SPACE-TIME RANDOM WALK FOR STOCHASTIC DIFFERENTIAL EQUATIONS IN A BOUNDED DOMAIN

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Abstract. A mean-square approximation, which ensures boundedness of both time and space increments, is considered for stochastic differential equations in a bounded domain. The proposed algorithm is based on a space-time discretization using a random walk over boundaries of small space-time parallelepipeds. To realize the algorithm, exact distributions for exit points of the space-time Brownian motion from a space-time parallelepiped are given. Convergence theorems are stated for the proposed algorithm. A method of approximate searching for exit points of the space-time diffusion from the bounded domain is constructed. Results of several numerical tests are presented.

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1. **Introduction**

The paper is devoted to a mean-square approximation for a system of stochastic differential equations (SDE)

$$dX = \chi_{\tau_{t,x}>s} b(s,X)ds + \chi_{\tau_{t,x}>s}\sigma(s,X)dw(s), \quad X(t) = X_{t,x}(t) = x, \qquad (1.1)$$

in a space-time bounded domain $Q = [t_0, t_1) \times G \subset R^{d+1}$. Here X and b are d-dimensional vectors, σ is a $d \times d$-matrix, $(w(s), \mathcal{F}_s)$, $s \geq t_0$, is a d-dimensional standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$, G is a bounded open domain in R^d, and the Markov moment $\tau_{t,x}$ is the first-passage time of the process $(s, X_{t,x}(s))$, $s \geq t$, to $\Gamma = \overline{Q} \diagdown Q$. The set Γ is a part of the boundary ∂Q consisting of the lateral surface and the upper base of the cylinder $\overline{Q}$. We put $X_{t,x}(s) = X_{t,x}(\tau_{t,x})$ under $s \geq \tau_{t,x}$, and thus, the process $(s, X_{t,x}(s))$ is defined for all $t \leq s < t_1$. The coefficients $b^i(s,x)$ and $\sigma^{ij}(s,x)$, $(s,x) \in \overline{Q}$, and the boundary ∂G are assumed to be sufficiently smooth, while the strict ellipticity condition is imposed on the matrix $a(s,x) := \sigma(s,x)\sigma^\top(s,x)$.

The first numerical method concerning simulation of a diffusion process in a bounded domain is constructed in [27]. The method is based on a random walk over touching spheres and applied to solving the Dirichlet problem for elliptic equations with constant coefficients by a Monte Carlo technique.

Probabilistic methods for solving boundary value problems, which involve the numerical integration of ordinary SDE, are the main subject of the works [19, 20, 21, 24, 26]. These methods ensure that the proposed weak approximations belong to the bounded domain associated with a considered boundary value problem. Some other probabilistic approaches are also available in [5, 8, 16, 31].

A mean-square approximation for simulation of an autonomous diffusion process in a space bounded domain is considered in [23, 25]. The algorithm is based on a space discretization (quantization) using a random walk over small spheres. It gives the points which are close in the mean-square sense to the points of the real phase trajectory for SDE in the space bounded domain. To realize the algorithm, the exit point of the Wiener process from a d-dimensional ball has to be constructed at each step. Due to independence of the first exit time and the first exit point of the Wiener process from the ball, it is possible to simulate them separately. It is known, that the exit point is distributed uniformly on the sphere, but simulation of the exit time is a fairly laborious problem. Consequently, the algorithm gives only the phase component of the approximate trajectory without modelling the corresponding time component like the algorithm over touching spheres [27]. The space-time point lies on the d-dimensional lateral surface of a semi-cylinder with sphere base in the $(d+1)$-dimensional semi-space $[0,\infty) \times R^d$. The algorithm ensures smallness of the phase increments at each step, but the non-simulated time increments can take arbitrary large values with some probability.

As is well known, "ordinary" mean-square methods (see, e.g. [14, 18, 28]), intended to solve SDE on a finite time interval, are based on a time discretization (sampling). The space-time point, corresponding to an "ordinary" one-step approximation constructed at a time point t_k, lies on the d-dimensional plane $t = t_k$, which belongs to the $(d+1)$-dimensional semi-space $[t_0,\infty) \times R^d$. The "ordinary" mean-square methods give both time and phase components of the approximate trajectory. They ensure smallness of time increments at each step, but space increments can take arbitrary large values with some probability.

The mean-square approximation, which is the subject of the present paper, controls boundedness of both space increments and time increments at each step. In addition it gives approximate values for both time and phase components of the space-time diffusion in the space-time bounded domain Q. It is possible to solve this problem in a constructive manner by the implementation of a space-time discretization by a random walk over boundaries of small space-time parallelepipeds. It turns out that the first exit point $(\bar{\theta}, w(\bar{\theta}))$ of the space-time Brownian motion $(s, w(s))$, $s > 0$, from the space-time parallelepiped $\Pi_r = [0, lr^2) \times C_r$, $C_r \subset R^d$ is a cube with center at the origin and edge length equal to $2r$, can be easily simulated in a sufficiently easy way (some aspects of the space-time Brownian motion under $d = 1$ are considered in [11]). To construct a one-step approximation, we introduce the system with frozen coefficients (both t, x fixed)

$$d\bar{X} = b(t,x)ds + \sigma(t,x)dw(s), \quad \bar{X}(t) = x. \tag{1.2}$$

As an approximation of the point $(t+\bar{\theta}, X_{t,x}(t+\bar{\theta}))$ of the space-time diffusion $(s, X_{t,x}(s))$, $s \geq t$, we take the point $(t+\bar{\theta}, \bar{X}_{t,x}(t+\bar{\theta}))$, where $\bar{X}_{t,x}(t+\bar{\theta})$ is a solution of (1.2):

$$\bar{X}_{t,x}(t+\bar{\theta}) = x + b(t,x)\bar{\theta} + \sigma(t,x)(w(t+\bar{\theta}) - w(t)), \tag{1.3}$$

and $(\bar{\theta}, w(t+\bar{\theta}) - w(t))$ is the exit point of the space-time Brownian motion $(s-t, w(s) - w(t))$, $s > t$, from the space-time parallelepiped Π_r.

The point $(t+\bar{\theta}, \bar{X}_{t,x}(t+\bar{\theta}))$ lies on the lateral surface or on the upper base of a certain parallelepiped obtained from Π_r by a linear transformation, i.e., it is constructed on a bounded d-dimensional manifold in contrast to the "ordinary" mean-square approximations and to the approximations of [23, 25], which are constructed on the d-dimensional unbounded manifolds.

On the basis of the one-step approximation (1.3), we form a Markov chain $(\bar{\vartheta}_k, \bar{X}_k)$ which belongs to Q at each step and approximates the points $(\bar{\vartheta}_k, X(\bar{\vartheta}_k))$ of the trajectory $(s, X_{t,x}(s))$, $s \geq t$, in the mean-square sense.

2. Auxiliary knowledge

Let G be a bounded domain in R^d, $Q = [t_0, t_1) \times G$ be a cylinder in R^{d+1}, $\Gamma = \bar{Q} \backslash Q$. The set Γ is a part of the boundary of the cylinder Q consisting of the upper base and the lateral surface.

Consider the first boundary value problem for the equation of parabolic type

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t,x) \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(t,x) \frac{\partial u}{\partial x^i} + c(t,x)u + e(t,x) = 0, \ (t,x) \in Q, \tag{2.1}$$

with the initial condition on the upper base

$$u(t_1, x) = f(x), \ x \in \bar{G}, \tag{2.2}$$

and the boundary condition on the lateral surface

$$u(t,x) = g(t,x), \ t_0 \le t \le t_1, \ x \in \partial G. \tag{2.3}$$

Introduce the function φ defined on Γ such that it is equal to $f(x)$ on the upper base and it is equal to $g(t,x)$ on the lateral surface. Then the conditions (2.2)-(2.3) may be rewritten shortly as

$$u_{\mid \Gamma} = \varphi. \tag{2.4}$$

All the coefficients and the boundary ∂G of the domain G in (2.1)-(2.3) are assumed to satisfy the appropriate conditions of smoothness. Besides, the coefficients $a^{ij} = a^{ji}$ are such that the property of strong ellipticity in Q is fulfilled, i.e.,

$$\lambda_1^2 = \min_{(t,x) \in \bar{Q}} \max_{1 \le i \le d} \lambda_i^2(t,x) > 0,$$

where $\lambda_1^2(t,x) \le \lambda_2^2(t,x) \le \cdots \le \lambda_d^2(t,x)$ are eigenvalues of the matrix $a(t,x) = \{a^{ij}(t,x)\}$.

Let $\lambda_d^2 = \max_{(t,x) \in \bar{Q}} \lambda_d^2(t,x)$. Then for any $(t,x) \in \bar{Q}$ and $y \in \mathbf{R}^d$ the following inequality

$$\lambda_1^2 \sum_{i=1}^{d} y^{i^2} \le \sum_{i,j=1}^{d} a^{ij}(t,x) y^i y^j \le \lambda_d^2 \sum_{i=1}^{d} y^{i^2} \tag{2.5}$$

holds.

The solution to the problem (2.1), (2.4) has the following probabilistic representation [6]

$$u(t,x) = E\left[\varphi(\tau, X_{t,x}(\tau)) Y_{t,x,1}(\tau) + Z_{t,x,1,0}(\tau)\right], \tag{2.6}$$

where $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s \ge t$, is the solution of the Cauchy problem of the following system of stochastic differential equations

$$dX = b(s,X)ds + \sigma(s,X)dw(s), \ X(t) = x,$$

$$dY = c(s,X)Y ds, \ Y(t) = y,$$

$$dZ = e(s,X)Y ds, \ Z(t) = z. \tag{2.7}$$

Here the point (t,x) belongs to Q, $\tau = \tau_{t,x}$ is the first-passage time of the trajectory $(s, X_{t,x}(s))$ to the boundary Γ. In the system (2.7) Y and Z are scalars, $w(s) = (w^1(s), ..., w^d(s))^\top$ is a d-dimensional standard Wiener process, $b(s,x)$ is a column-vector of dimension d compounded from the coefficients $b^i(s,x)$, $\sigma(s,x)$ is a matrix of dimension $d \times d$ which is received from the equation

$$\sigma(s,x)\sigma^{\top}(s,x) = a(s,x),\ a(s,x) = \{a^{ij}(s,x)\}. \tag{2.8}$$

Setting in (2.1)-(2.7)

$$c = 0,\ e = 0,\ f = 0,\ g = \chi_{(\partial G)_0}(x), \tag{2.9}$$

where $(\partial G)_0 \subseteq \partial G$, we get the following formula

$$u(t,x) = P(\tau_{t,x} < t_1,\ X_{t,x}(\tau_{t,x}) \in (\partial G)_0),\ t_0 \le t < t_1, \tag{2.10}$$

where the time $\tau_{t,x}$ is the first-passage time of the trajectory $X_{t,x}(s)$ to the boundary ∂G.

In particular, if

$$c = 0,\ e = 0,\ f = 0,\ g = 1, \tag{2.11}$$

then

$$u(t,x) = P(\tau_{t,x} < t_1),\ t_0 \le t < t_1. \tag{2.12}$$

Setting in (2.1)-(2.7)

$$c = 0,\ e = 0,\ f = \chi_{G_0}(x),\ g = 0, \tag{2.13}$$

where $G_0 \subset G$, we get the following formula

$$u(t,x) = P(\tau_{t,x} \ge t_1,\ X_{t,x}(t_1) \in G_0). \tag{2.14}$$

In autonomous case (i.e., a^{ij}, b^i, c, e, g do not depend on t) we shall consider the first boundary value problem for parabolic equations in the following form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x)\frac{\partial u}{\partial x^i} + c(x)u + e(x),\ t > 0,\ x \in G, \tag{2.15}$$

$$u(0,x) = f(x),\ x \in \bar{G}, \tag{2.16}$$

$$u(t,x) = g(x),\ t > 0,\ x \in \partial G. \tag{2.17}$$

Using (2.9)-(2.10) and (2.13)-(2.14), it is not difficult to obtain that:
the function

$$u(t,x) = P(\tau_{0,x} < t,\ X_{0,x}(\tau_{0,x}) \in (\partial G)_0),\ t > 0, \tag{2.18}$$

is the solution of the problem (2.15)-(2.17) under (2.9);
the function

$$u(t,x) = P(\tau_{0,x} < t),\ t > 0, \tag{2.19}$$

is the solution of the problem (2.15)-(2.17) under (2.11);
the function

$$u(t,x) = P(\tau_{0,x} \ge t,\ X_{0,x}(t) \in G_0) \tag{2.20}$$

is the solution of the problem (2.15)-(2.17) under (2.13).

Here $X_{0,x}(s)$ is the solution to the Cauchy problem

$$dX = b(X)ds + \sigma(X)dw(s),\ X(0) = x, \tag{2.21}$$

and $\tau_{0,x}$ is the first-passage time of the trajectory $X_{0,x}(s)$ to the boundary ∂G.

3. Some distributions for one-dimensional Wiener process

A part of distributions for the Wiener process, which we give in the paper (see Sections 3, 4, 9), may be found in the literature. For instance, in [4, 7, 11] some distributions for the one-dimensional Wiener process are written down in a certain form. But we do not know whether all the distributions needed for our goals are available in the literature. Moreover, we need in various analytical forms of one and the same distribution due to computational aspects. That is why, for completeness of the exposition, we derive all the distributions here and give them in the forms, which are suitable for practical realization.

Introduce the first-passage time $\tau_x := \tau_{0,x}$ of the one-dimensional Wiener process $x + W(t)$, $-1 \le x \le 1$, $t > 0$, to the boundary of the interval $[-1, 1]$. Derive the formulas for

$$u(t,x) = P(\tau_x < t).$$

From (2.15)-(2.17) under (2.11) we obtain that the function (see (2.19))

$$v(t,x) = u(t,x) - 1 = P(\tau_x < t) - 1$$

satisfies the following boundary value problem

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2},\ t > 0,\ -1 < x < 1, \tag{3.1}$$

$$v(0,x) = -1,\ v(t,-1) = v(t,1) = 0. \tag{3.2}$$

By the method of separation of variables, we get the following distribution

$$P(\tau_x < t) = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\frac{\pi(2k+1)x}{2}\cdot e^{-\frac{1}{8}\pi^2(2k+1)^2 t}\,. \tag{3.3}$$

Further, extending the initial data in (3.1)-(3.2) by the odd way on the whole axis and solving the obtained Cauchy problem, we get another form for the same distribution

$$P(\tau_x < t) = 1 - \int_{-1}^{1} G(t,x,y)dy, \tag{3.4}$$

where

$$G(t,x,y) = \frac{1}{\sqrt{2\pi t}}\sum_{k=-\infty}^{\infty}\left(e^{-\frac{1}{2t}(x-4k-y)^2} - e^{-\frac{1}{2t}(x-(4k+2)+y)^2}\right). \tag{3.5}$$

We shall use the formulas (3.3) and (3.4) under $x = 0$. Denote $\tau = \tau_0$,

$$\mathcal{P}(t) := P(\tau < t),$$

and introduce the density $\mathcal{P}'(t)$. From (3.3) and (3.4) one can obtain the following lemma.

Lemma 3.1. *Let τ be the first-passage time of the one-dimensional standard Wiener process $W(t)$ to the boundary of the interval $[-1, 1]$. Then the following formulas for its distribution and density take place*

$$\mathcal{P}(t) = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cdot e^{-\frac{1}{8}\pi^2(2k+1)^2 t}\,,\ t > 0, \tag{3.6}$$

and

$$\mathcal{P}(t) = 2\sum_{k=0}^{\infty}(-1)^k \text{erfc}\frac{2k+1}{\sqrt{2t}}\ ,\ t > 0, \tag{3.7}$$

$$\mathcal{P}'(t) = \frac{\pi}{2}\sum_{k=0}^{\infty}(-1)^k(2k+1)e^{-\frac{1}{8}\pi^2(2k+1)^2 t}\ ,\ t > 0, \tag{3.8}$$

and

$$\mathcal{P}'(t) = \frac{2}{\sqrt{2\pi t^3}}\sum_{k=0}^{\infty}(-1)^k(2k+1)e^{-\frac{1}{2t}(2k+1)^2}\ ,\ t > 0. \tag{3.9}$$

Remember

$$\text{erfc}\ x = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-s^2}ds,\ \text{erfc}\ 0 = 1.$$

The formulas (3.6) and (3.8) are suitable for calculations under great t, and the formulas (3.7) and (3.9) are suitable under small t. The remainders of the series (3.8) and (3.9) are evaluated by the quantities

$$r_k(t) = \frac{\pi}{2}(2k+3)e^{-\frac{1}{8}\pi^2(2k+3)^2 t}$$

and

$$\rho_k(t) = \frac{2}{\sqrt{2\pi t^3}}(2k+3)e^{-\frac{1}{2t}(2k+3)^2}$$

correspondingly.

These quantities coincide under $t = \dfrac{2}{\pi}$ and

$$r_k(t) < r_k(\frac{2}{\pi}),\ t > \frac{2}{\pi},$$

$$\rho_k(t) < r_k(\frac{2}{\pi}),\ t < \frac{2}{\pi}.$$

If we take k, for example, equal to 2, then

$$r_2(\frac{2}{\pi}) = \frac{7\pi}{2}e^{-49\pi/4} < 2.13 \cdot 10^{-16},$$

and consequently,

$$\bar{\mathcal{P}}'(t) = \begin{cases} \dfrac{2}{\sqrt{2\pi t^3}}(e^{-1/2t} - 3e^{-9/2t} + 5e^{-25/2t})\ ,\ 0 < t < \dfrac{2}{\pi}, \\ \dfrac{\pi}{2}(e^{-\pi^2 t/8} - 3e^{-9\pi^2 t/8} + 5e^{-25\pi^2 t/8})\ ,\ t > \dfrac{2}{\pi}, \end{cases}$$

differs from $\mathcal{P}'(t)$ by a quantity of $2.13 \cdot 10^{-16}$ on the whole interval $[0, \infty)$.

It is not difficult to evaluate that

$$\bar{\mathcal{P}}(t) = \int_0^t \bar{\mathcal{P}}'(s)ds$$

differs from $\mathcal{P}(t)$ on the whole interval $[0, \infty)$ by $\dfrac{8}{7\pi}e^{-49\pi/4} < 7.04 \cdot 10^{-18}$. Such an exactness is quite sufficient for practical calculations. See the curves of the distribution $\mathcal{P}(t)$ and its density $\mathcal{P}'(t)$ on Figure 1.

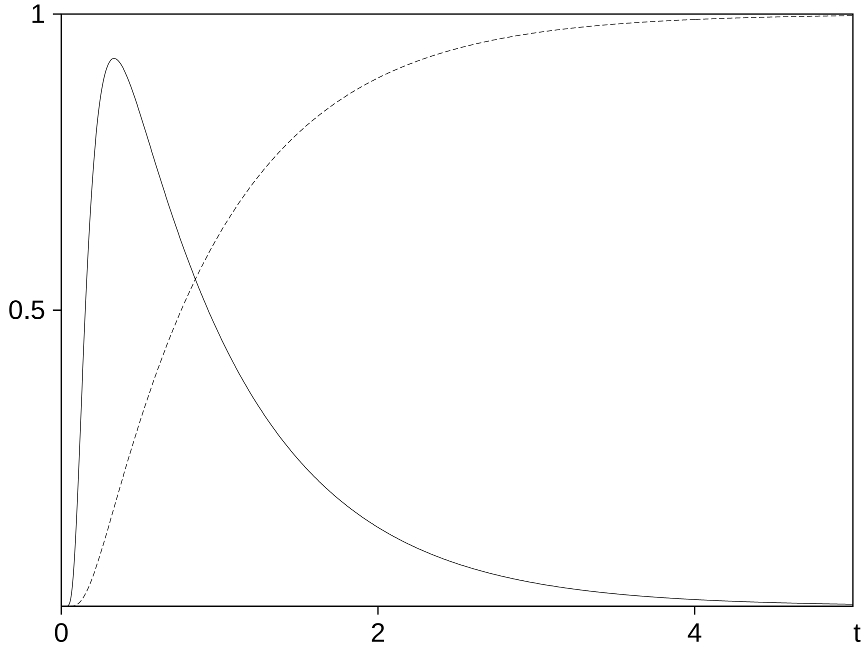

FIGURE 1. The distribution function $\mathcal{P}(t)$ and the density $\mathcal{P}'(t)$.

Denote the inverse function to $\mathcal{P}$ by $\mathcal{P}^{-1}$, and let γ be a uniformly distributed on $[0,1]$ random variable. Then the random variable

$$\tau = \mathcal{P}^{-1}(\gamma)$$

is distributed by the law $\mathcal{P}(t)$.

To simulate this law in practice, we have to solve the following equation

$$\bar{\mathcal{P}}(t) = \gamma \ . \tag{3.10}$$

Let us note that due to analytical simplicity of the function $\bar{\mathcal{P}}(t)$ it is natural to use the Newton method for solving the equation (3.10).

Lemma 3.2. *For the conditional probability*

$$\mathcal{Q}(\beta;t) := P(W(t) < \beta \diagup \, |W(s)| < 1,\ 0 < s < t) \ ,$$

where $-1 < \beta \le 1$, *the following equalities hold:*

$$\mathcal{Q}(\beta;t) = \frac{P(W(t) < \beta \ , \ \tau \ge t)}{P(\tau \ge t)}$$

$$= \frac{1}{1-\mathcal{P}(t)} \cdot \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \cdot ((-1)^k + \sin \frac{\pi(2k+1)\beta}{2}) \cdot e^{-\frac{1}{8}\pi^2(2k+1)^2 t} \ , \tag{3.11}$$

and

$$\mathcal{Q}(\beta;t) = \frac{1}{1-\mathcal{P}(t)} \times$$

$$\times \sum_{k=0}^{\infty} \frac{(-1)^k}{2} (\mathrm{erfc} \frac{2k-1}{\sqrt{2t}} - \mathrm{erfc} \frac{2k+\beta}{\sqrt{2t}} - \mathrm{erfc} \frac{2k+2-\beta}{\sqrt{2t}} + \mathrm{erfc} \frac{2k+3}{\sqrt{2t}}) \ . \tag{3.12}$$

Proof. The first equality in (3.11) flows out equivalence of the events $(|W(s)| < 1,\ 0 < s < t)$ and $(\tau \geq t)$. Let us prove the second one. To this end consider the probability

$$u(t,x) = P(\tau_x \geq t,\ \alpha \leq x + W(t) < \beta)$$

where $\alpha \geq -1$. Due to (2.15)-(2.17), (2.20) under (2.13), this probability is the solution of the following boundary value problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2},\ t > 0,\ -1 < x < 1, \tag{3.13}$$

$$u(0,x) = \chi_{[\alpha,\beta)}(x),\ u(t,-1) = u(t,1) = 0,\ t > 0. \tag{3.14}$$

Solving this problem, we get

$$u(t,x) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin\frac{\pi k(\alpha+\beta)}{2}\sin\frac{\pi k(\beta-\alpha)}{2}\sin\pi kx\cdot e^{-\frac{1}{2}\pi^2k^2t}$$

$$+\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\frac{\pi(2k+1)(\beta-\alpha)}{4}\cos\frac{\pi(2k+1)(\beta+\alpha)}{4}\cos\frac{\pi(2k+1)x}{2}\cdot e^{-\frac{1}{8}\pi^2(2k+1)^2t}.$$

As $P(W(t) < \beta\ ,\ \tau \geq t) = u(t,0)$ under $\alpha = -1,\ x = 0$, we obtain (3.11) from here. The equality (3.12) follows from

$$u(t,x) = \frac{1}{\sqrt{2\pi t}}\int_{\alpha}^{\beta} G(t,x,y)dy$$

obtained analogously to (3.4) . Lemma 3.2 is proved.

Let us note that the series (3.11) and (3.12) are of the Leibniz type, the formula (3.11) is convenient for calculations under great t, and the formula (3.12) is convenient under small t. We draw our attention to the denominator $(1-\mathcal{P}(t))$ in (3.11) which is close to zero for $t \gg 1$. But it is not difficult to transform (3.11) to the proper for calculations form. See the curves of the distribution $\mathcal{Q}(\beta;t)$ for some values of t on Figure 2.

Let the function $\mathcal{Q}^{-1}(\ \cdot\ ;t)$ for every fixed t be the inverse function to $\mathcal{Q}(\ \cdot\ ;t)$. Then the random variable

$$\xi = \mathcal{Q}^{-1}(\gamma;t)$$

has $\mathcal{Q}(\beta;t)$ as its distribution function.

4. Simulation of exit time and exit point of Wiener process from cube

Let $C \subset R^d$ be a d-dimensional cube with center at the origin and with edge length equal to 2. We suppose all the edges of the cube to be parallel to the coordinate axes, i.e., $C = \{x = (x^1,...,x^d) :\ |x^i| < 1,\ i = 1,...,d\}$. Let $W(s) = (W^1(s),...,W^d(s))^\top$ be a d-dimensional standard Wiener process, τ be the first-passage time of $W(s)$ to the boundary ∂C of the cube C.

Let us give the following evident result in the form of a lemma.

Lemma 4.1. *The distribution function $\mathcal{P}_d(t)$ for τ is equal to*

$$\mathcal{P}_d(t) = P(\tau < t) = 1 - (1-\mathcal{P}(t))^d \tag{4.1}$$

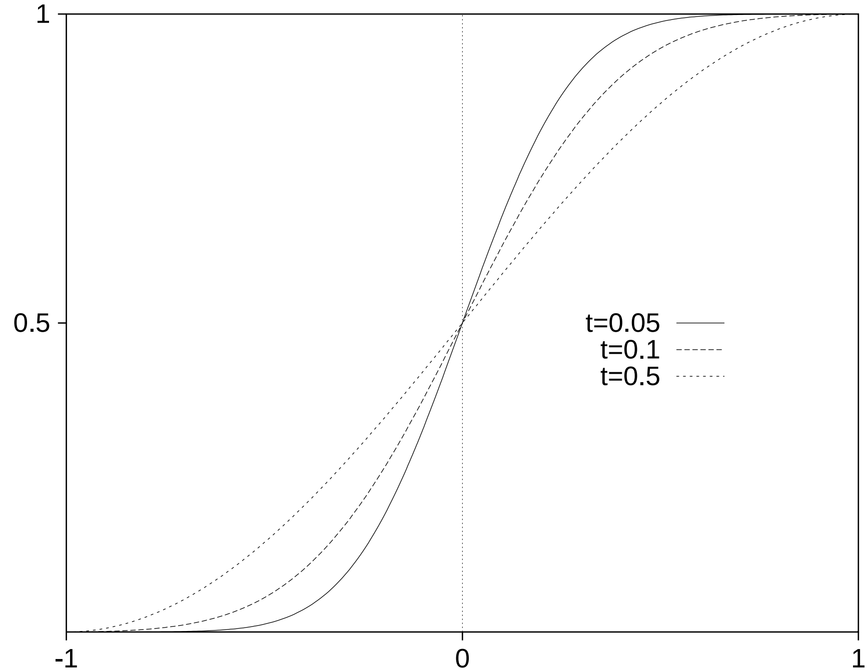

FIGURE 2. The distribution function $Q(\beta;\cdot)$; under $t \geq 0.5$ the curves coincide visually.

and the random variable

$$\tau = \mathcal{P}^{-1}(1-\gamma^{1/d}) \tag{4.2}$$

is distributed by the law $\mathcal{P}_d(t)$.

Our nearest goal is to construct an algorithm for simulation of the point $(\tau, W(\tau))$. To this end, let us obtain some distributions connected with the d-dimensional Wiener process.

Lemma 4.2. *Let τ^j be the first-passage time of the component $W^j(t)$ to the boundary of the interval $[-1,1]$. Then*

$$P\Big(\bigcap_{i\neq j}(W^i(\tau^j) < \beta^i \ , \ |W^i(s)| < 1, \ 0 < s < \tau^j)\big/\tau^j\Big)$$

$$= (1-\mathcal{P}(\tau^j))^{d-1} \cdot \prod_{i\neq j} \mathcal{Q}(\beta^i;\tau^j) \ . \tag{4.3}$$

Proof. We shall use an assertion of the following kind: if $\zeta \geq 0$ is $\tilde{\mathcal{F}}$-measurable (where $\tilde{\mathcal{F}}$ is a σ-subalgebra of a general σ-algebra $\mathcal{F}$), a random variable $\varphi(t,\omega)$ under every $t \geq 0$ does not depend on $\tilde{\mathcal{F}}$ ($\varphi(t,\omega)$ is supposed to be measurable on t), and $E\varphi(t,\omega) = h(t)$, then $E(\varphi(\zeta,\omega)\big/\tilde{\mathcal{F}}) = h(\zeta)$ (see [9, p. 67], [15, p. 158]).

Due to Lemma 3.2 and independence of the processes $W^i(s)$, we get for any $t \geq 0$

$$P\Big(\bigcap_{i\neq j}(W^i(t) < \beta^i \ , \ |W^i(s)| < 1, \ 0 < s < t)\Big) = (1-\mathcal{P}(t))^{d-1} \cdot \prod_{i\neq j} \mathcal{Q}(\beta^i;t) \ .$$

This equality implies (4.3) in accordance with the above-mentioned assertion because the processes $W^i(s)$, $i \neq j$, do not depend on the process $W^j(s)$. Lemma 4.2 is proved.

Introduce the random variable $\varkappa$ which takes the value j for $\omega \in \{\omega : W^j(\tau) = \pm 1\}$. This variable is defined uniquely with probability 1, and $P(\varkappa = j) = \frac{1}{d}$. Let $\nu := W^{\varkappa}(\tau)$. Clearly, the distribution law for ν is given by $P(\nu = -1) = P(\nu = 1) = \frac{1}{2}$.

Lemma 4.3. *The following equality takes place*

$$P(\varkappa = j,\ \tau < \theta,\ \bigcap_{i \neq j} (W^i(\tau) < \beta^i))$$

$$= \int_0^{\theta} (1 - \mathcal{P}(\vartheta))^{d-1} \cdot \prod_{i \neq j} \mathcal{Q}(\beta^i; \vartheta) \cdot \mathcal{P}'(\vartheta) d\vartheta \tag{4.4}$$

Proof. We have

$$P(\varkappa = j,\ \tau < \theta,\ \bigcap_{i \neq j} (W^i(\tau) < \beta^i))$$

$$= P(\bigcap_{i \neq j} (W^i(\tau^j) < \beta^i\ ,\ |W^i(s)| < 1,\ 0 < s < \tau^j),\ \tau^j < \theta)$$

$$= \int_0^{\theta} P(\bigcap_{i \neq j} (W^i(\tau^j) < \beta^i\ ,\ |W^i(s)| < 1,\ 0 < s < \tau^j) / \ \tau^j = \vartheta) d\mathcal{P}_{\tau^j}(\vartheta) \tag{4.5}$$

where $\mathcal{P}_{\tau^j}(\vartheta)$ is the distribution function for τ^j. Clearly $\mathcal{P}_{\tau^j}(\vartheta) = \mathcal{P}(\vartheta)$. Now the assertion (4.4) arises from Lemma 4.2. Lemma 4.3 is proved.

Lemma 4.4. *The following equality takes place*

$$P(\bigcap_{i \neq j} (W^i(\tau) < \beta^i) / \varkappa = j,\ \tau = \theta) = \prod_{i \neq j} \mathcal{Q}(\beta^i; \theta)\ . \tag{4.6}$$

Proof. The random variables $\varkappa$ and τ are independent. Indeed, $P(\varkappa = 1,\ \tau < \theta) = \cdots = P(\varkappa = d,\ \tau < \theta)$ on the strength of symmetry. Hence $P(\varkappa = i,\ \tau < \theta) = \frac{1}{d} P(\tau < \theta) = P(\varkappa = i) P(\tau < \theta)$. Further (see (4.1))

$$dP(\varkappa = j,\ \tau < \theta) = \frac{1}{d} d\mathcal{P}_d(\theta) = (1 - \mathcal{P}(\theta))^{d-1} \mathcal{P}'(\theta) d\theta.$$

From here we get

$$P(\varkappa = j,\ \tau < \theta,\ \bigcap_{i \neq j} (W^i(\tau) < \beta^i))$$

$$= \int_0^{\theta} P(\bigcap_{i \neq j} (W^i(\tau) < \beta^i) / \varkappa = j,\ \tau = \vartheta) \cdot (1 - \mathcal{P}(\vartheta))^{d-1} \mathcal{P}'(\vartheta) d\vartheta\ . \tag{4.7}$$

Comparing (4.4) with (4.7), we obtain (4.6). Lemma 4.4 is proved.

Let us note that the point $(\tau,\ W(\tau)) \in [0, \infty) \times \partial C$, i.e., this point belongs to the lateral surface of the unbounded semi-cylinder $[0, \infty) \times C$ with cubic base in $(d+1)$-dimensional space of variables $(t, x^1, ..., x^d)$.

Theorem 4.1 (Algorithm for simulating exit point to lateral surface of cylinder with cubic base). *Let $\varkappa$, $\nu, \gamma, \gamma^1, ..., \gamma^{d-1}$ be independent random variables. Let $\varkappa$ and ν be simulated by the laws $P(\varkappa = j) = \frac{1}{d}$, $j = 1, ..., d$; $P(\nu = \pm 1) = \frac{1}{2}$, and let $\gamma, \gamma^1, ..., \gamma^{d-1}$ be uniformly distributed on $[0,1]$. Then the point $(\tau,\ \xi) = (\tau, \xi^1, ..., \xi^d)$ with*

$$\tau = \mathcal{P}^{-1}(1-\gamma^{1/d}),\ \xi^1 = \mathcal{Q}^{-1}(\gamma^1;\tau), ...,\ \xi^{\varkappa-1} = \mathcal{Q}^{-1}(\gamma^{\varkappa-1};\tau),$$

$$\xi^{\varkappa} = \nu,\ \xi^{\varkappa+1} = \mathcal{Q}^{-1}(\gamma^{\varkappa};\tau), ...,\ \xi^d = \mathcal{Q}^{-1}(\gamma^{d-1};\tau) \tag{4.8}$$

has the same distribution as $(\tau,\ W(\tau))$.

Proof. This theorem is a simple consequence of Lemmas 4.1 and 4.4.

Corollary 4.1. *Let $C_r = \{x = (x^1, ..., x^d) :\ |x^i| < r,\ i = 1, ..., d\} \subset R^d$ be a d-dimensional cube with center at the origin and with edge length equal to $2r$. Let $\bar{\theta}$ be the first-passage time of the d-dimensional standard Wiener process $w(s)$ to the boundary ∂C_r of the cube C_r. Then the point*

$$(\bar{\theta}, \bar{w}) = (r^2\tau, r\xi),$$

where (τ, ξ) is simulated by the algorithm for simulating exit point to lateral surface of cylinder with the cubic base C, has the same distribution as $(\bar{\theta}, w(\bar{\theta}))$.

Proof. The proof easily follows from the fact: if $W(t)$ is a Wiener process, then $w(t) = rW(t/r^2)$ is a Wiener process as well.

Remark 4.1. The algorithm for simulating exit point to lateral surface of cylinder with parallelepiped base is more complicated because of dependence of $\varkappa$ and τ. This algorithm will be adduced later as a consequence of some next results.

5. Simulation of exit point of the space-time Brownian motion from space-time parallelepiped with cubic base

Now let us consider the space-time parallelepiped $\Pi = [0,l) \times C \subset R^{d+1}$, where the cube $C \subset R^d$ is defined as above, and construct an algorithm for simulating the exit point $(\tau(l), W(\tau(l)))$ from the parallelepiped Π. The random variable $\tau(l)$ is found as $\min(\tau, l)$, where τ is the first-passage time of $W(s)$ to the boundary ∂C as above, and the distribution function of $\tau(l)$ is equal to

$$P(\tau(l) < t) = \begin{cases} 1-(1-\mathcal{P}(t))^d,\ t \le l \\ 1,\ t > l \end{cases} \tag{5.1}$$

Theorem 5.1. **(Algorithm for simulating exit point from space-time parallelepiped with cubic base).**

Let ι, $\varkappa$, ν, γ, $\gamma^1, ...,$ γ^{d-1} be independent random variables. Let ι be simulated by the law

$$P(\iota = -1) = 1-(1-\mathcal{P}(l))^d,\ \ P(\iota = 1) = (1-\mathcal{P}(l))^d,$$

and the random variables $\varkappa$, ν, γ, $\gamma^1, ..., \gamma^{d-1}$ be simulated as in Theorem 4.1.

Then a random point $(\tau(l), \xi)$, distributed as the exit point $(\tau(l), W(\tau(l)))$, is simulated by the following algorithm:

If the simulated value of ι is equal to -1, then the point $(\tau(l),\xi)$ belongs to the lateral surface of Π, and

$$\tau(l)=\mathcal{P}^{-1}(1-[1-\gamma(1-(1-\mathcal{P}(l))^d]^{1/d}),$$

$$\begin{aligned}\xi^1 &= \mathcal{Q}^{-1}(\gamma^1;\tau(l)),\dots,\ \xi^{\varkappa-1}=\mathcal{Q}^{-1}(\gamma^{\varkappa-1};\tau(l)),\ \xi^{\varkappa}=\nu,\\ \xi^{\varkappa+1} &= \mathcal{Q}^{-1}(\gamma^{\varkappa};\tau(l)),\dots,\ \xi^{d}=\mathcal{Q}^{-1}(\gamma^{d-1};\tau(l));\end{aligned}$$

otherwise, when $\iota=1$, the point $(\tau(l),\xi)$ belongs to the upper base of Π, and

$$\tau(l)=l,$$

$$\xi^1=\mathcal{Q}^{-1}(\gamma;l),\xi^2=\mathcal{Q}^{-1}(\gamma^1;l),\dots,\xi^d=\mathcal{Q}^{-1}(\gamma^{d-1};l).$$

Proof. Using Lemma 4.1, we have

$$\begin{aligned}P(\tau(l) &< l)=P(\tau<l)=1-(1-\mathcal{P}(l))^d,\\ P(\tau(l) &= l)=P(\tau\ge l)=(1-\mathcal{P}(l))^d.\end{aligned}\tag{5.2}$$

The conditional probability $P(\tau(l)<t/\tau(l)<l)$ is equal to

$$P(\tau(l)<t/\tau(l)<l)=\frac{P((\tau(l)<t)\bigcap(\tau(l)<l))}{P(\tau(l)<l)}=\chi_{[l,\infty)}(t)+\chi_{[0,l)}(t)\frac{P(\tau<t)}{P(\tau<l)},$$

and the random variable $\mathcal{P}^{-1}(1-[1-\gamma(1-(1-\mathcal{P}(l))^d]^{1/d})$ is distributed by the law $P(\tau(l)<t/\tau(l)<l)$.

Carrying out reasoning similar to Lemmas 4.2, 4.3, and 4.4, we obtain

$$P(\bigcap_{i\neq j}(W^i(\tau(l))<\beta^i)/\varkappa=j,\tau(l)=\theta<l)=\chi_{[0,l)}(\theta)\prod_{i\neq j}\mathcal{Q}(\beta^i;\theta).\tag{5.3}$$

Further, the equality

$$P(\bigcap_{i=1}^{d}(W^i(\tau(l))<\beta^i)/\tau(l)=l)=P(\bigcap_{i=1}^{d}(W^i(l)<\beta^i)/\tau\ge l)$$

$$=\frac{1}{P(\tau\ge l)}\cdot P(\bigcap_{i=1}^{d}(W^i(l)<\beta^i,\ |W^i(s)|<1,\ 0<s<l))$$

$$=\frac{1}{\prod_{i=1}^d P(\tau^i\ge l)}P(\bigcap_{i=1}^{d}(W^i(l)<\beta^i,\tau^i\ge l))=\prod_{i=1}^{d}\mathcal{Q}(\beta^i;l)\tag{5.4}$$

holds due to the mutual independence of the components W^i, $i=1,\dots,d$, and Lemma 3.2.

Now the statement of the theorem easily follows from (5.2)-(5.4). Theorem 5.1 is proved.

The following corollary has the same proof as Corollary 4.1.

Corollary 5.1. *Let $\Pi_r=[0,lr^2)\times C_r=\{(t,x)=(t,x^1,...,x^d):\ 0\le t<lr^2,\ |x^i|<r,\ i=1,...,d\}\subset R^{d+1}$ be a space-time parallelepiped. Let $\bar\theta$ be the first-passage time of the process $(s,w(s))$, $s>0$, to the boundary $\partial\Pi_r$. Then the point*

$$(\bar\theta,\bar w)=(r^2\tau(l),r\xi),$$

where $(\tau(l),\xi)$ is simulated by the algorithm for simulating exit point from the space-time parallelepiped Π, has the same distribution as $(\bar\theta,w(\bar\theta))$.

6. Theorem on local mean-square approximation

Let us return to the problem (1.1). Introduce the space-time parallelepiped $U_r^{\sigma(t,x)}(x)$:

$$U_r^{\sigma(t,x)}(x) = \bigcup_{0\le s<lr^2} \{t+s\} \times C_r^{\sigma(t,x)}(x+b(t,x)s),$$

where $(t,x) \in Q$ and $C_r^{\sigma(t,x)}(x+b(t,x)s)$ is the space parallelepiped in R^d obtained from the open cube C_r by the linear transformation $\sigma(t,x)$ and the shift $x+b(t,x)s$, and as in the previous section, C_r is the cube with center at the origin and with edges of length $2r$ which are parallel to the coordinate axes.

Let Γ_δ be an intersection of a δ-neighborhood of the set Γ with the domain Q. Remember that the set Γ is a part of the boundary ∂Q consisting of the lateral surface and the upper base of the cylinder $\overline{Q}$. The size δ of the layer Γ_δ may depend on r. The condition of strict ellipticity ensures for any $\beta > 0$ the existence of a constant $\alpha > 0$ such that under all sufficiently small r for every point $(t,x) \in Q\backslash\Gamma_{\alpha r}$ the following relations take place:

$$U_r^{\sigma(t,x)}(x) \subset Q, \quad \min_{0\le s\le lr^2} \rho(\partial C_r^{\sigma(t,x)}(x+b(t,x)s), \partial G) \ge \beta r. \tag{6.1}$$

The values β, α, and r used below are such that these relations are fulfilled.

To construct a one-step approximation for the system (1.1), we consider the system with frozen coefficients

$$d\bar{X} = b(t,x)ds + \sigma(t,x)dw(s), \quad \bar{X}(t) = x, \ (t,x) \in Q\diagdown\Gamma_{\alpha r}. \tag{6.2}$$

Let $\bar{\theta}$ be the first-passage time of the process $(s-t, w(s)-w(t))$, $s > t$, to the boundary $\partial\Pi_r$ of the space-time parallelepiped $\Pi_r = [0, lr^2) \times C_r \subset R^{d+1}$. Clearly, $\bar{\theta} \le lr^2$. The point $(\bar{\theta}, w(t+\bar{\theta}) - w(t))$ is simulated in accordance with Corollary 5.1.

Let us take the point $(t+\bar{\theta},\ \bar{X}_{t,x}(t+\bar{\theta}))$ with $\bar{X}_{t,x}(t+\bar{\theta})$ calculated by

$$\bar{X}_{t,x}(t+\bar{\theta}) = x + b(t,x)\bar{\theta} + \sigma(t,x)(w(t+\bar{\theta}) - w(t)) \tag{6.3}$$

as an approximation of the point $(t+\bar{\theta},\ X_{t,x}(t+\bar{\theta})))$, $(t,x) \in Q\diagdown\Gamma_{\alpha r}$, where $X_{t,x}(s)$ is a solution of the system (1.1). Remember that if $t+\bar{\theta} \ge \tau_{t,x}$, then $X_{t,x}(t+\bar{\theta}) = X_{t,x}(\tau_{t,x})$.

The point $(t+\bar{\theta},\ \bar{X}_{t,x}(t+\bar{\theta}))$ belongs to the lateral surface or to the upper base of the space-time parallelepiped $U_r^{\sigma(t,x)}(x) \subset Q$.

It follows from (6.1) that

$$\rho(\bar{X}_{t,x}(t+s), \partial G) \ge \beta r, \ 0 \le s \le lr^2. \tag{6.4}$$

Theorem 6.1. *For every natural m there exists a constant $K > 0$ such that for any sufficiently small r and for any point $(t,x) \in Q\diagdown\Gamma_{\alpha r}$ the inequality*

$$E\left|X_{t,x}(t+\bar{\theta}) - \bar{X}_{t,x}(t+\bar{\theta})\right|^{2m} \le K\, r^{4m} \tag{6.5}$$

holds.

Proof. Below we use the same letter K without any index for various constants, which depend only on the system (1.1) and do not depend on (t,x), r, and so on. Thereby, we write K instead of, e.g., $K+K$, $2K$, K^2, etc.

We have (see (1.1)) that $\tau_{t,x} \le t_1$, $X_{t,x}(s) \in G$ under $s \in [t, \tau_{t,x})$, and $X_{t,x}(s) = X_{t,x}(\tau_{t,x})$ under $s \ge \tau_{t,x}$.

Let us rewrite the local error in the form

$$E\left|X_{t,x}(t+\bar{\theta}) - \bar{X}_{t,x}(t+\bar{\theta})\right|^{2m} =$$

$$E\Big|\int_t^{t+\bar\theta}\big(\chi_{\tau_{t,x}>s}b(s,X_{t,x}(s))-b(t,x)\big)\,ds+\int_t^{t+\bar\theta}\big(\chi_{\tau_{t,x}>s}\sigma(s,X_{t,x}(s))-\sigma(t,x)\big)\,dw(s)\Big|^{2m}$$

$$\le K\,E\Big|\int_t^{t+\bar\theta}\big(\chi_{\tau_{t,x}>s}b(s,X_{t,x}(s))-b(t,x)\big)\,ds\Big|^{2m}$$

$$+K\,E\Big|\int_t^{(t+\bar\theta)\wedge\tau_{t,x}}\left(\sigma(s,X_{t,x}(s))-\sigma(t,x)\right)dw(s)\Big|^{2m}$$

$$+K\,E\Big|\int_{(t+\bar\theta)\wedge\tau_{t,x}}^{t+\bar\theta}\sigma(t,x)\,dw(s)\Big|^{2m}. \tag{6.6}$$

We obtain for the first term in (6.6):

$$K\,E\Big|\int_t^{t+\bar\theta}\big(\chi_{\tau_{t,x}>s}b(s,X_{t,x}(s))-b(t,x)\big)\,ds\Big|^{2m}\le K\,E\bar\theta^{2m}\le K\,r^{4m} \tag{6.7}$$

because of boundedness of $b(s,x)$, $(s,x)\in\overline{Q}$, and $\bar\theta\le lr^2$.

Below we need the following inequality for Ito integrals in the case of scalar Wiener process (see, e.g., [9, p.26]):

$$E\Big(\int_t^{t+T}\varphi(s)\,dw(s)\Big)^{2m}\le\left(m(2m-1)\right)^{m-1}T^{m-1}\int_t^{t+T}E\varphi^{2m}(s)\,ds,\ m=1,2,... \tag{6.8}$$

Clearly, in the case of the d-dimensional Wiener process the inequality (6.8) implies

$$E\Big|\int_t^{t+T}\varphi(s)dw(s)\Big|^{2m}\le KT^{m-1}\int_t^{t+T}E\sum_{i,j=1}^d(\varphi^{ij}(s))^{2m}ds,\ m=1,2,\dots, \tag{6.9}$$

where the constant K depends on m, of course.

If φ is bounded, we also have

$$E\Big|\int_t^{t+T}\varphi(s)dw(s)\Big|^{2m}\le KT^m,\ m=1,2,... \tag{6.10}$$

Due to the inequality (6.9), smoothness of $\sigma(s,x)$, $(s,x)\in\overline{Q}$, and $(t+\bar\theta)\wedge\tau_{t,x}\le t+lr^2$, we obtain for the second term of (6.6):

$$KE\Big|\int_t^{(t+\bar\theta)\wedge\tau_{t,x}}\left(\sigma(s,X(s))-\sigma(t,x)\right)dw(s)\Big|^{2m}$$

$$=KE\Big|\int_t^{t+lr^2}\chi_{(t+\bar\theta)\wedge\tau_{t,x}>s}\left(\sigma(s,X_{t,x}(s))-\sigma(t,x)\right)dw(s)\Big|^{2m}$$

$$\le Kr^{2m-2}\int_t^{t+lr^2}E\Big(\chi_{(t+\bar\theta)\wedge\tau_{t,x}>s}\sum_{i,j=1}^d\left|\sigma^{ij}(s,X_{t,x}(s))-\sigma^{ij}(t,x)\right|^{2m}\Big)ds$$

$$\le Kr^{2m-2}\int_t^{t+lr^2}\left(E\chi_{\tau_{t,x}>s}\left|X_{t,x}(s)-x\right|^{2m}+(s-t)^{2m}\right)ds$$

$$\le Kr^{2m-2}\int_t^{t+lr^2}E\chi_{\tau_{t,x}>s}\left|X_{t,x}(s)-x\right|^{2m}ds+Kr^{6m}. \tag{6.11}$$

Further,

$$E\chi_{\tau_{t,x}>s}\,|X_{t,x}(s)-x|^{2m} = E\chi_{\tau_{t,x}>s}|\int_t^s b(s',X_{t,x}(s'))ds' + \int_t^s \sigma(s',X_{t,x}(s'))dw(s')|^{2m},$$

whence due to (6.10)

$$E\chi_{\tau_{t,x}>s}\,|X_{t,x}(s)-x|^{2m} \le K\cdot(s-t)^{2m} + K\cdot(s-t)^m.$$

Substituting this inequality in (6.11), we obtain

$$KE|\int_t^{(t+\bar\theta)\wedge\tau_{t,x}} \left(\sigma(s,X_{t,x}(s))-\sigma(t,x)\right)dw(s)|^{2m} \le Kr^{4m}. \tag{6.12}$$

It follows from the inequalities (6.7) and (6.12) that

$$E\left|X_{t,x}(\tau_{t,x}\wedge(t+\bar\theta)) - \bar X_{t,x}(\tau_{t,x}\wedge(t+\bar\theta))\right|^{2m} \le Kr^{4m}. \tag{6.13}$$

Now let us estimate the third term in (6.6). We have due to (6.9):

$$KE|\int_{(t+\bar\theta)\wedge\tau_{t,x}}^{t+\bar\theta} \sigma(t,x)\,dw(s)|^{2m} = KE|\int_t^{t+lr^2}\left(\chi_{(t+\bar\theta)>s} - \chi_{(t+\bar\theta)\wedge\tau_{t,x}>s}\right)\sigma(t,x)dw(s)|^{2m}$$

$$\le Kr^{2m-2}\int_t^{t+lr^2} E\left(\chi_{(t+\bar\theta)>s} - \chi_{(t+\bar\theta)\wedge\tau_{t,x}>s}\right)ds$$

$$= Kr^{2m-2}\cdot E\chi_{\tau_{t,x}<(t+\bar\theta)}\left((t+\bar\theta)-(t+\bar\theta)\wedge\tau_{t,x}\right) \le K\,r^{2m}\cdot P(\tau_{t,x}<t+\bar\theta). \tag{6.14}$$

Evaluate the probability $P(\tau_{t,x}<t+\bar\theta)$ using the reception from [25]. If $\tau_{t,x}<t+\bar\theta$, then $\tau_{t,x}<t_1$ and, consequently, $X_{t,x}(\tau_{t,x})\in\partial G$. At the same time due to (6.4)

$$\rho(\bar X_{t,x}(\tau_{t,x}\wedge(t+\bar\theta)),\partial G) \ge \beta r.$$

Therefore,

$$E\left(\chi_{\tau_{t,x}<t+\bar\theta}\left|X_{t,x}(\tau_{t,x}\wedge(t+\bar\theta)) - \bar X_{t,x}(\tau_{t,x}\wedge(t+\bar\theta))\right|^m\right)$$

$$\ge P(\tau_{t,x}<t+\bar\theta)\cdot(\beta r)^m,\ m=1,2,\dots\,.$$

From the other hand, due to (6.13) we have

$$P(\tau_{t,x}<t+\bar\theta)\cdot(\beta r)^m \le E\left(\chi_{\tau_{t,x}<t+\bar\theta}\left|X_{t,x}(\tau_{t,x}\wedge(t+\bar\theta)) - \bar X_{t,x}(\tau_{t,x}\wedge(t+\bar\theta))\right|^m\right)$$

$$\le \sqrt{P(\tau_{t,x}<t+\bar\theta)}\,\left[E\left|X_{t,x}(\tau_{t,x}\wedge(t+\bar\theta)) - \bar X_{t,x}(\tau_{t,x}\wedge(t+\bar\theta))\right|^{2m}\right]^{1/2}$$

$$\le Kr^{2m}\sqrt{P(\tau_{t,x}<t+\bar\theta)}.$$

Consequently,

$$P(\tau_{t,x}<t+\bar\theta) \le Kr^{2m},\ m=1,2,\dots\,. \tag{6.15}$$

Now the inequality (6.6) together with (6.7), (6.12), and (6.14) gives (6.5). Theorem 6.1 is proved.

7. Global algorithm and convergence theorems

Let us construct a random walk over small space-time parallelepipeds based on the one-step approximation (6.3) of the previous section. Let $(\bar\theta_1, w(t+\bar\theta_1)-w(t))$ be the first exit point of the process $(s-t, w(s)-w(t))$, $s>t$, from the parallelepiped Π simulated in accordance with Corollary 5.1, $(\bar\theta_2, w(t+\bar\theta_1+\bar\theta_2)-w(t+\bar\theta_1))$ be the exit point of the process $(s-t-\bar\theta_1, w(s)-w(t+\bar\theta_1))$, $s>t+\bar\theta_1$, from the parallelepiped Π, and so on.

Suppose that $(t,x)\in Q\diagdown\Gamma_{\alpha r}$. Then, we construct the recurrence sequence $(\bar\vartheta_k, \bar X_k)$, $k=0,1,\dots,\bar\nu$:

$$\begin{aligned}\bar\vartheta_0 &= t,\ \bar X_0 = x,\\ \bar\vartheta_k &= \bar\vartheta_{k-1}+\bar\theta_k,\\ \bar X_k &= \bar X_{k-1}+b(\bar\vartheta_{k-1},\bar X_{k-1})\bar\theta_k+\sigma(\bar\vartheta_{k-1},\bar X_{k-1})(w(\bar\vartheta_k)-w(\bar\vartheta_{k-1})),\quad k=1,\dots,\bar\nu,\end{aligned}$$

where the number $\bar\nu=\bar\nu_{t,x}$ is the first one for which $(\bar\vartheta_k,\bar X_k)\in\Gamma_{\alpha r}$.

If $(t,x)\in\Gamma_{\alpha r}$, we put $\bar\nu=0$.

Let $(\bar\vartheta_k,\bar X_k)=(\bar\vartheta_{\bar\nu},\bar X_{\bar\nu})$ under $k>\bar\nu$. The obtained sequence $(\bar\vartheta_k,\bar X_k)$, $k=0,1,\dots$, is a Markov chain stopping at the Markov moment $\bar\nu$. It is clear that the random number of steps $\bar\nu$ depends on the domain $Q\diagdown\Gamma_{\alpha r}$. That is why, the more rigorous notation for $\bar\nu$ is $\bar\nu_{t,x}(Q\diagdown\Gamma_{\alpha r})$.

At first we consider some average characteristics of $\bar\nu_{t,x}=\bar\nu_{t,x}(Q\diagdown\Gamma_{\alpha r})$ following the technique proposed in [19, 23, 24].

Define the operation P acting on functions $v(t,x), (t,\,x)\in\overline{Q}$, as

$$Pv(t,x)=Ev(\bar\vartheta_1,\bar X_1)$$

and the operator A :

$$Av(t,x)=Pv(t,x)-v(t,x)\tag{7.1}$$

which is called by generator of the chain.

The generator gives an average increment of the function v on the trajectory of the chain per step.

Lemma 7.1 ([32], see also [24]). *Let $v(t,x)$ be a solution to the boundary value problem*

$$qPv(t,x)-v(t,x)=-g(t,x),\ (t,x)\in Q\backslash\Gamma_{\alpha r},\tag{7.2}$$

$$v(t,x)=0,\ (t,x)\in\Gamma_{\alpha r},\tag{7.3}$$

where $q>0$ is a constant, $g(t,x)\geq 0$ is a continuous function on $Q\backslash\Gamma_{\alpha r}$.

Then for $(t,x)\in Q\backslash\Gamma_{\alpha r}$

$$v(t,x)=E\sum_{k=0}^{\bar\nu_{t,x}-1}g(\bar\vartheta_k,\bar X_k)\cdot q^k.\tag{7.4}$$

Proof. Let us complete the definition of the function g : $g(t,x)=0$ for $(t,x)\in\Gamma_{\alpha r}$. We have for $(t,x)\in Q\backslash\Gamma_{\alpha r}$:

$$v(t,x)=g(t,x)+qPv(t,x)=g(t,x)+qEv(\bar\vartheta_1,\bar X_1)$$

$$=g(t,x)+qE(g(\bar\vartheta_1,\bar X_1)+qPv(\bar\vartheta_1,\bar X_1))$$

$$= g(t,x) + qE\chi_{\bar{\nu}_{t,x}>1} g(\bar{\vartheta}_1, \bar{X}_1) + q^2 E\chi_{\bar{\nu}_{t,x}>1} E(v(\bar{\vartheta}_2, \bar{X}_2) / (\bar{\vartheta}_1, \bar{X}_1))$$

$$= g(t,x) + qE\chi_{\bar{\nu}_{t,x}>1} g(\bar{\vartheta}_1, \bar{X}_1) + q^2 E\chi_{\bar{\nu}_{t,x}>2} v(\bar{\vartheta}_2, \bar{X}_2)$$

$$= g(t,x) + qE\chi_{\bar{\nu}_{t,x}>1} g(\bar{\vartheta}_1, \bar{X}_1) + q^2 E\chi_{\bar{\nu}_{t,x}>2} g(\bar{\vartheta}_2, \bar{X}_2) + q^3 E\chi_{\bar{\nu}_{t,x}>3} v(\bar{\vartheta}_3, \bar{X}_3)$$

$$= \ldots = g(t,x) + qE\chi_{\bar{\nu}_{t,x}>1} g(\bar{\vartheta}_1, \bar{X}_1) + \ldots + q^N E\chi_{\bar{\nu}_{t,x}>N} g(\bar{\vartheta}_N, \bar{X}_N)$$

$$+ q^{N+1} E\chi_{\bar{\nu}_{t,x}>N+1} v(\bar{\vartheta}_{N+1}, \bar{X}_{N+1}).$$

As N goes to infinity, we obtain (7.4). Lemma 7.1 is proved.

Corollary. *If $q > 1$ and $g(t,x) \geq c$ for $(t,x) \in Q \backslash \Gamma_{\alpha r}$, then*

$$\frac{1}{q-1}(Eq^{\bar{\nu}_{t,x}} - 1) \leq \frac{1}{c} v(t,x).$$

Now consider the following boundary value problem in $\overline{Q}$

$$Av(t,x) = -g(t,x), \ (t,x) \in Q \backslash \Gamma_{\alpha r}, \tag{7.5}$$

$$v(t,x) = 0, \ (t,x) \in \Gamma_{\alpha r}, \tag{7.6}$$

which is connected with the chain $(\bar{\vartheta}_k, \bar{X}_k)$.

The solution of this problem $v(t,x)$ is equal to

$$v(t,x) = E \sum_{k=0}^{\bar{\nu}_{t,x}-1} g(\bar{\vartheta}_k, \bar{X}_k), \ (t,x) \in Q \backslash \Gamma_{\alpha r}.$$

Therefore, if $g \equiv 1$, then $v(t,x) = E\bar{\nu}_{t,x}$, and if

$$g(t,x) \geq 1,$$

then

$$E\bar{\nu}_{t,x} \leq v(t,x).$$

Theorem 7.1. *The mean number of steps $\bar{\nu}_{t,x}(Q \diagdown \Gamma_{\alpha r})$ is estimated as*

$$E\bar{\nu}_{t,x}(Q \diagdown \Gamma_{\alpha r}) \leq \frac{K}{r^2}, \tag{7.7}$$

where the positive constant K does not depend on r.

Proof. Introduce the function [19]

$$V(t,x) = \begin{cases} t_1 - t, \ (t,x) \in Q \backslash \Gamma_{\alpha r}, \\ 0, \ (t,x) \in \Gamma_{\alpha r}. \end{cases}$$

It is clear that $V(t,x) \geq 0$ for all $(t,x) \in \overline{Q}$ and it complies with the boundary condition (7.6).

At first consider the points (t,x) such that $U_r^\sigma(x) \subset Q \backslash \Gamma_{\alpha r}$, then $V(s,y) = t_1 - s$ for $(s,y) \in \partial U_r^\sigma(x)$ and

$$AV(t,x) = EV(t + \bar{\theta}_1, \bar{X}_1) - V(t,x) = -E\bar{\theta}_1.$$

Analogously to (5.1), the random variable $\bar{\theta}_1$ has the following distribution function (see also Corollary 4.1 and Lemma 4.1)

$$P(\bar{\theta}_1 < s) = \begin{cases} 1 - (1 - \mathcal{P}(s/r^2))^d, & s \le lr^2, \\ 1, & s > lr^2, \end{cases}$$

whence it is not difficult to obtain that

$$E\bar{\theta}_1 = \gamma r^2, \ \gamma = \int_0^l (1 - \mathcal{P}(s))^d \, ds.$$

Thus, for (t, x) mentioned above

$$AV = -\gamma r^2.$$

Let $(t, x) \in Q \backslash \Gamma_{\alpha r}$ be now such that a part of $U_r^\sigma(x)$ belongs to $\Gamma_{\alpha r}$. Introducing for a while the function $\bar{V}(s, y)$ which is equal to $t_1 - s$ on Q, we get as above:

$$A\bar{V} = -\gamma r^2.$$

Since $V(s, y) \le \bar{V}(s, y)$ on $\partial U_r^\sigma(x)$, we have due to (7.1)

$$AV \le -\gamma r^2. \tag{7.8}$$

Clearly, the inequality (7.8) is fulfilled for all $(x, t) \in Q \backslash \Gamma_{\alpha r}$, and the function

$$v(t, x) = \frac{V(t, x)}{\gamma r^2} \tag{7.9}$$

satisfies (7.5)-(7.6) with $g \ge 1$. Thus, we prove (7.7) with $K = (t_1 - t_0)/\gamma$. Theorem 7.1 is proved.

Remark 7.1. The statement of the theorem is also valid in the case of infinity t_1 and the bounded G if we assume that the coefficients of the system (1.1) are bounded in Q and the lowest eigenvalue of the matrix $a(t, x)$ is uniformly bounded with respect to $(t, x) \in Q$ by a positive constant from below. To prove the theorem in this case, one can use, for example, the following Lyapunov function analogously to [12, p.132]:

$$V(t, x) = \begin{cases} B^2 - |x + c|^{2n}, & (t, x) \in Q \backslash \Gamma_{\alpha r}, \\ 0, & (t, x) \in \Gamma_{\alpha r}, \end{cases}$$

where c is a vector such that $\min_{x \in \overline{G}} |x + c| \ge C > 0$, n is a sufficiently large natural number, the choice of which depends on bounds of $(b(t, x), y + c)$, $(t, x) \in \overline{Q}$, $y \in \overline{G}$, and B^2 is the constant equal to $\max_{x \in \overline{G}} |x + c|^{2n}$.

Theorem 7.2. (see [23, 24]). *For every $L > 0$ the inequality*

$$P\left\{ \bar{\nu}_{t,x}(Q \diagdown \Gamma_{\alpha r}) \ge \frac{L}{r^2} \right\} \le (1 + t_1 - t_0)\, e^{-c_r \frac{\gamma}{1 + t_1 - t_0} L}, \quad c_r \to 1 \text{ as } r \to 0, \tag{7.10}$$

is valid.

Proof. We have for the function $v(t, x)$ from (7.9):

$$(1 + \mu r^2) Pv - (1 + \mu r^2) v \le -(1 + \mu r^2), \ (t, x) \in Q \diagdown \Gamma_{\alpha r}.$$

Hence

$$(1 + \mu r^2) Pv - v \le \mu r^2 v - (1 + \mu r^2), \ (t, x) \in Q \diagdown \Gamma_{\alpha r},$$

$$v = 0,\ (t,x) \in \Gamma_{\alpha r}.$$

Thus, the function $v(t,x)$ is a solution to the problem (7.2)-(7.3) with $q = 1 + \mu r^2$ and with $g(t,x)$ satisfying the inequality

$$g(t,x) \geq 1 + \mu r^2 - \mu r^2 v = 1 + \mu r^2 - \frac{\mu V(t,x)}{\gamma}.$$

Then due to Corollary to Lemma 7.1 (remember $V \leq t_1 - t_0$), we have under $\mu = \dfrac{\gamma}{1 + t_1 - t_0}$:

$$\frac{1 + t_1 - t_0}{\gamma r^2}(E(1 + \frac{\gamma}{1 + t_1 - t_0} r^2)^{\bar{\nu}_{t,x}} - 1) \leq (1 + t_1 - t_0) v(t,x),$$

and, consequently (see (7.9)),

$$E(1 + \frac{\gamma}{1 + t_1 - t_0} r^2)^{\bar{\nu}_{t,x}} \leq 1 + t_1 - t_0,$$

whence by the Chebyshev inequality we obtain (7.10). Theorem 7.2 is proved.

We need in two auxiliary lemmas.

Lemma 7.2. *There exists a constant K such that for all r small enough and all $(t,x) \in Q \backslash \Gamma_{\alpha r}$ the inequality*

$$\left| E(X_{t,x}(t + \bar{\theta}_1) - \bar{X}_{t,x}(t + \bar{\theta}_1)) \right| \leq K r^4 \tag{7.11}$$

is valid.

Proof. By the Ito formula, smoothness of $b(s,x)$, and the inequality (6.15) under $m = 1$, we obtain

$$|E(X_{t,x}(t + \bar{\theta}_1) - \bar{X}_{t,x}(t + \bar{\theta}_1))|$$

$$= |E \int_t^{t+\bar{\theta}_1} \big(\chi_{\tau_{t,x} > s} b(s, X_{t,x}(s)) - b(t,x)\big)\, ds$$

$$+ E \int_t^{t+\bar{\theta}_1} \big(\chi_{\tau_{t,x} > s} \sigma(s, X_{t,x}(s)) - \sigma(t,x)\big)\, dw(s)|$$

$$= |E \int_t^{t+\bar{\theta}_1} \big(\chi_{\tau_{t,x} > s} b(s, X_{t,x}(s)) - b(t,x)\big)\, ds|$$

$$= |E(\int_t^{(t+\bar{\theta}_1)\wedge\tau_{t,x}} (b(s, X_{t,x}(s)) - b(t,x)) ds) - b(t,x) E((t + \bar{\theta}_1) - \tau_{t,x} \wedge (t + \bar{\theta}_1))|$$

$$\leq |E(\int_t^{(t+\bar{\theta}_1)\wedge\tau_{t,x}} \int_t^s Lb(s', X_{t,x}(s')) ds'\, ds)| + K E((t + \bar{\theta}_1) - \tau_{t,x} \wedge (t + \bar{\theta}_1))$$

$$\leq K E \bar{\theta}_1^2 + K r^2 P(\tau_{t,x} < t + \bar{\theta}_1) \leq K r^4,$$

where

$$L = \frac{\partial}{\partial s} + \frac{1}{2} \sum_{i,j=1}^d a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b^i \frac{\partial}{\partial x^i}.$$

Lemma 7.2 is proved.

Lemma 7.3. *Let the random variable Z be defined by the relation*

$$X_{t,x}(t+\bar\theta_1) - X_{t,y}(t+\bar\theta_1) = x - y + Z.$$

Then for every natural m there exists a positive constant K such that for any r small enough and all (t,x), $(t,y) \in Q \diagdown \Gamma_{\alpha r}$ the inequalities

$$E|Z|^m \le K\, r^m \cdot (|x-y|^m + r^m)\,, \tag{7.12}$$

$$|EZ| \le K\, r^2 \cdot \left(|x-y| + r^2\right). \tag{7.13}$$

hold.

Proof. We have due to (t,x), $(t,y) \in Q \diagdown \Gamma_{\alpha r}$ and (6.3):

$$\bar X_{t,x}(t+\bar\theta_1) = x + b(t,x)\bar\theta_1 + \sigma(t,x)(w(t+\bar\theta_1) - w(t))$$

and

$$\bar X_{t,y}(t+\bar\theta_1) = y + b(t,y)\bar\theta_1 + \sigma(t,y)(w(t+\bar\theta_1) - w(t)).$$

Then

$$Z = X_{t,x}(t+\bar\theta_1) - X_{t,y}(t+\bar\theta_1) - (x-y)$$

$$= (X_{t,x}(t+\bar\theta_1) - \bar X_{t,x}(t+\bar\theta_1)) - (X_{t,y}(t+\bar\theta_1) - \bar X_{t,y}(t+\bar\theta_1))$$

$$+(b(t,x) - b(t,y))\bar\theta_1 + (\sigma(t,x) - \sigma(t,y))(w(t+\bar\theta_1) - w(t)).$$

By Lemma 7.2 and smoothness of $b(s,x)$, $(s,x) \in \overline{Q}$, we get

$$|EZ| \le \left|E(X_{t,x}(t+\bar\theta_1) - \bar X_{t,x}(t+\bar\theta_1))\right| + \left|E(X_{t,y}(t+\bar\theta_1) - \bar X_{t,y}(t+\bar\theta_1))\right|$$

$$+ |b(t,x) - b(t,y)| \cdot E\bar\theta_1$$

$$\le Kr^4 + K|x-y| \cdot r^2,$$

that gives (7.13).

Now consider the $2n$-th moments of Z. Using Theorem 6.1, the property (6.9), boundedness of $b(s,x)$, $(s,x) \in \overline{Q}$, and smoothness of $\sigma(s,x)$, $(s,x) \in \overline{Q}$, we obtain

$$E\,|Z|^{2n} \le K\,E|X_{t,x}(t+\bar\theta_1) - \bar X_{t,x}(t+\bar\theta_1)|^{2n} + K\,E|X_{t,y}(t+\bar\theta_1) - \bar X_{t,y}(t+\bar\theta_1)|^{2n}$$

$$+K\,|b(t,x) - b(t,y)|^{2n} \cdot E\bar\theta_1^{2n} + E|\int_t^{t+lr^2} \chi_{t+\bar\theta_1 > s}(\sigma(t,x) - \sigma(t,y))\,dw(s)|^{2n}$$

$$\le Kr^{4n} + Kr^{2n}|x-y|^{2n}$$

that gives (7.12) in the case of the even m.

In the case of the odd m, we come to (7.12) using the Cauchy-Bunyakovskii inequality:

$$E|Z|^m \le (E|Z|^{2m})^{1/2} \le (Kr^{2m} \cdot (|x-y|^{2m} + r^{2m}))^{1/2} \le Kr^m \cdot (|x-y|^m + r^m).$$

Lemma 7.3 is proved.

For every $\varepsilon \in (0,1]$ and any $\beta > 0$ it is possible to introduce the layer $\Gamma_{\alpha r^{1-\varepsilon}}$ with a constant α such that under a sufficiently small r and for every $(t,x) \in Q \backslash \Gamma_{\alpha r^{1-\varepsilon}}$ the following relations together with the relations (6.1) take place:

$$U_r^{\sigma(t,x)}(x) \subset Q, \quad \min_{0 \le s \le lr^2} \rho(\partial C_r^{\sigma(t,x)}(x + b(t,x)s), \partial G) \ge \beta r^{1-\varepsilon}.$$

Clearly, $\Gamma_{\alpha r} \subset \Gamma_{\alpha r^{1-\varepsilon}}$.

The Markov moment $\bar{\nu}_{t,x}(Q \diagdown \Gamma_{\alpha r^{1-\varepsilon}})$, when the chain $(\bar{\vartheta}_k, \bar{X}_k)$ leaves the domain $Q \diagdown \Gamma_{\alpha r^{1-\varepsilon}}$, satisfies the inequality

$$\bar{\nu}_{t,x}(Q \diagdown \Gamma_{\alpha r^{1-\varepsilon}}) \le \bar{\nu}_{t,x}(Q \diagdown \Gamma_{\alpha r}).$$

We shall use the old notation $(\bar{\vartheta}_k, \bar{X}_k)$ for the new Markov chain, which is constructed by the same rules as above but stops in the layer $\Gamma_{\alpha r^{1-\varepsilon}}$ at the new Markov moment $\bar{\nu} = \bar{\nu}_{t,x}(Q \diagdown \Gamma_{\alpha r^{1-\varepsilon}})$. We believe that such a use of the same notation $(\bar{\vartheta}_k, \bar{X}_k)$ for various Markov chains and $\bar{\nu}$ for various stopping moments will cause no confusion below.

Consider the sequence $(\bar{\vartheta}_k, X_k)$, $k = 0, 1, \dots$:

$$\begin{aligned} X_0 &= x, \\ X_1 &= X_{t,x}(\bar{\vartheta}_1) \\ &\quad \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ X_k &= X_{t,x}(\bar{\vartheta}_k) = X_{\bar{\vartheta}_{k-1}, X_{k-1}}(\bar{\vartheta}_k) \\ &\quad \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \end{aligned}$$

connected with the system (1.1).

The sequence $(\bar{\vartheta}_k, X_k)$ is a Markov chain, which stops at the random moment $\bar{\nu}$ due to $\bar{\vartheta}_k = \bar{\vartheta}_{\bar{\nu}}$ under $k > \bar{\nu}$.

The following theorem states the closeness of X_k and $\bar{X}_k$ for $N = L/r^2$ steps.

Theorem 7.3. *Let $\bar{\nu} = \bar{\nu}_{t,x}(Q \diagdown \Gamma_{\alpha r^{1-\varepsilon}})$, $0 < \varepsilon \le 1$, be the first exit moment of the Markov chain $(\bar{\vartheta}_i, \bar{X}_i)$, $i = 1, 2, \dots$, from the domain $Q \diagdown \Gamma_{\alpha r^{1-\varepsilon}}$. Then, there exist constants $K > 0$ and $\gamma > 0$ such that for all r small enough the inequality*

$$\left(E \left| X_{N \wedge \bar{\nu}} - \bar{X}_{N \wedge \bar{\nu}} \right|^2 \right)^{1/2} = \left(E \left| X_N - \bar{X}_N \right|^2 \right)^{1/2} \le K \, e^{\gamma L} \, r$$

holds.

Proof. Below we use the same letter K for various constants (see the notice in Theorem 6.1).

Remember that $\bar{\vartheta}_k = \bar{\vartheta}_{k \wedge \bar{\nu}}$, $\bar{X}_k = \bar{X}_{k \wedge \bar{\nu}}$, and $X_k = X_{k \wedge \bar{\nu}} = X(\bar{\vartheta}_{k \wedge \bar{\nu}} \wedge \tau_{t,x})$.

Here we follow the proof of the corresponding theorem in [25].

Let ν be the first number at which $X_\nu \in \Gamma_{cr}$:

$$\nu = \begin{cases} \min\{k : X_k \in \Gamma_{cr}, k \le \bar{\nu}\}, \\ \quad \infty, \ X_k \notin \Gamma_{cr}, k \le \bar{\nu}, \end{cases}$$

where $c \le \dfrac{\beta}{2} r^{-\varepsilon}$ (here β is concerned to $\Gamma_{\alpha r^{1-\varepsilon}}$).

Then under $\nu \le \bar{\nu}$

$$|X_\nu - \bar{X}_\nu| \ge \frac{\beta}{2} r^{1-\varepsilon}. \tag{7.14}$$

We rewrite the global error in the form (l is a diameter of G) :

$$E\left|X_N-\bar{X}_N\right|^2=E\chi_{\nu\ge N\wedge\bar{\nu}}\left|X_N-\bar{X}_N\right|^2+E\chi_{\nu<N\wedge\bar{\nu}}\left|X_N-\bar{X}_N\right|^2$$

$$\le E\chi_{\nu\ge N\wedge\bar{\nu}}\left|X_{N\wedge\nu}-\bar{X}_{N\wedge\nu}\right|^2+l^2P(\nu<N\wedge\bar{\nu})$$

$$\le E\left|X_{N\wedge\nu}-\bar{X}_{N\wedge\nu}\right|^2+l^2P(\nu<N\wedge\bar{\nu}). \tag{7.15}$$

Due to (7.14), we have

$$E\chi_{\nu<N\wedge\bar{\nu}}\left|X_{N\wedge\nu}-\bar{X}_{N\wedge\nu}\right|^n\ge P(\nu<N\wedge\bar{\nu})\cdot(\frac{\beta}{2})^n\cdot r^{n-\varepsilon n},\ n=1,2,\dots .$$

From the other hand

$$E\chi_{\nu<N\wedge\bar{\nu}}\left|X_{N\wedge\nu}-\bar{X}_{N\wedge\nu}\right|^n$$

$$\le\ \sqrt{P(\nu<N\wedge\bar{\nu})}\cdot\left[E\left|X_{N\wedge\nu}-\bar{X}_{N\wedge\nu}\right|^{2n}\right]^{1/2}.$$

Consequently,

$$P(\nu<N\wedge\bar{\nu})\le K\,r^{-2n+2\varepsilon n}\cdot E\left|X_{N\wedge\nu}-\bar{X}_{N\wedge\nu}\right|^{2n}. \tag{7.16}$$

To prove the theorem, we need to find bounds for $E|d_k|^{2n},\ k=0,1,\dots,N$, where $d_k:=X_{k\wedge\nu}-\bar{X}_{k\wedge\nu}$. Note that the first term in (7.15) is equal to $E|d_N|^2$.

We have

$$d_k=X_{k\wedge\nu}-\bar{X}_{k\wedge\nu}=(X_{\bar{\vartheta}_{(k-1)\wedge\nu},X_{(k-1)\wedge\nu}}(\bar{\vartheta}_{k\wedge\nu})-X_{\bar{\vartheta}_{(k-1)\wedge\nu},\bar{X}_{(k-1)\wedge\nu}}(\bar{\vartheta}_{k\wedge\nu}))$$

$$+(X_{\bar{\vartheta}_{(k-1)\wedge\nu},\bar{X}_{(k-1)\wedge\nu}}(\bar{\vartheta}_{k\wedge\nu})-\bar{X}_{k\wedge\nu}).$$

Denote the second term by ρ_k and define Z_k similarly to Z in Lemma 7.3:

$$X_{\bar{\vartheta}_{(k-1)\wedge\nu},X_{(k-1)\wedge\nu}}(\bar{\vartheta}_{k\wedge\nu})-X_{\bar{\vartheta}_{(k-1)\wedge\nu},\bar{X}_{(k-1)\wedge\nu}}(\bar{\vartheta}_{k\wedge\nu})=X_{(k-1)\wedge\nu}-\bar{X}_{(k-1)\wedge\nu}+\chi_{\nu\wedge\bar{\nu}>k-1}Z_k.$$

Then

$$d_k=X_{(k-1)\wedge\nu}-\bar{X}_{(k-1)\wedge\nu}+\chi_{\nu\wedge\bar{\nu}>k-1}Z_k+\chi_{\nu\wedge\bar{\nu}>k-1}\rho_k=d_{k-1}+\chi_{\nu\wedge\bar{\nu}>k-1}(Z_k+\rho_k).$$

We have

$$E|d_k|^{2n}=E\left|d_{k-1}+\chi_{\nu\wedge\bar{\nu}>k-1}(Z_k+\rho_k)\right|^{2n}$$

$$=E[(d_{k-1},d_{k-1})+2(d_{k-1},\chi_{\nu\wedge\bar{\nu}>k-1}(Z_k+\rho_k))+\chi_{\nu\wedge\bar{\nu}>k-1}(Z_k+\rho_k,Z_k+\rho_k)]^n$$

$$\le E\left|d_{k-1}\right|^{2n}+2nE\left|d_{k-1}\right|^{2n-2}(d_{k-1},\chi_{\nu\wedge\bar{\nu}>k-1}(Z_k+\rho_k))$$

$$+K\sum_{m=2}^{2n}E\left|d_{k-1}\right|^{2n-m}\chi_{\nu\wedge\bar{\nu}>k-1}|Z_k+\rho_k|^m.$$

Due to $\mathcal{F}_{k-1}$-measurability (we denote $\mathcal{F}_m=\mathcal{F}_{\bar{\vartheta}_m}$) of d_{k-1} and $\chi_{\nu\wedge\bar{\nu}>k-1}$ and due to the conditional variants of (7.13) and (7.11), we get

$$E\left|d_{k-1}\right|^{2n-2}(d_{k-1},\chi_{\nu\wedge\bar{\nu}>k-1}(Z_k+\rho_k))$$

$$=E\left[|d_{k-1}|^{2n-2}(d_{k-1},\chi_{\nu\wedge\bar{\nu}>k-1}E\left((Z_k+\rho_k)/\mathcal{F}_{k-1}\right)\right]$$

$$\le E\left[|d_{k-1}|^{2n-1}\chi_{\nu\wedge\bar{\nu}>k-1}(|E(Z_k/\mathcal{F}_{k-1})|+|E(\rho_k/\mathcal{F}_{k-1})|\right]$$

$$\leq Kr^2E|d_{k-1}|^{2n} + Kr^4E|d_{k-1}|^{2n-1}.$$

Now consider $E\,|d_{k-1}|^{2n-m}\,\chi_{\nu\wedge\bar{\nu}>k-1}|Z_k+\rho_k|^m$. Using $\mathcal{F}_{k-1}$-measurability of d_{k-1} and $\chi_{\nu\wedge\bar{\nu}>k-1}$ and the conditional variants of (7.12) and (6.5), we obtain for $2 \leq m \leq 2n$:

$$E|d_{k-1}|^{2n-m}\chi_{\nu\wedge\bar{\nu}>k-1}|Z_k+\rho_k|^m = E\left[|d_{k-1}|^{2n-m}\chi_{\nu\wedge\bar{\nu}>k-1}E\left(|Z_k+\rho_k|^m \diagup \mathcal{F}_{k-1}\right)\right]$$

$$\leq KE\left[|d_{k-1}|^{2n-m}\chi_{\nu\wedge\bar{\nu}>k-1}E\left(|Z_k|^m \diagup \mathcal{F}_{k-1}\right)\right] + KE\left[|d_{k-1}|^{2n-m}\chi_{\nu\wedge\bar{\nu}>k-1}E\left(|\rho_k|^m \diagup \mathcal{F}_{k-1}\right)\right]$$

$$\leq KE(|d_{k-1}|^{2n}r^m + |d_{k-1}|^{2n-m}r^{2m}).$$

Then,

$$E|d_k|^{2n} \leq E\,|d_{k-1}|^{2n} + Kr^2E|d_{k-1}|^{2n} + Kr^4E|d_{k-1}|^{2n-1} + Kr^2\sum_{m=2}^{2n}E|d_{k-1}|^{2n-m}r^{2m-2}$$

$$\leq E\,|d_{k-1}|^{2n} + Kr^2\cdot E\sum_{m=0}^{2n}|d_{k-1}|^{2n-m}\cdot r^m.$$

Using the elementary inequality $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q},\ a,b>0,\ \ p,q>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1$, we get

$$|d_{k-1}|^{2n-m}r^m \leq \frac{|d_{k-1}|^{2n}}{2n/(2n-m)} + \frac{r^{2n}}{2n/m},\ 1 \leq m < 2n.$$

Hence

$$E|d_k|^{2n} \leq E|d_{k-1}|^{2n} + Kr^2E|d_{k-1}|^{2n} + Kr^{2n+2},\ d_0 = 0,$$

and we obtain for $N = L/r^2$:

$$E|d_N|^{2n} = E\left|X_{N\wedge\nu} - \bar{X}_{N\wedge\nu}\right|^{2n} \leq Ke^{2\gamma L}\cdot r^{2n}. \tag{7.17}$$

Taking $n \geq 1/\varepsilon$ and substituting (7.17) in (7.16), we get

$$P(\nu < N\wedge\bar{\nu}) \leq Ke^{2\gamma L}\cdot r^2. \tag{7.18}$$

Note that K and γ depend on ε.

Then, the inequality (7.15) together with (7.17) under $n=1$ and (7.18) gives the statement of the theorem. Theorem 7.3 is proved.

Theorem 7.4. *Let $\bar{\nu} = \bar{\nu}_{t,x}(Q\diagdown\Gamma_{\alpha r^{1-\varepsilon}}),\ 0 < \varepsilon \leq 1$, be the first exit moment of the Markov chain $(\bar{\vartheta}_i, \bar{X}_i),\ i = 1,2,\dots$, from the domain $Q\diagdown\Gamma_{\alpha r^{1-\varepsilon}}$. Then, there exist constants $K>0$ and $\gamma>0$ such that for all r small enough the inequality*

$$\left(E\left|X_{\bar{\nu}} - \bar{X}_{\bar{\nu}}\right|^2\right)^{1/2} \leq K\left(e^{\gamma L}\,r + e^{-c_r\gamma L/2}\right)$$

holds.

Proof. Introduce two sets $\mathcal{C} = \{\bar{\nu} \leq L/r^2\}$ and $\Omega\diagdown\mathcal{C} = \{\bar{\nu} > L/r^2\}$. Let l be a diameter of G. Using Theorems 7.2 and 7.3, we obtain

$$\begin{aligned}
E\left|X_{\bar{\nu}} - \bar{X}_{\bar{\nu}}\right|^2 &= E\left(\left|X_{\bar{\nu}} - \bar{X}_{\bar{\nu}}\right|^2;\mathcal{C}\right) + E\left(\left|X_{\bar{\nu}} - \bar{X}_{\bar{\nu}}\right|^2;\ \Omega\diagdown\mathcal{C}\right)\\
&= E\left(\left|X_{N\wedge\bar{\nu}} - \bar{X}_{N\wedge\bar{\nu}}\right|^2;\mathcal{C}\right) + E\left(\left|X_{\bar{\nu}} - \bar{X}_{\bar{\nu}}\right|^2;\ \Omega\diagdown\mathcal{C}\right)\\
&\leq E\left|X_N - \bar{X}_N\right|^2 + l^2\cdot P(\Omega\diagdown\mathcal{C}) \leq K\,e^{2\gamma L}\,r^2 + K\,e^{-c_r\gamma L}.
\end{aligned}$$

Theorem 7.4 is proved.

Remark 7.2. Let $\bar{\nu} = \bar{\nu}_{t,x}(Q\diagdown\Gamma_{\alpha r^{1-\varepsilon}})$, $0 < \varepsilon \le 1$, be the first exit moment of the Markov chain $(\bar{\vartheta}_i, \bar{X}_i)$, $i = 1, 2, \dots$, from the domain $Q\diagdown\Gamma_{\alpha r^{1-\varepsilon}}$. Then, for every natural m there exist constants $K > 0$ and $\gamma > 0$ such that for all r small enough the following inequalities

$$E\left|X_N - \bar{X}_N\right|^{2m} \le Ke^{2\gamma L}\, r^{2m}, \tag{7.19}$$

$$E\left|X_{\bar{\nu}} - \bar{X}_{\bar{\nu}}\right|^{2m} \le K(\, e^{2\gamma L}\, r^{2m} + e^{-c_r\gamma L}), \tag{7.20}$$

and

$$P(\tau_{t,x} < \bar{\vartheta}_N) \le Ke^{2\gamma L} r^{2n},\ n = 1, 2, \dots, \tag{7.21}$$

hold.

Indeed, taking $n \ge m/\varepsilon$ in (7.17) and in (7.16), we get

$$P(\nu < N \wedge \bar{\nu}) \le Ke^{2\gamma L}\cdot r^{2m} \tag{7.22}$$

instead of (7.18).

Similarly to (7.15), we have

$$E\left|X_N - \bar{X}_N\right|^{2m} \le E\left|X_{N\wedge\nu} - \bar{X}_{N\wedge\nu}\right|^{2m} + l^{2m}P(\nu < N \wedge \bar{\nu}).$$

Now the inequality (7.19) can be easily obtained from (7.17) under $n = m$.

The inequality (7.20) follows by the arguments as in the proof of Theorem 7.4.

Let us prove the inequality (7.21). Remember that $X_N = X_{t,x}(\tau_{t,x} \wedge \bar{\vartheta}_N) = X_{t,x}(\tau_{t,x})$ under $\tau_{t,x} < \bar{\vartheta}_N$, and $\rho(\bar{X}_N, \partial G) \ge \beta r^{1-\varepsilon}$. Therefore

$$E\chi_{\tau_{t,x}<\bar{\vartheta}_N}\left|X_{t,x}(\tau_{t,x}\wedge\bar{\vartheta}_N) - \bar{X}_N\right|^m \ge P(\tau_{t,x} < \bar{\vartheta}_N)\cdot\beta^m\cdot r^{m-\varepsilon m},\ m = 1, 2, \dots .$$

From the other hand,

$$E\chi_{\tau_{t,x}<\bar{\vartheta}_N}\left|X_{t,x}(\tau_{t,x}\wedge\bar{\vartheta}_N) - \bar{X}_N\right|^m \le \sqrt{P(\tau_{t,x} < \bar{\vartheta}_N)}\left[E\left|X_N - \bar{X}_N\right|^{2m}\right]^{1/2}.$$

Consequently, we get

$$P(\tau_{t,x} < \bar{\vartheta}_N) \le Kr^{2\varepsilon m - 2m}E\left|X_N - \bar{X}_N\right|^{2m}.$$

Using (7.19) under $m \ge n/\varepsilon$, we come to (7.21).

8. Approximation of exit point $(\tau, X(\tau))$

Here we are interesting in an approximation of the exit point $(\tau_{t,x}, X_{t,x}(\tau_{t,x}))$ of the space-time diffusion $(s, X_{t,x}(s))$, $s \ge t$, from the space-time domain Q.

We have $(\bar{\vartheta}_N, \bar{X}_N) = (\bar{\vartheta}_{\bar{\nu}}, \bar{X}_{\bar{\nu}}) \in \Gamma_{\alpha r^{1-\varepsilon}}$ on the set $\mathcal{C} = \{\bar{\nu} \le L/r^2\}$. Let $(\bar{\tau}_{t,x}, \xi_{t,x})(\omega)$, $\omega \in \mathcal{C}$, be a point on Γ defined as: if $\bar{\vartheta}_{\bar{\nu}} \ge t_1 - \alpha r^{1-\varepsilon}$ then $\bar{\tau}_{t,x} = t_1$ and $\xi_{t,x} = \bar{X}_{\bar{\nu}} \in G$, otherwise (i.e., when $\rho(\bar{X}_{\bar{\nu}}, \partial G) \le \alpha r^{1-\varepsilon}$) : $\bar{\tau}_{t,x} = \bar{\vartheta}_{\bar{\nu}}$ and a point $\xi_{t,x} \in \partial G$ is such that

$$\left|\bar{X}_{\bar{\nu}} - \xi_{t,x}\right| \le \alpha r^{1-\varepsilon},\ \omega \in \mathcal{C}. \tag{8.1}$$

To complete the definition of $(\bar{\tau}_{t,x}, \xi_{t,x})(\omega)$ on the set $\Omega\diagdown\mathcal{C}$, we put $\bar{\tau}_{t,x}$ be equal to $\bar{\vartheta}_N$ and $\xi_{t,x}$ be a point on ∂G nearest to $\bar{X}_N$.

It is natural to take the point $(\bar{\tau}_{t,x}, \xi_{t,x})$ as an approximate one to the exit point $(\tau_{t,x}, X_{t,x}(\tau_{t,x}))$.

Below we need the following lemma (it is analogous to the corresponding lemma from [23]).

Lemma 8.1. *There exists a constant $K > 0$ such that for all $(t, x) \in \overline{Q}$ and $y \in \partial G$ the inequalities*

$$E\left(X_{t,x}(\tau_{t,x}) - y\right)^2 \le K|x - y|,$$

$$E(\tau_{t,x} - t) \le K|x - y|$$

are valid.

Proof. Consider the Dirichlet problem

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\,\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(t,x)\frac{\partial u}{\partial x^i} = -g,\ (t,x) \in Q,$$

$$u(t,x)\mid_{\Gamma} = (x-y)^2,$$

where $g \ge 0$ is a constant.

The solution of the problem is

$$u_y(t,x) = E\left(X_{t,x}(\tau_{t,x}) - y\right)^2 + gE(\tau_{t,x} - t).$$

Due to the assumptions on the coefficients (see Introduction), u_y is a sufficiently smooth function on $\overline{Q}$. Since $u_y(t,y) = 0$, we have

$$u_y(t,x) = u_y(t,x) - u_y(t,y) \le K\,|x-y|\,.$$

Lemma 8.1 is proved.

Theorem 8.1. *Let $\bar\nu = \bar\nu_{t,x}(Q \diagdown \Gamma_{\alpha r^{1-\varepsilon}})$, $0 < \varepsilon \le 1$, be the first exit moment of the Markov chain $(\bar\vartheta_i, \bar X_i)$, $i = 1, 2, \dots$, from the domain $Q \diagdown \Gamma_{\alpha r^{1-\varepsilon}}$. Then, there exist positive constants K and γ such that for all r small enough the inequalities*

$$\left[E\left(\left|X_{t,x}(\tau_{t,x}) - \xi_{t,x}\right|^2; \mathcal{C}\right)\right]^{1/2} \le K\, r^{\frac{1-\varepsilon}{2}}, \tag{8.2}$$

and

$$\left[E\left|X_{t,x}(\tau_{t,x}) - \xi_{t,x}\right|^2\right]^{1/2} \le K(\, r^{\frac{1-\varepsilon}{2}} + e^{-c_r\gamma L/2}) \tag{8.3}$$

hold.

Proof. Consider the distance between $X_{t,x}(\tau_{t,x})$ and $\xi_{t,x}$ on $\mathcal{C}$:

$$E\left(\left|X_{t,x}(\tau_{t,x}) - \xi_{t,x}\right|^2; \mathcal{C}\right) = E\left(\chi_{\bar\vartheta_N \ge t_1 - \alpha r^{1-\varepsilon}}\left|X_{t,x}(\tau_{t,x}) - \xi_{t,x}\right|^2; \mathcal{C}\right)$$

$$+E\left(\chi_{\bar\vartheta_N < t_1 - \alpha r^{1-\varepsilon}}\left|X_{t,x}(\tau_{t,x}) - \xi_{t,x}\right|^2; \mathcal{C}\right). \tag{8.4}$$

We get for the first term of (8.4):

$$E\left(\chi_{\bar\vartheta_N \ge t_1 - \alpha r^{1-\varepsilon}}\left|X_{t,x}(\tau_{t,x}) - \xi_{t,x}\right|^2; \mathcal{C}\right) = E\chi_{\bar\vartheta_N \ge t_1 - \alpha r^{1-\varepsilon}}\left|X_{t,x}(\tau_{t,x}) - \bar X_N\right|^2$$

$$\le 2E\chi_{\bar\vartheta_N \ge t_1 - \alpha r^{1-\varepsilon}}\left|X_{t,x}(\tau_{t,x}) - X_N\right|^2 + 2E\left|X_N - \bar X_N\right|^2. \tag{8.5}$$

Due to Theorem 7.3, the second term of (8.5) is estimated by $Ke^{2\gamma L}\, r^2$. And we have for the first term of (8.5):

$$E\chi_{\bar\vartheta_N \ge t_1 - \alpha r^{1-\varepsilon}}\left|X_{t,x}(\tau_{t,x}) - X_N\right|^2 = E\left|\chi_{\bar\vartheta_N \ge t_1 - \alpha r^{1-\varepsilon}}(X_{t,x}(\tau_{t,x}) - X_N)\right|^2$$

$$\le 2E\left|\int_{\bar\vartheta_N \wedge \tau_{t,x}}^{\tau_{t,x}} \chi_{\bar\vartheta_N \ge t_1 - \alpha r^{1-\varepsilon}} b(s, X_{t,x}(s))ds\right|^2$$

$$+2E\left|\int_{\bar\vartheta_N\wedge\tau_{t,x}}^{\tau_{t,x}}\chi_{\bar\vartheta_N\geq t_1-\alpha r^{1-\varepsilon}}\sigma(s,X_{t,x}(s))dw(s)\right|^2$$

$$\leq KE\chi_{\bar\vartheta_N\geq t_1-\alpha r^{1-\varepsilon}}\left(\tau_{t,x}-\tau_{t,x}\wedge\bar\vartheta_N\right)^2+KE\chi_{\bar\vartheta_N\geq t_1-\alpha r^{1-\varepsilon}}\left(\tau_{t,x}-\tau_{t,x}\wedge\bar\vartheta_N\right)$$

$$\leq KE\chi_{\bar\vartheta_N\geq t_1-\alpha r^{1-\varepsilon}}(t_1-\bar\vartheta_N)^2+KE\chi_{\bar\vartheta_N\geq t_1-\alpha r^{1-\varepsilon}}(t_1-\bar\vartheta_N)\leq K\,r^{1-\varepsilon},$$

whence it follows that

$$E\left(\chi_{\bar\vartheta_N\geq t_1-\alpha r^{1-\varepsilon}}\left|X_{t,x}(\tau_{t,x})-\xi_{t,x}\right|^2;\mathcal{C}\right)\leq K\,r^{1-\varepsilon}. \tag{8.6}$$

Consider the second term of (8.4). Due to its definition, the point $\xi_{t,x}(\omega)$, $\omega\in\mathcal{C}$, belongs to ∂G if $\bar\vartheta_N<t_1-\alpha r^{1-\varepsilon}$. Then by the conditional version of Lemma 8.1, we get (note that $\xi_{t,x}$ is measurable with respect to $\mathcal{F}_N$)

$$E\left(\chi_{\bar\vartheta_N<t_1-\alpha r^{1-\varepsilon}}\left|X_{t,x}(\tau_{t,x})-\xi_{t,x}\right|^2;\mathcal{C}\right)$$

$$=E\left(\chi_{\bar\vartheta_N<t_1-\alpha r^{1-\varepsilon}}E\left(\left|X_{\bar\vartheta_N,X_N}(\tau_{\bar\vartheta_N,X_N})-\xi_{t,x}\right|^2\diagup\mathcal{F}_N\right);\mathcal{C}\right)$$

$$\leq KE\left(\chi_{\bar\vartheta_N<t_1-\alpha r^{1-\varepsilon}}\left|X_N-\xi_{t,x}\right|;\mathcal{C}\right).$$

Theorem 7.3 and the inequality (8.1) imply

$$E\left(\chi_{\bar\vartheta_N<t_1-\alpha r^{1-\varepsilon}}\left|X_N-\xi_{t,x}\right|;\mathcal{C}\right)\leq\left[E\left(\chi_{\bar\vartheta_N<t_1-\alpha r^{1-\varepsilon}}\left|X_N-\xi_{t,x}\right|^2;\mathcal{C}\right)\right]^{1/2}$$

$$\leq\left[2E\left|X_N-\bar X_N\right|^2+2\left(E\chi_{\bar\vartheta_N<t_1-\alpha r^{1-\varepsilon}}\left|\bar X_N-\xi_{t,x}\right|^2;\mathcal{C}\right)\right]^{1/2}$$

$$\leq K\,e^{\gamma L}r+2\alpha\,r^{1-\varepsilon}\leq Kr^{1-\varepsilon}. \tag{8.7}$$

Thus,

$$E\left(\chi_{\bar\vartheta_N<t_1-\alpha r^{1-\varepsilon}}\left|X_{t,x}(\tau_{t,x})-\xi_{t,x}\right|^2;\mathcal{C}\right)\leq K\;r^{1-\varepsilon}.$$

Substituting this inequality and the inequality (8.6) in (8.4), we get (8.2).

The inequality (8.3) is obtained by Theorem 7.2 analogously to the proof of Theorem 7.4. Theorem 8.1 is proved.

Theorem 8.2. *Under the assumptions of Theorem 8.1, the inequalities*

$$E\left(|\tau_{t,x}-\bar\tau_{t,x}|;\mathcal{C}\right)\leq K\,r^{1-\varepsilon}, \tag{8.8}$$

$$E|\tau_{t,x}-\bar\tau_{t,x}|\leq K(\,r^{1-\varepsilon}+e^{-\alpha_r\gamma L}) \tag{8.9}$$

hold.

Proof. Remember that $\tau_{t,x}\leq t_1$, $\bar\vartheta_N\leq t_1$. Further, $\bar\tau_{t,x}=t_1$ under $\bar\vartheta_N\geq t_1-\alpha r^{1-\varepsilon}$ and $\bar\tau_{t,x}=\bar\vartheta_N$ otherwise. Consequently, $\bar\tau_{t,x}\geq\bar\vartheta_N$. Let below $\tau:=\tau_{t,x}$, $\bar\tau:=\bar\tau_{t,x}$.

Consider the difference $|\tau-\bar\tau|$ on the set $\mathcal{C}$. We have

$$E\left(|\tau-\bar\tau|;\mathcal{C}\right)=E\left((\bar\tau-\tau\wedge\bar\tau);\mathcal{C}\right)+E\left((\tau-\tau\wedge\bar\tau);\mathcal{C}\right). \tag{8.10}$$

We get for the first term:

$$E\left((\bar\tau-\tau\wedge\bar\tau);\mathcal{C}\right)\leq E(\bar\tau-\tau\wedge\bar\tau)=E\chi_{\tau<\bar\tau}(\bar\tau-\tau\wedge\bar\tau)$$

$$=E\chi_{\tau<\bar\vartheta_N}(\bar\tau-\tau\wedge\bar\tau)+E\chi_{\bar\vartheta_N\leq\tau<\bar\tau}(\bar\tau-\tau\wedge\bar\tau)$$

$$\leq(t_1-t_0)\cdot P(\tau<\bar\vartheta_N)+E\chi_{t_1-\alpha r^{1-\varepsilon}\leq\tau<t_1}(t_1-\tau).$$

Then using (7.21) under $n = 1$, we obtain

$$E\left((\bar{\tau} - \tau \wedge \bar{\tau}); \mathcal{C}\right) \leq Ke^{2\gamma L} \cdot r^2 + \alpha r^{1-\varepsilon} \leq Kr^{1-\varepsilon}. \tag{8.11}$$

Consider the second term of (8.10). Due to $\xi_{t,x} \in \partial G$ under $\bar{\vartheta}_N < t_1 - \alpha r^{1-\varepsilon}$, Lemma 8.1, and the inequality (8.7), we get

$$E\left((\tau - \tau \wedge \bar{\tau}); \mathcal{C}\right) = E\left(\chi_{\bar{\tau}<\tau}(\tau - \tau \wedge \bar{\tau}); \mathcal{C}\right) = E\left(\chi_{\bar{\vartheta}_N<\tau}\chi_{\bar{\vartheta}_N<t_1-\alpha r^{1-\varepsilon}}(\tau - \tau \wedge \bar{\vartheta}_N); \mathcal{C}\right)$$

$$= E\left(\chi_{\bar{\vartheta}_N<t_1-\alpha r^{1-\varepsilon}}(\tau_{\bar{\vartheta}_N,X_N} - \bar{\vartheta}_N); \mathcal{C}\right) = E\left(\chi_{\bar{\vartheta}_N<t_1-\alpha r^{1-\varepsilon}}E(\tau_{\bar{\vartheta}_N,X_N} - \bar{\vartheta}_N / \mathcal{F}_N); \mathcal{C}\right)$$

$$\leq KE\left(\chi_{\bar{\vartheta}_N<t_1-\alpha r^{1-\varepsilon}}|X_N - \xi_{t,x}|; \mathcal{C}\right) \leq Kr^{1-\varepsilon}.$$

Substituting this inequality and the inequality (8.11) in (8.10), we get (8.8).

The inequality (8.9) is obtained by Theorem 7.2 analogously to the proof of Theorem 7.4. Theorem 8.2 is proved.

9. Simulation of Brownian motion with drift provided bounded space increment

Sections 6-8 are connected with the one-step approximation $(t+\bar{\theta}, \bar{X}_{t,x}(t+\bar{\theta}))$, $(t,x) \in Q\backslash\Gamma_{\alpha r}$ (see (6.3)), which is based on the simulation of the exit point $(\bar{\theta}, w(t+\bar{\theta}) - w(t))$ of the process $(s - t, w(s) - w(t))$, $s > t$, from the space-time parallelepiped $\Pi_r = [0, lr^2) \times C_r$ with the cubic base C_r. We can guarantee that $\bar{X}_{t,x}(t+\bar{\theta}))$ belongs to G due to the smallness of both C_r and lr^2. The smallness of the time-size lr^2 of Π_r ensures that the term $b(t,x) \cdot \bar{\theta}$ in (6.3) is not bigger than $b(t,x) \cdot lr^2$. Consequently, the projection of the space-time parallelepiped $\overline{U}_r^{\sigma}(x)$ on R^d differs not essentially from the space parallelepiped $\overline{C}_r^{\sigma}(x)$, to which the point $\sigma(t,x)(w(t+\bar{\theta}) - w(t))$ belongs. Remember that $(t+\bar{\theta}, \bar{X}_{t,x}(t+\bar{\theta})) \in \partial U_r^{\sigma}(x)$.

It is possible to derive other constructive one-step approximations. Let us consider a one-step approximation based on a simulation of exit points for the Brownian motion with drift $W_\mu(s)$:

$$W_\mu(s) = \mu s + W(s),\ W_\mu(0) = 0,$$

where μ is a d-dimensional fixed vector and $W(s)$ is a d-dimensional standard Wiener process.

If $(\bar{\theta}, w_\mu(t+\bar{\theta}) - w_\mu(t))$ is the first exit point of the process $(s-t, w_\mu(s) - w_\mu(t))$, $s > t$, under $\mu = \sigma^{-1}(t,x)b(t,x)$, $(t,x) \in Q\backslash\Gamma_{\alpha r}$, from the space-time parallelepiped $[0,l) \times C_r$, $l \leq t_1 - t$, then it is easy to see that the approximation

$$\bar{X}_{t,x}(t+\bar{\theta}) = x + \sigma(t,x)(w_\mu(t+\bar{\theta}) - w_\mu(t)) \tag{9.1}$$

belongs to the space parallelepiped $\overline{C}_r^{\sigma}(x)$ even under not small l.

Then we are able to ensure again belonging of $\bar{X}_{t,x}(t+\bar{\theta})$ to G, and, consequently, $(t+\bar{\theta}, \bar{X}_{t,x}(t+\bar{\theta}))$ to Q, but the smallness of time-size of the space-time parallelepiped $[0,l) \times C_r$ is already not required in contrast to the approximation (6.3).

The approximation (9.1) is more universal than the approximation (6.3). However, the approximation (6.3) is simpler in a computational sense than (9.1) and is quite appropriate for the majority of problems.

In this section we give algorithms on simulating exit points for the Brownian motion with drift $W_\mu(s)$. The theorems on local error and global convergence connected with the one-step approximation 9.1 can be done analogously to the corresponding theorems of Sections 6-8.

9.1. **Some distributions for one-dimensional Brownian motion with drift.**

Lemma 9.1. *Let τ be the first-passage time of the one-dimensional Brownian motion with drift $W_\mu(s) = \mu s + W(s)$, $W_\mu(0) = 0$, to the boundary of the interval $[-1, 1]$. Then its distribution $\mathcal{P}(t; \mu) = P(\tau < t)$ is equal to*

$$\mathcal{P}(t;\mu) = 1 - 2\pi e^{-\frac{1}{2}\mu^2 t}(e^{\mu} + e^{-\mu}) \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)}{\pi^2(2k+1)^2 + 4\mu^2}\, e^{-\frac{1}{8}\pi^2(2k+1)^2 t} \tag{9.2}$$

or

$$\mathcal{P}(t;\mu) = 1 - \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k e^{2\mu k}\Big(\operatorname{erfc}\frac{2k-1+\mu t}{\sqrt{2t}} - \operatorname{erfc}\frac{2k+1+\mu t}{\sqrt{2t}}\Big)$$

$$-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k e^{-2\mu k}\Big(\operatorname{erfc}\frac{2k-1-\mu t}{\sqrt{2t}} - \operatorname{erfc}\frac{2k+1-\mu t}{\sqrt{2t}}\Big)\,. \tag{9.3}$$

Proof. Due to (2.19), the distribution $P(\tau_x < t)$ is equal to $1 - v(t, x)$, where τ_x is the first exit time of $x + W_\mu(s) = x + \mu s + W(s)$, $-1 \le x \le 1$, to the boundary of the interval $[-1, 1]$, and $v(t, x)$ obeys the following boundary value problem

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \mu\frac{\partial v}{\partial x},\ t > 0,\ -1 < x < 1, \tag{9.4}$$

$$v(0, x) = 1,\ v(t, -1) = v(t, 1) = 0. \tag{9.5}$$

The function

$$u(t, x) = e^{\frac{1}{2}\mu^2 t + \mu x} v(t, x)$$

satisfies the following boundary value problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\ ,\ t > 0,\ -1 < x < 1, \tag{9.6}$$

$$u(0, x) = e^{\mu x},\ u(t, -1) = u(t, 1) = 0. \tag{9.7}$$

Solving this problem analogously to (3.1)-(3.2) and (3.13)-(3.14), we get two expressions for $v(t, x)$:

$$v(t,x) = e^{-\frac{1}{2}\mu^2 t - \mu x}\sum_{k=1}^{\infty}\frac{(-1)^k \pi k}{(\pi k)^2 + 4\mu^2}(e^{-\mu} - e^{\mu})\cdot \sin \pi k x \cdot e^{-\frac{1}{2}\pi^2 k^2 t}$$

$$+e^{-\frac{1}{2}\mu^2 t - \mu x}\sum_{k=0}^{\infty}\frac{(-1)^k 2\pi(2k+1)}{\pi^2(2k+1)^2 + 4\mu^2}(e^{-\mu} + e^{\mu})\cdot \cos\frac{\pi(2k+1)x}{2}\cdot e^{-\frac{1}{8}\pi^2(2k+1)^2 t}$$

and

$$v(t,x) = e^{-\frac{1}{2}\mu^2 t - \mu x}\int_{-1}^{1} G(t, x, y)\cdot e^{\mu y} dy.$$

The equality $\mathcal{P}(t; \mu) = P(\tau < t) = 1 - v(t, 0)$ gives (9.2) and (9.3). Lemma 9.1 is proved.

Remark 9.1. As earlier the formula (9.2) is convenient for calculations under great t, and the formula (9.3) is convenient under small t. It should be pointed out that if one of $2k\pm1\pm\mu t$ takes a negative value then the corresponding $\operatorname{erfc}\dfrac{2k \pm 1 \pm \mu t}{\sqrt{2t}} > 1$. Therefore it may be necessary to calculate more terms of the series in (9.3) in comparison with (3.7). But the number of the needed terms is not too large in practice due to very fast convergence of the series under small t.

Remark 9.2. Using the Laplace transform, it is possible to derive one more expression for $\mathcal{P}(t;\mu)$:

$$\mathcal{P}(t;\mu) = \frac{2(e^{-\mu}+e^{\mu})}{\sqrt{\pi}} \sum_{k=0}^{\infty}(-1)^k \int_{\frac{2k+1}{\sqrt{2t}}}^{\infty} \exp(-\frac{\mu^2(2k+1)^2}{4z^2} - z^2)dz.$$

It is clear that this expression is convenient for calculations under small t.

Lemma 9.2. *Let τ be the first-passage time of the one-dimensional Brownian motion with drift $W_\mu(s) = \mu s + W(s)$, $W_\mu(0) = 0$, to the boundary of the interval $[-1,1]$. Then the probabilities*

$$\mathcal{P}(t;-1;\mu) := P(\tau < t,\ W_\mu(\tau) = -1), \quad \mathcal{P}(t;1;\mu) := P(\tau < t,\ W_\mu(\tau) = 1)$$

are equal to

$$\mathcal{P}(t;-1;\mu) = \frac{1}{e^{2\mu}+1}\mathcal{P}(t;\mu),\ \ \mathcal{P}(t;1;\mu) = \frac{e^{2\mu}}{e^{2\mu}+1}\mathcal{P}(t;\mu)\ . \tag{9.8}$$

Proof. The probability $\mathcal{P}(t;-1;\mu)$ is equal to $v(t,0)$, where $v(t,x)$ is the solution of the equation (9.4) with the initial and boundary conditions: $v(0,x) = 0$, $v(t,-1) = 1$, $v(t,1) = 0$ (see the problem (2.15)-(2.17) under (2.9) and its solution (2.18)). The following change of variables

$$u(t,x) = e^{\frac{1}{2}\mu^2 t+\mu x}(v(t,x) + \frac{e^{-2\mu} - e^{-2\mu x}}{e^{2\mu} - e^{-2\mu}}) \tag{9.9}$$

leads to the problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\ ,\ t>0,\ -1<x<1,$$

$$u(0,x) = \frac{e^{-2\mu+\mu x} - e^{-\mu x}}{e^{2\mu} - e^{-2\mu}},\ \ u(t,-1) = u(t,1) = 0.$$

Solving this problem, we get (we restrict ourselves to writing $u(t,0)$ only)

$$u(t,0) = 2\pi e^{-\mu}\sum_{k=0}^{\infty}(-1)^{k+1}\frac{(2k+1)}{\pi^2(2k+1)^2+4\mu^2}\, e^{-\pi^2(2k+1)^2 t/8}$$

or

$$u(t,0) = \frac{e^{-2\mu}}{e^{2\mu}-e^{-2\mu}}\int_{-1}^{1} G(t,0,y)\cdot e^{\mu y}dy - \frac{1}{e^{2\mu}-e^{-2\mu}}\int_{-1}^{1} G(t,0,y)\cdot e^{-\mu y}dy.$$

Using (9.9), we obtain (9.8) for $\mathcal{P}(t;-1;\mu)$. The second formula in (9.8) is obtained analogously. Lemma 9.2 is proved.

Remark 9.3. Lemma 9.2 is a consequence of Reuter's theorem (see [30, p. 84]), which asserts that τ and $W_\mu(\tau) = \mu\tau + W(\tau)$ are independent random variables (it is not difficult to show that $P(W_\mu(\tau) = -1) = \dfrac{1}{e^{2\mu}+1}$, $P(W_\mu(\tau) = 1) = \dfrac{e^{2\mu}}{e^{2\mu}+1}$). But the given proof has an independent interest because it can be used for evaluation of some other probabilities, for example, like $P(\tau_x < t,\ W_\mu(\tau) = -1)$.

Lemma 9.3. *For the conditional probability*

$$\mathcal{Q}(\beta; t, \mu) := P(W_\mu(t) < \beta \big/ |W_\mu(s)| < 1,\ 0 < s < t),\ 1 < \beta \le 1$$

the following inequalities

$$\mathcal{Q}(\beta; t, \mu) = \frac{4}{1 - \mathcal{P}(t;\mu)} e^{-\mu^2 t/2} \sum_{k=0}^{\infty} \frac{1}{\pi^2(2k+1)^2 + 4\mu^2} \left((-1)^k \frac{\pi(2k+1)}{2} e^{-\mu} \right.$$

$$\left. + e^{\mu\beta} \left[\mu \cos \frac{\pi(2k+1)\beta}{2} + \frac{\pi(2k+1)}{2} \sin \frac{\pi(2k+1)\beta}{2} \right] \right) e^{-\frac{1}{8}\pi^2(2k+1)^2 t}, \tag{9.10}$$

$$\mathcal{Q}(\beta; t, \mu) = \frac{1}{2(1 - \mathcal{P}(t;\mu))} \sum_{k=-\infty}^{\infty} \left(e^{4\mu k} \left[\operatorname{erfc} \frac{4k - \beta + \mu t}{\sqrt{2t}} - \operatorname{erfc} \frac{4k + 1 + \mu t}{\sqrt{2t}} \right] \right.$$

$$\left. + e^{\mu(4k+2)} \left[\operatorname{erfc} \frac{4k + 3 + \mu t}{\sqrt{2t}} - \operatorname{erfc} \frac{4k + 2 - \beta + \mu t}{\sqrt{2t}} \right] \right) \tag{9.11}$$

hold.

Proof. We have (as in Lemma 3.2)

$$\mathcal{Q}(\beta; t, \mu) = \frac{P(W_\mu(t) < \beta,\ \tau \ge t)}{P(\tau \ge t)}.$$

Then to prove the lemma, we need in expressions for the probability $P(W_\mu(t) < \beta,\ \tau \ge t)$. This probability is equal to $v(t, 0)$, where $v(t, x)$ is the solution of the equation (9.4) with the initial and boundary conditions: $v(0, x) = \chi_{[-1,\beta)}(x)$, $v(t, -1) = v(t, 1) = 0$, $t > 0$ (see the function (2.20), which is the solution of the problem (2.15)-(2.17) under (2.13)). The following change of variables

$$u(t, x) = e^{\frac{1}{2}\mu^2 t + \mu x} v(t, x) \tag{9.12}$$

leads to the problem

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}\ ,\ t > 0,\ -1 < x < 1,$$

$$u(0, x) = e^{\mu x} \chi_{[-1,\beta)}(x),\ u(t, -1) = u(t, 1) = 0.$$

Solving this problem analogously to (3.13)-(3.14) and then using (9.12), we get the statement of the lemma. Lemma 9.3 is proved.

9.2. **Simulation of exit time and exit point of Brownian motion with drift from cube.** Let us consider a d-dimensional Brownian motion with drift $W_\mu(s)$ in the d-dimensional cube $C = \{x = (x^1, ..., x^d) :\ |x^i| < 1,\ i = 1, ..., d\} \subset R^d$, and let τ be the first-passage time of $W_\mu(s)$, $W_\mu(0) = 0$, to the boundary ∂C of the cube C.

Lemma 9.4. *The distribution function $\mathcal{P}_d(t;\mu)$ for τ is equal to*

$$\mathcal{P}_d(t;\mu) = P(\tau < t) = 1 - \prod_{i=1}^{d} (1 - \mathcal{P}(t;\mu^i)), \tag{9.13}$$

where μ^i, $i = 1, \ldots, d$, are components of the vector μ.

The proof is evident.

Introduce the random variable $\varkappa$, which takes the value j for $\omega \in \{\omega : W^j_\mu(\tau) = \pm 1\}$.

Lemma 9.5. *The conditional probability* $P(\varkappa = j \diagup \tau = \theta)$ *is equal to*

$$P(\varkappa = j \diagup \tau = \theta) = \frac{\mathcal{P}'(\theta;\mu^j)\prod\limits_{i\neq j}(1-\mathcal{P}(\theta;\mu^i))}{\sum\limits_{i=1}^{d}\mathcal{P}'(\theta;\mu^i)\prod\limits_{l\neq i}(1-\mathcal{P}(\theta;\mu^l))},\ j = 1,\dots,d. \tag{9.14}$$

Proof. To prove the lemma, we use two expressions for $P(\varkappa = j, \tau < \theta)$:

$$P(\varkappa = j, \tau < \theta) = \int_0^\theta P(\varkappa = j \diagup \tau = \vartheta)\, dP_\tau(\vartheta)$$

$$= \int_0^\theta P(\varkappa = j \diagup \tau = \vartheta)\sum_{i=1}^{d}\mathcal{P}'(\vartheta;\mu^i)\prod_{l\neq i}(1-\mathcal{P}(\vartheta;\mu^l))\, d\vartheta, \tag{9.15}$$

and

$$P(\varkappa = j, \tau < \theta) = P\Big(\bigcap_{i\neq j}(|W^i_\mu(s)| < 1,\ 0 < s < \tau^j), \tau^j < \theta\Big)$$

$$= \int_0^\theta P\Big(\bigcap_{i\neq j}(|W^i_\mu(s)| < 1,\ 0 < s < \tau^j) \diagup \tau^j = \vartheta\Big)\, dP_{\tau^j}(\vartheta)$$

$$= \int_0^\theta \prod_{i\neq j}(1-\mathcal{P}(\vartheta;\mu^i))\mathcal{P}'(\vartheta;\mu^j)\, d\vartheta. \tag{9.16}$$

The equality $P(\bigcap_{i\neq j}(|W^i_\mu(s)| < 1,\ 0 < s < \tau^j) \diagup \tau^j = \vartheta) = \prod\limits_{i\neq j}(1-\mathcal{P}(\vartheta;\mu^i))$ in (9.16) is proved similarly to Lemma 4.2. The expressions (9.15) and (9.16) imply (9.14). Lemma 9.5 is proved.

Lemma 9.6. *The following equalities*

$$P(W^j_\mu(\tau) = -1 \diagup \varkappa = j, \tau = \theta) = \frac{1}{e^{2\mu^j}+1}, \tag{9.17}$$

$$P(W^j_\mu(\tau) = 1 \diagup \varkappa = j, \tau = \theta) = \frac{e^{2\mu^j}}{e^{2\mu^j}+1}. \tag{9.18}$$

are true.

Proof. Due to Lemma 9.2 and Remark 9.3, which state the independence of τ^j and $W^j_\mu(\tau^j)$, we get

$$P(W^j_\mu(\tau) = -1 \diagup \varkappa = j, \tau = \theta) = P(W^j_\mu(\tau^j) = -1 \diagup \tau^j = \theta)$$

$$= P(W^j_\mu(\tau^j) = -1) = \frac{1}{e^{2\mu^j}+1}.$$

The formula (9.18) is obtained analogously. Lemma 9.6 is proved.

Lemma 9.7 *The following equality*

$$P(\bigcap_{i\neq j}(W_\mu^i(\tau) < \beta^i)/\varkappa = j, \tau = \theta) = \prod_{i\neq j} \mathcal{Q}(\beta^i; \theta, \mu^i). \tag{9.19}$$

is valid. In particular, the relation (9.19) *means that provided $\varkappa$ and τ been known, $W_\mu^i(\tau)$, $i \neq j$, are independent.*

Proof. Carrying out reasoning as in Lemma 4.2, we get

$$P(\bigcap_{i\neq j}(W_\mu^i(\tau^j) < \beta^i\ ,\ |W_\mu^i(s)| < 1,\ 0 < s < \tau^j)/\tau^j)$$

$$= \prod_{i\neq j}[(1 - \mathcal{P}(\tau^j; \mu^i)) \cdot \mathcal{Q}(\beta^i; \tau^j, \mu^i)],$$

whence, doing as in Lemma 4.3, we obtain

$$P(\varkappa = j,\ \tau < \theta,\ \bigcap_{i\neq j}(W_\mu^i(\tau) < \beta^i))$$

$$= \int_0^\theta \prod_{i\neq j}[(1 - \mathcal{P}(\vartheta; \mu^i)) \cdot \mathcal{Q}(\beta^i; \vartheta, \mu^i)] \cdot \mathcal{P}'(\vartheta; \mu^j)\, d\vartheta. \tag{9.20}$$

We have from (9.16):

$$dP(\varkappa = j,\ \tau < \theta) = \prod_{i\neq j}(1 - \mathcal{P}(\theta; \mu^i)) \cdot \mathcal{P}'(\theta; \mu^j)\, d\theta.$$

Then

$$P(\varkappa = j,\ \tau < \theta,\ \bigcap_{i\neq j}(W_\mu^i(\tau) < \beta^i))$$

$$= \int_0^\theta P(\bigcap_{i\neq j}(W_\mu^i(\tau) < \beta^i)/\varkappa = j, \tau = \vartheta) \prod_{i\neq j}(1 - \mathcal{P}(\vartheta; \mu^i)) \cdot \mathcal{P}'(\vartheta; \mu^j)\, d\vartheta. \tag{9.21}$$

Comparing (9.20) and (9.21), we come to (9.19). Lemma 9.7 is proved.

Let us note that the point $(\tau, W_\mu(\tau))$ belongs to the lateral surface of the unbounded semi-cylinder $[0, \infty) \times C \subset R^{d+1}$ with the cubic base C.

Theorem 9.1. (Algorithm for simulating exit point of the space-time Brownian motion with drift to lateral surface of cylinder with cubic base).

Let $\bar{\varkappa}$, $\bar{\nu}$, γ, $\gamma^1, \ldots, \gamma^{d-1}$ be independent, uniformly distributed on $[0,1]$ random variables. A random point (τ, ξ_μ), distributed as the first exit point $(\tau, W_\mu(\tau))$ of the process $(s, W_\mu(s))$ to the lateral surface of the cubic semi-cylinder, is simulated by the following algorithm:

$$\tau = \mathcal{P}_d^{-1}(\gamma; \mu),$$

where $\mathcal{P}_d^{-1}(\cdot; \mu)$ is the inverse function to $\mathcal{P}_d(t; \mu)$ with respect to t;
$\varkappa$ is found as

$$\varkappa = j \ \ if\ \bar{\varkappa} \in [\alpha_{j-1}, \alpha_j),\ j = 1, \ldots, d,$$

where

$$\alpha_0 = 0,\ \alpha_j = \alpha_{j-1} + \frac{\mathcal{P}'(\tau;\mu^j)\prod\limits_{i\neq j}(1-\mathcal{P}(\tau;\mu^i))}{\sum\limits_{i=1}^{d}\mathcal{P}'(\tau;\mu^i)\prod\limits_{l\neq i}(1-\mathcal{P}(\tau;\mu^l))};$$

ν is found as

$$\nu = \begin{cases} -1,\ \bar{\nu} \in [0, \dfrac{1}{e^{2\mu^{\varkappa}}+1}) \\ \\ 1,\ \bar{\nu} \in [\dfrac{1}{e^{2\mu^{\varkappa}}+1}, 1]; \end{cases}$$

and then the components ξ^i_μ, $i = 1,\dots,d$, of ξ_μ are simulated as

$$\begin{aligned} \xi^1_\mu &= \mathcal{Q}^{-1}(\gamma^1;\tau,\mu^1),\dots,\ \xi^{\varkappa-1}_\mu = \mathcal{Q}^{-1}(\gamma^{\varkappa-1};\tau,\mu^{\varkappa-1}),\ \xi^{\varkappa}_\mu = \nu, \\ \xi^{\varkappa+1}_\mu &= \mathcal{Q}^{-1}(\gamma^{\varkappa};\tau,\mu^{\varkappa+1}),\dots,\ \xi^d_\mu = \mathcal{Q}^{-1}(\gamma^{d-1};\tau,\mu^d). \end{aligned}$$

Proof. The statement of the theorem follows from Lemmas 9.4-9.7.

Corollary 9.1. *Let $C_r = \{x = (x^1,...,x^d) : \ |x^i| < r,\ i = 1,...,d\} \subset R^d$ be the d-dimensional cube with center at the origin and with edge length equal to $2r$. Let $\bar{\theta}$ be the first-passage time for the d-dimensional Brownian motion with drift $w_\mu(s) = \mu s + w(s)$ to the boundary ∂C_r of the cube C_r. Then the point*

$$(\bar{\theta}, \bar{w}_\mu) = (r^2\tau, r\xi_{r\mu}),$$

where $(\tau, \xi_{r\mu})$ is simulated by the algorithm for simulating exit point to lateral surface of cylinder with the cubic base C, has the same distribution as $(\bar{\theta}, w_\mu(\bar{\theta}))$.

Proof. We have

$$W_{r\mu}(\frac{t}{r^2}) = r\mu\cdot\frac{t}{r^2} + W(\frac{t}{r^2}).$$

Due to the fact that if $W(t)$ is a Wiener process, then $w(t) = rW(t/r^2)$ is also a Wiener process, we get

$$w_\mu(t) = \mu t + w(t) = rW_{r\mu}(\frac{t}{r^2}).$$

Evidently, the point $w_\mu(\bar{\theta})$ belongs to the boundary ∂C_r of the cube C_r and $w_\mu(s) \in C_r$ under $s \in [0,\bar{\theta})$. Corollary 9.1 is proved.

Remark 9.4. Consider an application of Theorem 9.1 in the case, when the domain G is bounded, $t_1 = \infty$, and the system (1.1) is autonomous. Then by Corollary 9.1, we are able to construct the following one-step approximation

$$\bar{X}_{t,x}(t+\bar{\theta}) = x + \sigma(x)(w_\mu(t+\bar{\theta}) - w_\mu(t)),$$

where $\bar{\theta}$ is the first passage time of the Brownian motion with drift $w_\mu(s) - w_\mu(t)$, $s \geq t$, $\mu = \sigma^{-1}(x)b(x)$, to the boundary of the cube $C_r \subset R^d$.

The approximation $\bar{X}_{t,x}(t+\bar{\theta})$ satisfies the equation with frozen coefficients (6.2). The point $(t+\bar{\theta}, \bar{X}_{t,x}(t+\bar{\theta}))$ belongs to the lateral surface of the semi-cylinder $[t_0,\infty)\times C_r^{\sigma(x)}(x) \subset R^{d+1}$, where the space parallelepiped $C_r^{\sigma(x)}(x)$ is obtained from the cube C_r

by the linear transformation $\sigma(x)$ and the shift x. Note that $\bar{\theta}$ can take arbitrary large values with some probability.

The point $(t+\bar{\theta}, \bar{X}_{t,x}(t+\bar{\theta}))$ approximates in the mean-square sense the point $(t+\bar{\theta}, X_{t,x}(t+\bar{\theta}))$. Theorems on the local mean-square error, global convergence, and on an approximation of the exit point of the autonomous diffusion process $X(s)$ from the domain G can be stated and proved analogously to the corresponding theorems of [25].

Remember that the algorithm of [25] gives only the phase component of the approximate trajectory (see also Introduction). Using a random walk over boundaries of small space parallelepipeds, we are able to simulate constructively both phase and time components of the approximate trajectory.

9.3. **Simulation of exit point of the space-time Brownian motion with drift from space-time parallelepiped with cubic base.** Analogously to Section 5, let us construct an algorithm for simulating the exit point $(\tau(l), W_\mu(\tau(l))$ of the process $(s, W_\mu(s))$ from the space-time parallelepiped $\Pi = [0,l) \times C \subset R^{d+1}$. The random variable $\tau(l)$ is found as $\min(\tau, l)$, where τ is the first-passage time of $W_\mu(s)$ to the boundary ∂C as above.

Theorem 9.2. (Algorithm for simulating exit point of the space-time Brownian motion with drift from space-time parallelepiped with cubic base). *Let ι, $\bar{\varkappa}$, $\bar{\nu}$, γ, $\gamma^1, \dots, \gamma^{d-1}$ be independent random variables. Let ι be simulated by the law*

$$P(\iota=-1) = \mathcal{P}_d(l;\mu),\ P(\iota=1) = 1-\mathcal{P}_d(l;\mu),$$

and the other random variables be uniformly distributed on $[0,1]$.

Then a random point $(\tau(l), \xi_\mu)$, simulated by the algorithm given below, is distributed as the exit point $(\tau(l), W_\mu(\tau(l)))$.

If the simulated value of ι is equal to -1, then the point $(\tau(l), \xi_\mu)$ belongs to the lateral surface of Π, *and*

$$\tau(l) = \mathcal{P}_d^{-1}(\gamma\mathcal{P}_d(l;\mu);\mu);$$

$\varkappa$ is found as

$$\varkappa = j \ \text{ if } \bar{\varkappa} \in [\alpha_{j-1}, \alpha_j),\ j=1,\dots,d,$$

where

$$\alpha_0 = 0,\ \alpha_j = \alpha_{j-1} + \frac{\mathcal{P}'(\tau(l);\mu^j)\prod\limits_{i\neq j}(1-\mathcal{P}(\tau(l);\mu^i))}{\sum\limits_{i=1}^{d}\mathcal{P}'(\tau(l);\mu^i)\prod\limits_{l\neq i}(1-\mathcal{P}(\tau(l);\mu^l))};$$

ν is found as

$$\nu = \begin{cases} -1, & \bar{\nu} \in [0, \dfrac{1}{e^{2\mu^\varkappa}+1}) \\ 1, & \bar{\nu} \in [\dfrac{1}{e^{2\mu^\varkappa}+1}, 1]; \end{cases}$$

and the components ξ_μ^i, $i=1,\dots,d$, of ξ_μ are simulated as

$$\begin{aligned} \xi_\mu^1 &= \mathcal{Q}^{-1}(\gamma^1;\tau(l),\mu^1),\dots,\xi_\mu^{\varkappa-1} = \mathcal{Q}^{-1}(\gamma^{\varkappa-1};\tau(l),\mu^{\varkappa-1}), \xi_\mu^\varkappa = \nu, \\ \xi_\mu^{\varkappa+1} &= \mathcal{Q}^{-1}(\gamma^\varkappa;\tau(l),\mu^{\varkappa+1}),\dots,\xi_\mu^d = \mathcal{Q}^{-1}(\gamma^{d-1};\tau(l),\mu^d); \end{aligned}$$

otherwise, when $\iota = 1$, *the point* $(\tau(l), \xi_\mu)$ *belongs to the upper base of* Π, *and*

$$\begin{aligned} \tau(l) &= l, \\ \xi_\mu^1 &= \mathcal{Q}^{-1}(\gamma; l, \mu^1), \xi_\mu^2 = \mathcal{Q}^{-1}(\gamma^1; l, \mu^2), \dots, \xi_\mu^d = \mathcal{Q}^{-1}(\gamma^{d-1}; l, \mu^d). \end{aligned}$$

Proof. The statement of the theorem follows from Lemmas 9.4-9.7 and reasoning similar to that done in the proof of Theorem 5.1.

The following corollary is proved as Corollary 9.1.

Corollary 9.2. *Let* $\Pi_r = [0, lr^2) \times C_r = \{(t, x) = (t, x^1, ..., x^d) : \ 0 \le t < lr^2, \ |x^i| < r, \ i = 1, ..., d\} \subset R^{d+1}$ *be a space-time parallelepiped. Let* $\bar{\theta}$ *be the first-passage time of the process* $(s, w_\mu(s)), \ s > 0$, *to the boundary* $\partial\Pi_r$.*Then the point*

$$(\bar{\theta}, \bar{w}_\mu) = (r^2\tau(l), r\xi_{r\mu}),$$

where $(\tau(l), \xi_{r\mu})$ *is simulated by the algorithm for simulating exit point from the space-time parallelepiped* Π, *has the same distribution as* $(\bar{\theta}, w_\mu(\bar{\theta}))$.

Remark 9.5. Let α be a d-dimensional vector, $C_\alpha = \{x = (x^1, ..., x^d) : \ |x^i| < \alpha^i, \ i = 1, ..., d\} \subset R^d$ be the d-dimensional parallelepiped, and $\Pi_\alpha = [0, l) \times C_\alpha \subset R^{d+1}$ be the corresponding space-time parallelepiped. By the results of Section 3 and reasoning of this section (see also Remark 4.1), we can prove lemmas, which are similar to Lemmas 9.4, 9.5, and 9.7, in the case, when τ is the exit time of the d-dimensional Wiener process $W(s)$, $W(0) = 0$, from the parallelepiped C_α. Then, it is not difficult to state the corresponding theorems on algorithms for simulating the exit points $(\tau, W(\tau))$ of the process $(s, W(s))$, $s > 0$, both to the lateral surface of the cylinder $[0, \infty) \times C_\alpha$ with parallelepiped base C_α and to the boundary of the space-time parallelepiped Π_α. Using these theorems, the corresponding one-step approximation can be constructed. Note that we are also able to write down the distributions for the exit points in the case when $W(0) = x$, $x \neq 0$, $x \in C_\alpha$.

10. **Numerical examples**

The numerical methods proposed in the paper are widely applicable. As it has been mentioned in Introduction, these methods are the first ones which can constructively approximate space-time trajectories of a space-time diffusion process. They can be also applied to solving boundary value problems through a Monte Carlo technique on a level with the weak methods. Let us underline that the proposed methods give an estimator for a solution to the Dirichlet problem for parabolic and elliptic equations with constant coefficients, which does not contain the error of numerical integration.

Here we give three numerical examples. The first and the second examples deal with solving boundary value problems. The third one essentially uses simulation of trajectories.

Example 1. Let us consider an application of random walks over touching space-time parallelepipeds to the Dirichlet problem for parabolic equation (2.1)-(2.3) in the case when the coefficients are constant. This problem has the probabilistic representation (2.6)-(2.7), which we use for the Monte Carlo procedure here.

Let $(\bar{\vartheta}_k, \bar{X}_k)$ be a Markov chain which is formed analogously to the one of Section 7 but wandering is realized over touching space-time parallelepipeds (instead of small space-time parallelepipeds in Section 7) and is finished in the layer Γ_δ at a random step $\bar{\nu}$, where $\delta > 0$ is a sufficiently small constant. The equation with frozen coefficients (6.3), which we are able to simulate exactly, coincides with the equation (2.7), when its coefficients are constant. Consequently, the chain $(\bar{\vartheta}_k, \bar{X}_k)$ coincides with the chain $(\bar{\vartheta}_k, X_k)$. In the considered case, the solution $u(t,x)$ to the Dirichlet problem (2.1)-(2.3) under $c = 0$ and $e = 0$ is simulated as (see (2.6))

$$u(t,x) \doteq \bar{u}(t,x) = \frac{1}{M}\sum_{m=1}^{M} \varphi(\bar{X}_{\bar{\nu}}^{(m)}) \pm 2[\bar{D}/M]^{1/2},$$

where

$$\varphi(\bar{X}_{\bar{\nu}}^{(m)}) = \begin{cases} f(\bar{X}_{\bar{\nu}}^{(m)}),\ \bar{\vartheta}_{\bar{\nu}}^{(m)} \in (t_1 - \delta, t_1], \\ g(\bar{X}_{\bar{\nu}}^{(m)}),\ \bar{\vartheta}_{\bar{\nu}}^{(m)} \notin (t_1 - \delta, t_1], \end{cases}$$

$$\bar{D} = \frac{1}{M}\sum_{m=1}^{M} \left[\varphi(\bar{X}_{\bar{\nu}}^{(m)}))\right]^2 - \left[\frac{1}{M}\sum_{m=1}^{M} \varphi(\bar{X}_{\bar{\nu}}^{(m)})\right]^2,$$

and M is a number of independent Markov chains $\left(\bar{\vartheta}_k^{(m)}, \bar{X}_k^{(m)}\right)$, $m = 1, \ldots, M$.

Because the simulated values $(\bar{\vartheta}_k, \bar{X}_k)$ coincide with the points of exact solution $(\bar{\vartheta}_k, X_k)$ here, the estimator $\bar{u}(t,x)$ does not contain the error of numerical integration (naturally, there are Monte Carlo error depending on M and the error due to approximation of the boundary conditions depending on δ).

The mean number of steps of the random walk over touching spheres up to the boundary of space domain G is estimated by $C \ln \frac{l}{2\delta}$ (see, e.g., [8, 31] and also [24]), if G is a convex and l is its diameter. In our case the value of $\bar{\nu}$ is also estimated by $C \ln \frac{l}{2\delta}$.

Another Monte Carlo approach, whereby a random walk is made on a maximum square and the differential Laplace operator is approximated by a difference one, was proposed in [10].

As an illustration, we take the following parabolic equation in the domain $Q = [0, t_1) \times G$, $G = \{x = (x_1, x_2) : |x_1| < 2,\ |x_2| < 1\}$ (this example is similar to one of [10]):

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u,\ t > 0,\ |x_1| < 2,\ |x_2| < 1, \tag{10.1}$$

with the initial and boundary conditions

$$u(0,x) = 2, \tag{10.2}$$

$$u(t,x) \mid_{\partial G} = 0,\ t > 0. \tag{10.3}$$

TABLE 1. Test results for the boundary value problem (10.1)-(10.3). The exact solution $u(1, 0.7, 0.4) = 0.4796$ ($\delta = 0.00001$).

M	$\bar{u}(1, 0.7, 0.4) \pm 2[\bar{D}/M]^{1/2}$	$E\bar{\nu}$
1000	0.4460 ± 0.0527	3.142
4000	0.4780 ± 0.0270	3.257
100000	0.4782 ± 0.0054	3.272

By changing of time $t = t_1 - s$ in (10.1)-(10.3), we obtain the corresponding boundary value problem (like (2.1)-(2.3)) with the initial condition on the upper base.

The results of numerical test are presented in Table 1.

Throughout our tests we use a generator of uniform random numbers from [29].

Example 2. Consider the boundary value problem for biharmonic equation

$$L^2 u + c_1(x) Lu + c_2(x) u = f(x), \; x \in G \subset R^d, \tag{10.4}$$

$$u \mid_{\partial G} = \varphi(x), \quad Lu \mid_{\partial G} = \psi(x), \tag{10.5}$$

where L is an operator of elliptic type:

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x) \frac{\partial}{\partial x^i},$$

and $c_1(x)$, $c_2(x)$, $f(x)$, $\varphi(x)$, and $\psi(x)$ are some known functions.

Introducing the function $v = Lu$, we obtain the system of elliptic equations

$$Lu - v = 0, \; x \in G, \; u \mid_{\partial G} = \varphi(x), \tag{10.6}$$

$$Lv + c_1(x) v + c_2(x) u = f(x), \; x \in G, \; v \mid_{\partial G} = \psi(x). \tag{10.7}$$

Let us give a probabilistic representation of the solution to the problem (10.6)-(10.7) (the first probabilistic representation for the problem (10.6)-(10.7) in the case of constant c_1 and c_2 is obtained in [13]). To this end introduce the system of SDE

$$dX = b(X)\, ds + \sigma(X)\, dw(s), \tag{10.8}$$

$$\frac{dY_1}{ds} = c_2(X) Y_2$$

$$\frac{dY_2}{ds} = -Y_1 + c_1(X) Y_2, \tag{10.9}$$

where $w(s)$ is a standard d-dimensional Wiener process, $b(x)$ is the d-dimensional vector with the components $b^i(x)$ introduced above, Y_1 and Y_2 are scalars, and $\sigma(x)$ is a matrix that is obtained from the equality

$$a(x) = \sigma(x)\sigma^\intercal(x), \quad a(x) = \{a^{ij}(x)\}.$$

Under some conditions on the coefficients of the problem (10.6)-(10.7), its solution $(u(x), v(x))$ has the following form (see [17]):

$$u(x) = E\left[\varphi(X_x(\tau)) Y_1^{(1)}(\tau) + \psi(X_x(\tau)) Y_2^{(1)}(\tau)\right] - E \int_0^\tau f(X_x(s)) Y_2^{(1)}(s)\, ds,$$

$$v(x) = E\left[\varphi(X_x(\tau)) Y_1^{(2)}(\tau) + \psi(X_x(\tau)) Y_2^{(2)}(\tau)\right] - E \int_0^\tau f(X_x(s)) Y_2^{(2)}(s)\, ds, \tag{10.10}$$

where τ is the first exit time of the process $X_x(s)$, $X(0) = x$, from the domain G, and $(Y_1^{(1)}, Y_2^{(1)})$ is the solution of the system (10.9) with the initial data: $Y_1^{(1)}(0) = 1$, $Y_2^{(1)}(0) = 0$, and $(Y_1^{(1)}, Y_2^{(1)})$ has the following initial data: $Y_1^{(2)}(0) = 0$, $Y_2^{(2)}(0) = 1$.

The probabilistic representation (10.8)-(10.10) for the boundary value problem (10.4)-(10.5) can be used for solving the problem (10.4)-(10.5) by implementation of the random walk over small space-time parallelepipeds through the Monte Carlo technique. If the coefficients of the elliptic operator L and the scalars c_1, c_2, f are constant, we can use the random walk over touching space parallelepipeds that gives an estimator, which is free from the error of numerical integration. Note that in this case the sufficient condition, under which the representation (10.10) is valid, consists in $c_1 \leq 0$, $c_2 \geq 0$.

As an illustration, consider the following two-dimensional problem in the square $G = \{x = (x_1, x_2) : |x_1| < 1, \ |x_2| < 1\}$:

$$\frac{1}{4}\Delta^2 u = 1, \ x \in G, \tag{10.11}$$

$$u \mid_{\partial G} = \varphi(x), \ \varphi(x_1, \pm 1) = \frac{1 + x_1^4}{12}, \ \varphi(\pm 1, x_2) = \frac{1 + x_2^4}{12},$$

$$\frac{1}{2}\Delta u \mid_{\partial G} = \psi(x), \ \psi(x_1, \pm 1) = \frac{1 + x_1^2}{2}, \ \psi(\pm 1, x_2) = \frac{1 + x_2^2}{2}. \tag{10.12}$$

Its exact solution is

$$u(x) = \frac{x_1^4 + x_2^4}{12}, \ \ u(x) = \frac{x_1^2 + x_2^2}{2}.$$

Introducing the function $v = \frac{1}{2}\Delta u$ as above, we obtain the system of elliptic equations

$$\frac{1}{2}\Delta u - v = 0, \ x \in G, \ u \mid_{\partial G} = \varphi(x) \tag{10.13}$$

$$\frac{1}{2}\Delta v = 1, \ x \in G, \ v \mid_{\partial G} = \psi(x). \tag{10.14}$$

Of course, one can solve the problem (10.13)-(10.14) sequentially: first find the function v from the problem (10.14) and then u from (10.13). But such an approach requires the knowledge of the function v in the whole domain G even if one needs the solution (u, v) only at individual points of the domain G. In the last case, the Monte Carlo approach is more preferable.

For the system (10.13)-(10.14), the formulas (10.8)-(10.10) acquire the form

$$u(x) = E\varphi(x + w(\tau)) - E[\tau\psi(x + w(\tau))] + \frac{1}{2}E\tau^2,$$

$$v(x) = E\psi(x + w(\tau)) - E\tau,$$

where τ is the first exit time of the process $x + w(s)$ from the domain G.

To simulate the point $(\tau, x + w(\tau))$, we use the random walk over touching space squares, which is finished in a δ-neighborhood of the boundary ∂G belonging to G. Remember that we are able to simulate both the exit point and the exit time of the Wiener process from a square exactly in accordance with Theorem 4.1. Then due to the same reasons as in Example 1, the corresponding estimator $(\bar{u}, \bar{v})$ does not contain the error of numerical integration. The notice on the mean number of steps $E\bar{\nu}$ from Example 1 is also valid here. Let us underline that the usual method of random walk over touching spheres in the space domain G cannot be applied to this problem, because we essentially use the simulation of both the exit point $x + w(\tau)$ and the exit time τ.

The results of numerical tests are given in Table 2.

TABLE 2. Test results for the boundary value problem (10.11)-(10.12) ($\delta = 0.00001$).

M	x_1	x_2	$u(x_1,x_2)$	$\bar{u}(x_1,x_2)$	$v(x_1,x_2)$	$\bar{v}(x_1,x_2)$	$E\bar{\nu}$
10000	0.3	0.5	0.00588	0.0065 ± 0.0038	0.17000	0.1700 ± 0.0082	4.01
100000				0.0058 ± 0.0012		0.1700 ± 0.0026	3.99
1000000				0.00586 ± 0.00039		0.17005 ± 0.00082	4.00
10000	0.7	0.8	0.05414	0.0531 ± 0.0020	0.56500	0.5637 ± 0.0061	3.98
100000				0.05378 ± 0.00061		0.5651 ± 0.0019	4.03
1000000				0.05419 ± 0.00020		0.56536 ± 0.00062	4.00
10000	0.9	0.9	0.10935	0.1088 ± 0.0010	0.81000	0.8070 ± 0.0038	3.05
100000				0.10918 ± 0.00033		0.8096 ± 0.0012	3.01

Example 3. Let us remind some needed facts concerning the stability analysis of stochastic equations. Consider the second-order Ito linear autonomous system of SDE

$$dX = AX\,dt + \sum_{i=1}^{2} B_i X\,dw_i(t), \tag{10.15}$$

where X is a two-dimensional vector, A and B_i, $i = 1, 2$, are constant 2×2 matrices, $w_i(t)$, $i = 1, 2$, are independent standard Wiener processes.

Various characteristic describing asymptotic behavior of solutions of the system (10.15), such as the Lyapunov exponent, moment Lyapunov exponents, the stability index, and some others, are considered in [1, 2, 12] (see also references therein). The Lyapunov exponent λ^* of system (10.15) (cf. [12]) is defined as

$$\lambda^* := \lim_{t\to\infty} \frac{1}{t} E \ln |X_x(t)| = \lim_{t\to\infty} \frac{1}{t} \ln |X_x(t)| \ a.s., \tag{10.16}$$

and the moment Lyapunov exponent $g(p)$ is defined as

$$g(p) := \lim_{t\to\infty} \frac{1}{t} E \ln |X_x(t)|^p, \ p \in R, \tag{10.17}$$

where $X_x(t)$, $t \geq 0$, is a nontrivial solution to system (10.15).

The limits λ^* and $g(p)$ exist, and they are independent of x, $x \neq 0$, in the ergodic case. The limit $g(p)$ is a convex analytic function of $p \in R$, $g(0) = 0$, $g(p)/p$ increases with growing p, and

$$g'(0) = \lim_{p\to 0} \frac{g(p)}{p} = \lambda^*. \tag{10.18}$$

If $\lambda^* < 0$ then the trivial solution to system (10.15) is a.s. asymptotically stable. It is well-known and follows from (10.18) that in this case $g(p)$ is negative for all sufficiently small $p > 0$, i.e., the solution $X = 0$ of (10.15) is p-stable for such p. If $g(p) \to +\infty$ as $p \to +\infty$, then the equation

$$g(p) = 0 \tag{10.19}$$

has the unique root $\gamma^* > 0$, which is known as the stability index.

It is clear that the solution $X = 0$ of (10.15) is p-stable for $0 < p < \gamma^*$ and p-unstable for $p > \gamma^*$. The stability index γ^* is connected with the asymptotic behavior of the probability $V_\delta(x)$ of the exit of $X_x(t)$ from the ball $|x| < \delta$ (see [3]): $V_\delta(x) :=$

$P\{\sup_{t\geq 0}|X_x(t)| > \delta\}$, $|x|/\delta \to 0$. It turns out that there exists a constant $K > 0$ such that for all $\delta > 0$ and $|x| < \delta$ the following inequality takes place:

$$\frac{1}{K}(|x|/\delta)^{\gamma^*} \leq V_\delta(x) \leq K \cdot (|x|/\delta)^{\gamma^*}. \tag{10.20}$$

The unstable case, when the equation (10.19) has a negative root γ^*, is considered analogously [3].

The stability properties of the system (10.15) can also be characterized by the exit time τ of $X_x(t)$ from a certain neighborhood of the origin. In [16] the value of $Ee^{-\mu\tau}$, $\mu > 0$, is simulated. By the algorithms proposed in the present paper, we are able to evaluate the distribution function $P(\tau < t)$, which may be a good characteristic for description of transient behavior related to the system (10.15). Naturally, we are also able to evaluate functionals on τ, e.g., $Ee^{-\mu\tau}$.

We take the following particular case of the two-dimensional system (10.15) for our numerical tests:

$$dX_1 = (aX_1 + cX_2)\,ds + b_1X_1\,dw_1(s) + b_2X_2\,dw_2(s)$$

$$dX_2 = (-cX_1 + aX_2)\,ds + b_1X_2\,dw_1(s) - b_2X_1\,dw_2(s),$$

$$X(0) = X_x(0) = x. \tag{10.21}$$

The function $g(p)$, the Lyapunov exponent λ^*, and the stability index γ^* for this system are equal to (cf. [22]):

$$g(p) = p\cdot(a + \frac{1}{2}(b_2^2 - b_1^2)) + \frac{1}{2}p^2b_1^2,$$

$$\lambda^* = g'(0) = a + \frac{1}{2}(b_2^2 - b_1^2),$$

$$\gamma^* = -\frac{2a + (b_2^2 - b_1^2)}{b_1^2}. \tag{10.22}$$

Here we evaluate the distribution function $P(\tau < t)$, where τ is the first exit time of $X_x(s)$ under $X(0) = (1,1)^\intercal$ from the square $G = \{(x_1,x_2) :\ |x_i| < 3,\ i = 1,2\}$. To simulate the system (10.21), we use the random walk over boundaries of small space-time parallelepipeds constructed in Section 7. The algorithm allows to find $\bar\tau$ (see Section 8), which is close to τ. The sampling distribution function $\bar P_M(t)$ is calculated as

$$\bar P_M(t) = \begin{cases} 0,\ t \leq \bar\tau_1^{(M)}, \\ m/M,\ \bar\tau_m^{(M)} < t \leq \bar\tau_{m+1}^{(M)}, \\ 1,\ t > \bar\tau_M^{(M)}, \end{cases}$$

where $\{\bar\tau_1^{(M)},\dots,\bar\tau_M^{(M)}\}$ is a sample point of size M sorting in the ascending order, it corresponds to the random variable $\bar\tau$.

The sampling function $\bar P_M(t)$ is close to the distribution function $\bar P(t) = P(\bar\tau < t)$ under a sufficiently big M, and $\bar P(t)$ is close to $P(\tau < t)$ under a sufficiently small r (remember that r is a distinctive size of the space-time parallelepipeds used in the algorithm for solving (10.21)). We control the accuracy of our simulations by increasing

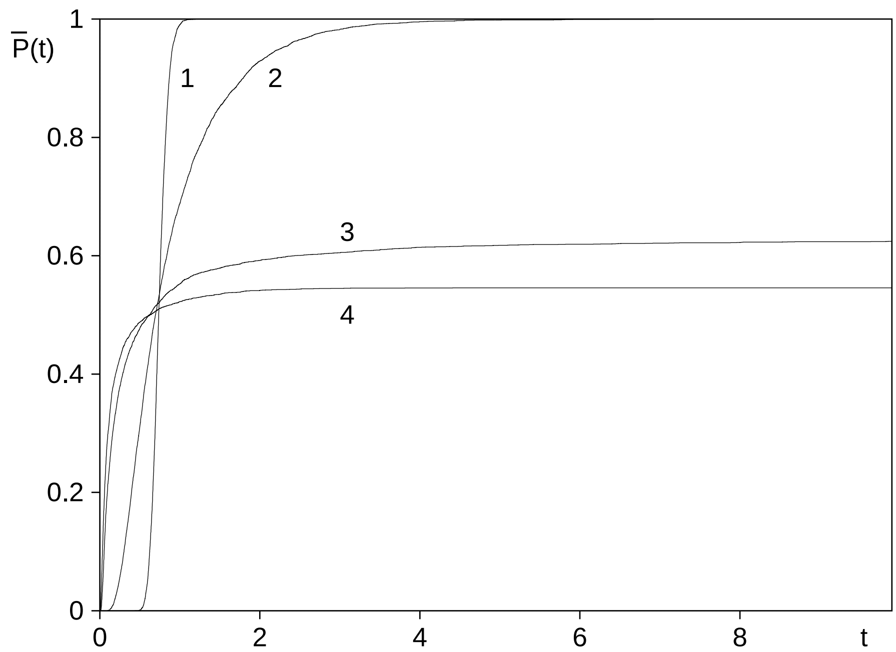

FIGURE 3. The distribution function $\bar{P}(t)$ for $a = -1$, $c = 1$, $b_2 = 2$, $X(0) = (1,1)^\intercal$, $r = 0.02$, $M = 5000$, and for various b_1 : (1) $b_1 = 0.1$ ($\lambda^* = 0.995$, $\gamma^* = -199$), (2) $b_1 = 0.6$ ($\lambda^* = 0.82$, $\gamma^* = -4.556$), (3) $b_1 = \sqrt{5}$ ($\lambda^* = -1.5$, $\gamma^* = 0.6$), and (4) $b_1 = 3$ ($\lambda^* = -3.5$, $\gamma^* = 0.778$).

M and decreasing r. We select M and r such that the curves $\bar{P}_M(t)$ are visually almost identical under larger values of M and smaller values of r.

Figure 3 presents the behavior $\bar{P}(t) \doteq \bar{P}_M(t)$ under fixed a, c, b_2, and various b_1. Increasing of b_1 leads to stabilization (see formulas (10.22)). It is interesting to note (see Figure 3) that the probability of the exit of $X_x(s)$ from G at small times t under $\lambda^* > 0$ (unstable case) is lower than the corresponding probability under $\lambda^* < 0$ (stable case). It may be explained in the following way. The radius $\rho(s) = \sqrt{X_1^2(s) + X_2^2(s)}$ satisfies the following equation

$$d\rho = (a + \frac{b_2^2}{2})\rho\, ds + b_1 \rho\, dw_1(s). \tag{10.23}$$

Due to the selection of the parameters, the Lyapunov exponent λ^* is positive (unstable case) under relatively small b_1 and large b_2. In this case the first term of (10.23) plays the main role and influence of noise is relatively small. So there is a lag time before the trajectory $X_x(s)$ leaves the domain G. In the stable case our parameters are such that b_1 is large and the second term of (10.23) plays an essential role. Then the trajectory $X_x(s)$ can leave the domain G during a small time interval with a rather large probability.

Figure 4 illustrates the behavior of $\bar{P}(t)$ under fixed a, c, and $\lambda^* = a + (b_2^2 - b_1^2)/2$ for various values of the stability index γ^* (see (10.22)). One can see that the probability of the exit of the trajectory $X_x(s)$ from G decreases with increasing of γ^* that is in accordance with (10.20).

Figures 3 and 4 also demonstrate that in the unstable case the trajectory leaves the neighborhood of the origin during a finite time interval with the probability equal to 1 (see the curves 1 and 2 on Figure 3). But in the stable case the probability $P(\tau < \infty)$,

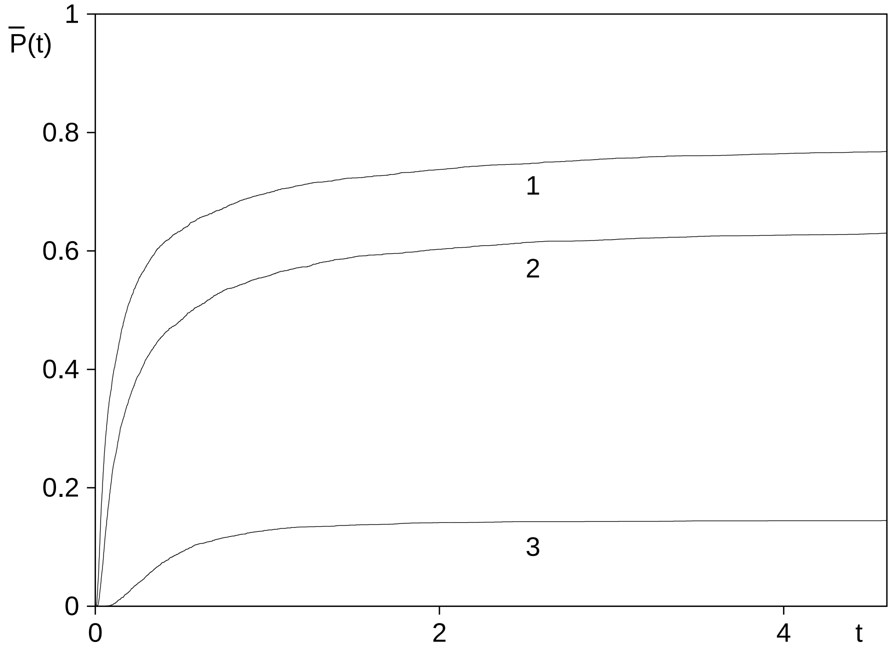

FIGURE 4. The distribution function $\bar{P}(t)$ for $a = -1$, $c = 1$, $X(0) = (1,1)^{\intercal}$, $\lambda^* = -1.5$, $M = 5000$, and for various γ^* : (1) $\gamma^* = 1/3$ ($b_1 = 3$, $b_2 = 2.828$, $r = 0.02$), (2) $\gamma^* = 0.6$ ($b_1 = \sqrt{5}$, $b_2 = 2$, $r = 0.02$), and (3) $\gamma^* = 2.479$ ($b_1 = 1.1$, $b_2 = 0.4683$, $r = 0.05$).

that the trajectory leaves the neighborhood of the origin, is less than 1. It decreases with decreasing of the Lyapunov exponent λ^* (see the curves 3 and 4 on Figure 3) and with increasing of the stability index γ^* (see Figure 4).

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