All-Optical Clock Recovery Using Multi-Section Distributed-Feedback Lasers

Daniela Peterhof and Björn Sandstede
Weierstraß-Institute for Applied Analysis and Stochastics
Mohrenstraße 39
10117 Berlin, Germany

Abstract

Recovering the frequency of incoming data sequences in optical transmission lines is important for signal processing. It has been suggested to use all-optical devices, for instance lasers diodes, for this purpose. Recently, self-pulsations have experimentally been discovered in multi-section distributed-feedback lasers. If a self-pulsating laser is exposed to an external data signal, it is expected that the frequency of the self-pulsation locks to the frequency of the data signal, and clock recovery would be obtained. Mathematically, this problem amounts to investigating frequency locking of periodic solutions, that is self-pulsating laser states, on invariant tori under external forcing which represents the external data sequence. In this article, Melnikov functions are derived for periodic forcing, and results on frequency locking under aperiodic forcing are given. The results are applied to a model describing the multi-section distributed-feedback lasers which have been used in experiments.

AMS subject classification. 35B10,34C15,78A60.

Keywords. Frequency entrainment, aperiodic forcing, laser.

1 Introduction

In recent years all-optical signal-processing systems have attracted much interest. Suppose that optical fibers are used to transmit a data signal. Typically, data signals are encoded by slow modulations of a rapidly-oscillating carrier wave traveling through the fiber. At the end of the fiber, the data sequence should be processed and regenerated. It is desirable to use optical devices for this purpose. Indeed, cumbersome electro-optic conversions could then be avoided. Figure 1 contains a schematic picture of an all-optical signal-processing device.

In a first step, the *power* frequency with which consecutive bits in the data sequence arrive at the processing device has to be recovered. The power frequency is only approximately known beforehand since it may change during transmission in the fiber due to dispersion and dependence of the refraction index on the intensity. It is necessary to obtain the precise value of the power frequency since otherwise the data signal cannot be recovered accurately. In fact, there are two frequencies which will play a role. The first is the aforementioned power frequency ω, that is the frequency of the data signal, the second is the *optical* frequency Ω which corresponds to the rapid oscillations of the carrier wave. All-optical clock recovery may be realized using lasers, see Figure 1. Suppose that the laser has a free-running quasiperiodic state which is given by

$$E(z,t) = e^{i\Omega_0 t}\Psi(z,\omega_0 t), \tag{1.1}$$

where $\Psi(z,\cdot)$ is 2π-periodic. In other words, the free-running state has power frequency ω_0 and optical frequency Ω_0 with $\Omega_0 \gg \omega_0$. Suppose that the laser can be tuned such that its frequencies are close to the one of the incoming data signal, that is $\omega_0 \approx \omega$ and $\Omega_0 \approx \Omega$. Clock recovery would work provided the frequencies of the laser would lock when exposed

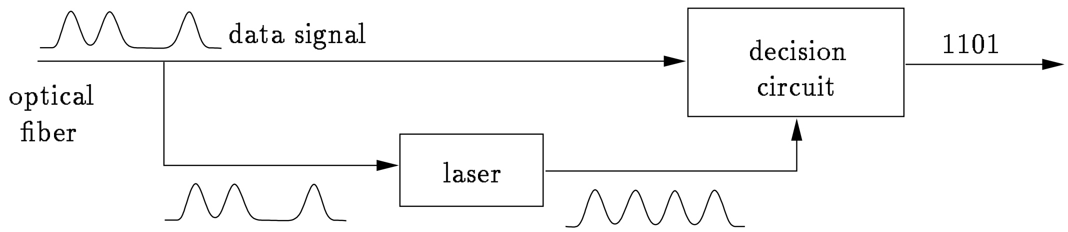

Figure 1: All-optical clock recovery using laser diodes. The self-pulsating laser locks to the frequency of the incoming data signal. The decision circuit uses the laser output to check whether a bit is set or not. If the frequency of the laser output is different from the frequency with which bits arrive at the decision circuit, the data signal is not regenerated exactly.

to a data signal

$$e^{i\Omega t}g(\omega t)$$

for some 2π-periodic function $g(\cdot)$. Note that this requires locking of two rather than one frequencies.

In the second step, the incoming periodic signal $g(\omega t)$ is changed to an arbitrary function $g_a(\omega t)$ encoding a bit sequence $a = (a_k)$ with $a_k \in \{0, 1\}$. The question is then whether the laser still stays sufficiently close to the locked state described above. Otherwise, clock recovery fails. This requires then a perturbation analysis of quasiperiodic solutions of the form (1.1) under aperiodic forcing.

Whether a laser is suitable for clock recovery depends on whether it supports a free-running state with frequencies close to those expected from the data signal. Distributed-feedback semiconductor lasers can be designed in that way. Their main feature is a spatial grating of the active waveguide. The spatial period of the grating selects the optical frequency Ω_0. Self-pulsations can be generated by decomposing the semiconductor into several sections with different optical properties. In addition, these sections are exposed to different injected currents.

In experiments, fast self-pulsations have recently been discovered in multi-section DFB lasers at the Heinrich-Hertz-Institute [14, 18]. These self-pulsations exhibit power frequencies from 10 to 80 GHz and are therefore of great technical interest. Within the range of about 10 to 20 GHz, their existence has been explained theoretically in [3, 4, 19]. Theory and experiment are compared in great detail in [3] for a 3-section DFB laser. Frequency locking of these self-pulsations to an injected signal with sinusoidal modulation has been demonstrated experimentally in [9]. The applicability of these self-pulsations for clock recovery has been shown experimentally at 18 GHz [8]. In [13], locking regions have been calculated numerically for a three-dimensional ODE model using direct simulations.

Based on a mathematical model of multi-section DFB lasers introduced recently by Bandelow *et al.* [4], we will derive conditions which will lead to frequency locking. Moreover, we investigate the dynamical behavior of the locked laser under aperiodic forcing. If certain conditions on the free-running laser state are satisfied, we will prove that the laser will remain near the locked state for low-powered external data signals. These conditions are formulated such that they can easily be verified numerically by computing only the self-pulsating state and certain solution of the adjoint linearization about this state. We will give an algorithm for the accurate numerical computation of the adjoint solutions.

Mathematically, we study a partial differential equation

$$u_t = Au + f(u) + \epsilon h(\omega t, u, \epsilon), \tag{1.2}$$

where A generates a C_0-semigroup. Here, u describes the amplitudes of the electric field and the carrier densities in the semiconductor laser. This equation has a S^1-symmetry since the electric field is only determined up to a phase. For $\epsilon = 0$, equation (1.2) has a solution $e^{\Omega_0 \Gamma t} p_0(\omega_0 t)$, where $p_0(t)$ is 2π-periodic in t and Γ generates the smooth group action $e^{\Gamma\beta}$ with $\beta \in S^1$. This solution corresponds to the free-running quasiperiodic state of the laser. We will derive bifurcation equations, the so-called Melnikov functions. A zero of the Melnikov functions corresponds to a quasiperiodic solution

$$e^{\Gamma(\Omega_0 + \epsilon\hat{\Omega})t} p_\epsilon((\omega_0 + \epsilon\hat{\omega})t)$$

of equation (1.2) for $\epsilon \neq 0$ and some 2π-periodic function $p_\epsilon(t)$.

Similar results have been obtained in [5, 6, 17] if (1.2) is an ordinary differential equation. In these articles, Ljapunov-Schmidt reduction has been used. Unfortunately, their proofs do not cover the case when A is a generator of a C_0-semigroup. Indeed, when applying Ljapunov-Schmidt reduction, time has to be rescaled such that the period is constant and the parameter ω appears as a factor on the right-hand side. However, taking derivatives with respect to ω results in a loss of regularity since $\frac{d}{d\omega} e^{\omega At} = Ate^{\omega At}$. If A were sectorial, this would be compensated by smoothing properties of the semigroup. C_0-semigroups, however, do not enjoy these properties.

Besides, we are mainly interested in a more geometric description in order to tackle aperiodic forcing which has not been addressed before. Therefore, and to avoid the aforementioned functional-analytical problems, we show that the invariant, attracting 2-torus

$$\left\{ e^{\Gamma\beta} p_0\left(\frac{\alpha}{\omega_0}\right) : (\alpha, \beta) \in S^1 \times S^1 \right\}$$

persists for $\epsilon \neq 0$. Therefore, we only have to consider suitable Poincare maps on the perturbed invariant torus. We will then prove that the Melnikov functions are the first-order terms in ϵ of the Poincare map restricted to the torus. Afterwards, aperiodic forcing is considered, and algorithms are presented which allow for the numerical computation of the Melnikov functions.

This paper is organized as follows. In Section 2, the mathematical model describing DFB lasers will be introduced. The main results are then given in Section 3 in an abstract setting. They are proved in Section 4. Finally, in Section 5, we apply the results to the model introduced in Section 2. Conclusions are also given.

2 The mathematical model

Here, we describe a model describing DFB lasers, which has been introduced in [4]. The laser consists of m sections given by intervals $S_j = [l_{j-1}, l_j]$, $j \in \{1, ..., m\}$, where $0 = l_0 < l_1 < ... < l_m = l$ and l is the longitudinal length of the laser. As mentioned above, the waveguide is corrugated, that is, it has a spatially periodic grating of period Λ.

Under reasonable assumptions, the amplitude of the main component of the electric field in laser is a superposition of forward and backward traveling waves

$$E(z,t) = e^{-i\frac{\pi z}{\Lambda}}\psi_+(z,t) + e^{i\frac{\pi z}{\Lambda}}\psi_-(z,t).$$

Here, time and spatial longitudinal variable are t and z, respectively. The spatial period of the grating of the waveguide is Λ. The grating will induce a feedback between forward and backward waves.

With this ansatz, the dynamics of an m-section laser can be described by the amplitudes

$$\psi(z,t) = (\psi_+(z,t), \psi_-(z,t)) \in \mathbb{C}^2$$

and the carrier densities

$$N(t) = (N_1(t), ..., N_m(t)) \in \mathbb{R}^m_+$$

of the electrons in the m sections of the laser. For the sake of simplicity, the space-dependence of the carrier densities will not be taken into account.

The time evolution of ψ is governed by the traveling-wave equation

$$\psi_t = H(N,z)\psi, \qquad 0 < z < l,$$

where $H(N,z)$ is the matrix operator

$$H(N,z) = v_g \begin{pmatrix} -\partial_z - i\beta(N,z) & i\kappa_+ \\ i\kappa_- & \partial_z - i\beta(N,z) \end{pmatrix}.$$

Here, v_g is the group velocity of light in the laser medium, and $\kappa_+, \kappa_- \in \mathbb{C}$ are the coupling coefficients due to the presence of the grating. The propagation constant $\beta(N,z) \in \mathbb{C}$ is constant in each section, that is, $\beta(N,z) = \beta_j(N_j)$ for $z \in S_j$, $j \in \{1, ..., m\}$. It is modeled by

$$\beta_j(N_j) = I_j\frac{\beta_{Ij}}{l_j} + \beta_{Nj}(\frac{N_j}{V_j} - N_{tj}) + \frac{i}{2}(G_j(N_j) - \alpha_{0j}) + \frac{\pi}{\Lambda},$$

see [2], using the gain model

$$G_j(N_j) = d_j N_{tj} \ln(\frac{N_j}{V_j N_{tj}}).$$

All parameters described above are positive and their value depend on the section which is indicated by the index j. The individual sections are exposed to the injection currents I_j. The currents may lead to heating which is accounted for by the parameters β_{Ij}. Similarly, the carrier contribution is subsumed in the parameters β_{Nj}, and V_j is the volume of the section. Absorption in the waveguide is denoted by α_{0j}. Finally, d_j and N_{tj} are gain and transparency density, respectively.

The dynamics of the carrier numbers is now governed by the balance equation

$$\dot{N}_j = F_j(N_j, \psi) \tag{2.1}$$

with

$$F_j(N_j,\psi) = \frac{I_j}{e} - \frac{N_j}{\tau_j} - \frac{G_j(N_j)}{\tilde{\Omega}\hbar}\int_{S_j}(|\psi_+|^2+|\psi_-|^2)\,dz, \tag{2.2}$$

where e is the elementary charge, $\tilde{\Omega}\hbar \approx \Omega_0\hbar$ is the energy of a single photon, and τ_j is the carrier lifetime in section S_j.

Summarizing we consider the equation

$$\begin{aligned} \dot{N} &= F(N,\psi), \\ \psi_t &= H(N,z)\psi, \quad z\in(0,l). \end{aligned} \tag{2.3}$$

where $F(N,\psi)_j = F_j(N_j,\psi)$.

At the facets of the laser, we assume reflection boundary conditions for the traveling-wave amplitudes

$$\begin{aligned} \psi_+(0,t) &= r_0\psi_-(0,t) + \epsilon\, e^{i\Omega t}\, g_a(\omega t), \\ \psi_-(l,t) &= r_l\psi_+(l,t), \end{aligned} \tag{2.4}$$

where r_0, $r_l \in \mathbb{C}$ are reflection coefficients with $|r_0|, |r_l| < 1$. The function $g_a : \mathbb{R} \to \mathbb{C}$ represents the data signal injected at the left facet of the laser and $\epsilon \geq 0$ is the power of this signal. The constants Ω and ω are optical and power frequencies of the data signal.

We will consider the following data signals. Choose any bit sequence $a_k \in \{0,1\}$ with $k \in \mathbb{N}$. The bit sequence $a = (a_k)$ is then encoded into a data signal according to

$$g_a(t) = a_k\hat{g}(t) \text{ for } t\in[2k\pi, 2(k+1)\pi), k\in\mathbb{N}_0,$$

where $\hat{g}(t)$ is a suitable 2π-periodic function.

We assume that for $\epsilon = 0$ and $I_j = I_j^0$ the DFB laser is in a stable self-pulsating state

$$(N(t),\psi(z,t)) = (N^p(t), e^{i\Omega_0 t}\Psi^p(z,t)),$$

where $N^p(t)$ and $\Psi^p(z,t)$ are $\frac{2\pi}{\omega_0}$-periodic in t. We remark that the existence of self-pulsations has been shown numerically in [4, 13, 20].

In order to describe equation (2.3)-(2.4) as an autonomous, semilinear evolution equation on an appropriate Hilbert space with a small periodic perturbation, we introduce the following transformation. For $\epsilon = 0$, the boundary-value problem (2.3)-(2.4) is equivariant with respect to the S^1-action

$$(N,\psi) \longrightarrow (N, e^{i\beta}\psi).$$

Therefore, we introduce new coordinates

$$\psi(z,t) = e^{i\Omega t}\,\tilde{\Psi}(z,t)$$

to have a pure periodic forcing. Furthermore, we set

$$\widetilde{\Psi}(z,t) = \Psi(z,t) + \epsilon G(z,\omega t),$$

where

$$G(z,\omega t) = \frac{l-z}{l} g_a(\omega t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This will transform the inhomogeneous boundary conditions into homogeneous conditions. We obtain

$$\begin{aligned} \dot{N} &= F(N, \Psi + G(z,\omega t)) \\ \Psi_t &= (H(N,z) - i\Omega)\Psi + \epsilon(H(N,z) - i\Omega - \partial_t)G(z,\omega t), \end{aligned} \tag{2.5}$$

with boundary conditions

$$\begin{aligned} \Psi_+(0,t) &= r_0 \Psi_-(0,t) \\ \Psi_-(l,t) &= r_l \Psi_+(l,t). \end{aligned} \tag{2.6}$$

Equation (2.5)-(2.6) then has a $\frac{2\pi}{\omega_0}$-periodic solution $(N^p(t), \Psi^p(z,t))$ for $\Omega = \Omega_0$ and $\epsilon = 0$.

In Section 5, we show that equation (2.5) is in the abstract class of equations which will be introduced in the next section.

3 Results for a general setting

We consider the abstract evolution equation

$$u_t = Au - \Omega\Gamma u + f(u) + \epsilon h(\omega t, u, \epsilon), \tag{3.1}$$

where u belongs to a real Hilbert space X equipped with the scalar product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$. Furthermore, $(\epsilon,\omega,\Omega) \in \mathbb{R}^3$ are parameters, and $t \geq 0$ denotes the time variable. We assume the following on A, Γ, f and h.

- (A1) $A : X \to X$ is a closed operator with dense domain $D(A)$ and generates a C_0-semigroup on X.
- (A2) $f \in C^k(X,X)$, $h \in C^k(\mathbb{R} \times X \times \mathbb{R}, X)$ for some $k \geq 2$ and $h(t,u,\epsilon)$ is 2π-periodic in t.
- (A3) The operator $\Gamma \in L(X)$ generates a smooth group action $e^{\Gamma\beta}$ with $\beta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $e^{2\pi\Gamma} = \mathrm{id}$. For $\epsilon = 0$, equation (3.1) commutes with this action.

We remark that the results presented here can be generalized to more general compact groups.

In addition, we assume the existence of a periodic solution for $\epsilon = 0$.

(P) Equation (3.1) has a periodic solution $p_0(t) \in D(A)$ of minimal period $T_0 = \frac{2\pi}{\omega_0}$ for $(\epsilon, \Omega) = (0, \Omega_0)$. We assume that the functions $\dot{p}_0(t)$ and $\Gamma p_0(t)$ are linearly independent.

The assumption on linear independence of $\dot{p}_0(t)$ and $\Gamma p_0(t)$ implies that $p_0(t)$ is not contained in the group orbit $\{e^{\Gamma\beta} p_0(0) : \beta \in S^1\}$ of $p_0(0)$. In other words, $p_0(t)$ is not a relative equilibrium.

We denote the evolution operator of the variational equation

$$v_t = (A - \Omega_0\Gamma + Df(p_0(t)))v \tag{3.2}$$

associated with $p_0(t)$ by $U(t, \tau) \in L(X)$ for $t \geq \tau$. It can be shown that the spectrum $\mathrm{spec}(U(t + T_0, t))\backslash\{0\}$ of the period maps is independent of t. Indeed, the proof of Henry's result [10, Lemma 7.2.2] for analytic semigroups works also for C_0-semigroups.

(S) The eigenvalue $\lambda = 1$ of $U(t+T_0, t)$ has multiplicity two and the rest of the spectrum

$$\sup\{|\lambda| : \lambda \in \mathrm{spec}(U(t + T_0, t)), \lambda \neq 1\} < 1$$

is strictly contained in the unit circle.

It is clear that $\lambda = 1$ has at least multiplicity two on account of Hypothesis (P) since $\dot{p}_0(t)$ and $\Gamma p_0(t)$ are T_0-periodic solutions of (3.2).

3.1 Frequency locking

In order to study the dynamics of (3.1) for non-zero ϵ, we introduce the variable $s = \omega t \in S^1$ and consider the autonomous equation

$$\begin{aligned} u_t &= Au - \Omega\Gamma u + f(u) + \epsilon h(s, u, \epsilon), \\ \dot{s} &= \omega, \end{aligned} \tag{3.3}$$

with $(u, s) \in X \times S^1$. Assumptions (P) and (S) imply that equation (3.3) has a semiflow-invariant and exponentially attracting 3-torus $\mathcal{T}$ in $D(A)$ for $(\epsilon, \omega, \Omega) = (0, \omega_0, \Omega_0)$. The 3-torus $\mathcal{T}$ is given by the union $W_{(0,\omega_0,\Omega_0)} \times \{s\}$ of 2-tori

$$W_{(0,\omega_0,\Omega_0)} = \Big\{e^{\Gamma\beta} p_0\Big(\frac{\alpha}{\omega_0}\Big) : (\alpha, \beta) \in S^1 \times S^1\Big\}.$$

It can be parametrized by

$$(\alpha, \beta, s) \longrightarrow \Big(e^{\Gamma\beta} p_0\Big(\frac{s + \alpha}{\omega_0}\Big), s\Big). \tag{3.4}$$

This parametrization corresponds to a transformation into a moving frame. The first result is straightforward. It states that the invariant torus persists upon varying $(\epsilon, \omega, \Omega)$.

Proposition 1 *Assume that (A1)-(A3), (P), and (S) are met. Then, for any* $(\epsilon, \omega, \Omega)$ *close to* $(0, \omega_0, \Omega_0)$*, equation (3.3) has a unique semiflow-invariant 3-torus close to* $\mathcal{T}$ *in* $D(A)$*. It depends smoothly on* $(\epsilon, \omega, \Omega)$ *and is exponentially attracting. Moreover, it can be parametrized by a smooth map*

$$(\alpha, \beta, s) \longmapsto (\theta(\alpha, \beta, s, \epsilon, \omega, \Omega), s) \in D(A) \times S^1$$

such that $\theta(\alpha, \beta, s, 0, \omega_0, \Omega_0)$ *coincides with (3.4). In addition, the 3-torus is given by a union* $W_{(\epsilon,\omega,\Omega)}(s) \times \{s\}$ *of 2-tori with* $s \in S^1$.

In the next step, we should determine the dynamics on the perturbed 3-torus. Note that the last equation in (3.3) can be solved explicitly. Moreover, the 2-tori $W_{(\epsilon,\omega,\Omega)}(s)$ are invariant under the time-$\frac{2\pi}{\omega}$ map. It suffices therefore to investigate the time-$\frac{2\pi}{\omega}$ map

$$\tilde{\Pi}_{(\epsilon,\omega,\Omega)} : W_{(\epsilon,\omega,\Omega)}(0) \longrightarrow W_{(\epsilon,\omega,\Omega)}(0) \tag{3.5}$$

associated with (3.1) restricted to one of the 2-tori.

Using the parametrization provided by Proposition 1, the map $\tilde{\Pi}$ can be represented by a map

$$\begin{array}{rcccc} \Pi_{(\epsilon,\omega,\Omega)} & : & S^1 \times S^1 & \longrightarrow & S^1 \times S^1 \\ & & (\alpha, \beta) & \longmapsto & \Pi_{(\epsilon,\omega,\Omega)}(\alpha, \beta). \end{array}$$

We have

$$\Pi_{(0,\omega,\Omega)}(\alpha, \beta) = (\alpha + 2\pi(\frac{\omega_0}{\omega} - 1), \beta + \frac{2\pi}{\omega}(\Omega - \Omega_0)), \tag{3.6}$$

and, in particular,

$$\Pi_{(0,\omega_0,\Omega_0)} = \mathrm{id}\,.$$

In order to calculate the perturbed Poincare map, we need to exploit the adjoint variational equation

$$w_t = -(A - \Omega_0 \Gamma + Df(p_0(t)))^* w. \tag{3.7}$$

Note that under the assumptions imposed on A the adjoint operator A^* satisfies (A1) since X is a Hilbert space, see [16, Section 1.10].

On account of (P), equation (3.7) has two T_0-periodic solutions w_1, w_2 which, after a suitable normalization, satisfy

$$\begin{array}{rclcrcl} \langle w_1(t), \dot{p}_0(t) \rangle & = & \omega_0, & \quad & \langle w_1(t), \Gamma p_0(t) \rangle & = & 0, \\ \langle w_2(t), \dot{p}_0(t) \rangle & = & 0, & \quad & \langle w_2(t), \Gamma p_0(t) \rangle & = & 1, \end{array} \tag{3.8}$$

for $t \in \mathbb{R}$. This normalization is possible since the scalar product $\langle w(t), v(t) \rangle$ of any two solutions $w(t)$ and $v(t)$ of (3.7) and (3.2), respectively, does not depend on time. We then define

$$\begin{aligned} M_1(\alpha,\beta,\hat{\omega},\hat{\Omega}) &= -T_0\hat{\omega} + \int_0^{T_0} \left\langle w_1(t), e^{-\Gamma\beta} h(\omega_0 t - \alpha, e^{\Gamma\beta} p_0(t), 0)\right\rangle dt \\ M_2(\alpha,\beta,\hat{\omega},\hat{\Omega}) &= -T_0\hat{\Omega} + \int_0^{T_0} \left\langle w_2(t), e^{-\Gamma\beta} h(\omega_0 t - \alpha, e^{\Gamma\beta} p_0(t), 0)\right\rangle dt, \end{aligned} \tag{3.9}$$

with $(\alpha,\beta) \in S^1 \times S^1$ and $(\hat{\omega},\hat{\Omega}) \in \mathbb{R}^2$. Let

$$M_{(\hat{\omega},\hat{\Omega})}(\alpha,\beta) = (M_1, M_2)(\alpha,\beta,\hat{\omega},\hat{\Omega}).$$

The functions M_j are the Melnikov integrals which will determine existence and stability of periodic solutions of (3.1) close to the group and time orbit of $p_0(t)$.

Theorem 1 *Assume that (A1)-(A3), (P), and (S) are met. Let $(\omega,\Omega) = (\omega_0 + \epsilon\hat{\omega}, \Omega_0 + \epsilon\hat{\Omega})$ for $(\hat{\omega},\hat{\Omega})$ in some compact set $K \subset \mathbb{R}^2$. There exists then an $\epsilon_0 > 0$ such that for any $|\epsilon| < \epsilon_0$ the Poincare map $\Pi_{(\epsilon,\omega,\Omega)}$ is given by*

$$\Pi_{(\epsilon,\omega,\Omega)}(\alpha,\beta) = (\alpha,\beta) + \epsilon M_{(\hat{\omega},\hat{\Omega})}(\alpha,\beta) + \mathrm{O}(\epsilon^2),$$

with the aforementioned relation between (ω,Ω) and $(\hat{\omega},\hat{\Omega})$. The remainder term is smooth and $\mathrm{O}(\epsilon^2)$ uniformly in $(\alpha,\beta,\hat{\omega},\hat{\Omega})$.

The proofs of the theorem and the following results are contained in Section 4.

We state the existence and stability of periodic solutions of (3.1) with frequency $\omega_0 + \epsilon\hat{\omega}$ as corollaries.

Corollary 1 *Suppose that there exists $(\hat{\omega}_0,\hat{\Omega}_0) \in K$ and (α_0,β_0) such that*

$$M_{(\hat{\omega}_0,\hat{\Omega}_0)}(\alpha_0,\beta_0) = 0 \text{ and } D_{(\alpha,\beta)} M_{(\hat{\omega}_0,\hat{\Omega}_0)}(\alpha_0,\beta_0) \text{ is invertible.}$$

There exist then numbers $\epsilon_0 > 0$ and $\delta_0 > 0$ with the following property: For any $(\epsilon,\hat{\omega},\hat{\Omega})$ with $0 < \epsilon < \epsilon_0$ and $|\hat{\omega} - \hat{\omega}_0| + |\hat{\Omega} - \hat{\Omega}_0| < \delta_0$, there exists a unique $\frac{2\pi}{\omega_0 + \epsilon\hat{\omega}}$-periodic solution of (3.1) close to $e^{\Gamma\beta_0} p_0(\frac{\alpha_0}{\omega_0})$. This solution depends smoothly on its arguments.

Corollary 2 *Suppose that in addition to the assumptions of Corollary 1 the matrix*

$$D_{(\alpha,\beta)} M_{(\hat{\omega}_0,\hat{\Omega}_0)}(\alpha_0,\beta_0)$$

has only eigenvalues with real part less than λ_0 for some $\lambda_0 < 0$. Choose $\eta_0 > 0$ such that

$$\|D^2_{(\alpha,\beta)} M_{(\hat{\omega}_0,\hat{\Omega}_0)}(\alpha,\beta)\| \leq \frac{|\lambda_0|}{\eta_0} \text{ for all } (\alpha,\beta) \text{ with } |\alpha - \alpha_0| + |\beta - \beta_0| < \eta_0$$

There exists then an $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ the associated $\frac{2\pi}{\omega_0 + \epsilon\hat{\omega}}$-periodic solution of (3.1) attracts an η_0-neighborhood with rate $1 + \frac{1}{2}\epsilon\lambda_0$.

In terms of the application, the above corollaries state that the laser locks to the injected periodic signal provided the assumptions of the corollaries are satisfied. These conditions are computable provided the self-pulsating laser state and the signature $\hat{g}(t)$ of the data signal are known beforehand.

Remark 3.1 *Suppose that the nonlinearity f depends on additional parameters $\mu \in \mathbb{R}^p$, that is $f = f(u, \mu)$, and the assumptions of the theorem and the corollaries are met for $\mu = \mu_0$. The statements of Theorem 1 and its corollaries remain then true even if μ_0 is changed to $\mu_0 + \epsilon\hat{\mu}$ provided the additional terms*

$$\int_0^{T_0} \left\langle w_j(t), D_\mu f(p_0(t), \mu_0)\hat{\mu} \right\rangle dt$$

are added to the Melnikov integrals in equation (3.9).

Remark 3.2 *Similarly, the forcing term $h = h(\omega t, u, \epsilon, \mu, \Omega)$ may depend on (μ, Ω) for $\mu \in \mathbb{R}^p$. The statements in this section remain then true without any change.*

Remark 3.1 can be used to characterize locking regions if other parameters, for instance the injected currents, are also varied upon exposing the laser to a periodic data signal.

3.2 Aperiodic forcing

We assume that the bit sequence $a_k = 1$ for all k generates a stable periodic solution of (3.1).

(B) Assume $h(t, u, \epsilon)$ satisfies (A2). Suppose that there are (α_0, β_0) and $(\hat{\omega}_0, \hat{\Omega}_0)$ such that $M_{(\hat{\omega}_0, \hat{\Omega}_0)}(\alpha_0, \beta_0) = 0$ and the eigenvalues of $D_{(\alpha,\beta)} M_{(\hat{\omega}_0, \hat{\Omega}_0)}(\alpha_0, \beta_0)$ are simple and have real part less than λ_0 for some $\lambda_0 < 0$.

Since h satisfies (A2), we may apply Corollary 1 and 2.

We will now consider general input data signals. Choose any bit sequence $a_k \in \{0, 1\}$ with $k \in \mathbb{N}$. The bit sequence $a = (a_k)$ is encoded into a data signal according to

$$h_a(t, u, \epsilon) = a_k h(t, u, \epsilon) \qquad \text{for } t \in [2\pi k, 2\pi(k+1))$$

with $k \in \mathbb{N}$. The semiflow of (3.1) with h replaced by h_a is then a suitable composition of the semiflows associated with

$$\begin{aligned} \dot{u} &= Au - (\Omega_0 + \epsilon\hat{\Omega})\Gamma u + f(u), \\ \dot{s} &= \omega_0 + \epsilon\hat{\omega}, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \dot{u} &= Au - (\Omega_0 + \epsilon\hat{\Omega})\Gamma u + f(u) + \epsilon h(s, u, \epsilon), \\ \dot{s} &= \omega_0 + \epsilon\hat{\omega}. \end{aligned} \tag{3.11}$$

We denote the corresponding time-$\frac{2\pi}{\omega_0+\epsilon\hat{\omega}}$ maps of (3.10) and (3.11) by Φ_0 and Φ_1, respectively. These maps are then composed in the order given by the sequence $a = (a_k)$. In other words,

$$u\Big(\frac{2\pi k}{\omega_0 + \epsilon\hat{\omega}}\Big) = \Phi_{a_k}\, \Phi_{a_{k-1}} \dots \Phi_{a_0}\, u(0)$$

for $k \in \mathbb{N}$ coincides with the solution of (3.1) with initial value $u(0)$ and with h replaced by h_a.

Clock recovery, as explained in the introduction, only works if $u(\frac{2\pi k}{\omega_0+\epsilon\hat{\omega}})$ stays in a small neighborhood of $e^{\Gamma\beta_0}p_0(\frac{\alpha_0}{\omega_0})$ for all $k \in \mathbb{N}$. This is clearly impossible if $a_k = 0$ for all k and $\epsilon\hat{\omega} \neq 0$. Indeed, the solution

$$u\Big(\frac{2\pi k}{\omega_0 + \epsilon\hat{\omega}}\Big) = (\Phi_0)^k\, u(0)$$

drifts away, see (3.6). To avoid this behavior, $a_k = 1$ is required for sufficiently many indices k. Therefore, for fixed $n \in \mathbb{N}$, we will consider bit sequences in the set

$$\Sigma_n = \{a : \text{ for any } k \in \mathbb{N}, \text{ there is at least one } j \in \{0, \dots, n-1\} \text{ with } a_{kn+j} = 1\}.$$

By Hypothesis (B), the matrix

$$D_{(\alpha,\beta)}M_{(\hat{\omega}_0,\hat{\Omega}_0)}(\alpha_0, \beta_0)$$

has simple eigenvalues. We denote the associated normalized eigenvectors by e_1 and e_2. Let V be the 2×2 matrix with columns given by e_1 and e_2. We define

$$\mathcal{N}(\eta, C) = \Big\{u = e^{\Gamma\beta}p_0(\frac{\alpha}{\omega_0}) + v \in X : \, |v| < C\epsilon \text{ and } |V^{-1}(\alpha - \alpha_0, \beta - \beta_0)| < \eta\Big\}.$$

Theorem 2 *Assume that (A1)-(A3), (P), (S), and (B) are met. Choose $\eta_1 > 0$ such that*

$$\|D^2_{(\alpha,\beta)}M_{(\hat{\omega}_0,\hat{\Omega}_0)}(\alpha, \beta)\| \leq \frac{|\lambda_0|\,|\det V|}{\eta_1} \text{ for all } (\alpha, \beta) \text{ with } |\alpha - \alpha_0| + |\beta - \beta_0| < \eta_1,$$

with V as defined right before the theorem. Suppose that $\eta < \frac{1}{2}\eta_1$ and

$$4(n-1)T_0 \sup(|\hat{\omega}|, |\hat{\Omega}|) < \eta|\lambda_0|\,|\det V| \tag{3.12}$$

for the constants appearing in (B). There is then a constant $C_0 > 0$ such that for any sequence $a \in \Sigma_n$, $C > C_0$, and $0 < \epsilon < \epsilon_0$ the following is true. If $u(0) \in \mathcal{N}(\eta, C)$, then $u(\frac{2\pi k}{\omega_0+\epsilon\hat{\omega}}) \in \mathcal{N}(\eta + \mathrm{O}(\epsilon), C)$ for all $k \in \mathbb{N}$.

This theorem is of relevance for clock recovery. Indeed, suppose that the external data signal begins with a periodic sequence. The self-pulsating laser would then lock to the power frequency of the signal provided the assumptions of Corollary 1 and 2 are met. Afterwards, the true data signal encoding a bit sequence is being sent. The laser should remain in the locked state, or at least stay close to it. Theorem 2 shows that it will provided (3.12) is satisfied. We point out that this condition is again computable provided the self-pulsating laser state and the signature $\hat{g}(t)$ of the data signal are known. The inequality (3.12) reflects a competition of the failure of frequency entrainment for $a_k = 0$ and the attraction to the locked state for $a_k = 1$.

3.3 Numerical computation of w_1 and w_2

Consider equation (3.1)

$$u_t = Au - \Omega\Gamma u + f(u) \tag{3.13}$$

for $\epsilon = 0$. We will give an algorithm for the numerical calculation of the solutions w_1 and w_2 mentioned above. Assume that a T_0-periodic solution of (3.13) has been computed.

A standard approach is to consider the operator

$$\begin{array}{rccc} \tilde{L}: & C^1((0,T_0),X) \cap C^0([0,T_0],D(A)) & \longrightarrow & C^0([0,T_0],X) \\ & w & \longmapsto & w_t + (A - \Omega_0\Gamma + Df(p_0(t)))^* w, \end{array}$$

together with boundary and normalization conditions. Unfortunately, this approach fails in the context of C_0-semigroups since the operator $\tilde{L}$ is not invertible. Indeed, even the inhomogeneous linear equation

$$w_t + A^* w = h(t), \qquad h \in C^0([0,T_0],X),$$

will not have a solution which is C^1 in time, see [16].

The evolution operator of the adjoint variational equation

$$w_t = -(A - \Omega_0\Gamma + Df(p_0(t)))^* w$$

is given by $U(\tau,t)^*$ for $t \le \tau$, see [10], where $U(\tau,t)$ denotes the evolution operator of the linearization about $p_0(t)$.

We then introduce the linear operator

$$L(w,\mu) = \begin{pmatrix} w(t) - U(T_0,t)^* w(T_0) \\ w(T_0) - w(0) + \mu_1 \dot{p}_0(0) + \mu_2 \Gamma p_0(0) \\ \int_0^{T_0} \langle w(t), \dot{p}_0(t) \rangle \, dt \\ \int_0^{T_0} \langle w(t), \Gamma p_0(t) \rangle \, dt \end{pmatrix}, \tag{3.14}$$

with $(w,\mu) \in C^0([0,T_0],X) \times \mathbb{R}^2$. Let

$$C^0_{T_0}([0,T_0],X) := \{h \in C^0([0,T_0],X) : \; h(T_0) = 0\}.$$

Note that the solution of

$$w_t + (A - \Omega_0 \Gamma + Df(p_0(t)))^* w = \tilde{h}(t)$$

is given by

$$w(t) = U(T_0, t)^* w(T_0) + \int_{T_0}^{t} U(\tau, t)^* \tilde{h}(\tau)\, d\tau =: U(T_0, t)^* w(T_0) + h(t),$$

and, in particular, $h(T_0) = 0$. Therefore, the above definition makes sense. We then have the following Theorem.

Theorem 3 *The operator*

$$L : C^0([0, T_0], X) \times \mathbb{R}^2 \to C^0_{T_0}([0, T_0], X) \times X \times \mathbb{R} \times \mathbb{R}$$

is an isomorphism. Moreover,

$$(w_1, 0) = L^{-1}(0, 0, 2\pi, 0), \qquad (w_2, 0) = L^{-1}(0, 0, 0, T_0)$$

are the T_0-periodic solutions of (3.7) satisfying (3.8).

Note that the artificial parameters $\mu \in \mathbb{R}^2$ are needed to guarantee that L is invertible. They can be used as error indicators if the linearization is not evaluated about $p_0(t)$ but about a slightly perturbed function, that is, if

$$w_t = -(A - \Omega_0 \Gamma + Df(p_0(t)))^* w + B(t)^* w$$

is solved with $\|B\|$ small. The operator L then becomes a operator called L_B with is given by (3.14) with the first component replaced by

$$w(t) - U(T_0, t)^* w(T_0) - \int_{T_0}^{t} U(\tau, t)^* B(\tau)^* w(\tau)\, d\tau.$$

If $\|B\| < \|L^{-1}\|^{-1}$, we can still invert the operator L_B and obtain the estimate

$$\|L_B^{-1} - L^{-1}\| \leq C\, \|B\|, \qquad C := \frac{\|L^{-1}\|^2}{1 - \|L^{-1}\|\, \|B\|}.$$

Self-pulsating solution can be found by solving the equation

$$\begin{pmatrix} u_t - \omega(Au - \Omega \Gamma u + f(u)) \\ u(2\pi) - u(0) \\ \langle \Gamma p_0(0), u(0) - p_0(0) \rangle \\ \int_0^{2\pi} \left\langle \dot{p}_0(\frac{t}{\omega_0}), u(t) - p_0(\frac{t}{\omega_0}) \right\rangle dt \end{pmatrix} = 0,$$

for (u, ω, Ω). We are, however, unable to prove convergence of this algorithm due to the functional-analytic difficulties mentioned before.

4 Proofs

The proof of Proposition 1 is straightforward and will be omitted, see, for instance, [11].

4.1 Proof of Theorem 1

The equation under consideration is

$$\begin{aligned} \dot{u} &= Au - \Omega\Gamma u + f(u) + \epsilon h(s,u,\epsilon), \\ \dot{s} &= \omega. \end{aligned} \tag{4.1}$$

Under the assumptions of Theorem 1, we have

$$(\omega, \Omega) = (\omega_0 + \epsilon\hat{\omega}, \Omega_0 + \epsilon\hat{\Omega}). \tag{4.2}$$

We shall prove the expansion

$$\Pi_{(\epsilon,\omega,\Omega)}(\alpha,\beta) = (\alpha,\beta) + \epsilon M_{(\hat{\omega},\hat{\Omega})}(\alpha,\beta) + \mathrm{O}(\epsilon^2),$$

where $M_{(\hat{\omega},\hat{\Omega})}$ has been given in (3.9). Note that $\Pi_{(\epsilon,\omega,\Omega)}$ represents the time-$\frac{2\pi}{\omega}$ map

$$\tilde{\Pi}_{(\epsilon,\omega,\Omega)} : W_{(\epsilon,\omega,\Omega)}(0) \longrightarrow W_{(\epsilon,\omega,\Omega)}(0),$$

see (3.5). In particular, we have $s(0) = 0$ in (4.1).

It follows from Proposition 1 that the center manifold of (4.1) is parametrized by

$$(\alpha,\beta,s) \longmapsto (\theta(\alpha,\beta,s,\epsilon,\omega,\Omega), s) \quad \text{for} \quad (\alpha,\beta,s) \in S^1 \times S^1 \times S^1.$$

Moreover, θ is smooth in all its arguments. In addition, we have

$$\theta_0(\alpha,\beta,s) := \theta(\alpha,\beta,s,0,\omega_0,\Omega_0) = e^{\Gamma\beta} p_0(\frac{s+\alpha}{\omega_0}). \tag{4.3}$$

Using (4.2), we may then write θ according to

$$\theta(\alpha,\beta,s,\epsilon,\omega,\Omega) = \theta_0(\alpha,\beta,s) + \epsilon\theta_1(\alpha,\beta,s,\epsilon,\omega,\Omega), \tag{4.4}$$

for some suitable smooth and bounded function θ_1. In particular, we have

$$\begin{aligned} D_\alpha\theta &= \frac{1}{\omega_0} e^{\Gamma\beta} \dot{p}_0(\frac{s+\alpha}{\omega_0}) + \mathrm{O}(\epsilon), \\ D_\beta\theta &= \Gamma e^{\Gamma\beta} p_0(\frac{s+\alpha}{\omega_0}) + \mathrm{O}(\epsilon), \\ D_s\theta &= \frac{1}{\omega_0} e^{\Gamma\beta} \dot{p}_0(\frac{s+\alpha}{\omega_0}) + \mathrm{O}(\epsilon), \end{aligned} \tag{4.5}$$

where the remainder terms are uniform in all arguments.

The flow on the center manifold can now be calculated by substituting $u = \theta(\alpha, \beta, s, \epsilon, \omega, \Omega)$ into equation (4.1). We obtain

$$\begin{aligned} \dot{\alpha} D_\alpha \theta + \dot{\beta} D_\beta \theta + \dot{s} D_s \theta &= A\theta - \Omega\Gamma\theta + f(\theta) + \epsilon h(s, \theta, \epsilon), \\ \dot{s} &= \omega, \end{aligned}$$

and, after substituting the expression for $\dot{s}$,

$$\dot{\alpha} D_\alpha \theta + \dot{\beta} D_\beta \theta + \omega D_s \theta = A\theta - \Omega\Gamma\theta + f(\theta) + \epsilon h(s, \theta, \epsilon). \tag{4.6}$$

This equation should be solved with values in X. Note that $A\theta \in X$ exists since θ is a smooth map into $D(A)$. For $\epsilon = 0$, we get

$$\begin{aligned} \dot{\alpha} D_\alpha \theta_0 + \dot{\beta} D_\beta \theta_0 + \omega_0 D_s \theta_0 &= A\theta_0 - \Omega_0 \Gamma\theta_0 + f(\theta_0), \\ \dot{s} &= \omega_0. \end{aligned}$$

Therefore, $s(t) = \omega_0 t$, and it remains to solve

$$\begin{aligned} &\dot{\alpha} D_\alpha \theta_0 + \dot{\beta} D_\beta \theta_0 + e^{\Gamma\beta} \dot{p}_0(t + \frac{\alpha}{\omega_0}) = \\ &\qquad (A - \Omega_0\Gamma) e^{\Gamma\beta} p_0(t + \frac{\alpha}{\omega_0}) + f(e^{\Gamma\beta} p_0(t + \frac{\alpha}{\omega_0})), \end{aligned}$$

which is only satisfied with $\dot{\alpha} = \dot{\beta} = 0$ by assumptions (A3) and (P). In particular, the (α, β) components of the vector field on the center manifold are of order ϵ. Using this fact together with (4.2) and (4.4), we can expand equation (4.6) to first order in ϵ:

$$\begin{aligned} &\dot{\alpha} D_\alpha \theta_0 + \dot{\beta} D_\beta \theta_0 + \epsilon \hat{\omega} D_s \theta_0 + \epsilon \omega_0 D_s \theta_1 = \\ &\qquad \epsilon\Big(A\theta_1 - \Omega_0 \Gamma\theta_1 + Df(\theta_0)\theta_1 - \hat{\Omega}\Gamma\theta_0 + h(s, \theta_0, 0)\Big) + \mathrm{O}(\epsilon^2), \end{aligned}$$

that is,

$$\begin{aligned} \dot{\alpha} D_\alpha \theta_0 + \dot{\beta} D_\beta \theta_0 &= \epsilon\Big(- \hat{\Omega}\Gamma\theta_0 + h(s, \theta_0, 0) - \hat{\omega} D_s \theta_0 + \\ &\qquad A\theta_1 - \Omega_0\Gamma\theta_1 + Df(\theta_0)\theta_1 - \omega_0 D_s \theta_1\Big) + \mathrm{O}(\epsilon^2). \end{aligned} \tag{4.7}$$

We recall that $w_1(t)$ and $w_2(t)$ satisfy the adjoint variational equation (3.7)

$$w_t = -(A - \Omega_0\Gamma + Df(p_0(t)))^* w$$

and are normalized according to (3.8). Taking the scalar product of equation (4.7) with

$$e^{-\Gamma^*\beta} w_1(\frac{s + \alpha}{\omega_0}) \text{ and } e^{-\Gamma^*\beta} w_2(\frac{s + \alpha}{\omega_0}),$$

and using (4.5) and the aforementioned normalization, we obtain

$$\begin{aligned} \dot{\alpha} &= \epsilon\Big\langle w_1(\frac{s + \alpha}{\omega_0}), e^{-\Gamma\beta} F\Big\rangle + \mathrm{O}(\epsilon^2), \\ \dot{\beta} &= \epsilon\Big\langle w_2(\frac{s + \alpha}{\omega_0}), e^{-\Gamma\beta} F\Big\rangle + \mathrm{O}(\epsilon^2), \\ \dot{s} &= \omega, \end{aligned}$$

where $F = F(\alpha, \beta, s, \hat{\omega}, \hat{\Omega})$ is defined by

$$F = h(s, \theta_0, 0) - \hat{\Omega}\Gamma\theta_0 - \hat{\omega} D_s\theta_0 + (A - \Omega_0\Gamma + Df(\theta_0))\theta_1 - \omega_0 D_s\theta_1. \tag{4.8}$$

Since $s(t) = \omega t$, we have

$$\begin{aligned}
\dot{\alpha} &= \epsilon\Big\langle w_1(\frac{\omega t + \alpha}{\omega_0}), e^{-\Gamma\beta} F(\alpha, \beta, \omega t, \hat{\omega}, \hat{\Omega})\Big\rangle + \mathrm{O}(\epsilon^2), \\
\dot{\beta} &= \epsilon\Big\langle w_2(\frac{\omega t + \alpha}{\omega_0}), e^{-\Gamma\beta} F(\alpha, \beta, \omega t, \hat{\omega}, \hat{\Omega})\Big\rangle + \mathrm{O}(\epsilon^2).
\end{aligned} \tag{4.9}$$

We have the following elementary averaging lemma.

Lemma 4.1 *Consider $\dot{x} = \epsilon f(t, x, \epsilon)$ for $x \in \mathbb{R}^n$. Assume that f is smooth in (t, x, ϵ). Then, for any fixed $T \in \mathbb{R}$,*

$$x(T) = x(0) + \epsilon \int_0^T f(t, x(0), 0)\, dt + \mathrm{O}(\epsilon^2)$$

uniformly for x in compact sets and ϵ sufficiently small.

Proof. We have

$$x(t) = x(0) + \epsilon \int_0^t f(\tau, x(\tau), \epsilon)\, d\tau.$$

Therefore,

$$\begin{aligned}
x(T) &= x(0) + \epsilon \textstyle\int_0^T f(t, x(t), \epsilon)\, dt \\
&= x(0) + \epsilon \textstyle\int_0^T f\Big(t, x(0) + \epsilon \int_0^t f(x(\tau), \epsilon)\, d\tau, \epsilon\Big) dt,
\end{aligned}$$

and expansion to first order in ϵ proves the lemma. ∎

Therefore, the time-$\frac{2\pi}{\omega}$ map associated with equation (4.9) is given by

$$\Pi_{(\epsilon,\omega,\Omega)}(\alpha, \beta) = \mathrm{id} + \epsilon \begin{pmatrix} \int_0^{\frac{2\pi}{\omega}} \Big\langle w_1(\frac{\omega t + \alpha}{\omega_0}), e^{-\Gamma\beta} F(\alpha, \beta, \omega t, \hat{\omega}, \hat{\Omega})\Big\rangle\, dt \\ \int_0^{\frac{2\pi}{\omega}} \Big\langle w_2(\frac{\omega t + \alpha}{\omega_0}), e^{-\Gamma\beta} F(\alpha, \beta, \omega t, \hat{\omega}, \hat{\Omega})\Big\rangle\, dt \end{pmatrix} + \mathrm{O}(\epsilon^2),$$

with F given by (4.8). It remains to calculate the integrals

$$\begin{aligned}
&\int_0^{\frac{2\pi}{\omega}} \Big\langle w_j(\frac{\omega t + \alpha}{\omega_0}), e^{-\Gamma\beta} F(\alpha, \beta, \omega t, \hat{\omega}, \hat{\Omega})\Big\rangle\, dt \\
&= \int_0^{\frac{2\pi}{\omega_0}} \Big\langle w_j(t + \frac{\alpha}{\omega_0}), e^{-\Gamma\beta} F(\alpha, \beta, \omega_0 t, \hat{\omega}, \hat{\Omega})\Big\rangle\, dt + \mathrm{O}(\epsilon) \\
&= \int_0^{\frac{2\pi}{\omega_0}} \Big\langle e^{-\Gamma^*\beta} w_j(t + \frac{\alpha}{\omega_0}), h(\omega_0 t, \theta_0, 0) - \hat{\Omega}\Gamma\theta_0 - \hat{\omega} D_s\theta_0 + \\
&\qquad (A - \Omega_0\Gamma + Df(\theta_0) - \partial_t)\theta_1\Big\rangle\, dt + \mathrm{O}(\epsilon).
\end{aligned}$$

Since w_j and θ_1 are $\frac{2\pi}{\omega_0}$-periodic in t up to order ϵ, w_j satisfies (3.7), and (3.7) is equivariant with respect to $e^{\Gamma^*\beta}$, we have

$$\int_0^{\frac{2\pi}{\omega_0}} \Big\langle e^{-\Gamma^*\beta} w_j(t + \frac{\alpha}{\omega_0}), (A - \Omega_0\Gamma + Df(\theta_0) - \partial_t)\theta_1\Big\rangle\, dt = \mathrm{O}(\epsilon).$$

For the remaining terms, we substitute θ_0 from (4.3). Exploiting equivariance, shifting time, and using periodicity of the integrand, we obtain

$$\int_0^{\frac{2\pi}{\omega_0}} \left\langle w_j(t), e^{-\Gamma\beta} h(\omega_0 t - \alpha, e^{\Gamma\beta} p_0(t), 0) - \hat{\Omega}\Gamma p_0(t) - \frac{\hat{\omega}}{\omega_0}\dot{p}_0(t) \right\rangle dt + \mathrm{O}(\epsilon).$$

Using the normalization (3.8), we get the expression

$$\Pi(\alpha, \beta) = (\alpha, \beta) + \epsilon M(\alpha, \beta) + \mathrm{O}(\epsilon^2),$$

with

$$M_{(\hat{\omega},\hat{\Omega})}(\alpha, \beta) = \begin{pmatrix} -T_0\hat{\omega} + \int_0^{\frac{2\pi}{\omega_0}} \left\langle w_1(t), e^{-\Gamma\beta} h(\omega_0 t - \alpha, e^{\Gamma\beta} p_0(t), 0)\right\rangle dt \\ -T_0\hat{\Omega} + \int_0^{\frac{2\pi}{\omega_0}} \left\langle w_2(t), e^{-\Gamma\beta} h(\omega_0 t - \alpha, e^{\Gamma\beta} p_0(t), 0)\right\rangle dt, \end{pmatrix}.$$

This proves Theorem 1.

4.2 Proofs of the Corollaries

In this paragraph, we will omit the subscript $(\hat{\omega}, \hat{\Omega})$. The corollaries follow from Theorem 1 upon examining the map

$$\Pi(\alpha, \beta) = (\alpha, \beta) + \epsilon M(\alpha, \beta) + \mathrm{O}(\epsilon^2)$$

for given $(\hat{\omega}, \hat{\Omega})$. Indeed, fixed points (α_0, β_0) of Π satisfy

$$M(\alpha, \beta) + \mathrm{O}(\epsilon) = 0,$$

which can then be solved using an implicit function theorem provided $M(\alpha_0, \beta_0) = 0$ and $DM(\alpha_0, \beta_0)$ is invertible.

If the eigenvalues of $DM(\alpha_0, \beta_0)$ have negative real part, then the map

$$\mathrm{id} + \epsilon\, DM(\alpha_0, \beta_0)$$

has eigenvalues of modulus strictly less than one. In particular, it is a contraction. The fixed point $x_*(\epsilon) = (\alpha_0, \beta_0) + \mathrm{O}(\epsilon)$ obtained by the implicit function theorem is then attracting. Indeed, using the coordinate y defined by $(\alpha, \beta) = x_*(\epsilon) + y$, the dynamics is given by

$$\begin{aligned} y \quad \longmapsto \quad & y + \epsilon DM(\alpha_0, \beta_0)y + \epsilon\Big(M((\alpha_0, \beta_0) + y) - M(\alpha_0, \beta_0) - DM(\alpha_0, \beta_0)y\Big) + \mathrm{O}(\epsilon^2) \\ & = y + \epsilon DM(\alpha_0, \beta_0)y + \mathrm{O}(\epsilon^2 + \epsilon|y|^2). \end{aligned}$$

This map is still a contraction for y in an η_0-neighborhood of zero independent of ϵ provided

$$||D^2M((\alpha, \beta) + y)|| \leq \frac{\lambda_0}{\eta_0}$$

for all y with $|y| < \eta_0$, and ϵ is sufficiently small. Indeed, then the Lipschitz constant of the remainder term is less than $\frac{1}{2}\lambda_0\epsilon + \mathrm{O}(\epsilon^2)$, and the map is a contraction.

4.3 Proof of Theorem 2

We will first study the time-$\frac{2\pi}{\omega_0+\epsilon\hat{\omega}}$ maps Φ_ϵ and $\tilde{\Phi}_\epsilon$ of equation (3.10) and (3.11), respectively. These maps restricted to the center manifold will be denoted by Π_ϵ and $\tilde{\Pi}_\epsilon$, respectively.

First, consider the autonomous equation (3.10)

$$u_t = Au - (\Omega_0 + \epsilon\hat{\Omega})\Gamma u + f(u). \tag{4.10}$$

The center manifold is given by

$$\left\{ e^{\Gamma\beta} p_0(\frac{\alpha}{\omega_0}) : (\alpha, \beta) \in S^1 \times S^1 \right\}$$

independently of ϵ. We set $\epsilon = 0$ and consider the time-$\frac{2\pi}{\omega_0}$ map Φ_0. We recall that $\Pi_0 = \text{id}$. By Hypothesis (S), the linearized map $D\Phi_0(e^{\Gamma\beta}p_0(\frac{\alpha}{\omega_0}))$ about a point in the center manifold has two eigenvalues equal to one and the rest of its spectrum is strictly inside the unit circle. Let $P_0(\alpha,\beta)$ denote the projection onto the generalized eigenspace associated with the latter stable spectral set of $D\Phi_0(e^{\Gamma\beta}p_0(\frac{\alpha}{\omega_0}))$ with kernel given by the tangent space of the center manifold. Note that $P_0(\alpha,\beta)$ depends smoothly on (α,β) by Dunford-Taylor calculus. For any u in a small neighborhood of the center manifold we then have

$$u = e^{\Gamma\beta}p_0(\frac{\alpha}{\omega_0}) + v, \qquad v \in R(P_0(\alpha,\beta))$$

for unique (α,β) and $v \in R(P_0(\alpha,\beta))$. Moreover, this decomposition is smooth, that is, (α,β,v) depends smoothly on u. We will use this parametrization from now on. Let

$$L_0(\alpha,\beta) = D\Phi_0(e^{\Gamma\beta}p_0(\frac{\alpha}{\omega_0}))|_{R(P_0(\alpha,\beta))}.$$

Expanding the map Φ_0 and using invariance of the center manifold, we obtain

$$\Phi_0(\alpha,\beta,v) = (\alpha,\beta,L_0(\alpha,\beta)v) + R_0(\alpha,\beta,v), \quad |R_0(\alpha,\beta,v)| \le C_0|v|^2,$$

for some $C_0 > 0$. For $\epsilon \neq 0$, we then have

$$\begin{aligned} \Phi_\epsilon(\alpha,\beta,v) &= (\alpha + 2\pi(\tfrac{\omega_0}{\omega_0+\epsilon\hat{\omega}} - 1), \beta + \tfrac{2\pi}{\omega_0+\epsilon\hat{\omega}}\epsilon\hat{\Omega}, L_0(\alpha,\beta)v) + R_\epsilon(\alpha,\beta,v) \\ &= \Big((\alpha - \tfrac{2\pi}{\omega_0}\epsilon\hat{\omega}, \beta + \tfrac{2\pi}{\omega_0}\epsilon\hat{\Omega}) + \mathrm{O}(|\epsilon|^2), L_0(\alpha,\beta)v\Big) + R_\epsilon(\alpha,\beta,v) \end{aligned} \tag{4.11}$$

with

$$|R_\epsilon(\alpha,\beta,v)| \le C_0|v|(|\epsilon| + |v|), \tag{4.12}$$

possibly after increasing C_0. Indeed, the center manifold is still invariant.

Next, we shall consider (3.11), that is,

$$u_t = Au - (\Omega_0 + \epsilon\hat{\Omega})\Gamma u + f(u) + \epsilon h((\omega_0 + \epsilon\hat{\omega})t, u, \epsilon). \tag{4.13}$$

Using the aforementioned coordinates (α, β, v) introduced for Φ_ϵ, we can parametrize the center manifold $\tilde{W}_{(\epsilon,\hat{\omega},\hat{\Omega})}$ associated with (4.13) by

$$(\alpha, \beta, v) \in \tilde{W}_{(\epsilon,\hat{\omega},\hat{\Omega})} \Longleftrightarrow v = \Theta_{(\epsilon,\hat{\omega},\hat{\Omega})}(\alpha, \beta),$$

with

$$|\Theta_{(\epsilon,\hat{\omega},\hat{\Omega})}(\alpha, \beta)| \leq C_1 \epsilon, \tag{4.14}$$

for some constant $C_1 > 0$, see, for instance, [11]. We introduce new coordinates $(\tilde{\alpha}, \tilde{\beta}, \tilde{v})$ by

$$(\tilde{\alpha}, \tilde{\beta}, \tilde{v}) := (\alpha, \beta, v - \Theta_{(\epsilon,\hat{\omega},\hat{\Omega})}(\alpha, \beta)). \tag{4.15}$$

In other words, $(\tilde{\alpha}, \tilde{\beta}, 0)$ parametrizes the center manifold $\tilde{W}_{(\epsilon,\hat{\omega},\hat{\Omega})}$. In these coordinates, we have

$$\tilde{\Phi}_\epsilon(\tilde{\alpha}, \tilde{\beta}, \tilde{v}) = \Big((\tilde{\alpha}, \tilde{\beta}) + \epsilon M(\tilde{\alpha}, \tilde{\beta}) + \mathrm{O}(|\epsilon|^2), L_0(\alpha, \beta)\tilde{v}\Big) + \tilde{R}_\epsilon(\tilde{\alpha}, \tilde{\beta}, \tilde{v}), \tag{4.16}$$

with

$$|\tilde{R}_\epsilon(\tilde{\alpha}, \tilde{\beta}, \tilde{v})| \leq C_0 |\tilde{v}|(|\epsilon| + |\tilde{v}|),$$

using (4.14) and after possibly changing C_0.

Before we compute the iterated map, a positively invariant neighborhood $\mathcal{N}$ of the center manifolds of (4.10) and (4.13) will be constructed. By assumption,

$$|\lambda| < \rho < 1 \text{ for any } \lambda \in \operatorname{spec}(L_0(\alpha, \beta))$$

for some $\rho < 1$, and

$$|L_0(\alpha, \beta)^n v| \leq C_2 \rho^n |v|$$

for some constant $C_2 > 0$ and $n \in \mathbb{N}$. Using the equivalent norm

$$|v|_{new} := \sup_{n \geq 0} |L_0(\alpha, \beta)^n v|, \quad |v| \leq |v|_{new} \leq C_2 |v|,$$

we have

$$|L_0(\alpha, \beta)^n v|_{new} \leq \rho^n |v|_{new}.$$

In particular, L_0 maps a ball of radius one in the new norm into a ball of radius ρ. Let

$$\mathcal{N}_{(\alpha,\beta)}(\epsilon) := \Big\{ v \in R(P_0(\alpha, \beta)) : |v|_{new} < \frac{2\rho C_1 C_2}{1 - \rho} \epsilon \Big\}.$$

As a consequence, we have the following.

Lemma 4.2 *Let* $v = v_1 + v_2$ *such that* $v_1 \in \mathcal{N}_{(\alpha,\beta)}(\epsilon)$ *and* $|v_2| \leq C_1 \epsilon$. *We then have* $L_0(\alpha, \beta) v \in \mathcal{N}_{(\alpha,\beta)}(\epsilon)$.

Proof. Indeed,

$$\begin{aligned}|L_0(\alpha,\beta)v|_{new} &\le \rho|v|_{new} \le \rho(|v_1|_{new}+|v_2|_{new}) \le \rho\epsilon(\tfrac{2\rho C_1C_2}{1-\rho}+C_2|v_2|)\\ &\le \rho\epsilon(\tfrac{2\rho C_1C_2}{1-\rho}+C_1C_2) \le \rho(1+\rho)\tfrac{C_1C_2}{1-\rho}\epsilon \le 2\rho\tfrac{C_1C_2}{1-\rho}\epsilon,\end{aligned}$$

which proves the claim. ∎

Let

$$\mathcal{N} = \{(\alpha,\beta,v):\ v\in\mathcal{N}_{(\alpha,\beta)}(\epsilon)\},$$

suppressing the dependence of $\mathcal{N}$ on ϵ. In order to prove Theorem 2, it suffices to show that

$$\tilde{\Phi}_\epsilon(\Phi_\epsilon)^j(\alpha,\beta,v)\in\mathcal{N}$$

for $(\alpha,\beta,v)\in\mathcal{N}$ and $j\le 2(n-1)$.

Therefore, let $(\alpha,\beta,v)\in\mathcal{N}$. In particular, $|v|=\mathrm{O}(\epsilon)$. Using (4.11) and (4.12), we obtain

$$(\Phi_\epsilon)^j(\alpha,\beta,v) = (\alpha-\frac{2\pi}{\omega_0}\epsilon j\hat{\omega},\beta+\frac{2\pi}{\omega_0}\epsilon j\hat{\Omega},L_0(\alpha,\beta)^jv)+\mathrm{O}(|\epsilon|^2).$$

Transforming into the other coordinates according to (4.15), we get

$$(\tilde{\alpha},\tilde{\beta},\tilde{v}) = (\alpha-\frac{2\pi}{\omega_0}\epsilon j\hat{\omega},\beta+\frac{2\pi}{\omega_0}\epsilon j\hat{\Omega},L_0(\alpha,\beta)^jv-\Theta_{(\epsilon,\hat{\omega},\hat{\Omega})}(\alpha,\beta))+\mathrm{O}(|\epsilon|^2). \qquad (4.17)$$

Finally, we apply the map $\tilde{\Phi}_\epsilon$ given in (4.16) to (4.17) and obtain

$$(\hat{\alpha},\hat{\beta},\hat{v}) := \Big((\tilde{\alpha},\tilde{\beta})+\epsilon M(\tilde{\alpha},\tilde{\beta}),L_0(\alpha,\beta)\tilde{v}\Big)+\mathrm{O}(|\epsilon|^2). \qquad (4.18)$$

We claim that $\hat{v}$ is contained in $\mathcal{N}_{(\hat{\alpha},\hat{\beta})}$. Indeed, this follows easily from (4.14) and Lemma 4.2 applied to

$$v_1 = L_0(\alpha,\beta)^jv, \quad \text{and} \quad v_2=\Theta_{(\epsilon,\hat{\omega},\hat{\Omega})}(\alpha,\beta).$$

It remains to show that

$$(\hat{\alpha},\hat{\beta}) = (\tilde{\alpha},\tilde{\beta})+\epsilon M(\tilde{\alpha},\tilde{\beta})+\mathrm{O}(|\epsilon|^2),$$

with

$$(\tilde{\alpha},\tilde{\beta}) = (\alpha-\frac{2\pi}{\omega_0}\epsilon j\hat{\omega},\beta+\frac{2\pi}{\omega_0}\epsilon j\hat{\Omega})+\mathrm{O}(|\epsilon|^2),$$

see (4.17) and (4.18), is close to (α_0,β_0). Therefore, consider

$$\begin{aligned}&(\hat{\alpha},\hat{\beta})-(\alpha_0,\beta_0) \qquad (4.19)\\ &= (\tilde{\alpha},\tilde{\beta})+\epsilon M(\tilde{\alpha},\tilde{\beta})-(\alpha_0,\beta_0)+\mathrm{O}(|\epsilon|^2)\\ &= (\alpha,\beta)+\frac{2\pi}{\omega_0}\epsilon j(-\hat{\omega},\hat{\Omega})+\epsilon M(\alpha,\beta)-(\alpha_0,\beta_0)+\mathrm{O}(|\epsilon|^2)\\ &= (\alpha,\beta)+\epsilon M(\alpha,\beta)-(\alpha_0,\beta_0)-\epsilon M(\alpha_0,\beta_0)+\frac{2\pi}{\omega_0}\epsilon j(-\hat{\omega},\hat{\Omega})+\mathrm{O}(|\epsilon|^2),\end{aligned}$$

using $M(\alpha_0, \beta_0) = 0$, see Hypothesis (B). We have to show that

$$\left|V^{-1}\Big((\hat{\alpha}, \hat{\beta}) - (\alpha_0, \beta_0)\Big)\right| \leq \eta,$$

that is, upon substituting (4.19),

$$\left|V^{-1}\Big((\alpha, \beta) + \epsilon M(\alpha, \beta) - (\alpha_0, \beta_0) - \epsilon M(\alpha_0, \beta_0) + \frac{2\pi}{\omega_0}\epsilon j(-\hat{\omega}, \hat{\Omega})\Big)\right| + \mathrm{O}(|\epsilon|^2) \leq \eta.$$

Denoting $x = (\alpha, \beta)$ and $x_0 = (\alpha_0, \beta_0)$, we have

$$x - x_0 + \epsilon M(x) - \epsilon M(x_0) = (\mathrm{id} + \epsilon DM(x_0))(x - x_0) + \epsilon \hat{R}_{x_0}(x - x_0),$$

with $|\hat{R}_{x_0}(x - x_0)| \leq \frac{1}{2}|\det V|\,|\lambda_0|\,|x - x_0|$ by assumption. In addition, since the columns of the matrix V are the eigenvectors of $DM(x_0)$, we have

$$|V^{-1}(\mathrm{id} + \epsilon DM(x_0))(x - x_0)| \leq (1 - \epsilon|\lambda_0|)\,|x - x_0|.$$

Therefore,

$$\left|V^{-1}\Big((\mathrm{id} + \epsilon DM(x_0))(x - x_0) + \hat{R}_{x_0}(x - x_0)\Big)\right| \leq (1 - \frac{1}{2}\epsilon|\lambda_0|)\,|x - x_0|,$$

and

$$\begin{aligned} &\left|V^{-1}\Big((\alpha, \beta) + \epsilon M(\alpha, \beta) - (\alpha_0, \beta_0) - \epsilon M(\alpha_0, \beta_0) + \frac{2\pi}{\omega_0}\epsilon j(-\hat{\omega}, \hat{\Omega})\Big)\right| + \mathrm{O}(|\epsilon|^2) \\ \leq\ & (1 - \frac{1}{2}\epsilon|\lambda_0|)\eta + \frac{1}{|\det V|}\epsilon j T_0 \sup(|\hat{\omega}|, |\hat{\Omega}|) + \mathrm{O}(|\epsilon|^2), \end{aligned}$$

which is smaller than η provided

$$4T_0(n-1)\sup(|\hat{\omega}|, |\hat{\Omega}|) < |\det V|\,|\lambda_0|\eta,$$

and $\epsilon > 0$ is sufficiently small. Here, we used the worst-case $j = 2(n-1)$. This completes the proof of Theorem 2.

4.4 Proof of Theorem 3

We shall prove that the operator

$$L : C^0([0, T_0], X) \times \mathbb{R}^2 \to C^0_{T_0}([0, T_0], X) \times X \times \mathbb{R} \times \mathbb{R}$$

defined in (3.14) is invertible. Therefore, consider

$$w(t) = U(T_0, t)^* w(T_0) + h(t) \tag{4.20}$$

$$w(T_0) - w(0) + \mu_1 \dot{p}_0(0) + \mu_2 \Gamma p_0(0) = b \tag{4.21}$$

$$\textstyle\int_0^{T_0} \langle w(t), \dot{p}_0(t)\rangle\, dt = a_1 \tag{4.22}$$

$$\textstyle\int_0^{T_0} \langle w(t), \Gamma p_0(t)\rangle\, dt = a_2, \tag{4.23}$$

for $(h, b, a_1, a_2) \in C^0([0, T_0], X) \times X \times \mathbb{R} \times \mathbb{R}$ with $h(T_0) = 0$.

For any $w(T_0) \in X$, equation (4.20) defines $w(t)$ for $t \in [0, T_0)$. This equation is also satisfied for $t = T_0$ since $h(T_0) = 0$. Substituting $w(0)$ into equation (4.21), we obtain

$$(\mathrm{id} - U(T_0, 0)^*)\, w(T_0) + h(0) + \mu_1 \dot{p}_0(0) + \mu_2 \Gamma p_0(0) = b. \tag{4.24}$$

On account of Hypothesis (S), the operator $U(T_0, 0)^*$ is invertible when restricted to the space

$$E_*^s = \{w \in X : \langle w, \dot{p}_0(0)\rangle = \langle w, \Gamma p(0)\rangle = 0\}.$$

In particular, we may write

$$w(T_0) = d_1 w_1(0) + d_2 w_2(0) + w_3$$

with $d_1,\ d_2 \in \mathbb{R}$ and $w_3 \in E_*^s$. Since $w_1(t)$ and $w_2(t)$ are T_0-periodic, equation (4.24) is given by

$$(\mathrm{id} - U(T_0, 0)^*)\, w_3 + h(0) + \mu_1 \dot{p}_0(0) + \mu_2 \Gamma p_0(0) = b,$$

which can be solved uniquely for $(\mu_1, \mu_2) \in \mathbb{R}^2$ and $w_3 \in E_*^s$. Also,

$$|(\mu_1, \mu_2, w_3)| \leq C_3(|b| + |h|),$$

for some positive constant C_3. Equations (4.22) and (4.23) are then given by

$$\begin{aligned} \int_0^{T_0} \langle d_1 w_1(t) + d_2 w_2(t), \dot{p}_0(t)\rangle\, dt &= a_1 + B_1(b, h) \\ \int_0^{T_0} \langle d_1 w_1(t) + d_2 w_2(t), \Gamma p_0(t)\rangle\, dt &= a_2 + B_2(b, h) \end{aligned}$$

for some bounded linear functional B_1 and B_2. Using the normalization (3.8), these equations are easily solved. Finally, the claim

$$(w_1, 0) = L^{-1}(0, 0, 2\pi, 0), \qquad (w_2, 0) = L^{-1}(0, 0, 0, T_0)$$

follows directly from the proof given above.

5 Multi-section DFB lasers

As mentioned in the introduction, the dynamics of multi-section DFB lasers can be modeled by equation (2.5) with boundary conditions (2.6).

We will first show that equation (2.5) is well-posed in a suitable Banach space. It is then shown that the Spectral Theorem is applicable to the linearization of the period map associated with a periodic solution of (2.5). Finally, the Melnikov integrals are simplified using the special structure of (2.5).

5.1 Existence and uniqueness

We consider equation (2.5) in the open subset

$$Y = \mathbb{R}_+^m \times L^2((0,l),\mathbb{C}^2)$$

of the real Hilbert space

$$X = \mathbb{R}^m \times L^2((0,l),\mathbb{C}^2)$$

with the usual scalar product

$$\Big\langle (N,\Psi),(M,\Phi)\Big\rangle = N\cdot M + \mathrm{Re}\,\frac{1}{l}\int_0^l \Psi(z)\cdot\overline{\Phi}(z)dz$$

for (N,Ψ), $(M,\Phi)\in X$. Here, $\mathbb{R}_+$ refers to the set of strictly positive real numbers. Equation (2.5) can be rewritten in the form

$$u_t = Au - \Omega\Gamma u + f(u) + \epsilon h(\omega t,u,\Omega,\epsilon), \tag{5.1}$$

where $A = A_d + A_r$ and

$$\begin{aligned}
u &= \begin{pmatrix} N \\ \Psi \end{pmatrix}, \qquad \Gamma = i\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_2 \end{pmatrix},\\
A_d &= \begin{pmatrix} -\mathrm{diag}(\frac{1}{\tau_j}) & 0 \\ 0 & \begin{pmatrix} -v_g\partial_z & 0 \\ 0 & v_g\partial_z \end{pmatrix} \end{pmatrix},\\
A_r &= \begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} 0 & iv_g\kappa_+ \\ iv_g\kappa_- & 0 \end{pmatrix} \end{pmatrix},\\
f(u) &= \begin{pmatrix} F(N,\Psi) \\ -i(v_g\beta(N,z)+\Omega)\Psi \end{pmatrix},\\
h(\omega t,u,\epsilon,\Omega) &= \begin{pmatrix} F(N,\Psi+G(z,t)) - F(N,\Psi) \\ (H(z,n) - i\Omega - \partial_t)G(z,t) \end{pmatrix}.
\end{aligned}$$

We then have the following result.

Lemma 5.1 *The operator A with domain*

$$D(A) = \{(N,\Psi)\in X : \Psi\in H^1((0,l),\mathbb{C}^2),\ \Psi_+(0) = r_0\Psi_-(0) \text{ and } \Psi_-(l) = r_l\Psi_+(l)\}$$

is the generator of a C_0-semigroup.

Proof. Since A_r is bounded from X into X, it suffices to consider A_d. The statement of the lemma follows from the Lumer-Phillips Theorem (see, for instance, [16, Theorem 1.4.3]). Indeed, the operator A_d is dissipative, and it is easy to check that $R(A_d) = X$ for $r_0r_l \neq 1$ and $R(A_d - \mathrm{id}) = X$ if $r_0r_l = 1$. ∎

The nonlinearities $u \mapsto f(u)$ and $(u, \Omega, \epsilon) \mapsto h(\omega t, u, \Omega, \epsilon)$ are analytic functions from Y into X and $X \times \mathbb{R} \times \mathbb{R}$ into X, respectively. Furthermore, $t \mapsto h(\omega t, u, \Omega, \epsilon)$ is smooth. Hence, we have the following existence and uniqueness result.

Theorem 4 *For every $u_0 = (N^0, \Psi^0) \in Y \cap D(A)$, there exists a unique strong and globally defined solution of equation (5.1) with $u|_{t=0} = u_0$, that is,*

$$u \in C^1(\mathbb{R}^+, Y) \cap C^0(\mathbb{R}^+, D(A))$$

and u satisfies (5.1) with values in X. In addition, the solution depends smoothly on its initial value u_0.

Proof. Local existence, uniqueness, and dependence on initial values follows immediately from [16, Theorem 6.1.5] and its proof. Note that $\mathbb{R}^m_+$ is positively invariant under the semiflow, see (2.1) and (4.9).

For global existence, it suffices to consider the equation for N. Indeed, for $\epsilon = 0$, the equation for Ψ is linear in Ψ and therefore solutions of equation (5.1) exist for all time. Using equation (2.3), we have

$$\begin{aligned}
&\frac{d}{dt}(N_j(t) - V_j N_{tj})^2 + \frac{2}{\tau_j}(N_j(t) - V_j N_{tj})^2 \\
&= \; 2\Big(\frac{I_j}{e} - \frac{V_j N_{tj}}{\tau_j}\Big)(N_j(t) - V_j N_{tj}) \\
&\quad -2\frac{d_j N_{tj}}{\hbar\tilde{\Omega}} \int_{S_j} \left(|\Psi_+(z,t)|^2 + |\Psi_-(z,t)|^2\right) dz \, \ln\Big(\frac{N_j}{V_j N_{tj}}\Big)(N_j(t) - V_j N_{tj}) \\
&\leq \; \frac{1}{\tau_j}(N_j(t) - V_j N_{tj})^2 + \tau_j\Big(\frac{I_j}{e} - \frac{V_j N_{tj}}{\tau_j}\Big)^2.
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\Big(e^{-\frac{1}{\tau_j}t}(N_j(t) - V_j N_{tj})^2\Big) \leq \tau_j e^{-\frac{1}{\tau_j}t}\Big(\frac{I_j}{e} - \frac{V_j N_{tj}}{\tau_j}\Big)^2,$$

and we obtain

$$(N_j(t) - V_j N_{tj})^2 \leq e^{-t/\tau_j}(N_j(0) - V_j N_{tj})^2 + \tau_j^2\Big(\frac{I_j}{e} - \frac{V_j N_{tj}}{\tau_j}\Big)^2.$$

■

5.2 The spectrum of the period map

Assume now that for $\Omega = \Omega_0$ and $\epsilon = 0$ equation (5.1) has a T_0-periodic solution

$$p_0(t) = (N^p(t), \Psi^p(t)) \in Y.$$

Equation (5.1) then fits into the setting of Section 3 provided the linearization about the periodic orbit

$$v_t = \Big(A - \Omega_0\Gamma + Df(p_0(t))\Big)v \tag{5.2}$$

satisfies condition (S). Denote the evolution operator of equation (5.2) by $U(t,s) \in L(X)$ for $t > s$. Before we describe the spectrum of the period map $U(T_0, 0)$, we introduce more notation. Let

$$\begin{aligned} \gamma_j &= \tfrac{1}{2}\max_{t\in[0,T_0]} G_j(N_j^p(t)), \quad k \in \{1,...,m\}, \\ \gamma &= \max\{\gamma_1,...,\gamma_m\}, \\ \alpha_0 &= \min\{\alpha_{01},...,\alpha_{0m}\}. \end{aligned}$$

Theorem 5 *The essential spectrum of $U(t,s)$ is contained in the circle of radius*

$$|r_0 r_l|^n \exp(v_g R(t-s)) \qquad \textit{for} \qquad t-s \in \Big(\frac{2ln}{v_g}, \frac{2l(n+1)}{v_g}\Big),$$

where

$$R = \gamma - \frac{\alpha_0}{2}.$$

The remainder part of the spectrum consists of isolated eigenvalues with finite multiplicity.

Theorem 5 is much weaker than assumption (S) since neither the multiplicity of the eigenvalue one nor the number of isolated eigenvalues outside the essential spectrum is known. The spectral estimate shows the balance between absorption, gain, and loss of energy through the facets of the laser measured, respectively, by α_0, γ, and (r_0, r_l).

Proof. Neves, Ribeiro and Lopes considered in [15] the spectrum of a general class of homogeneous hyperbolic systems which includes equation (5.2). In their Theorem A they compare the hyperbolic system with its diagonal part which is in our case

$$v_t = \Big(A_d - \Omega_0\Gamma + B_d(p_0(t))\Big)v, \tag{5.3}$$

where $B_d(p_0(t))$ is the diagonal part of $Df(p_0(t))$. Let $S_d(t,s) \in L(X)$ be the evolution operator of equation (5.3) for $t > s$. On account of [15, Theorem A], we have:

Lemma 5.2 *For any $t \geq s$, the difference $S(t,s) - S_d(t,s) : X \to X$ is a compact operator.*

It suffices to study the Ψ part of $S_d(t,s)$ in order to prove Theorem 5 since any two bounded operators whose difference is compact have the same essential spectral radius, see, for instance, [12]. We recall that the equation for $\Psi = (\Psi_+, \Psi_-)$ is given by

$$\begin{aligned} \partial_t \Psi_+ &= v_g(-\partial_z - i\Omega_0 - i\beta(N^p(t), z))\Psi_+, \\ \partial_t \Psi_- &= v_g(\partial_z - i\Omega_0 - i\beta(N^p(t), z))\Psi_-, \end{aligned}$$

with boundary conditions

$$\Psi_+(0,t) = r_0\Psi_-(0,t), \quad \Psi_-(l,t) = r_l\Psi_+(l,t).$$

For $\beta(N^p(t), z) \equiv 0$, the associated C_0-semigroup is given by

$$\Psi_+(z,t) = e^{-i\Omega_0 t} \begin{cases} r_0 (r_0 r_l)^n \, \Psi^0_-(-z+v_g t - 2nl) & 0 \le -z + v_g t - 2nl < l \\ (r_0 r_l)^n \, \Psi^0_+(z - v_g t + 2nl) & 0 \le z - v_g t + 2nl < l, \end{cases}$$
$$\Psi_-(z,t) = e^{-i\Omega_0 t} \begin{cases} r_0 (r_0 r_l)^n \, \Psi^0_-(z+v_g t - 2nl) & 0 \le z + v_g t - 2nl < l \\ (r_0 r_l)^{n+1} \, \Psi^0_+(-z - v_g t + (2n+1)l) & 0 \le z + v_g t - (2n+1)l < l, \end{cases}$$

with $t > 0$ and $n := [v_g t/l] = \max\{n \in \mathbb{N} : n \le v_g t/l\}$. In particular,

$$\|\Psi(t)\| \le |r_0 r_l|^n \qquad \text{for} \qquad t \in \Big(\frac{2ln}{v_g}, \frac{2l(n+1)}{v_g}\Big),$$

again only for $\beta \equiv 0$.

In case $\beta \not\equiv 0$, the method of characteristics may be used to derive the result. Indeed, $\beta(N(t), z)$ is piecewise constant in z. We will omit the tedious calculations. ■

5.3 The Melnikov functions

In this part, we compute the functions (M_1, M_2) associated with (2.5). On account of Lemma 5.1, Theorem 4 and Remark 3.2, the results in Section 3 can be applied to equation (5.1).

We denote the T_0-periodic solutions of the adjoint variational equation

$$w_t = -\Big(A - \Omega_0 \Gamma + Df(p_0(t))\Big)^* w$$

by

$$w_1(t) = \begin{pmatrix} \tilde N^1(t) \\ \xi^1_+(z,t) \\ \xi^1_-(z,t) \end{pmatrix} \text{ and } w_2(t) = \begin{pmatrix} \tilde N^2(t) \\ \xi^2_+(z,t) \\ \xi^2_-(z,t) \end{pmatrix}.$$

By the definitions of h and f, and with $\hat G(z,t) = (0, G(z,t))$, we obtain

$$\begin{aligned}
&\int_0^{T_0} \Big\langle w_j(t), e^{-\Gamma\beta} h(\omega_0 t - \alpha, e^{\Gamma\beta} p_0(t), 0)\Big\rangle \, dt \\
&= \int_0^{T_0} \Big\langle w_j(t), e^{-\Gamma\beta}(A + Df(p_0(t)) - \partial_t)\hat G(z, \omega_0 t - \alpha)\Big\rangle \, dt \\
&= \int_0^{T_0} \Big\langle (A^* + Df(p_0(t))^* + \partial_t) w_j(t), e^{-\Gamma\beta} \hat G(z, \omega_0 t - \alpha)\Big\rangle \, dt \\
&\quad + \frac{v_g}{l} \,\mathrm{Re} \int_0^{T_0} e^{i\beta} \xi^j_+(0,t)\, \overline{g}(\omega_0 t - \alpha)\, dt \\
&= \frac{v_g}{l} \,\mathrm{Re} \int_0^{T_0} e^{i\beta} \xi^j_+(0,t)\, \overline{g}(\omega_0 t - \alpha)\, dt,
\end{aligned}$$

for $j = 1, 2$. Here, we used equivariance and periodicity of w_j and $\hat G$ in time.

The Melnikov integrals are given by

$$\begin{aligned} M_1(\alpha,\beta,\hat\omega,\hat\Omega) &= -T_0\hat\omega + \frac{v_g}{l}\operatorname{Re}\int_0^{T_0} e^{i\beta}\xi^1_+(0,t)\,\overline{g}(\omega_0 t-\alpha)\,dt, \qquad (5.4)\\ M_2(\alpha,\beta,\hat\omega,\hat\Omega) &= -T_0\hat\Omega + \frac{v_g}{l}\operatorname{Re}\int_0^{T_0} e^{i\beta}\xi^2_+(0,t)\,\overline{g}(\omega_0 t-\alpha)\,dt, \end{aligned}$$

for $(\alpha,\beta)\in S^1\times S^1$, using the normalization (3.8)

$$\begin{aligned} \operatorname{Re}\Big(\tilde N^1(0)\,\frac{d}{dt}\overline{N^p}(0)+\frac{1}{l}\int_0^l \xi^1(z,0)\,\frac{\partial}{\partial t}\overline{\Psi^p}(z,0)\,dz\Big) &= \omega_0\\ \operatorname{Im}\frac{1}{l}\int_0^l \xi^1(z,0)\,\overline{\Psi^p}(z,0)\,dz &= 0,\\ \operatorname{Re}\Big(\tilde N^1(0)\,\frac{d}{dt}\overline{N^p}(0)+\frac{1}{l}\int_0^l \xi^2(z,0)\,\frac{\partial}{\partial t}\overline{\Psi^p}(z,0)\,dz\Big) &= 0,\\ \operatorname{Im}\frac{1}{l}\int_0^l \xi^2(z,0)\,\overline{\Psi^p}(z,0)\,dz &= -1. \end{aligned}$$

at $t=0$.

The equation $M=0$ can be written

$$\begin{aligned} \hat\omega &= \frac{v_g}{lT_0}\operatorname{Re}\int_0^{T_0} e^{i\beta}\xi^1_+(0,t)\,\overline{g}(\omega_0 t-\alpha)\,dt, \qquad (5.5)\\ \hat\Omega &= \frac{v_g}{lT_0}\operatorname{Re}\int_0^{T_0} e^{i\beta}\xi^2_+(0,t)\,\overline{g}(\omega_0 t-\alpha)\,dt. \end{aligned}$$

Since the right-hand side does not depend on $(\hat\omega,\hat\Omega)$, it suffices to compute the image of the left-hand side to obtain the range of admissible frequency shifts $(\hat\omega,\hat\Omega)$.

5.4 Conclusions

In this paper, we derived conditions guaranteeing frequency locking and clock recovery. The Melnikov functions which we derived depend only on the periodic solution and additional solutions of the adjoint linearization about the periodic orbit. They can therefore be verified numerically by solving boundary-value problems rather than using expensive direct simulations. Furthermore, we can calculate ranges of admissible frequency perturbations $(\hat\omega,\hat\Omega)$ at once by computing the image of the right-hand side of (5.5).

We proved that the model introduced in [4] fits into our setting. As remarked before, the existence of self-pulsations has been shown numerically in [4] and [20].

In [2], the results of this paper have already been compared with the experiments described in [3]. For that purpose, we used a one-mode approximation of the PDE for Ψ which has been developed in [19]. Self-pulsations and solutions to the associated adjoint variational equation were then computed numerically using the algorithm described in Section 3. This algorithm was implemented in AUTO97 [7]. Afterwards, using a sinusoidal modulation for the signal, we determined regions in $(\hat{\omega}, \hat{\Omega})$-space for which

$$M_{(\hat{\omega}_0, \hat{\Omega}_0)}(\alpha_0, \beta_0) = 0$$

with M given by (5.4) has a solution such that $DM_{(\hat{\omega}_0, \hat{\Omega}_0)}(\alpha_0, \beta_0)$ has negative spectrum. By Corollary 1, we then have frequency locking. Finally, we compared these locking regions with those obtained experimentally. Theory and experiment appeared to agree well even though we used only a one-mode approximation.

We hope to apply the conditions developed in this paper directly to the full PDE model. It should then be possible to optimize laser parameters such as lengths of the section, injection currents and optical properties of the waveguides such that locking regions are maximal. We would also like to test the predictions of Theorem 2 by simulating the PDE directly. This is work in progress.

Acknowledgement. It is a pleasure to thank Uwe Bandelow, Reiner Lauterbach, and Lutz Recke for many helpful discussion on DFB lasers.

References

[1] D.J. As, R. Eggemann, U. Feiste, M. Möhrle, E. Patzak, and K. Weich. *Clock recovery based on a new type of self-pulsation in a* 1.5 μm *two-section InGaAsP-InP DFB laser.* Electron. Lett. **29** (1993), 141–142.

[2] U. Bandelow, L. Recke, and B. Sandstede. *Frequency regions for forced locking of self-pulsating multi-section DFB lasers.* WIAS Preprint **345** (1997).

[3] U. Bandelow, H.-J. Wünsche, B. Sartorius, and M. Möhrle. *Dispersive self Q-switching in DFB lasers – theory versus experiment.* IEEE J. Select. Topics Quantum Electron., Special Issue on *Semiconductor lasers* (1997), in press.

[4] U. Bandelow, H.J. Wünsche, and H. Wenzel. *Theory of self pulsations in two-section DFB lasers.* IEEE Photon. Technol. Lett. **5** (1993), 1176–1179.

[5] C. Chicone. *Bifurcations of nonlinear oscillations and frequency entrainment near resonance.* SIAM J. Math. Anal. **23** (1992), 1577–1608.

[6] C. Chicone. *Lyapunov-Schmidt reduction and Melnikov integrals for bifurcation of periodic solutions in coupled oscillators.* J. Diff. Eqns. **112** (1994), 407–447.

[7] E.J. Doedel, A.R. Champneys, T.F. Fairgrieve, Y.A. Kuznetsov, B. Sandstede, and X. Wang. AUTO97*: Continuation and bifurcation software for ODEs (with HomCont).* Technical report (1997).

[8] U. Feiste, D.J. As, and A. Erhardt. *18 GHz all-optical frequency locking and clock recovery using a self-pulsating two-section DFB laser.* IEEE Photon. Technol. Lett. **6** (1994), 106–108.

[9] U. Feiste, M. Möhrle, B. Sartorius, J. Hörer, and R. Löffler. *12 GHz to 64 GHz continuous frequency tuning in self pulsating 1.55 μm quantum well DFB lasers.* IEEE 14th International Semiconductor Laser Conference, 1994.

[10] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Lecture Notes Math. **840**, Springer, New York, 1981.

[11] M.W. Hirsch, C. Pugh, and M. Shub. *Invariant Manifolds* Lecture Notes Math. **583**, Springer, New York, 1977.

[12] T. Kato. *Perturbation theory for linear operators.* Springer, New York, 1966.

[13] H. Kawaguchi and I.S. Hidayat. *Analysis of all-optical clock extraction using self-pulsating laser diodes.* In *Physics and simulation of optoelectronic devices III*, SPIE **2399** (1995).

[14] M. Möhrle, U. Feiste, R. Molt, and B. Sartorius. *Gigahertz self-pulsation in 1.5 μm wavelength multisection DFB lasers.* IEEE Photon. Technol. Lett. **4** (1992), 976–978.

[15] A.F. Neves, H.S. Ribeiro, and O. Lopes. *On the Spectrum of evolution operators generated by hyperbolic systems.* J. Func. Anal. **67** (1986), 320–344.

[16] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Springer, Berlin, New York, 1983.

[17] L. Recke and D. Peterhof. *Abstract forced symmetry breaking and forced frequency locking of modulated waves.* WIAS Preprint **256/257** (1996).

[18] B. Sartorius, M. Möhrle, and U. Feiste. *12 GHz to 64 GHz continuous frequency tuning in self-pulsating* 1.55 μ*m multi quantum well DFB lasers.* IEEE J. Select. Topics Quantum Electron. **1** (1995), 535–538.

[19] H. Wenzel, U. Bandelow, H.-J. Wünsche, and J. Rehberg. *Mechanisms of fast self pulsations in two-section DFB lasers.* IEEE J. Quantum Electron. **32** (1996), 69–78.

[20] H.-J. Wünsche, U. Bandelow, H. Wenzel, and D.D. Marcenac. *Self pulsations by mode degeneracy in two-section DFB lasers.* In *Physics and simulation of optoelectronic devices III*, SPIE **2399** (1995), 195–206.