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Fast periodic oscillations in singularly perturbed relay control systems and sliding modes

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Abstract

As a mathematical model of chattering in the small neighbourhood of switching surface in the sliding mode systems we examine the singularly perturbed relay control systems (SPRCS). The sufficient conditions for existence of fast periodic solutions in such systems are found. Their stability is investigated. It is proved that the slow motions in such SPRCS are approximately described with equations obtained from the equations for the slow variables of SPRCS by averaging along fast periodic motions. It is shown that in the case when the original SPRCS contains the relay control linearly the averaged equations and equations which describe the motions of the reduced system in the sliding mode are coincide. The algorithm is proposed which allows to solve the problem of eigenvalues assignment for averaged equations using the additional dynamics of fast actuator .

Introduction

The chattering phenomena is one of the actual problems in modern sliding mode control theory. The presence of actuators and measuring devices is one of basic reasons of chattering in sliding mode control systems ([17], [4]).

The behavior of such systems is described by the singularly perturbed relay control systems (SPRCS). Moreover, for such systems the conditions of dynamic uncertainty are held. This means that for original SPRCS there are no stable first order sliding modes, but for the reduced system the sufficient conditions for existence of the stable first order sliding mode are held ([4], [3] ,[8]).

If in the original SPRCS contains either a sliding mode of 3rd order and greater or positive feedback, then sliding modes are unstable [1],[9]. In such systems the fast periodic oscillations can occur [17],[8].

The general model of sliding mode control system with fast actuators and measuring devices has the following form (see [4])

$$\mu dz/dt = g(z, s, x, u(s)), \tag{0.1}$$

$$ds/dt = h_1(z, s, x, u(s)), \quad dx/dt = h_2(z, s, x, u(s))$$

where $z \in \mathbf{R}^m u(s) = sgn(s), g, h_1, h_2$ are smooth functions of their arguments.

Conditions of dynamic uncertainty for system (0.1) mean that letting $\mu = 0$ and expressing $z_0 = \varphi(s, x, u(s))$ from the equation

$$g(z_0, s, x, u(s)) = 0$$

according to the formula $z_0 = \varphi(s, x, u(s))$ we obtain the reduced system

$$ds/dt = h_1(\varphi(s, x, u(s)), s, x, u(s)) = H_1(s, x, u(s)), \tag{0.2}$$

$$dx/dt = h_2(\varphi(s, x, u(s)), s, x, u(s)) = H_2(s, x, u(s)).$$

It is assumed that

(i) almost everywhere on $s = 0$

$$h_1(z, 0, x, 1)h_1(z, 0, x, -1) > 0 \tag{0.3}$$

or

$$h_1(z, 0, x, 1) > 0,\ h_1(z, 0, x, -1) < 0;$$

(ii) the measure of domain

$$S = \{x : H_1(0, x, 1) < 0,\ H_1(0, x, -1) > 0, x \in \mathrm{R}^n\}$$

is nonzero and consequently S is the domain of stable first order sliding for system (0.2).

System (0.1) under such suppositions can describe for instance the behavior of control systems in which variables x, s describe plant behavior, vector z describes the behavior of the fast actuator.

Proposed paper is devoted to the investigation of fast periodic solutions in system (0.1). The paper consists of 3 section. Section 1 is devoted to development of mathematical apparatus for the investigation of periodic solution of SPRCS. In section 2 this apparatus is used for the investigation of behavior of the sliding mode control systems with fast actuators. In section 3 proposed approach is used for the design of desired averaged equation in sliding mode control system with fast actuators.

1. Mathematical apparatus

1.1 Problem Formulation

In this section we will consider the existence and stability of the fast periodic solutions for the singularly perturbed relay control system of the form

$$\mu dz/dt = g(z, \xi, x, u(\xi)), \tag{1.1}$$

$$\mu d\xi/dt = h_1(z, \xi, x, u(\xi)),\ dx/dt = h_2(z, \xi, x, u(\xi)),$$

where $z \in \mathrm{R}^m, \xi \in u(\xi) = sign\,(\xi), g, h_1, h_2$ are smooth functions of their arguments.

Introducing the "fast time" $\tau = t/\mu$ into (1.1), we will obtain

$$dz/d\tau = g(z, \xi, x, u(\xi)), \tag{1.2}$$

$$d\xi/d\tau = h_1(z, \xi, x, u(\xi)), \quad dx/d\tau = \mu h_2(z, \xi, x, u(\xi)).$$

For the smooth singularly perturbed system the existence and stability in the first approximation of the fast periodic solution was investigated by [13]. The existence and stability in the first approximation of fast periodic solution of (1.1) was investigated in [8].

It turns out that for the investigation of the fast periodic solutions of singularly perturbed system (1.2) it's impossible to use standard methods of small parameter

[7] for autonomous systems because setting $\mu = 0$ in (1.2) we will obtain degenerate equation for the slow variables x.

In this paper we develop the mathematical apparatus for investigation of the fast periodic oscillations (1.1),(1.2). For this end we employ the point mapping method (see [11],[12]). In section 1.2 the specific features of the point mapping which generated by system (1.2) is investigated. In section 1.3 the theorem about existence of fast periodic solution of system (1.1) is proved. A proof of the theorem about investigation of stability of this periodic solutio in the first approximation is given in section 1.4. In section 1.5 the auxiliary theorems about decomposition of two speed point mapping are formulated. The theorem about averaging is given in section 1.6. Section 1.7 is devoted to the algorithm of asymptotic representation for the fast periodic solution of system (1.1). The algorithm for correction of the averaged equation is suggested in section 1.8. The reduction principle theorem is given in section 1.9.

1.2 Some properties of the point mapping which made by SPRCS

Let us mark the variation domain as Z, X variables (z, s, x) and x.

Definition. *We shall call the surface $\xi = 0$ the surface without stable sliding towards trajectories of the system*

$$dz/d\tau = g(z, \xi, x, u(\xi)), \tag{1.3}$$

$$d\xi/d\tau = h_1(z, \xi, x, u(\xi))$$

if all the trajectories of (1.1) *which start outside the surface $\xi = 0$ cross it at the point $(z, 0, x)$ where the conditions* (0.3) *are fulfilled.*

Suppose that the following conditions are true:

1^0 $h_1, h_2, g \in \mathbf{C}^2[\bar{Z} \times [-1, 1]]$;

2^0 surface $\xi = 0$ under all $x \in \bar{X}$ is a surface without stable sliding towards trajectories of system (1.3);

3^0 system (1.3) for all $x \in \bar{X}$ has an isolated orbitally asymptotically stable solution $(z_0(\tau, x), \xi_0(\tau, x))$ with the period $T(x)$;

4^0 let $R(z, x)$ be a point mapping of the set $V = \{(z, x) : h_1(z, 0, x, 1) > 0\}$ on the surface $\xi = 0$ into itself, performed by system (3) which has a fixed point $z^*(x)$ corresponding to $(z_0(\tau, x), \xi_0(\tau, x))$;

5^0 suppose that for $\lambda_i(x_0)$ $(i = 1, ..., m)$ the eigenvalues of the matrix $\frac{\partial R}{\partial z}(z^*(x_0), x_0)$ the inequalities $|\lambda_i(x_0)| \neq 1$ are true;

6^0 the averaged system $dx/dt = `(x)$, where $p(x) =$

$$= \frac{1}{T(x)} \int_0^{T(x)} h_2(z_0(\tau, x), \xi_0(\tau, x), x, u(\xi_0(\tau, x))d\tau, \tag{1.4}$$

has an isolated equilibrium point x_0 such that

$$p(x_0) = 0, \quad det|\frac{dp}{dx}(x_0)| \neq 0.$$

Let us denote as $z^{\pm}(\tau, z, x, \mu), \xi^{\pm}(\tau, z, x, \mu)$ the solutions of system (1.2) with the initial conditions $z^{\pm}(0, z, x, \mu) = z, \xi^{\pm}(0, z, x, \mu) = 0$ for $\xi > 0$ and $\xi < 0$.

The point mapping of domain V of the surface $\xi = 0$ has the following form

$$\Phi(z, x, \mu) = (\Phi_1(z, x, \mu), \Phi_2(z, x, \mu)) =$$

$$(z^-(\Theta, z^+(\theta, z, x, \mu), x^+(\theta, z, x, \mu), \mu),$$

$$x^-(\Theta, z^+(\theta, z, x, \mu), x^+(\theta, z, x, \mu), \mu)),$$

where functions $\theta(z, x, \mu), \Theta(z, x, \mu)$ are determined by equations

$$\xi^+(\theta, z, x, \mu) = 0,$$

$$\xi^-(\Theta, z^+(\theta, z, x, \mu), x^+(\theta, z, x, \mu), \mu)) = 0.$$

This means that $\Phi_1(z, x, 0) = R(z, x)$.

The surface $\xi = 0$ is the surface without stable sliding for system (1.3). This means that there exists a neighbourhood of the point $(z^*(x_0), x_0)$ on the surface $\xi = 0$ for which

$$max\{|d\xi^+/d\theta|, |d\xi^-/d\Theta|\} > 0.$$

It follows from condition 1^0 and implicit function theorem that for some small μ_0 functions Φ, θ, Θ have the continuous derivatives into the some set $U \times [0, \mu_0]$ on the surface $\xi = 0$. This means that we can consider the function Φ as the point mapping of the set $U \times [0, \mu_0]$ on the surface $\xi = 0$ Moreover we can rewrite $\Phi(z, x, \mu)$ in the form

$$\Phi(z, x, \mu) = (\bar{R}(z, x, \mu), x + \mu\bar{Q}(z, x, \mu)),$$

where $\bar{R}(z, x, \mu), \bar{Q}(z, x, \mu)$ are the sufficiently smooth functions and $\bar{Q}(z^*(x_0), x_0, 0) = 0, \bar{R}(z^*(x), x, 0) = z^*(x)$.

Let's make in Φ the substitution of variables using the formula $\eta = z - z^*(x)$. Then the point mapping (1.4) takes the form

$$\Psi(\eta, x, \mu) = (\Psi_1(\eta, x, \mu), \Psi_2(\eta, x, \mu)) =$$

$$= (\bar{R}(\eta + z^*(x), x, \mu) - z^*(x), x + \mu\bar{Q}(\eta + z^*(x), x, \mu)), \tag{1.5}$$

and consequently $\Psi(0, x, 0) = (0, x)$.

1.3 Existence of the Fast Periodic Solution

Theorem 1.1. *Under conditions $1^0 - 6^0$ system (1.1) has the isolated periodic solution with the period $\mu(T(x_0) + O(\mu))$ near to the circle $(z_0(t/\mu, x_0), \xi_0(t/\mu, x_0), x_0)$.*

Proof. We will prove the existence of the periodic solution as the existence of the fixed point $(\eta^*(\mu), x^*(\mu))$ of the point mapping Ψ. Let's rewrite the conditions of existence of this fixed point in the form

$$G(\eta^*, x^*, \mu) = \begin{pmatrix} G_1(\eta^*, x^*, \mu) \\ G_2(\eta^*, x^*, \mu) \end{pmatrix} =$$

$$= \begin{pmatrix} \eta^* - \Psi_1(\eta^*, x^*, \mu) \\ \frac{1}{\mu}[x^* - \Psi_2(\eta^*, x^*, \mu)] \end{pmatrix} = 0. \tag{1.6}$$

It is necessary to take into account that for $\mu = 0$ $\eta^*(0) = 0, x^*(0) = x_0$ and $G_2(0, x_0, 0) = -T(x_0)`(x_0) = 0$ and consequently for $\mu = 0$ conditions (1.6) are fulfilled. Moreover, taking into account that for all $x \in \bar{X}$ $G_1(0, x, 0) = 0$ we can conclude $\frac{\partial G_1}{\partial x}(0, x_0, 0) = 0$. Let us compute the Jacobian of function G with respect by variables η, x at $\mu = 0$.

$$\frac{\partial G}{\partial(\eta, x)}(0, x_0, 0) =$$

$$= \begin{vmatrix} I_m - \frac{\partial R}{\partial z}(z^*(x_0), x_0) & 0 \\ \frac{\partial G_2}{\partial \eta}(0, x_0, 0) & -T(x_0)\frac{\partial p}{\partial x}(x_0) \end{vmatrix} \neq 0.$$

This means that there exists an isolated fixed point $(z^*(\mu), x^*(\mu))$ of point mapping G which corresponds to the periodic solution of systems (1.1) and (1.3) and in this case $z^*(\mu) = z^*(x_0) + O(\mu), x^*(\mu) = x_0 + O(\mu)$.

1.4 Stability in the First Approximation

Assume that

7^0 the eigenvalues $\lambda_i(x_0)\,(i = 1, m)$ of the matrix $\frac{\partial R}{\partial z}(z(x_0), x_0)$. satisfy the inequalities $|\lambda_i(x_0)| < 1\,(i = 1, m)$;

8^0 the eigenvalues $\nu_j(x_0)$, $j = 1, ..., n$ of matrix $\frac{dp}{dx}(x_0)$ satisfy the inequalities

$$\mathbf{Re}\,\nu_j(x_0) < 0.$$

Theorem 1.2. *Under conditions* $1^0 - 8^0$ *the periodic solution of* (1.1),(1.2) *is orbitally asymptotically stable.*

Proof. Let's find the derivatives Ψ by variables η, x

$$\frac{\partial \Psi}{\partial(\eta, x, \mu)} = \Gamma(\eta, x, \mu) =$$

$$\begin{bmatrix} I_m - \frac{\partial R}{\partial z}(x_0) + O(\mu) & O(\mu) \\ \frac{\partial \Psi_2}{\partial \eta}(0, x_0, 0) + O(\mu) & I_m + \mu T(x_0)\frac{\partial p}{\partial x}(x_0) + O(\mu) \end{bmatrix}.$$

Consequently the matrix $\Gamma(\eta, x, \mu)$ has at the small vicinity of $(0, x_0, 0)$ two groups of eigenvalues

$$\lambda_i(x_0) + O(\mu),\ i = 1, ..., m,$$

$$1 + \mu T(x_0)\nu_j(x_0) + o(\mu),\ j = 1, ..., n.$$

This means that under conditions of theorem 1.3 there exists some neighbourhood of $(0, x_0, 0)$ for which Ψ is contraction mapping and corresponding fast periodic solution of systems (1.1),(1.3) is orbitally asymptotically stable.

1.5 Some Auxiliary Theorems about Decomposition of Two - Speed Point Mappings

It is obvious that the problems of stability of fast periodic solution of system (1.1) is equivalent to the problem of stability of fixed point $\eta^*(\mu), x^*(\mu)$ of $\Psi(\mu)$. Let's introduce into Ψ the new variables according the formulae $\kappa = \eta - \eta^*(\mu), \chi = x - x^*(\mu)$. Then taking into account that $\partial\Psi(0, x_0, 0)/\partial x = 0$, we have

$$\Lambda_1(\kappa, \chi, \mu) = P\kappa + Q(\kappa, \chi, \mu),$$

$$\Lambda_2(\kappa, \chi, \mu) = \chi + \mu R(\kappa, \chi, \mu), \tag{D.1}$$

where Q, R - are smooth functions and under conditions $1^0 - 7^0$

$$P = \partial\bar{R}/\partial z(z(x_0), x_0), \; ||P|| < 1$$

$$Q(\kappa, \chi, \mu) = O(\mu)O(|\kappa| + |\chi|) + O(|\kappa|^2 + |\chi|^2),$$

$$R(\kappa, \chi, \mu) = O(|\kappa| + |\chi|).$$

Thus we can reduce Cauchy problem for system (1.1) with initial conditions

$$z(0, \mu) = z^0, \; s(0, \mu) = 0 \, x(0, \mu) = x^0 \tag{IC}$$

to the investigation of the two-speed discrete system

$$\kappa_{k+1} = P\kappa_k + Q(\kappa_k, \chi_k, \mu), \; \chi_{k+1} = \chi_k + \mu R(\kappa_k, \chi_k, \mu), \tag{D.2}$$

$$\kappa_0 = z^0 - z^*(x^*(\mu)), \; \chi_0 = x^0 - x^*(\mu).$$

Below we will use the following theorems about decomposition of point mappings (D.1),(D.2) (the proofs are in [2],[14]).

Theorem D.1. Assume, that for system (D.1) conditions (D.2) are held. Then system (D.1) has the slow motions integral manifold in the form $\kappa = V(\chi, \mu)$ for the small μ. Then there exist C_1, C_2 such that

$$||V(\chi, \mu)|| < C_1,$$

$$||V(\chi, \mu) - V(\bar{\chi}, \mu)|| < C_2||\chi - \bar{\chi}||.$$

The motion on manifold $\kappa = V(\chi, \mu)$ described by the equation

$$\Lambda_1(V(\chi, \mu), \chi, \mu) = \chi + \mu R(V(\chi, \mu), \chi, \mu). \tag{D.3}$$

For the slow coordinate of solution (D.2) $\chi_k(\chi_0)$ and $\bar{\chi}_k(\tilde{\chi})$ the solution of system (D.3) with initial condition $\bar{\chi}_0 = \tilde{\chi}$ there exist $c > 0$, $0 < q < 1$ and $\tilde{\chi} \in \mathrm{R}$ for which the inequality

$$|\chi_k(\chi_0) - \bar{\chi}_k(\tilde{\chi})| < cq^k$$

is true.

Theorem D.2. (reduction principle). If

$$Q(0,0,\mu)=0; \quad R(0,0,\mu)=0,$$

then the problem of stability of zero solutions of systems(D.1) and (D.3) are equivalent. This means that the zero solution of (D.1) are stable (asymptotically stable, unstable) if and only if the zero solution of (D.3) is stable (asymptotically stable,unstable).

The function $V(\chi,\mu)$ may be found from the equation

$$PV(\chi,\mu)+Q(V(\chi,\mu),\chi,\mu)=V(\chi+\mu R(V(\chi,\mu),\chi,\mu),\mu)$$

with any level of precision in form

$$V(\chi,\mu)=V_0(\chi)+\mu V_1(\chi)+\mu^2 V_2(\chi)+\cdots.$$

The function $V_0(\chi)$ is a solution of the equation

$$PV_0(\chi)+Q(V_0(\chi),\chi,0)=V_0(\chi).$$

Function $V_1(\chi)$ can be found from equation

$$PV_1(\chi)+Q'_\mu(V_0(\chi),\chi,0)=V_1(\chi).$$

An equation describing the flow on slow motions integral manifold have the form

$$\Lambda_2(V(\chi,\mu),\chi,\mu)=\chi+\mu R(V_0(\chi),\chi,\mu)+ \qquad (D.4)$$
$$+\mu^2(R'_\kappa(V_0(\chi),\chi,0)V_1(\chi)+R'_\mu(V_0(\chi),\chi,0))+O(\mu^3).$$

1.6 Theorem about averaging

Assume that

9^0 The solution $\bar{x}(t)$ of averaged system with initial conditions $\bar{x}(0)=x^0$ for $t\in[0,L]$ is situated into the closed subdomain $\bar{X}\in X$.

Theorem 1.3. *Under conditions* 1^0-7^0 *and* 9^0 *the slow coordinate* $x(t,\mu)$ *of solution* (1.1),(1.2) *and* $\bar{x}(t)$ *satisfy the inequality*

$$\sup_{t\in[0,L]}|x(t,\mu)-\bar{x}(t)|=O(\mu).$$

1.7 Founding of the periodic solution

Assume now that

1^0 $h_1,h_2.g\in C^{k+2}[\bar{Z}\times[-1,1]]$. We will find the period of the desired periodic solution of (1.2) in form

$$T(\mu)=T_0+\mu T_1+\mu^2 T_2+..., \qquad (1.7)$$

where $T_0 = T(x_0)$ and time interval for which $u = 1$ and $u = -1$ in form

$$\theta^{\pm}(\mu) = \theta_0^{\pm} + \mu\theta_1^{\pm} + \mu^2\theta_2^{\pm} + \cdots + \mu^k\theta_k^{\pm} + \cdots,$$

where $\theta_0 = \theta(x_0)$. Then the asymptotic representation of desired periodic solution on $[0, T(\mu)]$ takes the form

$$z(\tau,\mu) = z_0(\tau) + \mu z_1(\tau) + \mu^2 z_2(\tau) + \dots + \mu^k z_k(\tau) + \dots,$$

$$x(\tau,\mu) = \xi_0(\tau) + \mu\xi_1(\tau) + \mu^2\xi_2(\tau) + \dots + \mu^k\xi_k(\tau) + \dots,$$

$$x(\tau,\mu) = x_0 + \mu x_1(\tau) + \mu^2 x_2(\tau) + \dots + \mu^k x_k(\tau) + \dots \quad .$$

Denote

$$\tilde{T}_k(\mu) = T_0 + \mu T_1 + \mu^2 T_2 + \dots + \mu^k T_k,$$

$$\tilde{\theta}_k^{\pm}(\mu) = \theta_0^{\pm} + \mu\theta_0^{\pm} + \mu^2\theta_2^{\pm} + \dots + \mu^k\theta_k^{\pm}.$$

Let's find the $k-th$ approximation of asymptotic representation for desired periodic solution for $\tau \in [0, \tilde{T}_k(\mu)]$ in form

$$Z_k(\tau,\mu) = z_0(\tau) + \mu z_1(\tau) + \mu^2 z_2(\tau) + \dots + \mu^k z_k(\tau),$$

$$\Xi_k(\tau,\mu) = \xi_0(\tau) + \mu\xi_1(\tau) + \mu^2\xi_2(\tau) + \dots + \mu^k\xi_k(\tau),$$

$$X_k(\tau,\mu) = x_0 + \mu x_1(\tau) + \mu^2 x_2(\tau) + \dots + \mu^k x_k(\tau),$$

where continuous functions $z_i, \xi_i, x_i aresmoothon[0, \tilde{\theta}_k^+(\mu)) \cup (\theta_k^+(\mu), T_k(\mu)]$ but have the jumps in the derivatives at $\tau = \tilde{\theta}_k^+(\mu)$. Let's show that under conditions of theorem 1.1 the functions $z_i^{\pm}, \xi_i^{\pm}, x_i^{\pm}$ and constants θ_i, Θ_i for every $i = 1, ..., k$ can be uniquely found.

Let's introduce in system (1.2) two "new times" according the formulae

$$\tau^+ = \tau/\tilde{\theta}_k^+(\mu); \tau^- = (\tau - \tilde{\theta}_k^+(\mu))/\tilde{\theta}_k^-(\mu), \tau^{\pm} \in [0,1]$$

and the auxiliary functions $z_0^{\pm}(\tau^{\pm}), \xi_0^{\pm}(\tau^{\pm})$ as the solutions of systems

$$dz_0^{\pm}/d\tau^{\pm} = \theta_0^{\pm} g(z_0^{\pm}, \xi_0^{\pm}, x_0, \pm 1), \tag{1.3.$\pm$}$$

$$d\xi_0^{\pm}/d\tau = \theta_0^{\pm} h_1(z_0^{\pm}, \xi_0^{\pm}, x_0, \pm 1)$$

with initial and periodicity conditions

$$z_0^+(0) = z^*(x_0) = z_0^*, \quad \xi_0^+(0) = 0; \tag{1.8}$$

$$z_0^-(0) = z_0^+(1), \quad \xi_0^-(0) = \xi_0^+(1) = 0;$$

$$z_0^-(1) = z^*(x_0), \quad \xi_0^-(1) = 0.$$

From the periodicity of functions $z_0(\tau), \xi_0(\tau)$ it follows that system (1.3$\pm$),(1.8) has the unique solution.

Functions $x_1^{\pm}(\tau)$ are described by the equations

$$dx_1^{\pm}/d\tau^{\pm} = \theta_0^{\pm} h_2(z_0^{\pm}(\tau^{\pm}, x_0), \xi_0^{\pm}(\tau^{\pm}, x_0), x_0, \pm 1), \tag{1.4.1}$$

with initial and periodicity conditions given by

$$x_1^+(0) = x_1^*, \quad x_0^-(0) = x_1^+(1), \quad x_1^-(1) = x_1^*. \tag{1.9.1}$$

Moreover

$$[h_{20}](x_0) = \int_0^1 h_2(z_0(\tau^+, x_0), \xi_0^+(\tau^+, x_0), x_0, 1)d\tau^+ +$$

$$+ \int_0^1 h_2(z_0(\tau^-, x_0), \xi_0^-(\tau^-, x_0), x_0, -1)d\tau^- = 0, \tag{1.10}$$

$$det|\frac{d[h_{20}]}{dx}(x_0)| = T(x_0)\frac{d'p}{dx}(x_0) \neq 0.$$

This means that for every x_1^* there exists the unique solution of (1.4.1) and (1.9.1) for which $\int_0^1 \tilde{x}_1^+(\tau^+)d\tau^+ + \int_0^1 \tilde{x}_1^-(\tau^-)d\tau^- = 0$ and we can define function $x_1(\tau)$ in form

$$x_1(\tau) = x_1^* + \tilde{x}_1(\tau) =$$

$$= \begin{cases} x_1^* + \tilde{x}_1^+(\tau/\tilde{\theta}_k^+(\mu)) & \text{for} \quad \tau \in [0, \tilde{\theta}_k^+(\mu)] \\ x_1^* + \tilde{x}_1^-((\tau - \tilde{\theta}_k^+(\mu))/\tilde{\theta}_k^-(\mu)) \text{for} & \tau \in [\tilde{\theta}_k^+(\mu), \tilde{T}_k(\mu)]. \end{cases}$$

Functions $z_1^\pm(\tau^\pm, x_1^*), \xi_1^\pm(\tau^\pm, x_1^*)$ are defined by equations

$$dz_1^\pm/d\tau = g_z'^\pm z_1^\pm + g_\xi'^\pm \xi_1^\pm + g_x'^\pm x_1^\pm) + \theta_1^\pm g^\pm; \tag{1.3.1}$$

$$d\xi_1/d\tau = \theta_0^\pm(h_{1z}'^\pm z_1^\pm + h_{1\xi}'^\pm \xi_1^\pm + h_{1x}'^\pm x_1^\pm) + \theta_1^\pm h_1^\pm,$$

where the values of functions $g^\pm, h_1^\pm$ and its derivatives are calculated at the points $(z_0^\pm(\tau^\pm, x_0), \xi_0^\pm(\tau^\pm, x_0), x_0, \pm 1)$.

Initial and periodicity conditions for system (1.3.1) are defined by equations

$$z_1^+(0, x_1^*) = z_1^-(1, x_1^*); \; z_1^-(0, x_1^*) = z_1^+(1, x_1^*) \tag{1.8.1}$$

$$\xi_1^+(0, x_1^*) = \xi_1^+(1, x_1^*) = \xi_1^-(0, x_1^*) = \xi_1^-(1, x_1^*) = 0.$$

Equations (1.3.1) depend linearly on $z_1^\pm, \xi_1^\pm, \theta_1^\pm$ end consequently their solutions $z_1^\pm(\tau, x_1^*), \xi_1^\pm(\tau, x_1^*), \theta_1^\pm(x_1^*)$ are linearly dependent on the initial conditions $z_1^\pm(0, x_1^*)$. Expressing $z_1^\pm(\tau, x_1^*), \xi_1^\pm(\tau, x_1^*), \theta_1^\pm(x_1^*)$ through $z_1^+(0, x_1^*)$ end substitute the results in the first equation of (8.1) we have the linear on $z_i^+(0, x_1^*)$ system of algebraic equations which determinant coincides with $det|I_m - \partial R(z^*(x_0), x_0)/\partial z| \neq 0$.

Functions $x_2^\pm(\tau)$ are described by the equations

$$dx_2^\pm/d\tau = \theta_0^\pm(h_{2z}' z_1^\pm +$$

$$+ h_{2\xi}' \xi_1^\pm x_1^\pm + h_{2x}' x_1^\pm) + \theta_1^\pm h_2, \tag{1.4.2}$$

where the values of functions $h_2^\pm$ are calculated at the points

$$(z_0^\pm(\tau^\pm, x_0), \xi_0^\pm(\tau^\pm, x_0), x_0, \pm 1).$$

Initial and periodicity conditions are

$$x_2^+(0) = x_2^*, \quad x_2^-(0) = x_2^+(1), \quad x_2^-(1) = x_2^*. \tag{1.9.2}$$

The condition under which system (1.9.2) for every x_2^* have the periodic solution with zero averaged value takes the form

$$\int_0^1 [\theta_0^+(h'^+_{2z} z_1^+(\tau^+, x_1^*) + h'^+_{2\xi}\xi_1^+(\tau^+, x_1^*) +$$
$$+h'^+_{2x} x_1^+(\tau^+, x_1^*)) + \theta_1^+(\tau^+, x_1^*)h_2^+]d\tau^+ +$$
$$+\int_0^1 [\theta_0^-(h'^-_{2z} z_1^-(\tau^-, x_1^*) + h'^-_{2\xi}\xi_1^-(\tau^-, x_1^*) +$$
$$+h'^-_{2x} x_1^-(\tau^-, x_1^*)) + +\theta_1^-(x_1^*)h_2^-]d\tau^- = 0. \qquad (1.10.1)$$

Condition (1.10.1) is a system of linear equations for obtaining of x_1^*, whose determinant coincides with $\frac{dp}{dx}(x_0) \neq 0$. This means that we can find uniquely the function $x_2(\tau)$ in form $x_2(\tau) = x_2^* + \tilde{x}_2(\tau)$, where $\tilde{x}_2(\tau)$ is the function with zero averaged value.

Suppose now that functions $z_{i-1}(\tau), \xi_{i-1}(\tau), x_i(\tau)$ and constants $x_{i-1}^*, \theta_{i-1}^{\pm}$ are found, moreover, the periodic function $x_i(\tau)$ for every x_i^* can be represented in form of the sum of x_i^* and the function $\tilde{x}_i(\tau)$ with zero averaged value.

Then the functions $z_i^{\pm}(\tau^{\pm}, x_i^*), \xi_i^{\pm}(\tau^{\pm}, x_i^*), x_i^{\pm}(\tau^{\pm}, x_i^*)$ are defined by equations

$$dz_i/d\tau^{\pm} = \theta_0^{\pm}(g'^{\pm}_z z_i^{\pm} + g'^{\pm}_{\xi}\xi_i^{\pm} + g'^{\pm}_x x_i^{\pm}) +$$
$$+\theta_i^{\pm}(x_i^*)g^{\pm} + \Pi_{1i}^{\pm}(\tau^{\pm}); \qquad (1.3.i)$$
$$d\xi_i/d\tau^{\pm} = \theta_0^{\pm}(h'^{\pm}_{1z} z_i^{\pm} + h'^{\pm}_{1\xi}\xi_i^{\pm} + h'^{\pm}_{1x} x_i^{\pm}) +$$
$$\theta_i^{\pm}(x_i^*)h_1^{\pm} + \Pi_{2i}^{\pm}(\tau^{\pm}),$$

where the values of functions and its derivatives $g^{\pm}, h_1^{\pm}$ are calculated at the points $(z_0^{\pm}(\tau^{\pm}, x_0), \xi_0^{\pm}(\tau^{\pm}, x_0), x_0, \pm 1)$, and functions $\Pi_{ji}^{\pm}, j = 1, 2$ are uniquely defined functions containing the terms of order μ^i in asymptotic representations of $g^{\pm}, h_1^{\pm}$ depending from $z_j^{\pm}, \xi^{\pm}, x_j^{\pm}, x_j^*$, $j \leq i-1$. Initial and periodicity conditions for system (1.3.i) are defined by the equations

$$z_i^+(0, x_1^*) = z_i^-(1, x_i^*) = z_i^*,\ z_i^-(0, x_i^*) = z_i^+(1, x_i^*) \qquad (1.8.i)$$
$$\xi_i^+(0, x_i^*) = \xi_i^+(1, x_i^*) = \xi_i^-(0, x_i^*) = \xi_i^-(1, x_i^*) = 0.$$

Equations (1.3.i) depend linearly on $z_i^{\pm}, \xi_i^{\pm}, \theta_i^{\pm}$ and consequently their solutions $z_i^{\pm}(\tau, x_i^*), \xi_i^{\pm}(\tau, x_i^*), \theta_i^{\pm}(x_i^*)$ are linearly depend on the initial conditions $z_i^{\pm}(0, x_i^*)$. Expressing $z_i^{\pm}(\tau, x_i^*), \xi_i^{\pm}(\tau, x_i^*), \theta_i^{\pm}(x_i^*)$ through $z_i^+(0, x_i^*)$ and substituting the results in the first equation of (1.8.i) we have linear in $z_i^+(0, x_i^*)$ system of algebraic equations whose determinant is coincide with $det|I_m - \partial R(z^*(x_0), x_0)/\partial z| \neq 0$.

Functions $x_{i+1}(\tau)$ are described by the equations

$$dx_{i+1}^{\pm}/d\tau = \theta_0^{\pm}(h'_{2z} z_1^{\pm} + h'_{2\xi}\xi_1^{\pm} + h'_{2x} x_1^{\pm}) +$$
$$+\theta_1^{\pm}h_{i+1} + \pi_{3i}^{\pm}(\tau), \qquad (1.4.i+1)$$

where the values of functions $h_2^{\pm}$ and its derivatives are calculated at the

$$(z_0^{\pm}(\tau^{\pm}, x_0), \xi_0^{\pm}(\tau^{\pm}, x_0), x_0, \pm 1)$$

and functions $\pi_3^{\pm}, j = 1, 2$ are uniquely defined functions containing the terms of order μ^i in asymptotic representations of $h_2^{\pm}$ depending from $z_j^{\pm}, \xi^{\pm}, x_j^{\pm}, x_j^*$, $j \leq i-1$. Initial and periodicity conditions are

$$x_{i+1}^{+}(0) = x_{i+1}^{*},\ x_{i+1}^{-}(0) = x_{i+1}^{+}(1),\ x_{i+1}^{-}(1) = x_{i+1}^{*}. \qquad (1.9.i+1)$$

The condition under which system (1.9.i+1) have a periodic solution with zero averaged value for every x_{i+1}^* takes the form

$$\int_0^1 [\theta_0^{+}(h_{2z}'^{+} z_i^{+}(\tau^{+}, x_i^{*}) + h_{2\xi}'^{+}\xi_i^{+}(\tau^{+}, x_i^{*}) +$$
$$+ h_{2x}'^{+} x_i^{+}(\tau^{+}, x_i^{*})) + \theta_i^{+}(\tau^{+}, x_i^{*}) h_2^{+}] d\tau^{+} + \qquad (1.10.i+1)$$
$$+ \int_0^1 [\theta_0^{-}(h_{2z}'^{-} z_i^{-}(\tau^{-}, x_i^{*}) + h_{2\xi}'^{-}\xi_i^{-}(\tau^{-}, x_i^{*}) +$$
$$+ h_{2x}'^{-} x_i^{-}(\tau^{-}, x_i^{*})) + \theta_i^{-}(x_i^{*}) h_2^{-}] d\tau^{-} = 0.$$

Condition (1.10.i+1) is a system of linear equations for obtaining of x_1^*, whose determinant coincides with $\frac{dp}{dx}(x_0) \neq 0$. This means that we can uniquely find uniquely the function $x_{i+1}(\tau)$ in the form $x_{i+1}(\tau) = x_{i+1}^* + \tilde{x}_{i+1}(\tau)$, where $\bar{x}_{i+1}(\tau)$ is a function with zero averaged value. To finish the algorithm for design of desired asymptotic representation it is necessary to define

$$(z_i(\tau), \xi_i(\tau)) =$$

$$= \begin{cases} (z_i^{+}(\tau/\tilde{\theta}_k^{+}(\mu), x_i^{*}), \xi_i^{+}(\tau/\tilde{\theta}_k^{+}(\mu), x_i^{*})) \text{ for } \quad \tau \in [0, \tilde{\theta}_k^{+}(\mu)], \\ (z_i^{-}((\tau - \theta_k^{+}(\mu))/\tilde{\theta}_k^{-}(\mu), x_i^{*}), \xi_i^{+}((\tau - \theta_k^{+}(\mu))/\tilde{\theta}_k^{-}(\mu), x_i^{*})) \\ \text{for } \quad \tau \in [\tilde{\theta}_k^{+}(\mu), \tilde{T}_k(\mu)], j = 1, ..., k. \end{cases}$$

$$x_j(\tau) = \begin{cases} x_j^* + \tilde{x}_j^{+}(\tau/\tilde{\theta}_k^{+}(\mu)) \quad \text{for} \quad \tau \in [0, \tilde{\theta}_k^{+}(\mu)], \\ x_j^* + \tilde{x}_j^{-}((\tau - \tilde{\theta}_k^{+}(\mu))/\tilde{\theta}_k^{-}(\mu)) \\ \text{for} \quad \tau \in [\tilde{\theta}_k^{+}(\mu), \tilde{T}_k(\mu)], j = 1, ..., k. \end{cases}$$

1.8 Correction of Averaged Equations

Let us show how we can use the knowledge of the fast periodic solution for correction of averaged equations with any precision level according the small parameter degrees. The knowledge of such equations is necessary the case when the linear part of averaged equations (1.3) has the spectral points on the imaginary axis.

Assume that we have found the functions

$$\theta^{\pm}(x, \mu) = \theta_0^{\pm}(x) + \sum_{i=1}^{\infty} \mu^i \theta_i^{\pm}(x),$$

$$T(x,\mu) = \theta^+(x,\mu) + \theta^-(x,\mu)$$

and $z_i^{\pm}(\tau^{\pm},x), \xi_i^{\pm}(\tau^{\pm},x), x_j^{\pm}(\tau^{\pm},x) =$

$$= \begin{cases} (z_i^+(\tau/\theta^+(\mu,x),x), \xi_i^+(\tau/\theta^+(\mu,x),x), x^+(\tau/\theta^+(\mu,x),x)) \\ \text{for} \quad [0,\theta^+(\mu,x)], \\ (z_i^-((\tau-\bar{\theta}^+(\mu))/\theta^-(\mu,x),x), \xi_i^-((\tau-\theta^+(\mu,x))/\theta^-(\mu,x),x), \\ x_j^-((\tau-\bar{\theta}^+(\mu,x))/\theta^-(\mu,x),x)) \\ \text{for} \quad [\theta^+(\mu,x), T(\mu,x)]. \end{cases}$$

then

$$z(\tau,x,\mu) = z_0(\tau,x) + \mu z_1(\tau,x) + ... + \mu^i z_i(\tau,x) + ...,$$

$$\xi(\tau,x,\mu) = \xi_0(\tau,x) + \mu\xi_1(\tau,x) + ... + \mu^i\xi_i(\tau,x) + ...,$$

$$x(\tau,x,\mu) = \mu x_1(\tau,x) + ... + \mu^i x_i(\tau,x) + ... \quad .$$

Then the precise averaged equation has the form

$$dx/dt = \frac{1}{T(x\mu)} \int_0^{T(x,\mu)} h_2(z(\tau,x,\mu), \xi(\tau,x,\mu), x+ \\ + \tilde{x}(\tau,\mu), u(\xi(\tau,x,\mu)))d\tau. \qquad (PAE)$$

Equations (PAE) correspond to the system (D.3) which describes a flow on the slow motion manifold in system (D.1). In this case the first order approximation of (PAE) has the form

$$dx/dt = \frac{1}{T_0(x)} \Bigg\{ (1-\mu T_1(x)) \int_0^{T_0(x)} h_2 d\tau + \\ +\mu \Bigg[\int_0^{T_0(x)} \Big(h'_{2z} z_1(\tau,x) + h'_{2\xi}\xi_1(\tau,x) + h'_{2x}\tilde{x}_1(\tau) \Big) d\tau + \\ +\theta_1^+(x) h_2(z_0(\theta_0(x),x), \xi_0(\theta_0(x),x), x, 1) + \\ +\theta_1^-(x) h_2(z_0(T_0(x),x), \xi_0(T_0(x),x), x, -1) \Bigg] \Bigg\}, \qquad (FAAE)$$

where the values of functions h_2 and it's derivatives in integral terms are calculated at the points $(z_0(\tau,x), \xi_0(\tau,x), x, u(\xi_0(\tau,x))$. Analogously we can obtain the averaged equations with any precision level expanding in powers of the small parameter.

1.9 Investigation of Stability in Critical Case

Theorem 1.4 (Reduction Principle). *Under conditions $1^0 - 7^0$ the periodic solution for original system* (1.1) *is stable (asymptotically stable, unstable) if and only if the equilibrium point of system* (PAE) *is stable (asymptotically stable, unstable).*

Corollary. *Assume that for system (1.1) conditions $1^0 - 7^0$ are true. If the equilibrium point of system* (FAAE) *is asymptotically stable (unstable) in the first approximation than the periodic solution for original system* (1.1) *is asymptotically stable (unstable).*

2. Analysis of Averaged Equations in Sliding Mode Control Systems with Fast Actuators

2.1. Averaged Equations of Systems which Linearly Depend on Relay Control

In this section we will consider the SPRCS which linearly depend on relay control. We will show that the averaged equations whose describe the slow motions in such SPRCS and the equations whose describe the sliding motion in the reduced systems are coincide.

Let's consider the system

$$\begin{aligned} \mu dz/dt &= A(s,x)z + f_1(s,x) + K_1(s,x)u(s), \\ ds/dt &= B(s,x)z + f_2(s,x) + K_2(s,x)u(s), \\ dx/dt &= D(s,x)z + f_3(s,x) + K_3(s,x)u(s), \end{aligned} \tag{2.1}$$

where $z \in \mathbf{R}^m, s \in R, x \in \mathbf{R}^n, u(s) = sgn\,(s), f_i, K_i\,(i = 1,2,3)$ are smooth functions of their arguments. Accepting $\mu = 0$ and expressing z_0 from the first equation of system (2.1) according to the formula $z_0 = -A^{-1}(s,x)[f_1(s,x) + K_1(s,x)u(s)]$ we obtain the reduced system

$$\begin{aligned} ds/dt &= -B(s,x)A^{-1}(s,x)f_1(s,x) + f_2(s,x)- \\ &-[B(s,x)A^{-1}(s,x)K_1(s,x) - K_2(s,x)]u(s), \\ dx/dt &= D(s,x)A^{-1}(s,x)f_1(s,x) + f_3(s,x)- \\ &-[D(s,x)A^{-1}(s,x)K_1(s,x) - K_3(s,x)]u(s). \end{aligned}$$

Suppose that for original system (2.1) the conditions of dynamic uncertainty are held which means that

$$K_2(0,x) \geq 0, \quad B(0,x)A^{-1}(0,x)K_1(0,x) - K_2(0,x) > 0. \tag{CDU}$$

The equations which describe the motions in sliding modes in the reduced system have the form

$$\begin{aligned} dx/dt &= -D(0,x)A^{-1}(0,x)f_1(0,x) + f_3(0,x)- \\ &-[D(0,x)A^{-1}(0,x)K_1(0,x) - K_3(0,x)](u(s) - u_{eq}(x)). \\ u_{eq}(x) &= [B(0,x)A^{-1}(0,x)K_1(0,x) - K_2(0,x)]^{-1}\times \\ &\times[-B(0,x)A^{-1}(0,x)f_1(0,x) + f_2(0,x)]. \end{aligned} \tag{2.2}$$

Let's show that the averaged equations for original system (2.1) are coincide with system (2.2).

Suppose that for all $x \in \bar{X}$ the following conditions are true:

(*) $Re \quad Spec\,A(0,x) < 0$;

(**) $|u_{eq}(x)| < 1$.

It is obvious that if conditions of dynamical uncertainty are true it is reasonable to consider only solutions of system (2.1) with initial conditions

$$z(0,\mu) = z^0,\ s(0,\mu) = \mu s^0,\ x(0,\mu) = x^0$$

which are situated in the $O(\mu)$ vicinity of switching surface. Following [3],[8] let us increase in $1/\mu$ times the neighbourhood of discontinuity surface $s = 0$ in system (0.1) and introduce into it the fast time $\tau = t/\mu$ and the variable $\xi = s/\mu$. Then we will rewrite the system (2.1) in form

$$\begin{aligned} \mu dz/dt &= A(\mu\xi, x)z + f_1(\mu\xi, x) + K_1(\mu\xi, x)u(\xi), \\ \mu d\xi/dt &= B(\mu\xi, x)z + f_2(\mu\xi, x) + K_2(\mu\xi, x)u(\xi), \\ dx/dt &= D(\mu\xi, x)z + f_3(\mu\xi, x) + K_3(\mu\xi, x)u(\xi). \end{aligned} \tag{2.3}$$

In this case the system which describes the fast motions in system (2.1) has analogously (1.3) the form

$$\begin{aligned} dz/d\tau &= A(0,x)z + f_1(0,x) + K_1(0,x)u(\xi), \\ d\xi/d\tau &= B(0,x)z + f_2(0,x) + K_2(0,x)u(\xi), \\ &(x - parameter). \end{aligned} \tag{2.4}$$

Introducing into system (2.4) the new variables $\eta = z + A^{-1}(0,x)[f_1(0,x) + K_1(0,x)u_{eq}(x)]$ we will have

$$\begin{aligned} d\eta/d\tau &= A(0,x)\eta + K_1(0,x)\bar{u}(\xi,x), \\ d\xi/d\tau &= B(0,x)\eta + K_2(0,x)\bar{u}(\xi,x), \\ \bar{u}(\xi,x) &= u(\xi) - u_{eq}(x). \end{aligned} \tag{2.5}$$

Let's consider the point mapping of surface $\xi = 0$ into itself which made by system (2.5). The solution of system (2.5) with initial conditions

$$\eta^+(0,\mu) = \eta, \quad \xi^+ = 0;$$

$$\eta \in \Omega^+ = \{(\eta,\mu) : B(0,x)\eta + K_2^+(0,x)\bar{u}(\xi,x) > 0\}$$

$$K_i^+ = K_i(1 - u_{eq}), i = 1,2$$

has the form

$$\begin{aligned} \eta^+(\tau,\eta,x) &= e^{A\tau}(\eta + A^{-1}K_1^+) - A^{-1}K_1^+, \\ \xi^+(\tau,\eta,\mu) &= BA^{-1}(e^{A\tau} - I)(\eta + A^{-1}K_1^+) - \\ &-(BA^{-1}K_1^+ - K_2^+)\tau. \end{aligned}$$

Here and always below the functions A, B, K_1, K_2 are computed at the point $(0,x)$. For $\tau = 0$ $d\xi/d\tau = B(0,x)\eta + K_2^+(0,x)\bar{u}_{eq}(\xi,x)$ and consequently $\xi^+(\tau,\eta,\mu) > 0$ at list for the small $\tau > 0$. From the other hand from the condition (i) it follows that

$$\lim_{\tau \to -\infty} \xi^+(\tau,\eta,\mu) = -\infty.$$

This means that there exists $T_1(\tau,\eta,\mu)$ the smallest root of equation $\xi^+(T_1(\tau,\eta,\mu),\eta,\mu) = 0$. Let's rewrite this equation in form

$$BA^{-1}(e^{AT_1} - I)(\eta + A^{-1}K_1^+) =$$

$$= (BA^{-1}K_1^+ - K_2^+)T_1.$$

It follows from the definition of T_1 that $d\xi^+/d\tau(T_1)) \leq 0$. This means that we can define the point mapping of the set Ω^+ into the set

$$\Omega^- = \{(\eta,\mu) : B(0,x)\eta^+(T_1,\eta,x) - K_2^-(0,x)\bar{u}_{eq}(\xi,x) < 0\}$$

where $K_i^- = K_i(1+u_{eq}), i = 1,2$. Analogously the point mapping

$$\eta^-(T_2,\eta,x) = e^{AT_2}(\eta - A^{-1}K_1^-) + A^{-1}K_1^-,$$

$$BA^{-1}(e^{AT_2} - I)(\eta - A^1K_1^- =$$

$$= -(BA^{-1}K_1^- - K_2^-)T_2.$$

transforms the set Ω^+ into the set

$$\Omega^* = \{(\eta,\mu) : B(0,x)\eta - K_2^-(0,x) > 0,\ x \in \bar{X}\}.$$

This means that the point mapping $\eta^-(T_2,\eta^+(T_1,\eta,x),x)$ describing by formula

$$\Phi(\eta,x) = e^{AT_2}(e^{AT_1}(\eta + A^{-1}K_1^+) - 2A^{-1}K_1) + A^{-1}K_1^-,$$

transforms the set Ω^* into itself. Let's mark $\eta^*(x)$ the fixed point of the point mapping $\Phi(\eta,x)$ which corresponds to the periodic solution $(z_0(\tau,x),\xi_0(\tau,x))$. For $\eta^*(x)$ we have the formula

$$\eta^*(x) = \{2[I - e^{A(T_1+T_2)}]^{-1}[I - e^{AT_2}] - -(1-u_{eq})\}A^{-1}K_1.$$

Let's study the properties of averaged values of the periodic solutions $\eta_0(\tau,x), \xi_0(\tau,x)$

$$I(x) = \int_0^{T(x)} \eta_0(\tau,x)d\tau = [(1+u_{eq})T_2 - (1-u_{eq})T_1]A^{-1}K_1.$$

Taking into account that $\xi_0(T(x),x) = 0$ we will have

$$\xi_0(T(x),x) =$$

$$= \int_0^{T(x)} B\eta_0(\tau,x)d\tau - K_2[(1+u_{eq})T_2 - (1-u_{eq})T_1] = 0.$$

This means that $(1+u_{eq})T_2 = (1-u_{eq})T_1$. The following lemma is true.

Lemma 2.1. *If there exists the $T(x)$ periodic solution of system (2.5) then*

$$\int_0^{T(x)} \eta_0(\tau,x)d\tau = 0.$$

$$\int_0^{T(x)} u(\eta_0(\tau, x))d\tau = \frac{T_2(x) - T_1(x)}{T(x)} = u_{eq}(x).$$

Remark. This lemma was obtained at the first time in [10] by using of transfer functions method.

Let's turn back to the system (2.1). If for system (2.3) the conditions of Theorem 1.2 are true there exists the isolated periodic solution $(z(\tau, \mu), \xi(\tau, \mu), x(\tau, \mu))$ which corresponds to the periodic solution $(\eta_0(\tau, x), \xi_0(\tau, x))$ of system (2.5). Moreover

$$\int_0^{T(x)} z_0(\tau, x)d\tau = A^{-1}(0, x)(f_1(0, x) + K_1(0, x)u_{eq}(x)).$$

This means that the averaged equations which approximately describe the behavior of the slow motions in the system (2.1) are coincide with equations (2.2) for the sliding motions in the reduced system .

2.2 Example

Suppose that mathematical model of control system taking account of actuator behavior has the form

$$\mu dz/dt = -z - u, ds/dt = z + (\alpha + x)u, \ \alpha > 0 \tag{2.6}$$

$$dx/dt = -z + x - u. \tag{2.7}$$

$z, s, x \in R, u(s) = sgns, \mu$ - actuator time constant. Fast motions taking place in (2.6), (2.7) are described with the system

$$dz/d\tau = -z - u, \ d\xi/d\tau = z + (\alpha + x)u, \ u = sgn\xi. \tag{2.8}$$

System (2.8) is symmetric relatively to the point $z = \xi = 0$ so we shall consider as point mapping $R(z, x)$ of domain $z + \alpha > 0$ on the switching line $\xi = 0$ into domain $z + \alpha < 0$ performed by system (2.8) with $\xi > 0$. Then $\Psi(z) = -1 + e^{-T}(z + 1)$ where τ is the smallest root for equation

$$(1 - e^{-T})(z + 1) = (1 - \alpha - x)\tau.$$

The fixed point $z^* = \Psi(z^*(x), x)$ corresponding to periodic solution (2.8) is determined by the equation $\Psi(z^*(x), x) = -z^*(x)$. Then the fixed point $z^*(x)$ (amplitude) and the semiperiod $T(x))$ of the periodic solution are determined by equations

$$2th(T/2) = (1 - \alpha - x)T, \ z^*(x) = th(T/2). \tag{2.9}$$

Equations (2.9) with $0 < \alpha + x < 1$ have positive solution which corresponds to existence of $2T$ periodic solution in system (2.8). The slow motions averaged equation for system (2.7) assumes the form

$$dx/dt = -x$$

This equation has the asymptotically stable equilibrium point $x = 0$. For $x = 0 \quad \alpha = \frac{1}{2}\frac{3-e}{3+e}$. It follows from (2.9) that $T \approx 3.83, \lambda \approx -0.07$ and so system (2.6), (2.7) has an orbitally asymptotically stable periodic solution which situated into the $O(\mu)$ neighbourhood of the switching surface.

Changing positive feedback parameter α in system (2.6) we can design the fast periodic oscillations with desired frequency. Thus if we want to obtain in system (2.6) the periodic oscillations with frequency $\frac{1}{2\mu(T+O(\mu))}$ and amplitude $th(T/2)$ then from (2.9) we should choose the value of α according the formula $\alpha = 1 - 2\frac{th(T/2)}{T}$.

2.3 The System Containing The Relay Control Nonlinearly

Consider the control system which described by the equations

$$\mu dz/dt = -z - u, ds/dt = z + \alpha u,$$
$$dx/dt = (z^4 - z^2 + \beta)x, \tag{2.11}$$

where $x, s, z \in u(s) = sgn\,(s)$, $0 < \alpha, \beta < 1, \mu$ is the actuator time constant. If we take $\mu = 0$ in system (2.11) we will have

$$ds/dt = (\alpha - 1)u,\ dx/dt = (u^4 - u^2 + \beta)x \tag{2.12}$$

In system (2.12) the stable sliding mode exists. For motions in sliding mode in (2.12) both classical definitions of solutions according to the equivalent control method and Filippov are coincide. This motions are described by the equation $dx/dt = \beta x$. They are unstable for $\beta > 0$.

At the same time if $0 < \alpha < 1$ in system (2.11) the fast periodic solutions occur. Let us mark $z(\tau)$ the first coordinate of periodic solution (2.8) by $x = 0$. If α and β are selected so that

$$-\gamma = \int_0^{T(x_0)} [z^4(\tau) - z^2(\tau)]d\tau < -\beta < 0$$

the averaged equation has the form $dx/dt = -(\gamma - \beta)x$. Consequently system (2.11) has stable periodic solution in $O(\mu)$ neighbourhood of the point $s = x = 0$. This means that the general case averaged equations not coincide with equations of the equivalent control method and Filippov determination of solution. Moreover the introduction of positive feedback was used for transition from one vector of convex closure of the right hand part to the other one and for giving the system desired dynamic properties.

3. Eigenvalues Assignment in Averaged Equations using Dynamics of Actuators

3.1 Problem Formulation

Let us suppose that the behavior of control system is described with the state vector (s, x) $(s \in R, x \in R^n)$ is described with equations

$$\dot{s} = A_1 s + A_2 x + b_1 u(s), \dot{x} = A_3 s + A_4 x + b_2 u(s), \tag{3.1}$$

where the discontinuous control law has been designed in form $u(s) = sgn\,(s)$ Let us suppose that this control law ensure the stable sliding mode on surface $s = 0$.

Then the motions in sliding mode in system (3.1) are described by the equations

$$\dot{x} = (A_4 - b_2 b_1^{-1} A_3)x. \tag{3.2}$$

For eigenvalues assignment in system (3.2) in [17] was proposed to extend the state space by using of additional dynamics and then to solve the problem of eigenvalues assignment in extended state space.

In [15] was considered the problem of sintrol for the slow motions equations for systems which describe the behavior of the sliding mode systems with fast actuators. It was suggested to introduce the fast variable in equation for switching surface. This approach ensures the existence of first order sliding mode in complete system. For such systems in [5] the composite control method (see [6]) was used.

We suggest to use the dynamics of actuators which is present in system for control of motions in (3.1). Proposed algorithm is based on the theorem 1.2. It is necessary to mark that in this case we can use only slow coordinates of state-vector for control design and we can solve the eigenvalues assignment problem in space of sliding mode equations.

3.2 The Eigenvalues Assignment in Averaged Equations

Let us suppose that the complete model of control system allowing to take into account the fast actuator dynamics has the form

$$\mu\dot{z} = B_1 z + B_2 s + B_3 x + d_1 v$$

$$\dot{s} = B_4 z + B_5 s + B_6 x + d_2 v, \quad \dot{x} = B_7 z + B_8 s + B_9 x + d_3 v. \tag{3.3}$$

where $z \in R^m$, $v \in R^l$, μ is actuator time constant. Now we suppose that the conditions

$$rank \begin{pmatrix} B_4 \\ B_7 \end{pmatrix} \geq 2,$$

$$rank\, d_1 \geq 2, \quad m \geq l \geq 2 \tag{3.4}$$

are held.

The conditions (3.4) mean that the discontinuous control is transmitted to the plant through actuators and the dimension of state vector of actuators is more than dimension of control vector.

Ignoring actuator dynamics , having accepted that $\mu = 0$ and expressing z according to the formula $z = -B_1^{-1}(B_2 s + B_3 x + d_1 v)$ we obtain

$$\dot{s} = (B_5 - B_4 B_1^{-1} B_2)s + (B_6 - B_4 B_1^{-1} B_3)x + (d_2 - B_4 B_1^{-1} d_1)v \tag{3.5}$$

$$\dot{x} = (B_8 - B_7 B_1^{-1} B_2)s + (B_9 - B_7 B_1^{-1} B_3)x + (d_3 - B_7 B_1^{-1} d_1)v.$$

Let us suppose that in the case when the control law has been designed in the form $v = Ku(s)$ $(K, u(s) = sgn\, s$ is constant vector), systems (3.5) and (3.1) coincide.

Proposed algorithm uses theorem 1.2. We propose to use the control law in the form

$$v = Ku(s) + w. \tag{3.6}$$

¿From theorems 1.2 it follows that slow motions in (3.3) are described by the equations

$$\dot{x} = (A_4 - b_2 b_1^{-1} A_2)x - [d_3 - B_7 b_1^{-1}(d_2 - B_4 B_1^{-1} d_1)]w \tag{3.7}$$

Assume that

D.1. *Matrices*

$$(A_4 - b_2 b_1^{-1} A_2)\, and\, [d_3 - B_7 b_1^{-1}(d_2 - B_4 B_1^{-1} d_1)]$$

are controllable.

$$w = Lx$$

Then, choosing the control vector in form the $w = Lx$, we can solve the eigenvalues assignment problem for (3.7). Then if the conditions of Lemma 2.1 are fulfilled the equations (3.7) coincide with averaged equations which approximately describe the slow motions in a small neighbourhood of the switching surface of system (3.3). To use this algorithm it is necessary to ensure existence and stability in the first approximation of periodic solutions of the system

$$dz/d\tau = B_1 z + d_1 K u(s),\; ds/d\tau = B_4 z + d_2 K u(s) \tag{3.8}$$

describing the fast motions in (3.3) in the small neighbourhood of the equilibrium point $x = 0$ of averaged equation (3.7).

To formulate those sufficient conditions consider the point mapping $R(z)$ of the domain

$$\Omega^* = \{z \,:\, B_4 z - d_2 K > 0, z \in R^n\}$$

on the surface $s = 0$ into itself, given by the formulae

$$R^+(z) = e^{B_1 \tau_1}(z + B_1^{-1} d_1 K) - B_1^{-1} d_1 K,$$

$$R(z) = e^{B_1 \tau_2}(R^+(z) - B_1^{-1} d_1 K) + B_1^{-1} d_1 K,$$

where τ_1, τ_2 are the smallest positive roots of the equations

$$B_4 B_1^{-1}(e^{B_1 \tau_1} - I)(z + B_1^{-1} d_1 K) = (B_4 B_1^{-1} d_1 - d_2) K \tau_1,$$

$$B_4 B_1^{-1}(e^{B_1 \tau_2} - I)[e^{B_1 \tau_1}(z + B_1^{-1} d_1 K) - -2 B_1^{-1} d_1 K] = -(B_4 B_1^{-1} d_1 - d_2) K \tau_2.$$

Taking into account the symmetry of system (3.1) for $u(s) = sign\,(s)$ we can rewrite the conditions of existence of the fixed point in the form $R^+(z^*) = -z^*$. Then

$$z^* = [I + e^{B_1 T}]^{-1}(I - e^{B_1 T}) B_1^{-1} d_1 K,$$

where a semiperiod of desired periodic solution $T > 0$ is the smallest root of the equation

$$B_4 B_1^{-1}(e^{B_1 T} - I)(z^* + B_1^{-1} d_1 K) = (B_4 B_1^{-1} d_1 - d_2) K T$$

Then from theorems 1.2 it follows

Theorem 3.1. *Assume that condition D.1 is true and the conditions*

D.2 *Re Spec* $B_1 < 0$.

D.3. *The point mapping* $R(z)$ *has an isolated fixed point* $z^* \in \Omega^*$.

D.4. *For* $\lambda_i(x_0)$ $(i = 1, ..., m)$ *the eigenvalues of the matrix* $\frac{\partial R}{\partial z}(z^*)$ *the inequalities* $|\lambda_i| \neq 1$

are true.

Then the slow motions in system (3.3) within the accuracy $O(\mu)$ *are described by equation (3.7) and there exists a matrix* L *which provides that the characteristic polygon of the matrix*

$$(A_4 - b_2 b_1^{-1} A_2) - [d_3 - B_7 b_1^{-1}(d_2 - B_4 B_1^{-1} d_1)]L$$

has desired form.

3.3 Example

Let us suppose that the state vector of control system is described by the equations

$$\dot{s} = -u(s)/2, \qquad \dot{x}_1 = x_2,$$

$$\dot{x}_2 = -x_1, \qquad s, x_1, x_2 \in R \tag{3.9}$$

and discontinuous control $u(s) = -sign(s)$ has been designed. The motions in sliding mode in (3.9) are described by the equations

$$\dot{x}_1 = x_2, \dot{x}_2 = -x_1. \tag{3.10}$$

Spectrum of matrix in (3.10) is imaginary. Let us suppose that the discontinuous control $u(s)$ is transmitted to the plant with the help of actuators which behavior is described by variables z_1, z_2 and the complete model of the system has the following form

$$\mu\dot{z}_1 = -z_1 + v_1 - x_1, \mu\dot{z}_2 = -z_1 + v_2,$$

$$\dot{s} = z_2 + v_2/2, \ \dot{x}_1 = x_2, \ \dot{x}_2 = z_1. \tag{3.11}$$

It can be easily seen that in the case where we suppose that $v_1 = v_2 = -signs$, system (3.11) takes the form

$$\mu\dot{z}_1 = -z_1 - sign\, s - x_1, \quad \mu\dot{z}_2 = -z_2 - sign\, s,$$

$$\dot{s} = z_2 + 1/2 sign\, s, \ \dot{x}_1 = x_2, \ \dot{x}_2 = z_1.$$

and slow motions in it are described by system (3.10) within accuracy $O(\mu)$.

Let's show that for system (3.11) the conditions of theorem 3.1 are fulfilled. Consider the point mapping $R^+(z)$ of the domain

$$\Omega^* = \{(z, x) : z - 1/2 > 0\}$$

on the surface $s = 0$ into the domain $\Omega^- = \{(z, x) \,:\, z_2 - 1/2 > 0,\}$ made by the system

$$dz_1/d\tau = -z_1 - sign\, s, \quad dz_2/d\tau = -z_2 - sign\, s, \quad d\xi/d\tau = z_2 + 1/2 sign\, s. \quad (3.12)$$

The point mapping $R^+(z)$ of the domain Ω^* into the domain $\Omega^- = \{z : z_2 + 1/2 < 0\}$ made by the system (3.12) has the form

$$R^+(z) = \{R_1^+(z), R_2^+(z)\} = \{-1 + e^{-\tau}(z_1 + 1), -1 + e^{-\tau}(z_2 + 1)\},$$

where τ is the smallest root of the equation $(1 - e^{-\tau})(z_2 + 1) = \tau/2$. System (3.12) is symmetric with respect to the point $(0, 0)$ and consequently the condition of existence of fixed point z^* corresponding to desired periodic solution of (3.12) takes the form $R^+(z^*) = -z^*$. Then z^* and the semiperiod T satisfy the equations $z_2^* = th(T/2)$, $2th(T/2) = T/2$ with the solution $z_2^* = z_1^* \approx 0,95$, $T \approx 3,83$. Moreover

$$\frac{\partial R^+}{\partial(z_1, z_2)}(z^*) = \begin{pmatrix} --0,07 & 0 \\ 0 & -0,07 \end{pmatrix}$$

This means that for system (3.12) the conditions of theorem 1.2 are fulfilled and the slow motions in (3.12) are described by averaged equations (3.10). This means that for eigenvalues assignment in system can use the control law in the form

$$v_1 = -signs + l_1 x_1 + l_2 x_2, v = -signs.$$

Assume that for our goal the desired characteristic polygon of averaged equations

$$\dot{x}_1 = x_2,\ \dot{x}_2 = (l_1 - 1)x_1 + l_2 x_2 \quad (3.13)$$

has the form

$$\lambda^2 + \alpha\lambda + \beta,\ \alpha, \beta \,\text{are constants.} \quad (3.14)$$

This means that choosing $l_1 = 1 - \beta, l_2 = -\alpha$, we can ensure that the characteristic polygon of averaged system (3.13) has the form (3.14).

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References

[1] D.V. Anosov, *On stability of equilibrium points of relay systems,* Automatica i telemechanica (Automation and Remote Control), (1959),no. 2, 135-149 (Russian).

[2] J.P. Barbot, S. Monaco, D. Normand -Cyrot and N.Pantalos, *Discretization schemes for nonlinear singulerly perturbed systems*, Proc. of the 30th IEEE CDC, Brighton,(1991),pp.443-448.

[3] S.V Bogatyrev and L.M.Fridman, *Singular correction of the equivalent control method* , Differentialnye uravnenija (Differential equations),**28**(1992),no. 6,740-751.

[4] A.G. Bondarev, S.A.Bondarev, N.Ye.Kostylyeva and V.I. Utkin, *Sliding Modes in Systems with Asymptotic State Observers,* Automation and Remote Control,**46** (1985), 679-684.

[5] B. Heck, *Sliding mode control for singularly perturbed system,* International J. of Control, **50**(1991),no. 4, 985 - 1001.

[6] P.V. Kokotovic,H.K. Khalil and J. O'Railly, *Singular Perturbation Methods in Control: Analysis and Design.* Academic Press,1986.

[7] E.A. Coddington and N. Levinson, *Theory of ordinary differential equations*,McGraw - Hill,New-York, 1955.

[8] L.M. Fridman, Singular extension definition of discontinuous control systems and stability,*Differential equation*,**26** (1990), 1759-1764.

[9] L. Fridman and A. Levant, *Higher order sliding modes as the natural phenomenon in control theory.* in: Robust Control via variable structure and Lyapunov techniques ,Ed. F. Garafalo and L. Glielmo, Springer - Verlag, Berlin,1996, 103-130.

[10] Y.I.Neimark, About periodic solutions of relay systems. In.:*Memorize of A.A. Andronov*, Nauka,Moscow (Russian),1955.

[11] Y.I. Neimark, The method of averaging from the viewpoint of the point mapping method, *Izvestija Vusov, Radiophysika*, **VI** (1963), no. 5, 1023-1032 (Russian).

[12] Y.I. Neimark, *The point mapping method in the theory of nonlinear oscillations,* Nauka, Moscow, 1972 (Russian).

[13] L.S. Pontriagin and L.V. Rodygin, *Periodic solution of one system of differential equation with small parameter near the derivative,* Doklady Academii Nauk,**132 (1960)**,no. 3,537-540 (Russian).

[14] D. Ruelle,*Elements of differentiable dynamics and bifurcations theory*,Academic Press,1992.

[15] H. Sira-Ramires, *Sliding Regimes on Slow Manifolds of Systems with fast Actuators,* International J. of System Science,**37**(1988),no. 6.,875-887.

[16] V.A. Sobolev, *Integral manifold and decomposition of singularly perturbed systems,* Systems & Control Letters, **5**(1984),169-179.

[17] V.I. Utkin, *Sliding Modes in Control Optimization*, Springer Verlag,Berlin, 1992.

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