ON THE MEAN-SQUARE APPROXIMATION OF A DIFFUSION PROCESS IN A BOUNDED DOMAIN

G.N. MILSTEIN

25. July 1997

1991 *Mathematics Subject Classification.* 60H10, 60J15, 65U05.
Key words and phrases. mean-square approximation, random walk, exit point.

domain is considered. For systems with zero drift the next approximate point on the phase trajectory is found as a solution of the system with coefficients frozen at the previous point by a random walk over the boundary of a small ellipsoid. Theorems on mean-square order of accuracy for such an approximation are proved. An algorithm for approximate construction of exit points from the bounded domain is given.

Key words: mean-square approximation, random walk, exit point.

AMS Subject Classification: 60H10, 60J15, 65U05.

1. **Introduction**

The present preprint adjoints the paper [8] and essentially strengthens its results. We try, on the one hand, to make the exposition here as self-contained as possible and, on the other hand, to give only new results. That is why several necessary results from [8] are presented without proofs, and systems with drift (see [8]) are not considered here.

Let us remind about the main motivation and notation of the paper [8]. Consider an autonomous system of stochastic differential equations

$$dX = \chi_{\tau_x > t}\sigma(X)dw(t),\ X(0) = x, \tag{1.1}$$

in a bounded domain $G \subset R^d$ with a boundary ∂G.

Here $w(t) = (w^1(t), ..., w^d(t))^\top$, $t \geq 0$, is a standard $\mathcal{F}_t$-measurable Wiener process of dimension d defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}_t$ is a non-increasing family of sub-σ-algebras of $\mathcal{F}$; $X = (X^1, ..., X^d)^\top$ is a vector of dimension d, $\sigma(x) = \{\sigma^{ij}(x)\}$ is a matrix of dimension $d \times d$, τ_x is a random time at which the path $X_x(t)$ leaves the region G.

The following conditions are assumed to be satisfied:

(i) G is a convex open bounded set with twice continuously differentiable boundary ∂G;

(ii) the coefficients $\sigma^{ij}(x)$ belong to the class $C^{(2)}\left(\bar{G}\right)$;

(iii) the matrix

$$a(x) = \sigma(x)\sigma^\top(x),\ a(x) = \left\{a^{ij}(x)\right\},$$

satisfies the strict ellipticity condition, i.e.,

$$\lambda_1^2 = \min_{x\in\bar{G}} \min_{1\leq i\leq d} \lambda_i^2(x) > 0,$$

where $\lambda_1^2(x) \leq \lambda_2^2(x) \leq \cdots \leq \lambda_d^2(x)$ are eigenvalues of the matrix $a(x)$.

Let $\lambda_d^2 = \max\limits_{x\in\bar{G}} \lambda_d^2(x)$. Then for any $x \in \bar{G}$, $y \in R^d$ the following inequality

$$\lambda_1^2 \sum_{i=1}^d y^{i^2} \leq \sum_{i,j=1}^d a^{ij}(x)\, y^i y^j \leq \lambda_d^2 \sum_{i=1}^d y^{i^2} \tag{1.2}$$

holds.

Due to (1.2) τ_x is finite with probability one. We shall consider the process $X_x(t)$ defined on $0 \leq t < \infty$ regarding it as the stopped one after τ_x.

In addition to (1.1), we introduce the system with coefficients frozen at x

$$d\bar{X} = \sigma(x)dw(t),\ \ \bar{X}(0) = x. \tag{1.3}$$

Let $r > 0$ be a small number, $U_r \subset R^d$ be an open sphere of radius r with centre at the origin and with the boundary ∂U_r. Let $\bar{\theta}$ be the first time at which the process $w(t)$ leaves the sphere U_r. Clearly, $w(\bar{\theta})$ has the uniform distribution on ∂U_r. Let $U_r^\sigma(x)$ be an open ellipsoid with the boundary $\partial U_r^\sigma(x)$ obtained from the sphere U_r with the help of the linear transformation $\sigma(x)$ and the shift x. It is assumed that r is small enough to satisfy the including $U_r^\sigma(x) \subset G$. The solution $\bar{X}_x(t)$ of the problem (1.3) at the time $\bar{\theta}$ is equal to

$$\bar{X}_x(\bar{\theta}) = x + \sigma(x)w(\bar{\theta}), \tag{1.4}$$

$\bar{X}_x(\bar{\theta}) \in \partial U_r^\sigma(x)$ and $\bar{\theta}$ is the first exit time from $U_r^\sigma(x)$ for the trajectory $\bar{X}_x(t)$.

Consider the point $X_x(\bar{\theta})$ (of course if $\tau_x \le \bar{\theta}$ then $X_x(\bar{\theta}) = X_x(\tau_x)$). It turns out that $\bar{X}_x(\bar{\theta})$ is close to $X_x(\bar{\theta})$ in the mean-square sense. Thus, the point $\bar{X}_x(\bar{\theta})$ is an approximation of a point which belongs to the phase trajectory starting at x.

Note that the construction of the point $(\bar{\theta}, \bar{X}_x(\bar{\theta}))$ amounts to modeling $\bar{\theta}$ and $\bar{X}_x(\bar{\theta})$ separately because of their independence. It is important to underline that if we are interested only in phase trajectories, it is possible to simulate them without modeling $\bar{\theta}$, which is a rather difficult problem. To simulate $\bar{X}(\bar{\theta})$, we need only in $w(\bar{\theta})$ which has the uniform distribution on ∂U_r, i.e., modeling of the point $\bar{X}_x(\bar{\theta}) \in \partial U_r^\sigma(x)$ is a fairly simple problem.

Denote $\bar{X}_0 = x$, $\bar{X}_1 = \bar{X}_x(\bar{\theta})$. We shall find the point $\bar{X}_2$ on the boundary $\partial U_r^\sigma(\bar{X}_1)$ by the same way as we found $\bar{X}_1$ coming from $\bar{X}_0 = x$. Then we construct $\bar{X}_3$ and so on until a point $\bar{X}_{\bar{\nu}}$ with a random subscript $\bar{\nu}$. As a result the sequence $\bar{X}_0, ..., \bar{X}_{\bar{\nu}}$ is obtained which can be considered as a mean-square approximation of the phase trajectory of the solution $X_x(t)$. If the point $\bar{X}_{\bar{\nu}}$ is sufficiently close to the boundary ∂G, it is possible to simulate the exit point $X_x(\tau_x)$.

In comparison with [8], we give a more strong version of the local approximation theorem here. In addition, the adduced proof of this theorem is essentially simpler than in [8]. Further, we give two different convergence theorems with complete proofs. One of these theorems is devoted to approximation properties of the sequence $\bar{X}_0, ..., \bar{X}_{\bar{\nu}}$ till leaving an open domain $D \subset G$ with $\rho(\partial D, \partial G) > 0$ which does not depend on r. In the second convergence theorem the point $\bar{X}_{\bar{\nu}}$ belongs to a boundary layer which decreases in a definite way with decreasing r, i.e., $\bar{X}_{\bar{\nu}}$ becomes sufficiently close to ∂G with decreasing r (more exactly, $\rho(\bar{X}_{\bar{\nu}}, \partial G) = O(r^{1-\varepsilon})$ with a sufficiently small $\varepsilon > 0$). In the both situations the mean-square order of accuracy is equal to $O(r)$. The second theorem is important for approximation of the exit point $X_x(\tau_x)$. It is shown that this point can be approximated by $\bar{X}_{\bar{\nu}}$ with the mean-square order which is close to $O(\sqrt{r})$. Such a lowering of exactness can be explained in the following way. Because $\rho(\bar{X}_{\bar{\nu}}, \partial G) = O(r^{1-\varepsilon})$ and $\rho(X_{\bar{\nu}}, \bar{X}_{\bar{\nu}}) = O(r)$ in the mean-square sense, the distance $\rho(X_{\bar{\nu}}, \partial G)$, which is evaluated by $O(r^{1-\varepsilon})$, is comparatively big. As a result the point $X_x(\tau_x)$ may be far from $X_{\bar{\nu}}$ and, consequently, far from $\bar{X}_{\bar{\nu}}$. Let us note in passing that the proof of the convergence theorem in [8] contains a mistake which is eliminated now.

In conclusion we note that the weak approximation with restrictions is regarded in [5 – 7, 9, 10]. The main aim of these works consists in development of probabilistic methods using the numerical integration of ordinary stochastic differential equations [2, 4, 12] for solving boundary value problems. Another approach is available in [3].

Everywhere below $X_x(t)$ is the solution of the problem (1.1), $X_{t_0,x}(t),\ \ t \geq t_0$, is the solution of the equation (1.1) with initial data $X(t_0) = x$, $\bar{X}_x(t)$ is found from (1.3). Let Γ_δ be the interior of a δ-neighborhood of the boundary ∂G belonging to G. Obviously, if $x \in G \backslash \Gamma_{2\lambda_d r}$, then the inclusion $U_r^\sigma(x) \subset U_{2r}^\sigma(x) \subset G$ holds for sufficiently small r.

Theorem 1. *For every natural number n there exists a constant $K > 0$ such that for any sufficiently small $r > 0$ and for any $x \in G \backslash \Gamma_{2\lambda_d r}$ the following inequality*

$$E|X_x(\bar{\theta}) - \bar{X}_x(\bar{\theta})|^{2n} \leq K r^{4n} \tag{2.1}$$

is fulfilled.

Proof. Introduce the Markov moment θ as the first time at which the process $X_x(t)$ leaves the ellipsoid $U_{2r}^\sigma(x)$. (In order to avoid an ambiguity, let us note that in [8] θ means the moment at which the process $X_x(t)$ leaves the ellipsoid $U_r^\sigma(x)$). At the beginning let us prove the theorem for $n = 1$. We have

$$E|X_x(\bar{\theta}) - \bar{X}_x(\bar{\theta})|^2 =$$

$$E|\int_0^{\bar{\theta}} (\chi_{\tau_x > s}\sigma(X_x(s)) - \sigma(x))dw(s)|^2 = E\int_0^{\bar{\theta}} |\chi_{\tau_x > s}\sigma(X_x(s)) - \sigma(x)|^2 ds$$

$$= E\int_0^{\bar{\theta}\wedge\theta} |\sigma(X_x(s)) - \sigma(x)|^2 ds + E\int_{\bar{\theta}\wedge\theta}^{\bar{\theta}} |\chi_{\tau_x > s}\sigma(X_x(s)) - \sigma(x)|^2 ds$$

$$\leq E\int_0^{\bar{\theta}\wedge\theta} |\sigma(X_x(s)) - \sigma(x)|^2 ds + K \cdot E(\bar{\theta} - \bar{\theta}\wedge\theta). \tag{2.2}$$

Here the notation $|x|$ means the Euclidean norm for a vector x and $|\sigma|$ means $(\mathrm{tr}\sigma\sigma^\top)^{1/2}$ for a matrix σ. Note that the various constants which depend only on the system (1.1) and do not depend on x, r and so on are given by the same letter K without any index. In connection with this, instead of, e.g., $K + K$, $2K$, K^2, etc., we write K.

Since $E\bar{\theta} = \dfrac{r^2}{d}$, then $E(\bar{\theta}\wedge\theta) \leq \dfrac{r^2}{d}$. Further, on the interval $(0, \bar{\theta}\wedge\theta)$ we have $X_x(s) \in U_{2r}^\sigma(x)$. Therefore

$$E|X_x(\bar{\theta}\wedge\theta) - \bar{X}_x(\bar{\theta}\wedge\theta)|^2$$

$$= E\int_0^{\bar{\theta}\wedge\theta} |\sigma(X_x(s)) - \sigma(x)|^2 ds \leq K r^2 \cdot E(\bar{\theta}\wedge\theta) \leq K r^4. \tag{2.3}$$

Due to (1.2) it is easy to show that if $\xi \in \overline{U}_r^\sigma(x)$, $\eta \in \partial U_{2r}^\sigma(x)$, then $|\xi - \eta| \geq \lambda_1 r$. Because $\bar{X}_x(\bar{\theta}\wedge\theta) \in \overline{U}_r^\sigma(x)$, $X_x(\theta) \in \partial U_{2r}^\sigma(x)$, we have for every $m > 0$

$$E(\chi_{\theta < \bar{\theta}}|X_x(\bar{\theta}\wedge\theta) - \bar{X}_x(\bar{\theta}\wedge\theta)|^m)$$

$$= E(\chi_{\theta < \bar{\theta}}|X_x(\theta) - \bar{X}_x(\bar{\theta}\wedge\theta)|^m) \geq P(\theta < \bar{\theta}) \cdot \lambda_1^m r^m. \tag{2.4}$$

On the other hand,

$$E(\chi_{\theta < \bar{\theta}}|X_x(\bar{\theta}\wedge\theta) - \bar{X}_x(\bar{\theta}\wedge\theta)|^m)$$

$$\leq (P(\theta < \bar{\theta}))^{\frac{1}{2}} \cdot (E|X_x(\bar{\theta}\wedge\theta) - \bar{X}_x(\bar{\theta}\wedge\theta)|^{2m})^{\frac{1}{2}}$$

$$= (P(\theta < \bar\theta))^{\frac12} \cdot (E|\int_0^{\bar\theta\wedge\theta} (\sigma(X_x(s)) - \sigma(x))dw(s)|^{2m})^{\frac12}. \qquad (2.5)$$

Let i be one of the indices $1, ..., d$. Introduce the variable

$$Z(t) = X_x^i(\bar\theta\wedge\theta\wedge t) - \bar X_x^i(\bar\theta\wedge\theta\wedge t)$$

$$= \int_0^{\bar\theta\wedge\theta\wedge t} \sum_{j=1}^d (\sigma^{ij}(X_x(s)) - \sigma^{ij}(x))dw^j(s) = \int_0^t \chi_{\bar\theta\wedge\theta\geq s}\varphi(s)dw(s),$$

where $\varphi(s)$ is the i-th row vector of the matrix $\sigma(X_x(s)) - \sigma(x)$. We do not write the index i under Z and φ because it does not lead to any misunderstanding.

Clearly, $Z(t),\ t\geq 0$, is a uniformly bounded scalar, and

$$|\varphi(s)| \leq |\sigma(X_x(s)) - \sigma(x)| \leq Kr,\ 0 \leq s \leq \bar\theta\wedge\theta\ .$$

We have for every natural $m\geq 1$

$$dZ^{2m}(t) = 2mZ^{2m-1}(t)\chi_{\bar\theta\wedge\theta\geq t}\varphi(t)dw(t) + m(2m-1)Z^{2m-2}(t)\chi_{\bar\theta\wedge\theta\geq t}|\varphi(t)|^2dt\ .$$

From here

$$EZ^{2m}(t) = m(2m-1)E\int_0^t Z^{2m-2}(s)\chi_{\bar\theta\wedge\theta\geq s}|\varphi(s)|^2ds$$

$$\leq Km(2m-1)r^2\cdot E(\bar\theta\wedge\theta\cdot\max_{0\leq s\leq t}|Z(s)|^{2m-2})\ .$$

Applying the Hölder inequality with $p = \dfrac{2m}{2m-2}$ (see such a reception, for instance, in [1] and in [9]) and taking into account that (see [9])

$$E(\bar\theta\wedge\theta)^m \leq E\bar\theta^m \leq \frac{m!}{d^m}r^{2m},$$

we get

$$E|Z(t)|^{2m} \leq Km(2m-1)r^2\cdot(E\max_{0\leq s\leq t}|Z(s)|^{2m})^{\frac{2m-2}{2m}}\cdot(E(\bar\theta\wedge\theta)^m)^{\frac1m}$$

$$\leq Km(2m-1)r^4\cdot(E\max_{0\leq s\leq t}|Z(s)|^{2m})^{\frac{2m-2}{2m}}. \qquad (2.6)$$

As $Z(t)$ is a martingale, we can use the Doob inequality

$$E\max_{0\leq s\leq t}|Z(s)|^{2m} \leq (\frac{2m}{2m-1})^{2m}E|Z(t)|^{2m}.$$

Now we obtain from (2.6)

$$E|Z(t)|^{2m} \leq Kr^{4m},$$

where K does not depend on t (of course, K depends on m).

Hence

$$E|Z(\bar\theta\wedge\theta)|^{2m} \leq Kr^{4m}$$

and, consequently,

$$E|\int_0^{\bar\theta\wedge\theta}(\sigma(X_x(s)) - \sigma(x))dw(s)|^{2m} \leq Kr^{4m}. \qquad (2.7)$$

The inequalities (2.4) and (2.5) imply

$$P(\theta < \bar\theta)\cdot\lambda_1^m r^m \leq K\cdot(P(\theta < \bar\theta))^{\frac12}\cdot r^{2m}.$$

$$P(\theta < \bar{\theta}) \leq K r^{2m}. \tag{2.8}$$

Further,

$$E(\bar{\theta} - \bar{\theta} \wedge \theta) = E\chi_{\theta<\bar{\theta}}(\bar{\theta} - \bar{\theta} \wedge \theta) \leq (P(\theta < \bar{\theta}))^{\frac{1}{2}} \cdot (E(\bar{\theta} - \bar{\theta} \wedge \theta)^2)^{\frac{1}{2}}$$

$$\leq (P(\theta < \bar{\theta}))^{\frac{1}{2}} \cdot (E\bar{\theta}^2)^{\frac{1}{2}} \leq K(P(\theta < \bar{\theta}))^{\frac{1}{2}} \cdot r^2,$$

whence

$$E(\bar{\theta} - \bar{\theta} \wedge \theta) \leq K r^{m+2}. \tag{2.9}$$

Using this inequality for $m = 2$ together with (2.2) and (2.3), we arrive at (2.1) for $n = 1$. Thus the theorem is proved for $n = 1$.

Further, we get

$$E|X_x(\bar{\theta}) - \bar{X}_x(\bar{\theta})|^{2n}$$

$$= E|\int_0^{\bar{\theta}\wedge\theta}(\sigma(X_x(s)) - \sigma(x))dw(s) + \int_{\bar{\theta}\wedge\theta}^{\bar{\theta}}(\chi_{\tau_x>s}\sigma(X_x(s)) - \sigma(x))dw(s)|^{2n}$$

$$\leq KE|\int_0^{\bar{\theta}\wedge\theta}(\sigma(X_x(s)) - \sigma(x))dw(s)|^{2n}$$

$$+KE|\int_{\bar{\theta}\wedge\theta}^{\bar{\theta}}(\chi_{\tau_x>s}\sigma(X_x(s)) - \sigma(x))dw(s)|^{2n}, \tag{2.10}$$

where the constant K depends on n only.

The first term from the right is bounded by Kr^{4n} due to (2.7). The second term can be bounded in the following way (see (2.2) and (2.9) under $m = 4n - 2$) :

$$E|\int_{\bar{\theta}\wedge\theta}^{\bar{\theta}}(\chi_{\tau_x>s}\sigma(X_x(s)) - \sigma(x))dw(s)|^{2n}$$

$$= E|\int_{\bar{\theta}\wedge\theta}^{\bar{\theta}}(\chi_{\tau_x>s}\sigma(X_x(s)) - \sigma(x))dw(s)|^2 \cdot |X_x(\bar{\theta}) - X_x(\bar{\theta}\wedge\theta) - \bar{X}_x(\bar{\theta}) + \bar{X}_x(\bar{\theta}\wedge\theta)|^{2n-2}$$

$$\leq KE|\int_{\bar{\theta}\wedge\theta}^{\bar{\theta}}(\chi_{\tau_x>s}\sigma(X_x(s)) - \sigma(x))dw(s)|^2 \leq KE(\bar{\theta} - \bar{\theta}\wedge\theta) \leq Kr^{4n}.$$

Now (2.10) implies (2.1). Theorem 1 is proved fully.

Remark 1. Clearly, the inequality (2.8) remains true if θ is the first time at which the process $X_x(t)$ leaves the ellipsoid $U^{\sigma}_{(1+\alpha)r}(x)$ for any $\alpha > 0$. Therefore, the condition $x \in G\backslash\Gamma_{2\lambda_d r}$ in Theorem 1 may be interchanged by $x \in G\backslash\Gamma_{(1+\alpha)\lambda_d r}$, $\alpha > 0$. Moreover, it is not difficult to show that the theorem remains true under the condition $x \in G\backslash\Gamma_{(1+r^{\beta})\lambda_d r}$ if only $0 \leq \beta < 2$. But for definiteness we take here and in what follows the layer $\Gamma_{2\lambda_d r}$.

Let $\bar{\theta}_1$ be the first time at which the Wiener process $w(t)$ leaves the sphere U_r, $\bar{\theta}_1+\bar{\theta}_2$ be the first time at which the process $w(t)-w(\bar{\theta}_1)$, $t\geq\bar{\theta}_1$, leaves the same sphere U_r and so on. Let $x\in G\backslash\Gamma_{2\lambda_d r}$. We construct a recurrence sequence of random vectors $\bar{X}_k$, $k=0,1,...,\bar{\nu}$:

$$
\begin{aligned}
&\bar{X}_0 = x\\
&\bar{X}_1 = \bar{X}_0+\sigma(\bar{X}_0)w(\bar{\theta}_1)\\
&\dots\dots\dots\\
&\bar{X}_{k+1} = \bar{X}_k+\sigma(\bar{X}_k)(w(\bar{\theta}_1+\dots+\bar{\theta}_{k+1})-w(\bar{\theta}_1+\dots+\bar{\theta}_k)),\\
&\dots\dots\dots
\end{aligned}
$$

where $\bar{\nu}=\bar{\nu}_x$ is the first number for which $\bar{X}_k\in\overline{\Gamma}_{2\lambda_d r}$.

Of course, the random moment $\bar{\nu}$ also depends on the domain $G\backslash\Gamma_{2\lambda_d r}$ which is left by $\bar{X}_{\bar{\nu}}$. Therefore, the more detailed notation for $\bar{\nu}=\bar{\nu}_x$ is $\bar{\nu}=\bar{\nu}_x(G\backslash\Gamma_{2\lambda_d r})$.

Let us set $\bar{\theta}_k=0$ and $\bar{X}_k=\bar{X}_{\bar{\nu}}$ for $k>\bar{\nu}$.

We have obtained a random walk

$$\bar{X}_0,...,\bar{X}_k,...,$$

which stops at a random step $\bar{\nu}$. It is a Markov chain.

Let us present some average characteristics of $\bar{\nu}=\bar{\nu}_x$.

Lemma 1. *There exists a constant $C>0$ depending only on a diameter of the domain G such that the inequality*

$$E\bar{\nu}_x\leq\frac{C}{\lambda_1 r^2} \tag{3.1}$$

takes place.

Lemma 2. *The probability $P(\bar{\nu}_x\geq L/r^2)$ decreases exponentially as L increases. More exactly, for every $L>0$ the inequality*

$$P(\bar{\nu}_x\geq\frac{L}{r^2})\leq(1+C)e^{-\alpha_r\frac{\lambda_1}{1+C}L}, \tag{3.2}$$

where $\alpha_r\to 1$ as $r\to 0$, is valid. The constant C in (3.2) is the same as in (3.1).

Proofs of these lemmas are available in [8].

Lemma 3. *For every natural number n there exists a constant $K>0$ such that for any sufficiently small $r>0$ and for any $x,y\in G\backslash\Gamma_{2\lambda_d r}$ the inequality*

$$E|\int_0^{\bar{\theta}}(\chi_{\tau_x>s}\sigma(X_x(s))-\chi_{\tau_y>s}\sigma(X_y(s)))dw(s)|^{2n}\leq K|x-y|^{2n}r^{2n}+Kr^{4n} \tag{3.3}$$

holds.

Proof. We have

$$\int_0^{\bar{\theta}}(\chi_{\tau_x>s}\sigma(X_x(s))-\chi_{\tau_y>s}\sigma(X_y(s)))dw(s)$$

$$=\int_0^{\bar{\theta}}(\chi_{\tau_x>s}\sigma(X_x(s))-\sigma(x))dw(s)-\int_0^{\bar{\theta}}(\chi_{\tau_y>s}\sigma(X_y(s))-\sigma(y))dw(s)$$

$$+\int_0^{\bar{\theta}}(\sigma(x)-\sigma(y))dw(s)=(X_x(\bar{\theta})-\bar{X}_x(\bar{\theta}))-(X_y(\bar{\theta})-\bar{X}_y(\bar{\theta}))$$

$$+(\sigma(x)-\sigma(y))\cdot w(\bar{\theta})\ .$$

$$|\int_0^{\bar\theta}(\chi_{\tau_x>s}\sigma(X_x(s))-\chi_{\tau_y>s}\sigma(X_y(s)))dw(s)|^{2n}$$

$$=|(X_x(\bar\theta)-\bar X_x(\bar\theta))-(X_y(\bar\theta)-\bar X_y(\bar\theta))+(\sigma(x)-\sigma(y))\cdot w(\bar\theta)|^{2n}$$

$$\leq K|X_x(\bar\theta)-\bar X_x(\bar\theta)|^{2n}+K|X_y(\bar\theta)-\bar X_y(\bar\theta)|^{2n}+K|\sigma(x)-\sigma(y)|^{2n}\cdot|w(\bar\theta)|^{2n},$$

where the constant K depends on n only.

Now Theorem 1 and the relations

$$|\sigma(x)-\sigma(y)|\leq K|x-y|,\ |w(\bar\theta)|^{2n}=r^{2n}$$

imply (3.3). Lemma 3 is proved.

Let D be an open domain such that $\bar D\subset G$. Let $\Delta=\rho(\partial D,\partial G)$. We consider $r\ll\Delta$ so that $D\subset G\backslash\Gamma_{2\lambda_d r}$. Let $x\in D$ and $\bar\nu=\bar\nu_x=\bar\nu_x(D)$ be the first moment at which $\bar X_{\bar\nu}\in G\backslash D$. For brevity, we conserve the old notation $\bar\nu$ for the new Markov moment $\bar\nu_x(D)$ as this does not cause any confusion. As earlier we set $\theta_k=0$ and $\bar X_k=\bar X_{\bar\nu}$ for $k>\bar\nu$, i.e., we stop the above constructed trajectory $\bar X_k$ at the moment $\bar\nu=\bar\nu_x(D)<\bar\nu_x(G\backslash\Gamma_{2\lambda_d r})$. Therefore, the inequality (3.1) is fulfilled for the moment $\bar\nu=\bar\nu_x(D)$ as well.

Consider now the sequence

$$\begin{aligned}
&X_0=x\\
&X_1=X_x(\bar\theta_1)\\
&\ldots\ldots\ldots\ldots\ldots\\
&X_{k+1}=X_x(\bar\theta_1+\cdots+\bar\theta_{k+1})=X_{\bar\theta_1+\ldots+\bar\theta_k,X_k}(\bar\theta_1+\cdots+\bar\theta_{k+1})\\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

which is connected with the solution of the system (1.1).

If $\bar\theta_1+\cdots+\bar\theta_k\geq\tau_x$ then naturally $X_k=X_x(\tau_x)$ and if $k>\bar\nu=\bar\nu_x(D)$ then $X_k=X_{\bar\nu}$ as $\bar\theta_{\bar\nu+1}=\ldots=\bar\theta_k=0$. Thus, X_k stops at a random step $\bar\nu\wedge\kappa$, where $\kappa=\min\{k:\ \bar\theta_1+\cdots+\bar\theta_k>\tau_x\}$ if $\tau_x<\bar\theta_1+\cdots+\bar\theta_{\bar\nu}$ and $\kappa=\bar\nu$ otherwise. The sequence X_k, just as $\bar X_k$, is a Markov chain. Furthermore both $\bar X_k$ and X_k are martingales over σ-algebras $\mathcal F_0=\{\emptyset,\Omega\}$, $\mathcal F_k=\mathcal F_{\bar\theta_1+\cdots+\bar\theta_k}$, $k=1,2,\ldots$.

Consider sequences $\bar X_k$, X_k for $N=L/r^2$ steps.

The closeness of $\bar X_k$ to X_k for N steps is established in the following theorem.

Theorem 2. *Let $\bar\nu=\bar\nu_x(D)$ be the first exit time of the approximate trajectory $\bar X_k$ from the domain D. There exist constants $K>0$ and $\gamma>0$ (which do not depend on x, r, L, and Δ) such that for any $x\in D$ and for any sufficiently small $r>0$ the inequality*

$$(E\max_{1\leq k\leq\bar\nu\wedge N}\left|X_k-\bar X_k\right|^2)^{\frac12}=(E\max_{1\leq k\leq N}\left|X_k-\bar X_k\right|^2)^{\frac12}\leq\frac{K}{\Delta}e^{\gamma L}\cdot r \qquad (3.4)$$

holds.

Proof. Let ν be the first number at which $X_\nu\in\Gamma_{2\lambda_d r}$. More exactly,

$$\nu=\begin{cases}\min\{k:X_k\in\Gamma_{2\lambda_d r}\ ,\ k\leq\bar\nu\},\\ \infty,\ X_k\notin\Gamma_{2\lambda_d r},\ k=1,\ldots,\bar\nu\ .\end{cases} \qquad (3.5)$$

Clearly, for a sufficiently small r (if only $D \subset G \setminus \Gamma_{2\lambda_d r}$ and $6\lambda_d r \leq \frac{\Delta}{2}$)

$$|X_\nu - \bar{X}_\nu| \geq \frac{\Delta}{2} \ , \ \text{if } \nu \leq \bar{\nu} \ . \tag{3.6}$$

Introduce the stopped at ν Markov chains $\bar{X}_{\nu\wedge m}, \ X_{\nu\wedge m}$ and the differences

$$d_m = X_{\nu\wedge m} - \bar{X}_{\nu\wedge m}, \ m = 0, 1, \dots \ .$$

As ν is a Markov moment with respect to the system of σ-algebras $(\mathcal{F}_m)$, the stopped sequences $(\bar{X}_{\nu\wedge m}, \mathcal{F}_m)$, $(X_{\nu\wedge m}, \mathcal{F}_m)$ and $(d_m, \mathcal{F}_m)$ are martingales. $\bar{X}_{\nu\wedge m}$ $(X_{\nu\wedge m})$ is the stopped at the moment ν Markov chain $\bar{X}_m$ (X_m). This is equivalent to the fact that $\bar{\theta}_m = 0$ not only for $m > \bar{\nu}$ but also for $m > \nu$, i.e., we may consider $\bar{\theta}_m = 0$ for $m > \bar{\nu} \wedge \nu$. Consequently, if $\bar{\nu} \wedge \nu = k$ then $d_k = d_{k+1} = \dots = d_N$. This implies $d_k^2 = d_{k+1}^2 = \dots = d_N^2$.

We have

$$d_m = d_1 \chi_{\bar{\nu}\wedge\nu=1} + \cdots + d_{m-1}\chi_{\bar{\nu}\wedge\nu=m-1} + d_m \chi_{\bar{\nu}\wedge\nu\geq m} \ ,$$

$$d_{m-1} = d_1 \chi_{\bar{\nu}\wedge\nu=1} + \cdots + d_{m-2}\chi_{\bar{\nu}\wedge\nu=m-2} + d_{m-1}\chi_{\bar{\nu}\wedge\nu=m-1} + d_{m-1}\chi_{\bar{\nu}\wedge\nu\geq m} \ ,$$

and therefore

$$d_m = d_{m-1} + (d_m - d_{m-1})\chi_{\bar{\nu}\wedge\nu\geq m} \ . \tag{3.7}$$

Analogously,

$$d_m^2 = d_{m-1}^2 + (d_m^2 - d_{m-1}^2)\chi_{\bar{\nu}\wedge\nu\geq m} \ .$$

We get

$$\begin{aligned} d_m \ &= X_m - \bar{X}_m = X_x(\bar{\theta}_1 + \cdots + \bar{\theta}_m) - \bar{X}_m \\ &= X_{\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}, X_{m-1}}(\bar{\theta}_1 + \cdots + \bar{\theta}_m) - \bar{X}_m \\ &= X_{\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}, X_{m-1}}(\bar{\theta}_1 + \cdots + \bar{\theta}_m) - X_{\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}, \bar{X}_{m-1}}(\bar{\theta}_1 + \cdots + \bar{\theta}_m) \\ &\quad + X_{\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}, \bar{X}_{m-1}}(\bar{\theta}_1 + \cdots + \bar{\theta}_m) - \bar{X}_m \ . \end{aligned} \tag{3.8}$$

The first difference at the right-hand side of (3.8) is the error of the solution because of the error in the initial data at the time $(\bar{\theta}_1 + \cdots + \bar{\theta}_{m-1})$, accumulated to the $(m-1)$-st step. The second difference is the one-step error at the m-th step.

For $m \leq \bar{\nu} \wedge \nu$ the vectors $\bar{X}_{m-1}$ and X_{m-1} belong to $G \setminus \Gamma_{2\lambda_d r}$ and we obtain from the equality (3.8)

$$\begin{aligned} \chi_{\bar{\nu}\wedge\nu\geq m} d_m = \chi_{\bar{\nu}\wedge\nu\geq m}(X_{m-1} + \int_{\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}}^{\bar{\theta}_1+\cdots+\bar{\theta}_m} \chi(s) \cdot \sigma(X_{\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}, X_{m-1}}(s)) dw(s)) \\ -\chi_{\bar{\nu}\wedge\nu\geq m}(\bar{X}_{m-1} + \int_{\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}}^{\bar{\theta}_1+\cdots+\bar{\theta}_m} \bar{\chi}(s) \cdot \sigma(X_{\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}, \bar{X}_{m-1}}(s)) dw(s)) \\ + \chi_{\bar{\nu}\wedge\nu\geq m}(X_{\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}, \bar{X}_{m-1}}(\bar{\theta}_1 + \cdots + \bar{\theta}_m) - \bar{X}_m) \ . \end{aligned} \tag{3.9}$$

Here

$$\chi(s) := \chi_{\tau(\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}, X_{m-1}) > s} \ , \ \bar{\chi}(s) := \chi_{\tau(\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}, \bar{X}_{m-1}) > s} \ ,$$

where $\tau(\bar{\theta}_1 + \cdots + \bar{\theta}_{m-1}, x)$ is a random time at which the path $X_{\bar{\theta}_1+\cdots+\bar{\theta}_{m-1}, x}(t)$ leaves the region G.

$$\sigma(s) := \sigma(X_{\bar\theta_1+\cdots+\bar\theta_{m-1},X_{m-1}}(s)),\ \bar\sigma(s) := \sigma(X_{\bar\theta_1+\cdots+\bar\theta_{m-1},\bar X_{m-1}}(s))\ .$$

From (3.9) and (3.7) we have

$$d_m - d_{m-1} = (d_m - d_{m-1})\chi_{\bar\nu\wedge\nu\geq m}$$

$$= \chi_{\bar\nu\wedge\nu\geq m}\int_{\bar\theta_1+\cdots+\bar\theta_{m-1}}^{\bar\theta_1+\cdots+\bar\theta_m}(\chi(s)\cdot\sigma(s) - \bar\chi(s)\cdot\bar\sigma(s))dw(s)$$

$$+\chi_{\bar\nu\wedge\nu\geq m}(X_{\bar\theta_1+\cdots+\bar\theta_{m-1},\bar X_{m-1}}(\bar\theta_1+\cdots+\bar\theta_m) - \bar X_m)\ . \tag{3.10}$$

Due to $\mathcal{F}_{m-1}$-measurability of the random variable $\chi_{\bar\nu\wedge\nu\geq m}$, the equality (3.10) implies

$$E\left(d_m - d_{m-1}\right)^2$$

$$\leq 2E\chi_{\bar\nu\wedge\nu\geq m}E(|\int_{\bar\theta_1+\cdots+\bar\theta_{m-1}}^{\bar\theta_1+\cdots+\bar\theta_m}(\chi(s)\cdot\sigma(s) - \bar\chi(s)\cdot\bar\sigma(s))dw(s)|^2 \mid \mathcal{F}_{m-1})$$

$$+2E\chi_{\bar\nu\wedge\nu\geq m}E(|X_{\bar\theta_1+\cdots+\bar\theta_{m-1},\bar X_{m-1}}(\bar\theta_1+\cdots+\bar\theta_m) - \bar X_m|^2 \mid \mathcal{F}_{m-1})\ .$$

By the conditional versions of Lemma 3 and Theorem 1 under $n=1$, we obtain

$$E(d_m - d_{m-1})^2 \leq Kr^2E(\chi_{\bar\nu\wedge\nu\geq m}d_{m-1}^2) + Kr^4 \leq Kr^2Ed_{m-1}^2 + Kr^4, \tag{3.11}$$

where the constant K does not depend on $x,\ r,\ L$, and Δ.

Because $(d_m, \mathcal{F}_m)$ is a martingale, we have

$$Ed_m^2 = Ed_{m-1}^2 + E\left(d_m - d_{m-1}\right)^2. \tag{3.12}$$

The relations (3.11) and (3.12) imply

$$Ed_m^2 \leq Ed_{m-1}^2 + Kr^2Ed_{m-1}^2 + Kr^4,\ \ d_0 = 0.$$

From here we get for $N = L/r^2$

$$Ed_N^2 = E|X_{\nu\wedge N} - \bar X_{\nu\wedge N}|^2 \leq ((1+Kr^2)^{L/r^2} - 1)\cdot Kr^2 \leq Ke^{2\gamma L}\cdot r^2, \tag{3.13}$$

where the constant $\gamma > 0$ does not depend on x, r, L, and Δ.

Further, $X_{\bar\nu\wedge\nu\wedge N} = X_{\nu\wedge N}$, $\bar X_{\bar\nu\wedge\nu\wedge N} = \bar X_{\nu\wedge N}$. Indeed, it is evident for $\bar\nu \geq \nu\wedge N$. For $\bar\nu < \nu\wedge N$ it is also valid because both X and $\bar X$ stop after the moment $\bar\nu$. Hence,

$$E|X_{\bar\nu\wedge\nu\wedge N} - \bar X_{\bar\nu\wedge\nu\wedge N}|^2 \leq Ke^{2\gamma L}\cdot r^2. \tag{3.14}$$

Let us prove now that

$$P(\nu \leq \bar\nu\wedge N) \leq K\frac{e^{2\gamma L}}{\Delta^2}\cdot r^2. \tag{3.15}$$

In fact, due to (3.6) we have

$$E\chi_{\nu\leq\bar\nu\wedge N}|X_{\bar\nu\wedge\nu\wedge N} - \bar X_{\bar\nu\wedge\nu\wedge N}| = E\chi_{\nu\leq\bar\nu\wedge N}|X_\nu - \bar X_\nu| \geq P(\nu\leq\bar\nu\wedge N)\cdot\frac{\Delta}{2}\ . \tag{3.16}$$

On the other hand, using (3.14) we get

$$E\chi_{\nu\leq\bar\nu\wedge N}|X_{\bar\nu\wedge\nu\wedge N} - \bar X_{\bar\nu\wedge\nu\wedge N}|$$

$$\leq (P(\nu\leq\bar\nu\wedge N))^{\frac12}\cdot(E|X_{\bar\nu\wedge\nu\wedge N} - \bar X_{\bar\nu\wedge\nu\wedge N}|^2)^{\frac12}$$

The relations (3.16) and (3.17) imply (3.15).

Since $X_{\bar{\nu}\wedge N} = X_N$, $\bar{X}_{\bar{\nu}\wedge N} = \bar{X}_N$, we obtain from (3.14) and (3.15):

$$E|X_N - \bar{X}_N|^2 = E|X_{\bar{\nu}\wedge N} - \bar{X}_{\bar{\nu}\wedge N}|^2$$

$$= E\chi_{\nu \geq \bar{\nu}\wedge N}|X_{\bar{\nu}\wedge N} - \bar{X}_{\bar{\nu}\wedge N}|^2 + E\chi_{\nu < \bar{\nu}\wedge N}|X_{\bar{\nu}\wedge N} - \bar{X}_{\bar{\nu}\wedge N}|^2$$

$$= E\chi_{\nu \geq \bar{\nu}\wedge N}|X_{\bar{\nu}\wedge\nu\wedge N} - \bar{X}_{\bar{\nu}\wedge\nu\wedge N}|^2 + E\chi_{\nu < \bar{\nu}\wedge N}|X_{\bar{\nu}\wedge N} - \bar{X}_{\bar{\nu}\wedge N}|^2$$

$$\leq E|X_{\bar{\nu}\wedge\nu\wedge N} - \bar{X}_{\bar{\nu}\wedge\nu\wedge N}|^2 + KP(\nu \leq \bar{\nu}\wedge N) \leq K\frac{e^{2\gamma L}}{\Delta^2}\cdot r^2. \tag{3.18}$$

Using Doob's inequality for the martingale $(X_m - \bar{X}_m,\ \mathcal{F}_m)$, we arrive at (3.4). Theorem 2 is proved.

Remark 2. We pay attention to the proof of this theorem which uses only the mean-square versions of Theorem 1 and Lemma 3. The more complicated versions are needed later.

Remark 3. It will be proved later (see Remark 4) that it is possible to avoid the multiplier $1/\Delta$ in (3.4), i.e., in reality, the following inequality

$$(E\max_{1\leq k\leq \bar{\nu}\wedge N}\left|X_k - \bar{X}_k\right|^2)^{\frac{1}{2}} = (E\max_{1\leq k\leq N}\left|X_k - \bar{X}_k\right|^2)^{\frac{1}{2}} \leq Ke^{\gamma L}\cdot r \tag{3.19}$$

is valid.

Theorem 3. *Let $\bar{\nu} = \bar{\nu}_x(D)$. The inequality*

$$(E\max_{1\leq k\leq \bar{\nu}}\left|X_k - \bar{X}_k\right|^2)^{\frac{1}{2}} \leq K(\frac{1}{\Delta}e^{\gamma L}\cdot r + e^{-\frac{1}{2}\alpha_r\frac{\lambda_1}{1+C}L}) \tag{3.20}$$

is valid.

Proof. Introduce two sets $\mathcal{C} = \{\bar{\nu} \leq L/r^2\}$ and $\Omega\setminus\mathcal{C} = \{\bar{\nu} > L/r^2\}$. In view of (3.2) and (3.4) we have (let l be the diameter of G)

$$E\left|X_{\bar{\nu}} - \bar{X}_{\bar{\nu}}\right|^2 = E(\left|X_{\bar{\nu}} - \bar{X}_{\bar{\nu}}\right|^2; \mathcal{C}) + E(\left|X_{\bar{\nu}} - \bar{X}_{\bar{\nu}}\right|^2; \Omega\setminus\mathcal{C})$$

$$= E(\left|X_{\bar{\nu}\wedge N} - \bar{X}_{\bar{\nu}\wedge N}\right|^2; \mathcal{C}) + E(\left|X_{\bar{\nu}} - \bar{X}_{\bar{\nu}}\right|^2; \Omega\setminus\mathcal{C})$$

$$\leq E(\left|X_{\bar{\nu}\wedge N} - \bar{X}_{\bar{\nu}\wedge N}\right|^2) + l^2\cdot P(\Omega\setminus\mathcal{C})$$

$$\leq K\frac{e^{2\gamma L}}{\Delta^2}\cdot r^2 + l^2\cdot(1+C)e^{-\alpha_r\frac{\lambda_1}{1+C}L} \tag{3.21}$$

from which (3.20) follows. Theorem 3 is proved.

The domain D in Theorems 2 and 3 is not changed with decreasing r. Now consider the domain $G\setminus\Gamma_{cr^{1-\frac{1}{n}}}$, where $c > 0$ is a certain number and $n \geq 2$ is a natural number. Let $x \in G$. Consider r to be sufficiently small such that $\Gamma_{cr^{1-\frac{1}{n}}} \supset \Gamma_{2\lambda_d r}$ and $x \in G\setminus\Gamma_{cr^{1-\frac{1}{n}}}$. We construct the approximate phase trajectory $\bar{X}_k$ till its exit into the layer $\overline{\Gamma}_{cr^{1-\frac{1}{n}}}$, i.e., we stop the approximate trajectory, which was constructed in the satisfies the inequality

$$\bar{\nu}_x(G\backslash\Gamma_{cr^{1-\frac{1}{n}}}) < \bar{\nu}_x(G\backslash\Gamma_{2\lambda_d r}) \ .$$

As before we conserve the same notation both for $\bar{X}_k$ with the new stopping moment and for the very stopping moment $\bar{\nu} = \bar{\nu}_x(G\backslash\Gamma_{cr^{1-\frac{1}{n}}})$ as there is no risk of ambiguity. And as before $N = L/r^2$.

Theorem 4. *Let $\bar{\nu} = \bar{\nu}_x(G\backslash\Gamma_{cr^{1-\frac{1}{n}}})$ be the first exit time of the approximate trajectory $\bar{X}_k$ from the domain $G\backslash\Gamma_{cr^{1-\frac{1}{n}}}$. There exist constants $K > 0$* and *$\gamma > 0$ (which do not depend on x, r,* and *L) such that for any sufficiently small $r > 0$ the inequality*

$$(E \max_{1\le k\le\bar{\nu}\wedge N} \left|X_k - \bar{X}_k\right|^2)^{\frac{1}{2}} = (E \max_{1\le k\le N} \left|X_k - \bar{X}_k\right|^2)^{\frac{1}{2}} \le Ke^{\gamma L}\cdot r \qquad (3.22)$$

holds.

Proof. Introduce the number ν analogously to (3.5) (emphasize that now $\bar{\nu}$ is equal to $\bar{\nu}_x(G\backslash\Gamma_{cr^{1-\frac{1}{n}}})$):

$$\nu = \begin{cases} \min\{k : X_k \in \Gamma_{2\lambda_d r} \ , \ k \le \bar{\nu}\}, \\ \infty, \ X_k \notin \Gamma_{2\lambda_d r}, \ k = 1, ..., \bar{\nu} \ . \end{cases}$$

and the sequences $\bar{X}_{\nu\wedge m}, \ X_{\nu\wedge m}$.

Clearly, for sufficiently small r (if only $G\backslash\Gamma_{cr^{1-\frac{1}{n}}} \subset G\backslash\Gamma_{2\lambda_d r}$ and $3\lambda_d r \le \frac{c}{2} r^{1-\frac{1}{n}}$)

$$|X_\nu - \bar{X}_\nu| \ge \frac{c}{2} r^{1-\frac{1}{n}} \ , \ \text{if } \nu \le \bar{\nu} \ . \qquad (3.23)$$

We have

$$E\chi_{\nu\le\bar{\nu}\wedge N}|X_{\bar{\nu}\wedge\nu\wedge N} - \bar{X}_{\bar{\nu}\wedge\nu\wedge N}|^n = E\chi_{\nu\le\bar{\nu}\wedge N}|X_\nu - \bar{X}_\nu|^n$$

$$\ge (\frac{c}{2})^n P(\nu \le \bar{\nu}\wedge N)\cdot r^{n-1} \qquad (3.24)$$

and

$$E\chi_{\nu\le\bar{\nu}\wedge N}|X_{\bar{\nu}\wedge\nu\wedge N} - \bar{X}_{\bar{\nu}\wedge\nu\wedge N}|^n$$

$$\le (P(\nu \le \bar{\nu}\wedge N))^{\frac{1}{2}} \cdot (E|X_{\bar{\nu}\wedge\nu\wedge N} - \bar{X}_{\bar{\nu}\wedge\nu\wedge N}|^{2n})^{\frac{1}{2}}. \qquad (3.25)$$

Let us bound the mathematical expectation $E|X_{\nu\wedge N} - \bar{X}_{\nu\wedge N}|^{2n}$. To this end let us return to the proof of Theorem 2. All the reasonings can be repeated without any change. Of course, $\bar{\nu}$ and ν are the others now.

From (3.10) we have for any natural number l :

$$|d_m - d_{m-1}|^{2l} = |d_m - d_{m-1}|^{2l}\chi_{\bar{\nu}\wedge\nu\ge m} = \chi_{\bar{\nu}\wedge\nu\ge m}\cdot$$

$$|\int_{\bar{\theta}_1+\dots+\bar{\theta}_{m-1}}^{\bar{\theta}_1+\dots+\bar{\theta}_m}(\chi(s)\sigma(s) - \bar{\chi}(s)\bar{\sigma}(s))dw(s) + X_{\bar{\theta}_1+\dots+\bar{\theta}_{m-1},\bar{X}_{m-1}}(\bar{\theta}_1 + \dots + \bar{\theta}_m) - \bar{X}_m|^{2l}$$

$$\le K\cdot\chi_{\bar{\nu}\wedge\nu\ge m}\cdot|\int_{\bar{\theta}_1+\dots+\bar{\theta}_{m-1}}^{\bar{\theta}_1+\dots+\bar{\theta}_m}(\chi(s)\sigma(s) - \bar{\chi}(s)\bar{\sigma}(s))dw(s)|^{2l}$$

$$+K\cdot\chi_{\bar{\nu}\wedge\nu\ge m}\cdot|X_{\bar{\theta}_1+\dots+\bar{\theta}_{m-1},\bar{X}_{m-1}}(\bar{\theta}_1 + \dots + \bar{\theta}_m) - \bar{X}_m|^{2l},$$

where the constant K depends on n only.

$$E|d_m - d_{m-1}|^{2l} \le$$

$$KE\chi_{\bar{\nu}\wedge\nu\ge m}\cdot E(|\int_{\bar{\theta}_1+\dots+\bar{\theta}_{m-1}}^{\bar{\theta}_1+\dots+\bar{\theta}_m}(\chi(s)\sigma(s)-\bar{\chi}(s)\bar{\sigma}(s))dw(s)|^{2l} \mid \mathcal{F}_{m-1})$$

$$+KE\chi_{\bar{\nu}\wedge\nu\ge m}\cdot E(|X_{\bar{\theta}_1+\dots+\bar{\theta}_{m-1},\bar{X}_{m-1}}(\bar{\theta}_1+\dots+\bar{\theta}_m)-\bar{X}_m|^{2l}\mid \mathcal{F}_{m-1})$$

$$\le KE\chi_{\bar{\nu}\wedge\nu\ge m}(|d_{m-1}|^{2l}r^{2l}+Kr^{4l})+KE\chi_{\bar{\nu}\wedge\nu\ge m}r^{4l}\le Kr^{2l}E|d_{m-1}|^{2l}+Kr^{4l}. \tag{3.26}$$

We get

$$|d_m|^{2l} = |d_{m-1}+(d_m-d_{m-1})|^{2l}$$

$$=(d_{m-1}+(d_m-d_{m-1}),\ d_{m-1}+(d_m-d_{m-1}))^l$$

$$=(|d_{m-1}|^2+2(d_{m-1},\ d_m-d_{m-1})+|d_m-d_{m-1}|^2)^l,$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the scalar product of d-dimensional vectors.

Further

$$|d_m|^{2l}=|d_{m-1}|^{2l}+\sum_{k=1}^{l}C_l^k|d_{m-1}|^{2(l-k)}(2(d_{m-1},\ d_m-d_{m-1})+|d_m-d_{m-1}|^2)^k$$

$$=|d_{m-1}|^{2l}+2l|d_{m-1}|^{2l-2}(d_{m-1},\ d_m-d_{m-1})+l|d_{m-1}|^{2l-2}\cdot|d_m-d_{m-1}|^2$$

$$+\sum_{k=2}^{l}C_l^k|d_{m-1}|^{2(l-k)}(2(d_{m-1},\ d_m-d_{m-1})+|d_m-d_{m-1}|^2)^k. \tag{3.27}$$

As d_m is a martingale, we have

$$E|d_{m-1}|^{2l-2}(d_{m-1},\ d_m-d_{m-1})$$

$$=E|d_{m-1}|^{2l-2}(d_{m-1},E(\ d_m-d_{m-1}\mid\mathcal{F}_{m-1}))=0\ . \tag{3.28}$$

Since

$$(2(d_{m-1},\ d_m-d_{m-1})+|d_m-d_{m-1}|^2)^k$$

$$\le K(|d_{m-1}|^k\cdot|d_m-d_{m-1}|^k+|d_m-d_{m-1}|^{2k})\le K(|d_{m-1}|^{2k}+|d_m-d_{m-1}|^{2k})\ ,$$

we obtain from (3.27) and (3.28)

$$E|d_m|^{2l}\le E|d_{m-1}|^{2l}+KE\sum_{k=1}^{l}|d_{m-1}|^{2(l-k)}|d_m-d_{m-1}|^{2k}. \tag{3.29}$$

Hölder's inequality with $p=\dfrac{l}{l-k}$, $q=\dfrac{l}{k}$ and then (3.26) imply

$$E|d_{m-1}|^{2(l-k)}|d_m-d_{m-1}|^{2k}\le(E|d_{m-1}|^{2l})^{\frac{l-k}{l}}\cdot(E|d_m-d_{m-1}|^{2l})^{\frac{k}{l}}$$

$$\le(E|d_{m-1}|^{2l})^{\frac{l-k}{l}}\cdot(Kr^{2l}E|d_{m-1}|^{2l}+Kr^{4l})^{\frac{k}{l}}$$

$$\le K(E|d_{m-1}|^{2l})^{\frac{l-k}{l}}\cdot(r^{2k}(E|d_{m-1}|^{2l})^{\frac{k}{l}}+r^{4k})$$

$$=K(E|d_{m-1}|^{2l}\cdot r^{2k}+(E|d_{m-1}|^{2l})^{\frac{l-k}{l}}\cdot r^{4k})\ . \tag{3.30}$$

Using the elementary inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ with

$$a = (E|d_{m-1}|^{2l})^{\frac{l-k}{l}} \cdot r^{\frac{2(l-k)}{l}}, \; b = r^{4k-\frac{2(l-k)}{l}}, \; p = \frac{l}{l-k} \; , \; q = \frac{l}{k} \; ,$$

we obtain for $k = 1, ..., l$

$$(E|d_{m-1}|^{2l})^{\frac{l-k}{l}} \cdot r^{4k} \le K(E|d_{m-1}|^{2l} \cdot r^2 + r^{4l+2-\frac{2l}{k}}) \le$$

$$KE|d_{m-1}|^{2l} \cdot r^2 + Kr^{4l+2-\frac{2l}{k}}). \tag{3.31}$$

The relations (3.29)-(3.31) give

$$E|d_m|^{2l} \le E|d_{m-1}|^{2l} + Kr^2 E|d_{m-1}|^{2l} + Kr^{2l+2}, \; |d_0|^{2l} = 0 \; .$$

From here we get for $N = L/r^2$

$$E|d_N|^{2l} = E|X_{\nu\wedge N} - \bar{X}_{\nu\wedge N}|^{2l} \le Ke^{2\gamma L} \cdot r^{2l},$$

where the constants $K > 0$ and $\gamma > 0$ depend on l but do not depend on $x, \; r, \; L$.

Just as in Theorem 2 $X_{\bar{\nu}\wedge\nu\wedge N} = X_{\nu\wedge N}$, $\bar{X}_{\bar{\nu}\wedge\nu\wedge N} = \bar{X}_{\nu\wedge N}$. Hence

$$E|X_{\bar{\nu}\wedge\nu\wedge N} - \bar{X}_{\bar{\nu}\wedge\nu\wedge N}|^{2l} \le Ke^{2\gamma L} \cdot r^{2l}. \tag{3.32}$$

Now from (3.24), (3.25) and (3.32) under $l = n$ we get

$$P(\nu \le \bar{\nu} \wedge N) \cdot r^{n-1} \le K(P(\nu \le \bar{\nu} \wedge N))^{\frac{1}{2}} \cdot Ke^{\gamma L} \cdot r^n,$$

whence

$$P(\nu \le \bar{\nu} \wedge N) \le Ke^{2\gamma L} \cdot r^2. \tag{3.33}$$

Finishing the proof in just the same way as in Theorem 2, we arrive at (3.22). Theorem 4 is proved.

Remark 4. Now the inequality (3.19) which reinforces Theorem 2 can be approved in the following way. Instead of (3.16) let us write the following inequality

$$E\chi_{\nu\le\bar{\nu}\wedge N}|X_{\bar{\nu}\wedge\nu\wedge N} - \bar{X}_{\bar{\nu}\wedge\nu\wedge N}|^2$$

$$= E\chi_{\nu\le\bar{\nu}\wedge N}|X_\nu - \bar{X}_\nu|^2 \ge P(\nu \le \bar{\nu} \wedge N) \cdot \frac{\Delta^2}{4} \; . \tag{3.34}$$

Then instead of (3.17) due to (3.32) under $l = 2$ (clearly, the same inequality (3.32) is true on condition of Theorem 2) we obtain

$$E\chi_{\nu\le\bar{\nu}\wedge N}|X_{\bar{\nu}\wedge\nu\wedge N} - \bar{X}_{\bar{\nu}\wedge\nu\wedge N}|^2$$

$$\le (P(\nu \le \bar{\nu} \wedge N))^{\frac{1}{2}} \cdot (E|X_{\bar{\nu}\wedge\nu\wedge N} - \bar{X}_{\bar{\nu}\wedge\nu\wedge N}|^4)^{\frac{1}{2}}$$

$$\le K(P(\nu \le \bar{\nu} \wedge N))^{\frac{1}{2}} \cdot e^{\gamma L} \cdot r^2 \; . \tag{3.35}$$

The inequalities (3.34) and (3.35) imply

$$P(\nu \le \bar{\nu} \wedge N) \le Ke^{2\gamma L} \cdot \frac{r^2}{\Delta^4} r^2$$

and (3.19) follows from (3.18) if only $r \le \Delta^2$ in addition to the previous restrictions to smallness of r.

The following theorem is proved in the same way as Theorem 3.

$$(E\max_{1\le k\le\bar\nu}\left|X_k-\bar X_k\right|^2)^{\frac12}\le K(e^{\gamma L}\cdot r+e^{-\frac12\alpha_r\frac{\lambda_1}{1+C}L})$$

is valid.

Theorem 6. *Let $n>1,\ l\ge1$ be some natural numbers and $\bar\nu=\bar\nu_x(G\backslash\Gamma_{cr^{1-\frac1n}})$ be the first exit moment of the approximate trajectory $\bar X_k$ from the domain $G\backslash\Gamma_{cr^{1-\frac1n}}$. There exist constants $K>0$ and $\gamma>0$ (which do not depend on $x,\ r,\ L$) such that for any sufficiently small $r>0$ the inequality*

$$(E\max_{1\le k\le\bar\nu\wedge N}\left|X_k-\bar X_k\right|^{2l})^{\frac1{2l}}=(E\max_{1\le k\le N}\left|X_k-\bar X_k\right|^{2l})^{\frac1{2l}}\le Ke^{\gamma L}\cdot r \tag{3.36}$$

is fulfilled.

Proof. First let us show that for every $l\ge1$ the following inequality

$$P(\nu\le\bar\nu\wedge N)\le Ke^{2\gamma L}\cdot r^{2l} \tag{3.37}$$

holds. We can come to (3.37) in the same way as to (3.33). To this end let us write

$$E\chi_{\nu\le\bar\nu\wedge N}|X_{\bar\nu\wedge\nu\wedge N}-\bar X_{\bar\nu\wedge\nu\wedge N}|^{ln}=E\chi_{\nu\le\bar\nu\wedge N}|X_\nu-\bar X_\nu|^{ln}$$

$$\ge(\frac{c}{2})^{ln}P(\nu\le\bar\nu\wedge N)\cdot r^{l(n-1)} \tag{3.38}$$

instead of (3.24).

As l in (3.32) is arbitrary, we have (of course, with another K and another γ)

$$E|X_{\bar\nu\wedge\nu\wedge N}-\bar X_{\bar\nu\wedge\nu\wedge N}|^{2ln}\le Ke^{2\gamma L}\cdot r^{2ln}.$$

Therefore

$$P(\nu\le\bar\nu\wedge N)\cdot r^{l(n-1)}\le KE\chi_{\nu\le\bar\nu\wedge N}|X_{\bar\nu\wedge\nu\wedge N}-\bar X_{\bar\nu\wedge\nu\wedge N}|^{ln}$$

$$\le K(P(\nu\le\bar\nu\wedge N))^{\frac12}\cdot e^{\gamma L}\cdot r^{ln},$$

whence the inequality (3.37) follows.

Now we get

$$E|X_N-\bar X_N|^{2l}=E|X_{\bar\nu\wedge N}-\bar X_{\bar\nu\wedge N}|^{2l}$$

$$=E\chi_{\nu\ge\bar\nu\wedge N}|X_{\bar\nu\wedge\nu\wedge N}-\bar X_{\bar\nu\wedge\nu\wedge N}|^{2l}+E\chi_{\nu<\bar\nu\wedge N}|X_{\bar\nu\wedge N}-\bar X_{\bar\nu\wedge N}|^{2l}$$

$$\le E|X_{\bar\nu\wedge\nu\wedge N}-\bar X_{\bar\nu\wedge\nu\wedge N}|^{2l}+KP(\nu\le\bar\nu\wedge N)\le Ke^{2\gamma L}\cdot r^{2l}.$$

The relation (3.36) follows from here due to the Doob inequality. Theorem 6 is proved.

We have obtained the point $\bar{X}_N = \bar{X}_{\bar{\nu}\wedge N}$, where $N = L/r^2$, $\bar{\nu} = \bar{\nu}_x(G\backslash\Gamma_{cr^{1-\frac{1}{n}}})$. What distance is between $\bar{X}_N$ and exit point $X_x(\tau_x)$? What point on ∂G can we take as an approximation for $X_x(\tau_x)$?

On the set $\mathcal{C} = \{\bar{\nu} \le L/r^2\}$ we have $\bar{X}_N = \bar{X}_{\bar{\nu}} \in \bar{\Gamma}_{cr^{1-\frac{1}{n}}}$.

Let $\xi_x(\omega)$, $\omega \in \mathcal{C}$, be a point on ∂G such that

$$\left|\bar{X}_N - \xi_x\right| \le cr^{1-\frac{1}{n}},\ \omega \in \mathcal{C}. \tag{4.1}$$

It is natural to take this point as an approximate point for exit point $X_x(\tau_x)$ if $\bar{X}_N \in \bar{\Gamma}_{cr^{1-\frac{1}{n}}}$. Due to Theorem 4 and (4.1) we obtain

$$E(|X_N - \xi|^2; \mathcal{C}) \le K(c^2 + e^{2\gamma L}) \cdot r^{2-\frac{2}{n}}. \tag{4.2}$$

Lemma 4. *There exists a constant K such that for any $x \in \bar{G}$, $y \in \partial G$ the inequality*

$$E(X_x(\tau_x) - y)^2 \le K|x - y|$$

is fulfilled.

Proof. Consider the Dirichlet problem

$$\frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} = 0,\ \ x \in G,$$

$$u\ |_{\partial G} = (x - y)^2.$$

The solution of the problem is

$$u_y(x) = E(X_x(\tau_x) - y)^2.$$

From the conditions $(i)-(iii)$ it follows that $u_y \in C^{(4)}\left(\bar{G}\right)$ (see [11]). Since $u_y(y) = 0$, we have

$$u_y(x) = u_y(x) - u_y(y) \le K|x - y|\ .$$

Lemma 4 is proved.

We have defined the variable $\xi_x(\omega)$ only on $\mathcal{C}$. To complete the definition of $\xi_x(w)$ on the set $\Omega \setminus \mathcal{C}$, let us take as $\xi_x(\omega)$, e.g., the nearest point to $\bar{X}_N$ on ∂G in the case when $\omega \in \Omega \setminus \mathcal{C}$.

By Lemma 4 we have

$$E((X_x(\tau_x) - \xi_x)^2 \mid \mathcal{F}_N) = E((X_{X_N}(\tau_{X_N}) - \xi_x)^2 \mid \mathcal{F}_N) \le K|X_N - \xi_x|$$

Since $\mathcal{C} \in \mathcal{F}_N$, from the above inequality and (4.2) we get

$$E((X_x(\tau_x) - \xi_x)^2; \mathcal{C}) \le KE(|X_N - \xi_x|; \mathcal{C})$$

$$\le K(E(|X_N - \xi_x|^2; \mathcal{C}))^{\frac{1}{2}} \le K\left(c + e^{\gamma L}\right) \cdot r^{1-\frac{1}{n}}.$$

We can also evaluate the mathematical expectation $E\left(X_x\left(\tau_x\right) - \xi_x\right)^2$ analogously to (3.21). As a result we obtain the following theorem.

Theorem 7. *Let $\xi_x(\omega) \in \partial G$ be the nearest point to $\bar{X}_N$. Then (for clearness we reduce some non-essential constants)*

$$(E((X_x(\tau_x) - \xi_x)^2; \mathcal{C}))^{\frac{1}{2}} \le Ke^{\frac{\gamma L}{2}} \cdot r^{\frac{1}{2}-\frac{1}{2n}},$$

$$(E(X_x(\tau_x) - \xi_x)^2)^{\frac{1}{2}} \leq Ke^{\frac{\gamma L}{2}} \cdot r^{\frac{1}{2}-\frac{1}{2n}} + Ke^{-\frac{1}{2}\alpha_r \frac{\lambda_1}{1+C}L}.$$

5. Acknowledgments

The author would like to express his gratitude to Dr. M.V. Tret'yakov for fruitful discussion.

References

[1] I.I. Gichman, A.V. Skorochod. Stochastic Differential Equations. Naukova Dumka, Kiev, 1968.

[2] P.E. Kloeden, E. Platen. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1992.

[3] H.J. Kushner. Probability Methods for Approximations in Stochastic Control and for Elliptic Equations. Academic Press, New York, 1977.

[4] G.N. Milstein. Numerical Integration of Stochastic Differential Equations (engl. transl.). Kluwer Academic Publishers, 1995.

[5] G.N. Milstein. Solving the first boundary value problem of parabolic type by numerical integration of stochastic differential equations. Theory Prob. Appl., 40(1995), 657-665.

[6] G.N. Milstein. Solving boundary value problems by numerical integration of stochastic equations. Mathematics and Computers in Simulation, 38(1995), 77-85.

[7] G.N. Milstein. Application of numerical integration of stochastic equations for solving boundary value problems with the Neumann boundary conditions. Theory Prob. Appl., 41(1996), 210-218.

[8] G.N. Milstein. The simulation of phase trajectories of a diffusion process in a bounded domain. Stochastics and Stochastic Reports, 56(1996), 103-126.

[9] G.N. Milstein. Weak approximation of a diffusion process in a bounded domain. Stochastics and Stochastic Reports, in print.

[10] G.N. Milstein and N.F. Rybkina. An algorithm for random walks over small ellipsoids for solving the general Dirichlet problem. Comput. Maths Math. Phys. 33(1993), No. 5, 631-647.

[11] C. Miranda. Partial Differential Equations of Elliptic Type. Springer, 1970.

[12] E. Pardoux and D. Talay. Discretization and simulation of stochastic differential equations. Acta Appl. Math., 3(1985), 23-47.