extension of f. For example, one can take the following function

$$\check{f}(\theta, \lambda) = f(\theta, \frac{\lambda}{|\lambda|} - \frac{a_0(\cdot)}{|a_0(\cdot)|^2}(\frac{\lambda}{|\lambda|}, a_0(\cdot))), \ a_0(\cdot) := a_0(\xi(\theta)), \ (\theta, \lambda) \in \check{\mathbf{D}},$$

as such an extension because under any $(\theta, \lambda) \in \mathbf{R}^{d+1}$, $|\lambda| \neq 0$, the point (θ, μ) with

$$\mu = \frac{\lambda}{|\lambda|} - \frac{a_0(\cdot)}{|a_0(\cdot)|^2}(\frac{\lambda}{|\lambda|}, a_0(\cdot))$$

belongs to $\mathbf{D}$: $(\theta, \mu) \in \mathbf{D}$.

The next theorem gives a formula for the generator $A(p)$ of the semigroup $T_t(p)$.

Theorem 3.1. *Let L be the infinitesimal generator of the diffusion process $(\Theta_\theta(t), \Lambda_{\theta,\lambda}(t))$, $(\theta, \lambda) \in \check{\mathbf{D}}$. Let $\check{f}$ be a twice continuously differentiable extension of a function $f \in \mathbf{C}^2(\mathbf{D})$. Then*

$$A(p)f(\theta, \lambda) = L\check{f}(\theta, \lambda) + p\frac{\partial \check{f}}{\partial \theta}\sum_{r=1}^{q}\beta_r\gamma_r + p\sum_{r=1}^{q}\gamma_r(\frac{\partial \check{f}}{\partial \lambda}, \beta_r b_0 + b_r)$$

$$+p \cdot f \cdot (\gamma_0 + \frac{1}{2}\sum_{r=1}^{q}(\frac{\partial \gamma_r}{\partial \lambda}, \beta_r b_0 + b_r) + \frac{1}{2}\sum_{r=1}^{q}\frac{\partial \gamma_r}{\partial \theta}\beta_r + \frac{1}{2}p\sum_{r=1}^{q}\gamma_r^2), \ (\theta, \lambda) \in \mathbf{D}, \tag{3.9}$$

where

$$\gamma_0 = \gamma_0(\theta, \lambda) := (B_0(\theta)\lambda, \lambda), \ \gamma_r = \gamma_r(\theta, \lambda) := \beta_r(\theta)\gamma_0(\theta, \lambda) + (B_r(\theta)\lambda, \lambda), \ r = 1, ..., q, \tag{3.10}$$

and L is defined by the formula (3.5).

The following formula holds

$$df(\Theta_\theta(t), \Lambda_{\theta,\lambda}(t))|\Delta_{\theta,\lambda}(t)|^p = A(p)f(\Theta_\theta(t), \Lambda_{\theta,\lambda}(t)) \cdot |\Delta_{\theta,\lambda}(t)|^p dt +$$

$$+\frac{\partial \check{f}}{\partial \theta}\sum_{r=1}^{q}\beta_r \cdot |\Delta|^p dw_r(t) + \sum_{r=1}^{q}(\frac{\partial \check{f}}{\partial \lambda}, \beta_r b_0 + b_r) \cdot |\Delta|^p dw_r(t)$$

where the function $\check{f}$ with its derivatives and the coefficients b_r, γ_r have $\Theta_\theta(t), \Lambda_{\theta,\lambda}(t)$ as their arguments, β_r have $\Theta_\theta(t)$ as their argument, and Δ is the abridged notation for $\Delta_{\theta,\lambda}(t)$.

Proof. Since the manifold $\mathbf{D}$ is invariant for the process $(\Theta_\theta(t), \Lambda_{\theta,\lambda}(t))$, we have

$$f(\Theta_\theta(t), \Lambda_{\theta,\lambda}(t)) = \check{f}(\Theta_\theta(t), \Lambda_{\theta,\lambda}(t)),\ t \geq 0,\ (\theta, \lambda) \in \mathbf{D}. \tag{3.12}$$

Let us adduce the stochastic system (3.2), (3.3), (3.7) to Ito's form

$$d\Lambda = b_0 dt + \frac{1}{2}\sum_{r=1}^{q}(\beta_r \frac{\partial b_0}{\partial \lambda} + \frac{\partial b_r}{\partial \lambda})(\beta_r b_0 + b_r)dt + \frac{1}{2}\sum_{r=1}^{q}\beta_r \frac{\partial}{\partial \theta}(\beta_r b_0 + b_r)dt$$

$$+\sum_{r=1}^{q}(\beta_r b_0 + b_r)dw_r(t)\ , \tag{3.13}$$

$$d\Theta = dt + \frac{1}{2}\sum_{r=1}^{q}\beta_r\beta_r' dt + \sum_{r=1}^{q}\beta_r dw_r(t)\ , \tag{3.14}$$

$$d|\Delta|^p = p\gamma_0 \cdot |\Delta|^p dt + \frac{1}{2}p\sum_{r=1}^{q}(\frac{\partial \gamma_r}{\partial \lambda}, \beta_r b_0 + b_r) \cdot |\Delta|^p dt + \frac{1}{2}p\sum_{r=1}^{q}\frac{\partial \gamma_r}{\partial \theta}\beta_r \cdot |\Delta|^p dt$$

$$+\frac{1}{2}p^2\sum_{r=1}^{q}\gamma_r^2 \cdot |\Delta|^p dt + p\sum_{r=1}^{q}\gamma_r \cdot |\Delta|^p dw_r(t)\ , \tag{3.15}$$

where all the functions have Θ, Λ as their arguments.

Now one can evaluate (denote for a while the right side of the formula (3.9) by $\tilde{A}(p)\check{f}(\theta, \lambda)$)

$$df(\Theta_\theta(t), \Lambda_{\theta,\lambda}(t))|\Delta_{\theta,\lambda}(t)|^p = d\check{f}(\Theta_\theta(t), \Lambda_{\theta,\lambda}(t))|\Delta_{\theta,\lambda}(t)|^p$$

$$= \tilde{A}(p)\check{f}(\Theta_\theta(t), \Lambda_{\theta,\lambda}(t)) \cdot |\Delta_{\theta,\lambda}(t)|^p dt$$

$$+\frac{\partial \check{f}}{\partial \theta}\sum_{r=1}^{q}\beta_r \cdot |\Delta|^p dw_r(t) + \frac{\partial \check{f}}{\partial \lambda}\sum_{r=1}^{q}(\beta_r b_0 + b_r) \cdot |\Delta|^p dw_r(t)$$

$$+f \cdot p\sum_{r=1}^{q}\gamma_r \cdot |\Delta|^p dw_r(t),\ (\theta, \lambda) \in \mathbf{D}. \tag{3.16}$$

From (3.8), (3.12), and (3.16) it follows

$$T_t(p)f(\theta, \lambda) - f(\theta, \lambda) = E\check{f}(\Theta_\theta(t), \Lambda_{\theta,\lambda}(t))|\Delta_{\theta,\lambda}(t)|^p - \check{f}(\theta, \lambda) =$$

$$E\int_0^t \tilde{A}(p)\check{f}(\Theta_\theta(s), \Lambda_{\theta,\lambda}(s)) \cdot |\Delta_{\theta,\lambda}(s)|^p ds,$$

whence the formula (3.9) runs out.

Now the equation (3.16) can be rewritten in the form (3.11). Theorem 3.1 is proved.

$$h_0(\theta,\lambda) = \begin{bmatrix} 1 \\ b_0(\theta,\lambda) \end{bmatrix}, \ h_r(\theta,\lambda) = \begin{bmatrix} \beta_r(\theta) \\ \beta_r(\theta)b_0(\theta,\lambda) + b_r(\theta,\lambda) \end{bmatrix}, \ r = 1,...,q, \ (\theta,\lambda) \in \mathbf{D}.$$

They touch the manifold $\mathbf{D}$ and generate the corresponding vector fields on $\mathbf{D}$.

The following condition of nondegeneracy is supposed to be fulfilled:

$$\dim LA(h_1,...,h_q) = d-1 \ \text{ for any } (\theta,\lambda) \in \mathbf{D}. \tag{3.17}$$

Here LA denotes the Lie algebra generated by the vector fields $h_1,...,h_q$.

A simple sufficient condition of nondegeneracy consists in

$$\dim L(h_1,...,h_q) = d-1 \ \text{ for any } (\theta,\lambda) \in \mathbf{D}, \tag{3.18}$$

where L denotes the linear hull spanned by the given vectors.

For many situations the weaker condition,

$$\dim LA(h_0,h_1,...,h_q) = d-1 \ \text{ for any } (\theta,\lambda) \in \mathbf{D}, \tag{3.19}$$

would be sufficient but in order to avoid some complications we impose (3.17) as a rule.

As in [4] under the Lie algebra condition (3.17), any operator $T_t(p)$, $t > 0$, $-\infty < p < \infty$, is compact and irreducible (even strongly positive). We recall that a positive operator T in $\mathbf{C}(\mathbf{D})$ is called irreducible if $\{0\}$ and $\mathbf{C}(\mathbf{D})$ are the only T-invariant closed ideals, and T is called strongly positive if $Tf(\theta,\lambda) > 0$, $(\theta,\lambda) \in \mathbf{D}$, for any $f \geq 0$, $f \neq 0$. Under each $p \in \mathbf{R}$, the generalized Perron-Frobenius theorem ensures the existence of a strictly positive eigenfunction $e_p(\theta,\lambda)$ of $T_t(p)$ (and, consequently, for $A(p)$) corresponding to the principal eigenvalue. It is known that

$$A(p)e_p(\theta,\lambda) = g(p)e_p(\theta,\lambda), \ e_p(\theta,\lambda) > 0, \ (\theta,\lambda) \in \mathbf{D}, \tag{3.20}$$

where the eigenvalue $g(p)$ is simple and it strictly dominates the real part of any other point of the spectrum of $A(p)$.

Remark 3.1. It should be noted that in the case of vanishing diffusion on the very orbit the condition (3.17) is not fulfilled. For such systems all the scalars β are equal to zero and $\dim LA(h_1,...,h_q)$ cannot be more than $d-2$. In [12] precisely this case is considered under the condition

$$\dim L(h_1,...,h_q) = d-2 \ \text{ for any } (\theta,\lambda) \in \mathbf{D}. \tag{3.21}$$

Clearly, from this condition it follows that

$$\dim L(h_0,h_1,...,h_q) = d-1 \ \text{ for any } (\theta,\lambda) \in \mathbf{D}.$$

In contrast to the nondegeneracy case (3.17), any operator $T_t(p)$, $t > 0$, $-\infty < p < \infty$, is non-compact and there are values t for which $T_t(p)$ is non-irreducible. But provided the condition (3.21) is fulfilled, the whole semigroup $T_t(p)$ is irreducible (we recall that a positive semigroup $T_t(p)$ in $\mathbf{C}(\mathbf{D})$ is called irreducible if $\{0\}$ and $\mathbf{C}(\mathbf{D})$ are the only invariant closed ideals for all $T_t(p)$, $t \geq 0$, at once), and the relation (3.20) holds. However, the eigenvalue $g(p)$, remaining real and simple, is only more than or equal to the real part of any other point of the spectrum of $A(p)$. We underline that the noted distinction is not any obstacle for carrying over the theory of moment Lyapunov exponent for the case of vanishing diffusion under (3.21) (see [12]).

Now we are ready to formulate a number of theorems relating to stability properties of the linearized orthogonal system (2.11)–(2.12). These theorems are analogous to

The following theorem is an analogue of the Khasminskii theorem (see [8], [9]).

Theorem 3.2. *Assume* (3.17). *Then the process* (Θ, Λ) *on* $\mathbf{D}$ *is ergodic, there exists an invariant measure* $\mu(\theta, \lambda)$ *and, for any* (θ, δ), $\delta \neq 0$, *with* $a_0^\top(\xi(\theta))\delta = 0$, *there exists the limit* (*which does not depend on* θ, δ)

$$P\text{-a.s.} \lim_{t\to\infty} \frac{1}{t} \ln |\Delta_{\theta,\delta}(t)| = \lim_{t\to\infty} \frac{1}{t} E \ln |\Delta_{\theta,\delta}(t)| = \int_{\mathbf{D}} Q(\theta, \lambda) d\mu(\theta, \lambda) : \;\; = \lambda^*, \tag{3.22}$$

where

$$Q(\theta, \lambda) = \gamma_0 + \frac{1}{2} \sum_{r=1}^{q} (\frac{\partial \gamma_r}{\partial \lambda}, \beta_r b_0 + b_r) + \frac{1}{2} \sum_{r=1}^{q} \frac{\partial \gamma_r}{\partial \theta} \beta_r. \tag{3.23}$$

The limit λ^* is called Lyapunov exponent of the system (2.11)–(2.12).

The following theorem is an analogue of the Arnold-Oeljeklaus-Pardoux theorem (see [4]).

Theorem 3.3. *Assume* (3.17). *Then for all* $(\theta, \delta), \delta \neq 0$, *with* $a_0^\top(\xi(\theta))\delta = 0$ *the limit* (*which is called* p-th-*moment Lyapunov exponent for* (2.11)–(2.12))

$$\lim_{t\to\infty} \frac{1}{t} \ln E|\Delta_{\theta,\delta}(t)|^p = g(p) \tag{3.24}$$

exists for any $p \in \mathbf{R}$ *and it is independent of* (θ, δ). *The limit* $g(p)$ *is a convex analytic function of* $p \in \mathbf{R}$, $g(p)/p$ *is increasing,* $g(0) = 0$, *and* $g'(0) = \lambda^*$.

Further, the moment Lyapunov exponent $g(p)$ *is an eigenvalue of* $A(p)$ *with a strictly positive eigenfunction* $e_p(\theta, \lambda)$, *i.e., the relation (3.20) is fulfilled. The eigenvalue* $g(p)$ *is simple and* $g(p)$ *is more than or equal to the real part of any other point of the spectrum of* $A(p)$.

These results can be applied (as in the case of stationary point) to study the behavior of $P\{\sup_{t\geq 0} |\Delta_{\theta,\delta}(t)| > \rho\}$, $|\delta| \ll \rho$, for asymptotically stable systems ($\lambda^* < 0$), and of $P\{\inf_{t\geq 0} |\Delta_{\theta,\delta}(t)| < \rho\}$, $|\delta| \gg \rho$, for unstable systems ($\lambda^* > 0$) (of course, it is supposed that $a_0^\top(\xi(\theta))\delta = 0$, $\rho > 0$ is a certain number).

The following theorem is an analogue of the Baxendale theorem (see [6]).

Theorem 3.4. *Assume* (3.17). *If* $g'(0) = \lambda^* < 0$ *and the equation*

$$g(p) = 0 \tag{3.25}$$

has a root $\gamma^* > 0$, *then there exists* $K \geq 1$ *such that for all* $\rho > 0$ *and for all* δ *with* $|\delta|<\rho$ and $a_0^\top(\xi(\theta))\delta = 0$

$$\frac{1}{K}(|\delta|/\rho)^{\gamma^*} \leq P\{\sup_{t\geq 0} |\Delta_{\theta,\delta}(t)| > \rho\} \leq K(|\delta|/\rho)^{\gamma^*}. \tag{3.26}$$

If $g'(0) = \lambda^* > 0$ *and the equation* (3.25) *has a root* $\gamma^* < 0$, *then there exists* $K \geq 1$ *such that for all* $\rho > 0$ *and for all* δ *with* $|\delta|>\rho$ and $a_0^\top(\xi(\theta))\delta = 0$

$$\frac{1}{K}(|\delta|/\rho)^{\gamma^*} \leq P\{\inf_{t\geq 0} |\Delta_{\theta,\delta}(t)| < \rho\} \leq K(|\delta|/\rho)^{\gamma^*}. \tag{3.27}$$

4. Orbital stability index

The root γ^* of the equation (3.25) is called stability index of the linearized orthogonal system (2.11)–(2.12). Theorem 3.4 establishes that the probability with which a solution of the linearized orthogonal system exceeds a threshold is controlled by the number γ^*. It turns out that the estimates (3.26)-(3.27) remain true for the nonlinear system (2.2), (2.10) as well. This fact is an analogue of the Arnold-Khasminskii theorem for the case of stationary points [3]. Such a theorem is proved in [12] for systems with vanishing diffusion on the invariant orbit. The idea of proving the next theorem is close to the adduced one in [3] and [12]. However, there are some distinctions of a technique nature. In view of importance of the following theorem its proof is given completely.

Theorem 4.1. *Let the linearized orthogonal system* (2.11)–(2.12) *for the system* (1.1) *be such that* (3.17) *is fulfilled. Assume that the stability index* γ^* *of* (2.11)–(2.12) *does not vanish,* $\gamma^* \neq 0$.

Then

1. Case $\gamma^* > 0$: *There exists a sufficiently small* $\rho > 0$ *and positive constants* c_1, c_2 *such that for all* $x : |\delta(x)| < \rho$ *the solution* $X_x(t)$ *of* (1.1) *satisfies the inequalities*

$$c_1(|\delta(x)|/\rho)^{\gamma^*} \leq P\{\sup_{t\geq 0} |\delta(X_x(t))| > \rho\} \leq c_2(|\delta(x)|/\rho)^{\gamma^*}. \tag{4.1}$$

2. Case $\gamma^* < 0$: *There exists a sufficiently small* $r > 0$, *positive constants* c_3, c_4, *and a constant* $0 < \alpha < 1$ *such that for any* $\rho \in (0, \alpha r)$ *and all* $x : \rho < |\delta(x)| < \alpha r$

$$c_3(|\delta(x)|/\rho)^{\gamma^*} \leq P\{\inf_{0\leq t<\tau} |\delta(X_x(t))| < \rho\} \leq c_4(|\delta(x)|/\rho)^{\gamma^*}. \tag{4.2}$$

Here $\tau := \inf\{t : |\delta(X_x(t))| > r\}$.

Proof. Due to the notation (2.13)-(2.15), the system (2.10), (2.2) (with respect to $\vartheta = \vartheta(X_x(t))$, $\delta = \delta(X_x(t))$) can be rewritten in the form

$$d\vartheta = dt + \sum_{r=1}^{q} \beta_r \circ dw_r(t) + O(|\delta|)dt + \sum_{r=1}^{q} O(|\delta|) \circ dw_r(t) \tag{4.3}$$

$$d\delta = B_0\delta dt + \sum_{r=1}^{q} (\beta_r B_0 + B_r)\delta \circ dw_r(t) + O(|\delta|^2)dt + \sum_{r=1}^{q} O(|\delta|^2) \circ dw_r(t), \tag{4.4}$$

where B_0, β_r, B_r, $r = 1, ..., q$, have $\vartheta(X_x(t))$ as their argument.

Because of the supposed smoothness of the coefficients of the system (1.1) in $\mathbf{U}$ (see the beginning of Section 2) the terms $O(|\delta|)$ and $O(|\delta|^2)$ in (4.3) and (4.4) being depended on δ and ϑ are sufficiently smooth as well. Moreover, for example, the derivatives $\partial O(|\delta|)/\partial\theta$, $\partial O(|\delta|)/\partial\delta^i$ are $O(|\delta|)$, $O(1)$ correspondingly, and these $O(|\delta|)$, $O(1)$ are uniform with respect to the points from the closure of $\mathbf{U}_r$ under a sufficiently small r. We need such claims to reduce a number of Stratonovich equations to the Ito form. In turn, the latter is necessary in this proof for the separation of martingale terms.

The system (4.3)-(4.4) has the following Ito form

$$d\vartheta = (1 + \frac{1}{2}\sum_{r=1}^{q} \beta_r^{'}\beta_r)dt + \sum_{r=1}^{q} \beta_r dw_r(t) + O(|\delta|)dt + \sum_{r=1}^{q} O(|\delta|)dw_r(t) \tag{4.5}$$

$$+\sum_{r=1}^{q}(\beta_r B_0 + B_r)\delta dw_r(t) + O(|\delta|^2)dt + \sum_{r=1}^{q} O(|\delta|^2)dw_r(t). \tag{4.6}$$

Introduce

$$\Gamma(X_x(t)) = \delta(X_x(t))/|\delta(X_x(t))|.$$

Clearly, $(\vartheta, \Gamma) = (\vartheta(X_x(t)), \Gamma(X_x(t))) \in \mathbf{D}$. In view of (4.4) and (4.6) it is not difficult to obtain

$$d\Gamma = b_0 dt + \sum_{r=1}^{q}(\beta_r b_0 + b_r)\circ dw_r(t) + O(|\delta|)dt + \sum_{r=1}^{q} O(|\delta|)\circ dw_r(t) \tag{4.7}$$

and

$$d\Gamma = (b_0 + \frac{1}{2}\sum_{r=1}^{q}(\beta_r \frac{\partial b_0}{\partial \lambda} + \frac{\partial b_r}{\partial \lambda})(\beta_r b_0 + b_r) + \frac{1}{2}\sum_{r=1}^{q}\beta_r \frac{\partial}{\partial \theta}(\beta_r b_0 + b_r))dt$$

$$\sum_{r=1}^{q}(\beta_r b_0 + b_r)dw_r(t) + O(|\delta|)dt + \sum_{r=1}^{q} O(|\delta|)dw_r(t). \tag{4.8}$$

Here the functions $\beta_r(\theta)$, the vectors $b_r(\theta,\lambda)$, $\frac{\partial}{\partial \theta}(\beta_r b_0 + b_r)$, the matrices $\frac{\partial b_0}{\partial \lambda}$, $\frac{\partial b_r}{\partial \lambda}$ have $\vartheta(X_x(t))$, $\Gamma(X_x(t))$ as their arguments. Finally, using (3.5) and (3.9), after fairly long but routine calculations we get for $f(\theta,\lambda) \in C^2(\mathbf{D})$ (compare with (3.11)):

$$df(\vartheta,\Gamma)|\delta|^p = A(p)f\cdot|\delta|^p dt + \frac{\partial \check{f}}{\partial \theta}\sum_{r=1}^{q}\beta_r\cdot|\delta|^p dw_r(t) + \sum_{r=1}^{q}(\frac{\partial \check{f}}{\partial \lambda}, \beta_r b_0 + b_r)\cdot|\delta|^p dw_r(t)$$

$$+pf\sum_{r=1}^{q}\gamma_r\cdot|\delta|^p dw_r(t) + O(|\delta|^{p+1})dt + \sum_{r=1}^{q} O(|\delta|^{p+1})dw_r(t). \tag{4.9}$$

As the specific form of the martingale terms has not any meaning in what follows, we shall use the same notation $\sum_{r=1}^{q} m_r dw_r(t)$ for different martingale terms. For example, the equation (4.9) acquires the form

$$df(\vartheta,\Gamma)|\delta|^p = A(p)f\cdot|\delta|^p dt + O(|\delta|^{p+1})dt + \sum_{r=1}^{q} m_r dw_r(t). \tag{4.10}$$

Case 1. Let $\gamma^* > 0$ be the stability index for (2.11)–(2.12), $0 < c < 1$ be a positive constant, and $e_{\gamma^*}(\theta,\lambda)$, $e_{\gamma^*+c}(\theta,\lambda)$ be strictly positive solutions of the equations (see Theorem 3.3, the formula (3.20) and remember that $g(\gamma^*) = 0$)

$$A(\gamma^*)e_{\gamma^*}(\theta,\lambda) = 0, \tag{4.11}$$

$$A(\gamma^* + c)e_{\gamma^*+c}(\theta,\lambda) = g(\gamma^* + c)e_{\gamma^*+c}(\theta,\lambda), \tag{4.12}$$

where $g(\gamma^* + c) > 0$.

Introduce the following functions

$$V_{\mp}(x) = e_{\gamma^*}(\vartheta(x), \delta(x)/|\delta(x)|)\cdot|\delta(x)|^{\gamma^*} \mp e_{\gamma^*+c}(\vartheta(x), \delta(x)/|\delta(x)|)\cdot|\delta(x)|^{\gamma^*+c}. \tag{4.13}$$

$$dV_{\mp}(X_x(t)) = \mp g(\gamma^* + c)e_{\gamma^*+c}(\vartheta, \Gamma) \cdot |\delta(X_x(t))|^{\gamma^*+c}dt + O(|\delta|^{\gamma^*+1})dt + \sum_{r=1} m_r dw_r(t). \tag{4.14}$$

Let the eigenfunctions e_{γ^*} and e_{γ^*+c} have already been chosen. It is clear from (4.13) and (4.14) that there exists a sufficiently small $\rho > 0$ such that $V_-(x) > 0$ for all x with $0 < |\delta(x)| < \rho$ and $V_-(X_x(t \wedge \tau_{x,\rho}))$ with

$$\tau_{x,\rho} : \ = \inf\{t : \ |\delta(X_x(t))| > \rho\}$$

is a supermartingale.

Hence there exist positive constants a_1 and a_2 such that the following inequalities hold:

$$a_1|\delta(x)|^{\gamma^*} \geq V_-(x) \geq EV_-(X_x(t \wedge \tau_{x,\rho})) \geq a_2\rho^{\gamma^*} P\{\sup_{0\leq s\leq t} |\delta(X_x(s))| > \rho\}$$

and therefore

$$P\{\sup_{t\geq 0} |\delta(X_x(t))| > \rho\} = \lim_{t\to\infty} P\{\sup_{0\leq s\leq t} |\delta(X_x(s))| > \rho\} \leq \frac{a_1}{a_2}(|\delta(x)|/\rho)^{\gamma^*}. \tag{4.15}$$

As $V_+(x) > 0$ (see (4.13)) and $V_+(X_x(t \wedge \tau_{x,\rho}))$ is a submartingale for a sufficiently small ρ (see (4.14)), we get

$$a_3|\delta(x)|^{\gamma^*} \leq V_+(x) \leq EV_+(X_x(\tau_{x,\varepsilon} \wedge \tau_{x,\rho})) \leq a_4\rho^{\gamma^*} P\{\sup_{t>0} |\delta(X_x(t))| > \rho\} + a_5\varepsilon^{\gamma^*}, \tag{4.16}$$

where a_3, a_4, a_5 are some positive constants which do not depend on ε, $\varepsilon < |\delta(x)| < \rho$, and

$$\tau_{x,\varepsilon} : \ = \inf\{t : \ |\delta(X_x(t))| < \varepsilon\}.$$

The relations (4.15) and (4.16) give (4.1) provided ρ is the smallest among (4.15) and (4.16). Case 1 is proved.

Case 2. Let $\gamma^* < 0$. Then there exists a sufficiently small c, $0 < c < 1$, such that $g(\gamma^* + c) < 0$ in (4.12). Now $V_+(X_x(t \wedge \tau_{x,r}))$ is a supermartingale for a sufficiently small r and for x with $0 < |\delta(x)| < r$.

We have for some positive a_1, a_2 and for x with $\rho < |\delta(x)| < r$:

$$a_1|\delta(x)|^{\gamma^*} \geq V_+(x) \geq EV_+(X_x(t \wedge \tau_{x,r})) \geq a_2\rho^{\gamma^*} P\{\inf_{0\leq t\leq \tau_{x,r}} |\delta(X_x(t))| < \rho\}. \tag{4.17}$$

Relation (4.17) implies the second part of (4.2).

Further, $V_-(X_x(t \wedge \tau_{x,r}))$ is a submartingale for a sufficiently small r and there exist positive constants a_3, a_4, a_5 such that for all x with $\rho < |\delta(x)| < r$:

$$a_3|\delta(x)|^{\gamma^*} \leq V_-(x) \leq EV_-(X_x(\tau_{x,\rho_0} \wedge \tau_{x,\rho})) \leq a_4\rho^{\gamma^*} P\{\inf_{0\leq t\leq \tau_{x,r}} |\delta(X_x(t))| < \rho\} + a_5 r^{\gamma^*},$$

where a_3, a_4, a_5 do not depend on ρ and r.

If $\rho < |\delta(x)| < \alpha r$, then

$$a_4\rho^{\gamma^*} P\{\inf_{0\leq t\leq \tau_{x,r}} |\delta(X_x(t))| < \rho\} \geq a_3|\delta(x)|^{\gamma^*} - a_5 r^{\gamma^*} \geq$$

$$\frac{1}{2}a_3|\delta(x)|^{\gamma^*} + \frac{1}{2}a_3|\alpha r|^{\gamma^*} - a_5 r^{\gamma^*}. \tag{4.18}$$

first part of (4.2). Theorem 4.1 is proved.

The root γ^* is called stability index of the orbit $\mathbf{O}$ of the system (1.1).

Let us give a summary with a comment on the procedure of searching for γ^*. We start from the fact that the deterministic system (1.3) has a T-periodic solution $X = \xi(t)$ which orbit $\mathbf{O} : x = \xi(\theta),\ 0 \le \theta < T$, is invariant for the stochastic system (1.1). To this end we suppose $a_r(x),\ r = 1, ..., q$, to be equal to zero at $\mathbf{O}$, i.e., $a_r(\xi(\theta)) = 0,\ 0 \le \theta < T$. We consider $\xi(\theta),\ -\infty < \theta < \infty$, as a T-periodic vector function. We introduce the scalar multifunction $\vartheta(x)$ for all sufficiently close to orbit $\mathbf{O}$ points $x :\ \vartheta(x)$ is such that the belonging to $\mathbf{O}$ point $\xi(\vartheta(x))$ is the nearest one to x. Clearly, the vector

$$\delta(x) = x - \xi(\vartheta(x))$$

is a displacement from the orbit which is normal to the orbit $\mathbf{O}$. Our most important aim is an investigation of asymptotic behavior of the displacement $\delta(X(t))$ for the solution $X(t)$ of the considered stochastic system provided $X(0)$ is sufficiently close to $\mathbf{O}$. With that end in view we derive the system (2.2), (2.10) for $\delta(X(t))$, $\vartheta(X(t))$. Then we linearize this system and obtain the linearized orthogonal system (2.11)–(2.12) for $\Delta(t),\ \Theta(t)$, where $\Delta(t)$ corresponds to $\delta(X(t))$ and $\Theta(t)$ corresponds to $\vartheta(X(t))$. Underline that the coefficients of the system (2.11)–(2.12) are found explicitly. Solutions of the linearized system repeat the orthogonal property for $\delta(X(t)),\ \ \vartheta(X(t))$: if $\Delta(t_0)$ is orthogonal to $a_0(\xi(\Theta(t_0)))$ then $\Delta(t)$ is orthogonal to $a_0(\xi(\Theta(t))),\ t \ge t_0$ (for $\delta(X(t)),\ \ \vartheta(X(t))$ this property flows out the very definition of $\delta,\ \vartheta$).

The most important characteristics of asymptotic behavior of Δ are the Lyapunov exponent λ^*, the moment Lyapunov function $g(p)$, and the stability index γ^*. To investigate them, we consider the Khasminskii-type system (3.2)–(3.3) with the invariant compact manifold $\mathbf{D}$. After that we can introduce on $\mathbf{C}(\mathbf{D})$ the strongly continuous semigroup $T_t(p)$ analogously to [1], [4]. The definition (3.8) of the semigroup is connected both with the linearized orthogonal system (2.11)–(2.12) and with the Khasminskii-type system (3.2)–(3.3). But because the equation (3.7) is linear with respect to $|\Delta(t)|^p$, it is not difficult to define the semigroup $T_t(p)$ only in terms of the system (3.2)–(3.3). Underline that the formula (3.9) for the infinitesimal generator of the semigroup $T_t(p)$ is obtained in explicit form. Theorems 3.2 and 3.3 prove the existence of the Lyapunov exponents λ^* and $g(p)$ and give the important formulas for them in the nondegenerate case (3.17). Theorem 3.4 explains meaning of the stability index γ^* in the asymptotic analysis of the linearized orthogonal system. Finally, Theorem 4.1 answers the question about stability of the orbit $\mathbf{O}$ for the input system (1.1). To emphasize the significance of this theorem, let us note that in contrast to the deterministic case when solutions of a nonlinear system and solutions of the corresponding linearized system usually have many common features in their asymptotic behavior, the stochastic case is far intricate. Consider, for example, a possible situation for the system (1.1) when all its solutions are uniformly bounded. Then the limit

$$g_\delta(p) := \lim_{t\to\infty} \frac{1}{t} \ln |\delta(X(t))|^p$$

cannot be positive for any $p > 0$.

At the same time, the moment Lyapunov function $g(p)$ for the linearized orthogonal system (see the formula (3.24)) is usually positive for a sufficiently large $p > 0$ because system (1.1). But the stability index γ^*, which is defined only by the linearized system, repeats the very important properties both of the system (2.2), (2.10) and of (2.11)–(2.12).

We also turn shortly our attention to computational aspects. A use of the formulas (3.22), (3.24) together with the Monte-Carlo evaluation of the mathematical expectations $E\ln|\Delta_{\theta,\delta}(t)|$ and $E|\Delta_{\theta,\delta}(t)|^p$ by virtue of the linearized orthogonal system (2.11)–(2.12) gives one of possible ways. An implementation of such a way requires numerical integration of the system (2.11)–(2.12) on large intervals of time. Because of unboundedness of Δ, such a problem is connected with serious computational difficulties. Apparently, a numerical integration of the Khasminskii-type system (3.2)–(3.3) is more preferable in view of compactness of the manifold $\mathbf{D}$ (we point out that for any p the equation (3.7) is the linear scalar one with respect to $|\Delta(t)|^p$ with coefficients depending only on the solution of the system (3.2)–(3.3)). Clearly, such an approach will require methods of numerical search for solutions which belong to a known invariant manifold.

Another way is analytical. It consists, for example, in a use of the formula (3.20) or of the last part of the formula (3.22). Such a way is effective for systems of not large dimension. It is fully realized for two-dimensional systems in the case of stationary point in [11] and in the case of orbit with vanishing diffusion in [12]. Below we extend this approach to systems with nonvanishing diffusion.

5. **Orbital stability on the plane**

Consider the input system (1.1) in two-dimensional case ($d = 2$). The equations (3.6) in this case define $\Lambda(t)$ in the following way:

$$\Lambda^1(t) = \mp\frac{a_0^2(\xi(\Theta(t)))}{|a_0(\xi(\Theta(t)))|},\ \Lambda^2(t) = \pm\frac{a_0^1(\xi(\Theta(t)))}{|a_0(\xi(\Theta(t)))|}\,, \tag{5.1}$$

i.e., the vector $\Lambda(t)$ is identically determined to within a sign by the values of $\Theta(t)$. For definiteness, let us choose minus for Λ^1 and plus for Λ^2 in the expressions (5.1).

Let us introduce the vector

$$\lambda(\theta) := \frac{1}{|a_0(\xi(\theta))|}\begin{bmatrix} -a_0^2(\xi(\theta)) \\ a_0^1(\xi(\theta)) \end{bmatrix}$$

and the coefficients (see the formulas (3.10) together with the condition $(\theta,\lambda)\in\mathbf{D}$)

$$\gamma_0 = \gamma_0(\theta) := (B_0(\theta)\lambda(\theta),\lambda(\theta)),$$

$$\gamma_r = \gamma_r(\theta) := \beta_r(\theta)\gamma_0(\theta) + (B_r(\theta)\lambda(\theta),\lambda(\theta)),\ r = 1,...,q. \tag{5.2}$$

These coefficients can be simplified in two-dimensional case. In fact, $(a_0(\xi(\theta)),\lambda(\theta)) = \lambda^\top(\theta)a_0(\xi(\theta)) = 0$, and we obtain

$$(a_0(\xi(\theta))a_0^\top(\xi(\theta))A_r(\xi(\theta))\lambda(\theta),\lambda(\theta)) = \lambda^\top(\theta)a_0(\xi(\theta))a_0^\top(\xi(\theta))A_r(\xi(\theta))\lambda(\theta) = 0.$$

Hence (see the formulas (2.14))

$$(B_r(\theta)\lambda(\theta),\lambda(\theta)) = (A_r(\xi(\theta))\lambda(\theta),\lambda(\theta)),\ r = 1,...,q. \tag{5.3}$$

Analogously

$$(B_0(\theta)\lambda(\theta),\lambda(\theta)) = (A_0(\xi(\theta))\lambda(\theta),\lambda(\theta)). \tag{5.4}$$

$$(A_r(\xi(\theta))\lambda(\theta),\lambda(\theta)) = \frac{1}{|a_0|^2}(\frac{\partial a_r}{\partial x^1}a_0^2a_0^2 - \frac{\partial a_r}{\partial x^2}a_0^1a_0^2 - \frac{\partial a_r}{\partial x^1}a_0^1a_0^2 + \frac{\partial a_r}{\partial x^2}a_0^1a_0^1). \tag{5.5}$$

Further, because $a_r(\xi(\theta)) \equiv 0$, $r = 1,...,q$, we have $A_r(\xi(\theta))a_0(\xi(\theta)) \equiv 0$ and, consequently,

$$\frac{1}{|a_0|^2}(A_ra_0,a_0) = \frac{1}{|a_0|^2}(\frac{\partial a_r^1}{\partial x^1}a_0^1a_0^1 + \frac{\partial a_r^1}{\partial x^2}a_0^1a_0^2 + \frac{\partial a_r^2}{\partial x^1}a_0^1a_0^2 + \frac{\partial a_r^2}{\partial x^2}a_0^2a_0^2) = 0, \quad r = 1,...,q. \tag{5.6}$$

From (5.5) and (5.6) we get

$$(A_r(\xi(\theta))\lambda(\theta),\lambda(\theta)) = \mathrm{tr}A_r(\xi(\theta)),\ r = 1,...,q. \tag{5.7}$$

Therefore due to (5.2), (5.3), (5.4), and (5.7)

$$\gamma_0(\theta) = (A_0(\xi(\theta))\lambda(\theta),\lambda(\theta)),\ \gamma_r(\theta) = \beta_r(\theta)\gamma_0(\theta) + \mathrm{tr}A_r(\xi(\theta)),\ r = 1,...,q. \tag{5.8}$$

Adduce also the formula

$$\mathrm{tr}A_r^2(\xi(\theta)) = \mathrm{tr}^2 A_r(\xi(\theta)),\ r = 1,...,q. \tag{5.9}$$

Indeed, the direct calculations give

$$\lambda\lambda^\top + \frac{1}{|a_0|^2}a_0a_0^\top = I\ . \tag{5.10}$$

Using (5.7), (5.10), and the relation $A_r(\xi(\theta))a_0(\xi(\theta)) \equiv 0$, we get

$$\mathrm{tr}^2 A_r(\xi(\theta)) = \lambda^\top A_r\lambda\lambda^\top A_r\lambda = \lambda^\top A_r^2\lambda - \frac{1}{|a_0|^2}\lambda^\top A_ra_0a_0^\top A_r\lambda = \lambda^\top A_r^2\lambda.$$

Further, we can show that $\lambda^\top A_r^2\lambda = \mathrm{tr}A_r^2$ in the same way as (5.7) was obtained because $A_r^2(\xi(\theta))a_0(\xi(\theta)) \equiv 0$. Thus the formula (5.9) is proved.

We have the following system for two scalar variables $\Theta(t)$ and $|\Delta(t)|^p$:

$$d\Theta = dt + \sum_{r=1}^q \beta_r(\Theta)\circ dw_r(t),\ \Theta(0) = \theta, \tag{5.11}$$

$$d|\Delta(t)|^p = p\gamma_0(\Theta)|\Delta(t)|^p dt + p\sum_{r=1}^q \gamma_r(\Theta)\cdot|\Delta(t)|^p\circ dw_r(t), \quad |\Delta(0)|^p = 1. \tag{5.12}$$

The strongly continuous semigroup $T_t(p)$ on $\mathbf{C}(\mathbf{O})$ is defined by the formula

$$T_t(p)f(\theta) = Ef(\Theta_\theta(t))|\Delta(t)|^p,\ |\Delta(0)|^p = 1,\ f\in\mathbf{C}(\mathbf{O}). \tag{5.13}$$

The system (5.11)-(5.12) has the following Ito form:

$$d\Theta = (1 + \frac{1}{2}\sum_{r=1}^q \beta_r'(\Theta)\beta_r(\Theta))dt + \sum_{r=1}^q \beta_r(\Theta)dw_r(t) \tag{5.14}$$

$$d|\Delta(t)|^p = p(\gamma_0(\Theta) + \frac{1}{2}\sum_{r=1}^q \gamma_r'(\Theta)\beta_r(\Theta) + \frac{1}{2}p\sum_{r=1}^q \gamma_r^2(\Theta))\cdot|\Delta(t)|^p dt$$

$$+p\sum_{r=1}^q \gamma_r(\Theta)\cdot|\Delta(t)|^p dw_r(t),\ |\Delta(0)|^p = 1. \tag{5.15}$$

$$A(p)f(\theta) = \frac{1}{2}\sum_{r=1}^{q}\beta_r^2(\theta)\cdot f''(\theta) + (1 + \frac{1}{2}\sum_{r=1}^{q}\beta_r'(\theta)\beta_r(\theta) + p\sum_{r=1}^{q}\gamma_r(\theta)\beta_r(\theta))\cdot f'(\theta)$$

$$+p(\gamma_0(\theta) + \frac{1}{2}\sum_{r=1}^{q}\gamma_r'(\theta)\beta_r(\theta) + \frac{1}{2}p\sum_{r=1}^{q}\gamma_r^2(\theta))\cdot f(\theta)$$

$$:= \frac{1}{2}k^2(\theta)\cdot f''(\theta) + b(\theta;p)\cdot f'(\theta) + c(\theta;p)\cdot f(\theta). \qquad (5.16)$$

Clearly, all the coefficients $k^2(\theta)$, $b(\theta;p)$, $c(\theta;p)$, which are defined by the relation (5.16), are T-periodic functions with respect to θ.

As it was mentioned, the case of vanishing diffusion on the very orbit has been considered in [12]. Remember the main formulas in this case (to avoid a confusion let us note that in [12] the input system was considered in the Ito sense). Because $\alpha_r(x) \equiv 0$, $r = 1,...,q$, we have $\beta_r = 0$, $r = 1,...,q$, and the Khasminskii system becomes extremely simple

$$d\Theta = dt,$$

i.e., Θ is deterministic: $\Theta_\theta(t) = \theta + t$.

The equation (5.15) acquires the form

$$d|\Delta(t)|^p = (p\gamma_0(\theta+t) + \frac{1}{2}p^2\sum_{r=1}^{q}\gamma_r^2(\theta+t))\cdot|\Delta(t)|^p dt$$

$$+p\sum_{r=1}^{q}\gamma_r(\theta+t)\cdot|\Delta(t)|^p dw_r(t),\ |\Delta(0)|^p = 1. \qquad (5.17)$$

Hence the semigroup $T_t(p)$ is defined by the formula

$$T_t(p)f(\theta) = Ef(\Theta_\theta(t))|\Delta(t)|^p = f(\theta+t)E|\Delta(t)|^p$$

$$= f(\theta+t)\exp\left\{\int_0^t(p\gamma_0(\theta+s) + \frac{1}{2}p^2\sum_{r=1}^{q}\gamma_r^2(\theta+s))ds\right\},\quad f\in\mathbf{C}(\mathbf{O}), \qquad (5.18)$$

and its generator $A(p)$ has the form

$$A(p)f(\theta) = f'(\theta) + (p\gamma_0(\theta) + \frac{1}{2}p^2\sum_{r=1}^{q}\gamma_r^2(\theta))f(\theta),\quad f\in\mathbf{C}(\mathbf{O}). \qquad (5.19)$$

Due to the formulas (5.8), and the relations $\beta_r = 0$, $r = 1,...,q$, we have

$$\gamma_0(\theta) = (A_0(\xi(\theta))\lambda(\theta),\lambda(\theta)),\quad \gamma_r(\theta) = \mathrm{tr}A_r(\xi(\theta)),\ r = 1,...,q.$$

From the equation

$$A(p)e_p(\theta) = g(p)e_p(\theta),\ e_p\in\mathbf{C}(\mathbf{O}),\ e_p(\theta) > 0,\ 0\le\theta<T,$$

we obtain the eigenfunction

$$e_p(\theta) = \exp\left\{g(p)\theta - \int_0^\theta(p\gamma_0(s) + \frac{1}{2}p^2\sum_{r=1}^{q}\gamma_r^2(s))ds\right\},$$

where the eigenvalue $g(p)$ is equal to

$$g(p) = \frac{1}{2T}\int_0^T\sum_{r=1}^{q}\gamma_r^2(s)ds\cdot p^2 + \frac{1}{T}\int_0^T\gamma_0(s)ds\cdot p\ . \qquad (5.20)$$

$$\int_0^T (A_0(\xi(s))\lambda(s),\lambda(s))ds = \int_0^T \mathrm{tr}A_0(\xi(s))ds. \tag{5.21}$$

We have (for the sake of simplicity we omit the argument $\xi(s)$ in writing)

$$\int_0^T (A_0(\xi(s))\lambda(s),\lambda(s))ds = \int_0^T \frac{1}{|a_0|^2}(\frac{\partial a_0^1}{\partial x^1}a_0^2a_0^2 - \frac{\partial a_0^1}{\partial x^2}a_0^1a_0^2 - \frac{\partial a_0^2}{\partial x^1}a_0^1a_0^2 + \frac{\partial a_0^2}{\partial x^2}a_0^1a_0^1)ds.$$

Further, due to periodicity of the considered functions we get

$$0 = \frac{1}{2}\int_0^T d\ln[(a_0^1(\xi(s))^2 + (a_0^2(\xi(s))^2]ds$$

$$= \int_0^T \frac{1}{|a_0|^2}(\frac{\partial a_0^1}{\partial x^1}a_0^1a_0^1 + \frac{\partial a_0^1}{\partial x^2}a_0^1a_0^2 + \frac{\partial a_0^2}{\partial x^1}a_0^1a_0^2 + \frac{\partial a_0^2}{\partial x^2}a_0^2a_0^2)ds.$$

Summarizing these two equalities, we obtain (5.21).

Therefore

$$\lambda^* = g'(0) = \frac{1}{T}\int_0^T \gamma_0(s)ds = \frac{1}{T}\int_0^T \mathrm{tr}A_0(\xi(s))ds.$$

The condition

$$\int_0^T \mathrm{tr}A_0(\xi(s))ds < 0$$

is a sufficient condition of orbital stability for deterministic systems in two-dimensional case (the Poincare criterion). Thus, the noise in the sense of Stratonovich does not make worse stability properties of a system with respect to the Lyapunov exponent λ^* of the linearized orthogonal system.

If $\int_0^T \sum_{r=1}^q \mathrm{tr}^2 A_r(\xi(s))ds \neq 0$, $\int_0^T \mathrm{tr}A_0(\xi(s))ds \neq 0$, then the stability index is equal to

$$\gamma^* = -2 \cdot \frac{\int_0^T \mathrm{tr}A_0(\xi(s))ds}{\int_0^T \sum_{r=1}^q \mathrm{tr}^2 A_r(\xi(s))ds} \neq 0\ . \tag{5.22}$$

So, all the characteristics in two-dimensional case with vanishing diffusion on the invariant orbit can be evaluated in explicit form.

In connection with Remark 3.1 we can note that as it obviously follows from the formula (5.18), any operator $T_t(p),\ 0 < t < \infty,\ -\infty < p < \infty$, is noncompact and, for instance, for $t_k = kT,\ k = 0, 1, \ldots$, the operator $T_{t_k}(p)$ is not irreducible. We also note that the spectrum $\sigma(A(p))$ consists of the eigenvalues $g(p) + 2\pi ik/T,\ k = 0, \pm1, \pm2, \ldots$.

Let us turn to the case of nonvanishing diffusion on the invariant orbit. In what follows the nondegeneracy condition

$$k^2(\theta) = \sum_{r=1}^q \beta_r^2(\theta) = \sum_{r=1}^q \alpha_r^2(\xi(\theta)) \neq 0 \text{ for any } -\infty < \theta < \infty \tag{5.23}$$

is supposed to be fulfilled.

Clearly, under the nondegeneracy condition (5.23) the process $\Theta(t)$ defined by the equation (5.14) is ergodic, and the equation for the density $\mu(\theta)$ of the invariant measure has the form

$$\frac{1}{2}(k^2(\theta)\mu)'' - ((1 + \frac{1}{2}\sum_{r=1}^q \beta_r'(\theta)\beta_r(\theta))\mu)' = 0,\ \mu(0) = \mu(T),\ \int_0^T \mu(\theta)d\theta = 1, \tag{5.24}$$

$$\mu(\theta) = C[1 + \frac{\cdots}{\int_0^T b(s)ds}\int_0^{\cdots} b(s)ds]\cdot(k^2(\theta)b(\theta))^{\cdots},$$

where

$$b(\theta) = \exp\left\{-2\int_0^\theta \frac{1+\frac{1}{2}\sum_{r=1}^q \beta_r'(s)\beta_r(s)}{k^2(s)}ds\right\}$$

and the constant C has to be found in accord with the second condition from (5.24).

Due to (3.22) the Lyapunov exponent λ^* (as in [9]) can be found explicitly

$$\lambda^* = \int_0^T (\gamma_0(\theta) + \frac{1}{2}\sum_{r=1}^q \gamma_r'(\theta)\beta_r(\theta))\cdot\mu(\theta)d\theta, \tag{5.25}$$

where $\gamma_r(\theta),\ r = 0, 1, ..., q$, are from (5.8).

One can take advantage of the results [11] for search for the Lyapunov moment function $g(p)$. The paper [11] is devoted to Lyapunov exponents of stationary points. But the offered there methods are connected with a boundary value problem for a second order deterministic linear differential equation. Here we have the problem (see Theorem 2.3 and (5.16))

$$A(p)f(\theta) \equiv \frac{1}{2}k^2(\theta)\cdot f''(\theta) + b(\theta;p)\cdot f'(\theta) + c(\theta;p)\cdot f(\theta) = g(p)f(\theta),$$

$$f(0) = f(T),\ f'(0) = f'(T),\ f(\theta) > 0,\ 0 \le \theta < T, \tag{5.26}$$

which is similar to the considered one in [11].

Let us give the main algorithm of solution of the problem (5.26) (proofs and more details see in [11]). To this end, introduce another boundary value problem on $[-T, T]$

$$A(p)y - \nu y = 0, \tag{5.27}$$

$$y(-T;p,\nu) = 1,\ y(T;p,\nu) = 1. \tag{5.28}$$

Let $\nu_0 = \nu_0(p)$ be the maximal eigenvalue for Sturm-Liouville's problem

$$A(p)y - \nu y = 0,\ y(-T;p) = y(T;p) = 0. \tag{5.29}$$

We note that $\nu_0(p) < \max_{0\le\theta\le T} c(\theta;p)$. For all $\nu > \nu_0$ solutions of the equation (5.27) are non oscillating on $[-T, T]$, and therefore the solution $y(\theta;p,\nu)$ of the problem (5.27)-(5.28) exists and is unique. It can be found in the following way. Let $y_1(\theta;p,\nu)$, $y_2(\theta;p,\nu)$ be the solutions of (5.27) with the initial data

$$y_1(-T;p,\nu) = 0,\ y_1'(-T;p,\nu) = 1,$$

$$y_2(T;p,\nu) = 0,\ y_2'(T;p,\nu) = -1.$$

It is clear (of course, we suppose $\nu > \nu_0$) that $y_1(\theta;p,\nu) > 0$ on $(-T, T]$ and $y_2(\theta;p,\nu) > 0$ on $[-T, T)$. Let us note in passing that if $y_1(\theta;p,\nu) > 0$ on $(-T, T]$ or $y_2(\theta;p,\nu) > 0$ on $[-T, T)$ for some ν, then $\nu > \nu_0$.

The solution $y(\theta;p,\nu)$ of (5.27)-(5.28)) is evidently expressed in the form

$$y(\theta;p,\nu) = \frac{y_1(\theta;p,\nu)}{y_1(T;p,\nu)} + \frac{y_2(\theta;p,\nu)}{y_2(-T;p,\nu)}\ . \tag{5.30}$$

Proposition 5.1. *The function $y(\theta;p,\nu)$ for any $-T < \theta < T$ and $p \in \mathbf{R}$ is a strongly monotonically decreasing convex function with respect to ν for $\nu > \nu_0(p)$, and the following relations* *are true.*

Proposition 5.2. *The eigenvalue $g(p)$ of the problem* (5.26) *is a root of the equation*

$$y(0;p,\nu) \equiv \frac{y_1(0;p,\nu)}{y_1(T;p,\nu)} + \frac{y_2(0;p,\nu)}{y_2(-T;p,\nu)} = 1, \tag{5.32}$$

$\nu_0(p) < g(p) < \infty$, *and the eigenfunction $f(\theta;p)$ is equal to*

$$f(\theta;p) = y(\theta;p,g(p)) = \frac{y_1(\theta;p,g(p))}{y_1(T;p,g(p))} + \frac{y_2(\theta;p,g(p))}{y_2(-T;p,g(p))}, \quad 0 \le \theta \le T. \tag{5.33}$$

Thanks to Propositions 5.1 and 5.2, the problem of evaluating $g(p)$ and $f(\theta;p)$ is sufficiently simple under any fixed p. In [11] several efficient numerical methods (and among them the Newton method) are obtained for searching for both $g(p)$ and $g'(p)$. Thus the evaluation of moment Lyapunov exponents becomes reliable and effective matter for orbits on the plane in the nondegenerate case (5.23).

6. Stability of orbits on the plane under small diffusion

Consider the two-dimensional perturbed Hamilton system with respect to $x = (x^1, x^2)$

$$dx^1 = -\frac{\partial H}{\partial x^2}dt + c_0^1(x)\cdot(H-C)dt + \sum_{r=1}^{q}(-\alpha_r(x)\frac{\partial H}{\partial x^2} + c_r^1(x)\cdot(H-C))\circ dw_r(t)$$

$$dx^2 = \frac{\partial H}{\partial x^1}dt + c_0^2(x)\cdot(H-C)dt + \sum_{r=1}^{q}(\alpha_r(x)\frac{\partial H}{\partial x^1} + c_r^2(x)\cdot(H-C))\circ dw_r(t). \tag{6.1}$$

Let $\mathbf{O} : H(p,q) = C$, where C is a constant, be the orbit of the Hamilton system

$$\frac{dx^1}{dt} = -\frac{\partial H}{\partial x^2},\ \frac{dx^2}{dt} = \frac{\partial H}{\partial x^1}.$$

Then the orbit $\mathbf{O}$ is invariant for the system (6.1). The noise in the system (6.1) is subdivided in two parts: the first one acts lengthwise to the directional field of the Hamilton system, and the second one vanishes on the orbit $\mathbf{O}$. Besides, the deterministic perturbations are present in the system (6.1). They are small nearby the orbit $\mathbf{O}$ and vanish on it. Let us note in passing that the $(2d-1)$-dimensional manifold $H = C$ is invariant for the $2d$-dimensional system of the form (6.1).

In what follows we restrict ourselves to the case $H = \frac{1}{2}(x^{1^2} + x^{2^2}) = \frac{1}{2}|x|^2$ and $c_0^1(x) = 0,\ c_0^2(x) = 0$. For convenience put $C = \frac{1}{2}\rho^2$. We come to the system of the form

$$dx^1 = -x^2dt + \sum_{r=1}^{q}(-\alpha_r\cdot x^2 + \frac{c_r^1}{2}\cdot(|x|^2-\rho^2))\circ dw_r(t)$$

$$dx^2 = x^1dt + \sum_{r=1}^{q}(\alpha_r\cdot x^1 + \frac{c_r^2}{2}\cdot(|x|^2-\rho^2))\circ dw_r(t), \tag{6.2}$$

where $c_r^i(x),\ i = 1,2;\ r = 0,1,\dots,q$, and $\alpha_r(x),\ r = 1,\dots,q$, are some scalar functions of x.

$$\mathbf{O}: x^1 = \xi^1(\theta) = \rho\cos\theta,\ x^2 = \xi^2(\theta) = \rho\sin\theta,\ 0 \le \theta < 2\pi,$$

$$\beta_r(\theta) = \alpha_r(\xi(\theta)) = \alpha_r(\rho\cos\theta,\ \rho\sin\theta),\ r = 1, ..., q, \tag{6.3}$$

$$a_0(\xi(\theta)) = \begin{bmatrix} -\rho\sin\theta \\ \rho\cos\theta \end{bmatrix},\ \lambda(\theta) = \begin{bmatrix} -\cos\theta \\ -\sin\theta \end{bmatrix},$$

$$A_0(\xi(\theta)) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ A_r(\xi(\theta)) = \begin{bmatrix} c_r^1(\xi(\theta))\cdot\rho\cos\theta & c_r^1(\xi(\theta))\cdot\rho\sin\theta \\ c_r^2(\xi(\theta))\cdot\rho\cos\theta & c_r^2(\xi(\theta))\cdot\rho\sin\theta \end{bmatrix},\ r = 1, ..., q,$$

$$\gamma_0(\theta) = 0,\ \gamma_r(\theta) = \rho\cdot(c_r^1(\xi(\theta))\cdot\cos\theta + c_r^2(\xi(\theta))\cdot\sin\theta),\ r = 1, ..., q. \tag{6.4}$$

If for every $r = 1, ..., q$ either $\gamma_r'(\theta) = 0$ or $\beta_r(\theta) = 0$, then the equation (5.15) acquires the form

$$d|\Delta(t)|^p = \frac{1}{2}p^2\sum_{r=1}^{q}\gamma_r^2(\Theta)\cdot|\Delta(t)|^p dt + p\sum_{r=1}^{q}\gamma_r(\Theta)\cdot|\Delta(t)|^p dw_r(t),\ |\Delta(0)|^p = 1.$$

From here

$$E|\Delta(t)|^p \ge 1,\ t \ge 0,$$

and consequently

$$g(p) = \lim_{t\to\infty}\frac{1}{t}\ln E|\Delta(t)|^p \ge 0,$$

i.e., in particular, the orbit $\mathbf{O}$ of the system (6.2) cannot be stabilized by noise if the every noise either vanishes on the orbit or acts only lengthwise to the directional field of the system

$$\frac{dx^1}{dt} = -x^2,\ \frac{dx^2}{dt} = x^1.$$

To investigate the possibility of stabilization by noise, consider the following system with small noise (we put in (6.2) $q = 1,\ \alpha_1(x^1, x^2) = \sqrt{\varepsilon}(\alpha_0 + \frac{\alpha}{\rho}x^1 + \frac{\beta}{\rho}x^2)$, where $\alpha,\ \beta,\ \alpha_0$ are some constants, $c_1^1 = const = \sqrt{\varepsilon}\frac{a}{\rho},\ c_1^2 = const = \sqrt{\varepsilon}\frac{b}{\rho}$)

$$dx^1 = -x^2dt + \sqrt{\varepsilon}[-(\alpha_0 + \frac{\alpha}{\rho}x^1 + \frac{\beta}{\rho}x^2)\cdot x^2 + \frac{a}{2\rho}\cdot(|x|^2 - \rho^2)] \circ dw(t)$$

$$dx^2 = x^1dt + \sqrt{\varepsilon}[(\alpha_0 + \frac{\alpha}{\rho}x^1 + \frac{\beta}{\rho}x^2)\cdot x^1 + \frac{b}{2\rho}\cdot(|x|^2 - \rho^2)] \circ dw(t). \tag{6.5}$$

Due to the formulas (5.16), (6.3), and (6.4), the boundary value problem (5.26) is of the form

$$(L_1 + \varepsilon L_2)f := \frac{1}{2}\varepsilon\beta^2(\theta)f'' + [1 + \frac{1}{2}\varepsilon\beta(\theta)\beta'(\theta) + \varepsilon p\beta(\theta)\gamma(\theta)]f'$$

$$+\frac{1}{2}\varepsilon[p\beta(\theta)\gamma'(\theta) + p^2\gamma^2(\theta)]f = g(p)f, \tag{6.6}$$

$$f(0) = f(2\pi) = 1,\ f'(0) = f'(2\pi),\ f(\theta) > 0,\ 0 \le \theta < 2\pi, \tag{6.7}$$

$$L_1 = \frac{d}{d\theta}\,,\ L_2 = \frac{1}{2}\beta^2(\theta)\frac{d^2}{d\theta^2} + [\frac{1}{2}\beta(\theta)\beta'(\theta) + p\beta(\theta)\gamma(\theta)]\frac{d}{d\theta} + \frac{1}{2}[p\beta(\theta)\gamma'(\theta) + p^2\gamma^2(\theta)].$$

We suppose that $\alpha_0^2 > \alpha^2 + \beta^2$, whence the nondegeneracy condition (5.23) runs out.

Let us use the formula (5.25). We have

$$\lambda^*(\varepsilon) = \frac{\varepsilon}{2}\int_0^{2\pi} \gamma'(\theta)\beta(\theta)\cdot\mu(\theta;\varepsilon)d\theta, \qquad (6.8)$$

where $\mu(\theta;\varepsilon)$ is the solution of the following problem (see (5.24))

$$\frac{1}{2}\varepsilon(\beta^2(\theta)\mu)'' - ((1+\frac{\varepsilon}{2}\beta'(\theta)\beta(\theta))\mu)' = 0,\ \mu(0;\varepsilon) = \mu(2\pi;\varepsilon),\ \int_0^{2\pi}\mu(\theta;\varepsilon)d\theta = 1. \qquad (6.9)$$

One can prove that

$$\lambda^*(\varepsilon) = \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots + \varepsilon^n\lambda_n + \varepsilon^{n+1}r_n(\varepsilon),\ |r_n(\varepsilon)| \le C_n,$$

where $\lambda_1, ..., \lambda_n, C_n$ are some constants. Moreover, one can adduce a procedure for finding these constants. The proof and the procedure are analogous to the ones from [5] (see also [13]).

Here we restrict ourselves to proving the following result.

Proposition 6.1. *Let $\alpha_0^2 > \alpha^2 + \beta^2$ and let*

$$0 < B_0 \le \beta^2(\theta) \le B_1,\ \frac{1}{2}|\beta'(\theta)\beta(\theta)| \le K, \qquad (6.10)$$

(*clearly, the constants $B_0,\ B_1,\ K$ can be indicated explicitly for $\beta(\theta) = \alpha_0 + \alpha\cos\theta + \beta\sin\theta$*).

Then under $\varepsilon \le 1/2K$

$$\mu(\theta;\varepsilon) = \frac{1}{2\pi} + \varepsilon\nu(\theta;\varepsilon),\ |\nu(\theta;\varepsilon)| \le \frac{4B_1K}{\pi B_0}, \qquad (6.11)$$

and

$$\lambda^*(\varepsilon) = \frac{\varepsilon}{4}(\alpha b - \beta a) + \varepsilon^2 r_1(\varepsilon),\ |r_1(\varepsilon)| \le \frac{4B_1K}{B_0}\cdot\sqrt{B_1(a^2+b^2)}. \qquad (6.12)$$

Proof. From (6.9) we have

$$\frac{1}{2}\varepsilon(\beta^2\mu)' - ((1+\frac{\varepsilon}{2}\beta'\beta)\mu) = A(\varepsilon).$$

Integrating this equality from zero to 2π, we get

$$-1 + \frac{\varepsilon}{2}\int_0^{2\pi}\beta'\beta\mu d\theta = 2\pi A(\varepsilon).$$

Because of (6.10) and the last condition of (6.9), we obtain for $1 + 2\pi A(\varepsilon) := \varepsilon C(\varepsilon)$

$$\varepsilon|C(\varepsilon)| = \frac{\varepsilon}{2}|\int_0^{2\pi}\beta'\beta\mu d\theta| \le \varepsilon K. \qquad (6.13)$$

Introduce the new function $\nu(\theta;\varepsilon)$ according to the equality

$$\mu(\theta;\varepsilon) = \frac{1}{2\pi} + \varepsilon\nu(\theta;\varepsilon).$$

where

$$\xi(\theta;\varepsilon)=\frac{2-\varepsilon\beta'(\theta)\beta(\theta)}{\varepsilon\beta^2(\theta)},\ \eta(\theta;\varepsilon)=\frac{2C(\varepsilon)-\beta'(\theta)\beta(\theta)}{2\varepsilon\pi\beta^2(\theta)}\ . \tag{6.15}$$

The problem (6.14) has the following solution

$$\nu(\theta;\varepsilon)=\frac{\exp(\int_0^{2\pi}\xi(s;\varepsilon)ds)}{1-\exp(\int_0^{2\pi}\xi(s;\varepsilon)ds)}\cdot\int_\theta^{2\pi}\exp(-\int_\theta^\varphi\xi(s;\varepsilon)ds)\eta(\varphi;\varepsilon)d\varphi$$

$$+\frac{\exp(\int_0^{\theta}\xi(s;\varepsilon)ds)}{1-\exp(\int_0^{2\pi}\xi(s;\varepsilon)ds)}\cdot\int_0^{\theta}\exp(-\int_0^\varphi\xi(s;\varepsilon)ds)\eta(\varphi;\varepsilon)d\varphi. \tag{6.16}$$

We have (see (6.15), (6.13), and (6.10))

$$|\eta|\le\frac{2K}{\pi\varepsilon B_0},\ \frac{1}{\xi(\varphi;\varepsilon)}\le\frac{\varepsilon B_1}{2-\varepsilon\cdot 2K}\ ,\ \varepsilon<\frac{1}{K}\ .$$

Therefore

$$|\int_\theta^{2\pi}\exp(-\int_\theta^\varphi\xi(s;\varepsilon)ds)\eta(\varphi;\varepsilon)d\varphi|\le\frac{2K}{\pi\varepsilon B_0}\cdot\int_\theta^{2\pi}\exp(-\int_\theta^\varphi\xi(s;\varepsilon)ds)d\varphi$$

$$=\frac{2K}{\pi\varepsilon B_0}\cdot\int_\theta^{2\pi}\frac{1}{\xi(\varphi;\varepsilon)}\exp(-\int_\theta^\varphi\xi(s;\varepsilon)ds)d(\int_\theta^\varphi\xi(s;\varepsilon)ds)$$

$$\le\frac{2K}{\pi\varepsilon B_0}\cdot\frac{\varepsilon B_1}{2(1-\varepsilon K)}\cdot(1-\exp(-\int_0^{2\pi}\xi(s;\varepsilon)ds)),$$

and consequently, the modulus of the first term in (6.16) is bounded from above by the number $\dfrac{KB_1}{\pi B_0(1-\varepsilon K)}$. The second term has the same bound. Hence the relation (6.11) is proved. The relations (6.12) easily follow from (6.8) and (6.11). Proposition 6.1 is proved.

Clearly, both g and f in (6.6)-(6.7) depend on p,ε : $g=g(p,\varepsilon)$, $f=f(\theta;p,\varepsilon)$. Let us give a procedure of asymptotic series expansion for $g(p,\varepsilon)$ and $f(\theta;p,\varepsilon)$. This procedure coincides with that one which is proposed in [10] for the moment Lyapunov exponent in the case of stationary points. After substituting the formal expressions

$$g(p,\varepsilon)=g_0(p)+\varepsilon g_1(p)+\cdots+\varepsilon^n g_n(p)+\cdots$$

$$f(\theta;p,\varepsilon)=f_0(\theta;p)+\varepsilon f_1(\theta;p)+\cdots+\varepsilon^n f_n(\theta;p)+\cdots$$

in (6.6), we obtain the following relations for $g_0,\ g_1, ...,\ g_n, ...$ and for 2π-periodic in θ functions $f_0,\ f_1, ...,\ f_n, ...$:

$$L_1f_0=g_0f_0,\ f_0(0;p)=f_0(2\pi;p)=1, \tag{6.17}$$

$$L_1f_1+L_2f_0=g_0f_1+g_1f_0,\ f_1(0;p)=f_1(2\pi;p)=0, \tag{6.18}$$

. . . .

$$L_1f_n+L_2f_{n-1}=g_0f_n+g_1f_{n-1}+\cdots+g_nf_0,\ f_n(0;p)=f_n(2\pi;p)=0. \tag{6.19}$$

$$g_0(p) = 0, \;\; f_0(\theta; p) = 1. \tag{6.20}$$

Let $g_0, \; g_1, ..., \; g_{n-1}$ and 2π-periodic in θ functions $f_0, \; f_1, ..., \; f_{n-1}$ be found. Due to (6.20) the equation (6.19) acquires the form

$$\frac{df_n}{d\theta} = -L_2 f_{n-1} + g_1 f_{n-1} + \cdots + g_{n-1} f_1 + g_n.$$

The function $-L_2 f_{n-1} + g_1 f_{n-1} + \cdots + g_{n-1} f_1$ is known and is evidently 2π-periodic in θ. The function f_n can be 2π-periodic if and only if

$$g_n = \frac{1}{2\pi} \int_0^{2\pi} (L_2 f_{n-1} - g_1 f_{n-1} - \cdots - g_{n-1} f_1) d\theta. \tag{6.21}$$

Provided (6.21)

$$f_n(\theta; p) = \int_0^{\theta} (-L_2 f_{n-1}(s; p) + g_1(p) f_{n-1}(s; p) + \cdots + g_{n-1}(p) f_1(s; p) + g_n(p)) ds.$$

Let us note that for any $f_n(\theta; p)$ the second condition in (6.7) $f'_n(0; p) = f'_n(2\pi; p)$ is also fulfilled. Thus, the formal asymptotic series expansions for $g(p, \varepsilon)$ and $f(\theta; p, \varepsilon)$ are obtained in the constructive manner.

The following theorem can be proved analogously to [10].

Proposition 6.2. *Let $\alpha_0^2 > \alpha^2 + \beta^2$. Let $g_0(p), ..., g_n(p)$ and $f_0(\theta; p), ..., f_n(\theta; p)$ be the functions obtained from the recursive procedure* (6.17)-(6.19). *Then for any $n > 0$*

$$g(p, \varepsilon) = g_0(p) + \varepsilon g_1(p) + \cdots + \varepsilon^n g_n(p) + O(\varepsilon^{n+1}), \tag{6.22}$$

where $O(\varepsilon^{n+1})$ is uniform with respect to $p \in B$, $B \subset \mathbf{R}$ is any bounded set.

The zero terms have already been found : $g_0(p) = 0, \; f_0(\theta; p) = 1$. From (6.21) we get

$$g_1(p) = \frac{1}{2\pi} \int_0^{2\pi} L_2 f_0 d\theta = \frac{1}{4} p(\alpha b - \beta a) + \frac{1}{4} p^2 (a^2 + b^2),$$

and consequently,

$$g(p, \varepsilon) = \frac{\varepsilon}{4} p(\alpha b - \beta a) + \frac{\varepsilon}{4} p^2 (a^2 + b^2) + O(\varepsilon^2).$$

The following formulas for the Lyapunov exponent and for the stability index can be proved analogously to [10]:

$$\lambda^*(\varepsilon) = \frac{\varepsilon}{4} (\alpha b - \beta a) + O(\varepsilon^2), \;\; \gamma^*(\varepsilon) = -\frac{\alpha b - \beta a}{a^2 + b^2} + O(\varepsilon).$$

Thus, the sufficient condition for stabilizing the orbit $|x|^2 = \rho^2$ of the system (6.5) by small noise is the fulfillment of the following inequality

$$\alpha b - \beta a < 0.$$

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