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Continuous and inverse shadowing

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Abstract

By the Shadowing Lemma we can shadow any sufficient accurate pseudo-trajectory of a hyperbolic system by a true trajectory of a hyperbolic system. If we are interested in finite trajectories, at least from one side, then a pseudo trajectory usually has many possible shadows. Here we show that we can choose a continuous single-valued selector from the corresponding multi-valued operator "pseudo-trajectory $\mapsto$ the totality of possible shadows". We do this in the context of Lipschitz mappings which are semi-hyperbolic on some compact subset, which need not be invariant. We also prove that semi-hyperbolicity implis inverse shadowing with respect to a very broad class of nonsmooth perturbations.

1 Introduction

Consider a discrete time dynamical system generated by a mapping f. The Shadowing Lemma tells us that any sufficiently accurate pseudo-trajectory can be shadowed by a true trajectory in the invariant set on which the mapping f is hyperbolic. If only bi-infinite trajectories are of interest, by the expansivity property of hyperbolic systems there is usually only one shadowing true trajectory for each pseudo-trajectory and the correspondence between a pseudo-trajectory and its unique shadow is continuous in a natural sense, see e.g. [10].

On the other hand, a pseudo-trajectory usually has many possible shadows when the trajectories are finite on at least one side. It is then useful to know if we can choose a *continuous* single-valued selector from the corresponding multi-valued operator "pseudo-trajectory $\mapsto$ the totality of possible shadows". This is of some interest when the dynamical system is simulated on a computer [2, 3, 5, 6, 8, 11], particularly in connection with an inverse shadowing property which says that for a given class of perturbations (for example, corresponding to certain types of arithmetical processes) it is possible to approximate any trajectory of the original system to a required accuracy by some trajectory generated by a sufficiently small perturbation belonging to this class. In fact, we will prove that continuous shadowing implies inverse shadowing with respect to a broad class of nonsmooth perturbations. In this paper the class consists of all continuous $(\beta, \mathcal{Y})$–methods introduced in [8] and contains trajectories generated by continuous functions, time varying numerical methods and hysteresis systems. Such "methods" are defined in terms of a mapping from the state space of the dynamical system to a space of sequences taking values in this state space, essentially assigning a pseudo-trajectory of the original system to a given initial state. Hence our results extend those of [3, 5, 6], where a combined shadowing and inverse shadowing property, bishadowing for short, was established for semi-hyperbolic mappings with respect to perturbations given by continuous functions and by hysteresis functions, and from [8] where systems generated by hyperbolic homeomorphisms on a compact manifold were considered.

The semi-hyperbolicity concept [3, 4, 5, 6] that we consider here is a far reaching generalization of the hyperbolicity concept for diffeomorphisms that encompasses nonsmooth and noninvertible dynamical systems which are not traditionally studied in dynamical systems theory [3, 4, 5]. In particular, the set on which it applies need not be invariant, thus allowing even wider application.

The paper is organised as follows. The definition and some basic properties of senmi-hyperbolic mappings are given in the next section. Section 3 contains the main result of the paper, a Continuous Shadowing Lemma, which is proved in Section 6. Results on continuous and inverse shadowing for the above mentioned methods are presented together with their proofs in Sections 4 and 5, respectively.

2 Semi-hyperbolicity

Denote by $\mathcal{L} = \mathcal{L}(\mathcal{X})$ the set of Lipschitz mappings $f : \mathcal{X} \mapsto \mathcal{X}$, where $\mathcal{X}$ is an open subset of $\mathbb{R}^d$, and call a four-tuple $\mathrm{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ of nonnegative real numbers a *split* if

$$\lambda_s < 1 < \lambda_u \quad \text{and} \quad (1-\lambda_s)(\lambda_u - 1) > \mu_s\mu_u. \tag{2.1}$$

Note that the inequalities (2.1) imply that the spectral radius of the *split matrix*

$$M(\mathrm{s}) = \begin{pmatrix} \lambda_s & \mu_s \\ \mu_u/\lambda_u & 1/\lambda_u \end{pmatrix} \tag{2.2}$$

given by

$$\sigma(\mathrm{s}) = \frac{1}{2}\left(\left(\frac{1}{\lambda_u} + \lambda_s\right) + \sqrt{\left(\frac{1}{\lambda_u} - \lambda_s\right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}}\right) \tag{2.3}$$

is strictly less than 1.

Definition 1. *Let $\mathcal{Y}$ be a compact subset of $\mathcal{X}$, let $\mathrm{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ be a split, and let h and δ be positive real numbers. A map $f \in \mathcal{L}(\mathcal{X})$ is said to be* (s, h, δ)–semi-hyperbolic on the set $\mathcal{Y}$ *if for each $x \in \mathcal{Y}$ there exists a decomposition* $\mathbb{R}^d = E_x^s \oplus E_x^u$ *with corresponding projectors P_x^s, P_x^u such that:*

SH0. $\dim(E_x^u) = \dim(E_y^u)$ *if* $x, y \in \mathcal{Y}$;

SH1. $\sup_{x\in\mathcal{Y}}\{|P_x^s|\,|P_x^u|\} \le h$;

SH2. *The inclusion $x + u + v \in \mathcal{X}$ and the inequalities*

$$\begin{aligned}
|P_y^s(f(x+u+v) - f(x+\tilde{u}+v))| &\le \lambda_s|u-\tilde{u}|,\\
|P_y^s(f(x+u+v) - f(x+u+\tilde{v}))| &\le \mu_s|v-\tilde{v}|,\\
|P_y^u(f(x+u+v) - f(x+\tilde{u}+v))| &\le \mu_u|u-\tilde{u}|,\\
|P_y^u(f(x+u+v) - f(x+u+\tilde{v}))| &\ge \lambda_u|v-\tilde{v}|
\end{aligned}$$

hold for all x, $y \in \mathcal{Y}$ with $|y - f(x)| \le \delta$ and all u, $\tilde{u} \in E_x^s$, v, $\tilde{v} \in E_x^u$ such that $|u|$, $|\tilde{u}|$, $|v|$, $|\tilde{v}| \le \delta$.

We shall call the semi-hyperbolic mapping *continuously semi-hyperbolic* if the decomposition can be chosen to be continuous in x. Let $\mathcal{Y}_\eta = \{x \in \mathcal{X} \; : \; \mathrm{dist}(x, \mathcal{Y}) \leq \eta\}$ be the closed η–neighbourhood of $\mathcal{Y}$.

Lemma 1. *Let f be continuously (s, h, δ)–semi-hyperbolic on $\mathcal{Y}$. Then for each sufficiently small $\varepsilon > 0$ there exists an $\eta(\varepsilon) > 0$ such that the system f is continuously semi-hyperbolic on $\mathcal{Y}_{\eta(\varepsilon)}$ with parameters $\delta - \varepsilon > 0$ and $h - \varepsilon > 0$ for the split $\mathrm{s}(\varepsilon) = (\lambda_s + \varepsilon, \lambda_u - \varepsilon, \mu_s + \varepsilon, \mu_u + \varepsilon)$.*

Proof: This is done using the Tietze Extension Theorem ([9], p. 127) in a standard way, so we omit the details. □

3 Continuous shadowing

Let $\mathcal{X}$ be an open subset of $\mathbb{R}^d$ and consider the discrete–time dynamical system generated by a continuous mapping $f : \mathcal{X} \mapsto \mathcal{X}$.

We will consider certain finite and infinite sequences $\mathrm{y} = \{y_i\}_0^n$ belonging to $\mathcal{X}$ as approximate trajectories of f. For such a sequence we define the *f-discrepancy* by

$$\mathcal{D}(\mathrm{y}; f) = \sup_{1 \leq i \leq n} |y_i - f(y_{i-1})|.$$

A sequence y with $\mathcal{D}(\beta; f) \leq \beta$ is called a *β-pseudo–trajectory* of the system f. Let $\mathrm{Tr}^n(f, \mathcal{Y}, \beta)$ denote the totality of such β–pseudo–trajectories belonging entirely to a subset $\mathcal{Y} \subset \mathcal{X}$. We write $\mathrm{Tr}^n(f, \mathcal{Y})$ for $\mathrm{Tr}^n(f, \mathcal{Y}, 0)$ and $\mathrm{Tr}^n(f)$ for $\mathrm{Tr}^n(f, \mathcal{X})$ and will use the uniform norm on these sets, that is

$$\|\mathrm{x} - \mathrm{y}\| = \sup_{0 \leq i \leq n} |x_i - y_i|.$$

The set $\mathrm{Tr}^\infty(f, \mathcal{Y}, \beta)$ will sometimes be endowed with the topology of coordinate-wise convergence when convenient.

A dynamical system generated by a mapping $f : \mathcal{X} \mapsto \mathcal{X}$ is said to be *continuously shadowing* with positive parameters α and β on a subset $\mathcal{Y}$ of $\mathcal{X}$ if for each given positive integer n there exists a continuous operator $W^n : \mathrm{Tr}^n(f, \mathcal{Y}, \beta) \to \mathrm{Tr}^n(f)$ such that

$$\|W^n(\mathrm{y}) - \mathrm{y}\| \leq \alpha \mathcal{D}(\mathrm{y}; f). \tag{3.1}$$

Theorem 1. *Let f be (s, h, δ)–semi-hyperbolic on the set $\mathcal{Y}$. Then it is continuously shadowing on $\mathcal{Y}$ with α and β given by*

$$\alpha(\mathrm{s}, h) = h \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}, \tag{3.2}$$

$$\beta(\mathrm{s}, h, \delta) = \delta h^{-1} \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{\max\{\lambda_u - 1 + \mu_s, 1 - \lambda_s + \mu_u\}}. \tag{3.3}$$

This theorem, which concerns the continuous shadowing of trajectories of a given length, is a special case of the next theorem. In applications it is often important to "glue together" continuous shadowing trajectories of different lengths. For such situations the following concretization of the above theorem is convenient. Define

$$r^u(\mathrm{s},\delta) = \delta \frac{1-\lambda_s+\mu_u}{\max\{\lambda_u - 1 + \mu_s, 1-\lambda_s+\mu_u\}} \tag{3.4}$$

and denote by $B^s_y(r)$ and $B^u_y(r)$ the closed balls in the linear subspaces E^s_y and E^u_y, respectively, of radius r and centered at y.

Theorem 2. **(Continuous Shadowing Lemma)** *Let f be (s,h,δ)–semi-hyperbolic on the set $\mathcal{Y}$. Then for each positive integer n there exists a continuous operator $W^n(\mathrm{y},z^u)$ defined for $\mathrm{y} \in \mathrm{Tr}^n(f,\mathcal{Y},\beta)$, $z^u \in B^u_{y_n}(r^u(\mathrm{s},\delta))$ and taking values in $\mathrm{Tr}^n(f)$ such that*

$$\|W^n(\mathrm{y},z^u) - \mathrm{y}\| \le \alpha\mathcal{D}(\mathrm{y};f) \tag{3.5}$$

and for each $1 \le m \le n$ the restriction of the trajectory $W^n(\mathrm{y},z^u)$ to the integer interval $[0,m]$ coincides with the trajectory

$$W^m(\mathrm{y},z^u_*) \ \ where\ z^u_* = P^u_{y_m}(x_m - y_m)\,. \tag{3.6}$$

There also exists an operator $W^\infty : \mathrm{Tr}^\infty(f,\mathcal{Y},\beta) \to \mathrm{Tr}^\infty(f)$ such that for each positive integer m the restriction of the trajectory $W^\infty(\mathrm{y})$ to the integer interval $[0,m]$ coincides with the trajectory (3.6). Moreover, the operator W^∞ is coordinate-wise continuous and the trajectories $W^n(\mathrm{y}^n,z^u_n)$ converge coordinate-wise to $W^\infty(\mathrm{y}^\infty)$ for each sequence $\mathrm{y}^n \in \mathrm{Tr}^n(f,\mathcal{Y},\beta)$ converging coordinate-wise to y^∞ and any $z^u_n \in B^u_{y^n_n}(r^u(\mathrm{s},\delta))$.

The proof will be presented in Section 6.

4 Continuous shadowing for methods

Denote by $\mathcal{T}$ the set of all possible (finite or infinite) sequences $\mathrm{y} = \{y_0,\dots,y_n,\dots\}$ belonging to $\mathcal{X}$ and for a sequence $\mathrm{y} \in \mathcal{T}$ define $\mathcal{D}(\mathrm{y};f,\mathcal{Y})$ to be equal to $\mathcal{D}(\mathrm{y}_*;f)$ where y_* is the maximal initial fragment of y which belongs entirely to $\mathcal{Y}$. We call a mapping $\varphi : \mathcal{Y} \mapsto \mathcal{T}$ a *method of accuracy β* for f on $\mathcal{Y}$, or just a $(\beta,\mathcal{Y})$–method for short, if $\varphi(y)_0 = y$ and $\mathcal{D}(\varphi(y);f,\mathcal{Y}) \le \beta$ for any $y \in \mathcal{Y}$. Note that the image of the set $\mathcal{Y}$ under a $(\beta,\mathcal{Y})$–method is a complete family of β–pseudo–trajectories in the terminology of Corless and Pilyugin [2, 11].

Example 1. *Let $\mathcal{Y} \subset \mathcal{X}$ and consider a sequences of maps $g_i : \mathcal{Y} \mapsto \mathcal{X}$, $i = 1,2,3,\dots$, with $D_\mathcal{Y}(g_i,f) = \sup\{|g_i(x) - f(x)| : x \in \mathcal{Y}\} \le \beta$. For each point $x \in \mathcal{Y}$ define a sequence (finite or infinite) $\varphi(x) \in \mathcal{T}$ by*

$$\varphi(x)_0 = x, \quad \varphi(x)_n = g_n(\varphi(x)_{n-1}), \quad for\ n > 0,$$

as long as successive points belong to $\mathcal{Y}$. Such a $(\beta,\mathcal{Y})$–method φ arises when a simulation of the forward orbits of the dynamical system generated by f involves procedures that may vary with time.

The above example is a special case of the following one which includes hysteresis perturbations [7] and multi-step procedures.

Example 2. *Let a family of maps $h_n : \mathcal{Y}^n \mapsto \mathcal{X}$ satisfy the following condition: For any $n > 0$ and any points $y_0, \ldots y_{n-1} \in \mathcal{Y}$*

$$|f(h_n(y_0, \ldots, y_{n-1})) - h_{n+1}(y_0, \ldots, y_{n-1}, h_n(y_0, \ldots, y_{n-1}))| \leq \beta$$

if $h_n(y_0, \ldots, y_{n-1}) \in \mathcal{Y}$. Define $\varphi : \mathcal{Y} \mapsto \mathcal{T}$ as

$$\varphi(y)_0 = y$$

for $n = 0$

$$\varphi(y)_n = h_n(\varphi(y)_0, \ldots, \varphi(y)_{n-1})$$

for $n > 1$.

We say that a $(\beta, \mathcal{Y})$–method is a continuous method if $\varphi : \mathcal{Y} \mapsto \mathcal{T}$ is continuous with respect to the coordinate-wise topology on $\mathcal{T}$. Let us note that, if the maps g_n and h_n are continuous, then the $(\beta, \mathcal{Y})$–methods in theabove examples are continuous methods.

Denote by $\mathbf{x}(x_0)$ the true trajectory $x_0, \ldots, x_n = f^n(x_0), \ldots$ of f starting at the point x_0 and let α, $\beta > 0$. We will say that the mapping f is *continuously $\mathcal{M}$-shadowing* with parameters $\alpha > 0$, $\beta > 0$ and $\eta \geq 0$ (by mentioning α and β only, we will mean the case $\eta = 0$) on a compact subset $\mathcal{Y}$ of $\mathcal{X}$ if for any continuous $(\beta, \mathcal{Y}_\eta)$–method φ there exists a continuous mapping $W : \mathcal{Y} \to \mathcal{X}$ such that for any positive integer n the inclusions

$$\varphi(y)_i \in \mathcal{Y}, \quad \text{for all} \quad i = 0, 1, \ldots, n \tag{4.1}$$

imply

$$|\varphi(y)_i - x_i| \leq \alpha \mathcal{D}(\varphi(y); f, \mathcal{Y}) \quad \text{for all} \quad i = 0, 1, \ldots, n \tag{4.2}$$

where $\mathbf{x} = \mathbf{x}(W(y))$.

We emphasize that we did not fix the length of trajectory under consideration in advance and, in fact, different trajectories are to be shadowed on quite different intervals. (Also, the mapping W here should be universal).

Theorem 3. *Let f be (s, h, δ)–semi-hyperbolic on the set $\mathcal{Y}$. Then it is continuously $\mathcal{M}$-shadowing on $\mathcal{Y}$ for any $\alpha > \alpha(\mathrm{s}, h)$, $\beta < \beta(\mathrm{s}, h, \delta)$ and a sufficiently small $\eta > 0$.*

Proof: Choose positive numbers $\alpha > \alpha(\mathrm{s}, h)$ and $\beta < \beta(\mathrm{s}, h, \delta)$. Then select a positive ε such that

$$\alpha(\mathrm{s}, h) < \alpha < \alpha(\mathrm{s}(\varepsilon), h - \varepsilon), \qquad \beta(\mathrm{s}, h\delta) < \beta < \beta(\mathrm{s}(\varepsilon), h - \varepsilon, \delta - \varepsilon)$$

and let $\eta = \eta(\varepsilon) > 0$ be that from Lemma 1. Now fix a continuous $(\beta, \mathcal{Y}_\eta)$–method φ and let $\mathcal{Y}_{\eta/n}$ denote the closed η/n neighbourhood of $\mathcal{Y}$. Let $\widehat{\mathcal{Y}}_0 = \mathcal{Y}_\eta$ and for $n > 1$ denote by $\widehat{\mathcal{Y}}_n$ the totality of $y \in \mathcal{Y}_{\eta/2n}$ such that

$$\varphi(y)_i \in \mathcal{Y}_{\eta/2n}, \quad i = 0, 1, \ldots, n.$$

Clearly, each $\widehat{\mathcal{Y}}_n$ is a compact set and $\widehat{\mathcal{Y}}_n$ belongs to the interior of $\widehat{\mathcal{Y}}_{n-1}$. Let Γ_n denote the the boundary of $\widehat{\mathcal{Y}}_n$ and define on Γ_n the function

$$z_{n-1}(y) = P^u_{y_{n-1}}(x_{n-1} - y_{n-1})$$

where

$$\mathrm{x} = W^n(\mathrm{y}, 0, 0)$$

and $\mathrm{y} = \varphi(y)$. Now extend $z_{n-1}(\cdot)$ to a continuous function on the whole set $\widehat{\mathcal{Y}}_{n-1} \setminus \widehat{\mathcal{Y}}_n$ satisfying

$$z_{n-1}(y) \equiv 0, \; y \in \Gamma_{n-1}, \qquad |z_{n-1}(y)| \leq r^u(\mathrm{s}(\varepsilon), \delta - \varepsilon),$$

which is possible due to the Tietze Extension Theorem [9]. Finally define the mapping W on $\widehat{\mathcal{Y}}_{n-1} \setminus \widehat{\mathcal{Y}}_n$ by $W(y) = y^*_0$ where

$$\mathrm{y}^* = W^{n-1}(\mathrm{y}, 0, z_{n-1}(y))$$

and by

$$W(y) = W^\infty(\mathrm{y}, 0), \qquad y \in \widehat{\mathcal{Y}}_\infty = \bigcap_{0 \leq n < \infty} \widehat{\mathcal{Y}}_n.$$

By definition the mapping W is continuous on Y_η and has the required shadowing property. Hence the mapping W, or strictly speaking the restriction of this mapping to $\mathcal{Y}$, has the neccessary properties and the theorem is proved. □

5 Inverse shadowing for methods

As before, let $\mathbf{x}(x_0)$ be the true trajectory of f starting at the point x_0 and let $\mathcal{Y}$ be compact subset of $\mathcal{X}$. A map f is said to be *inverse $\mathcal{M}$-shadowing* on $\mathcal{Y}$ with positive parameters α, β and η if for any continuous $(\beta, \mathcal{Y}_\eta)$-method φ and any $x_0 \in \mathcal{Y}$ there exists a point $y \in \mathcal{X}$ such that the inclusions

$$x_i \in \mathcal{Y} \quad \text{for all } i = 0, 1, \ldots, n \tag{5.1}$$

imply

$$|\varphi(y)_i - x_i| \leq \alpha \mathcal{D}(\varphi(y); f, \mathcal{Y}_\eta), \qquad \text{for all } i = 0, 1, \ldots, n \tag{5.2}$$

where $\mathbf{x} = \mathbf{x}(x_0)$.

This definition generalizes the "inverse POTP" concept considered by Corless and Pilyugin in [2] for diffeomorphisms on compact manifolds and bi-finite orbits. It is also differs slightly from the definitions of α-robustness and (α, β)–inverse shadowing investigated the papers [3, 5, 6, 8] where a whole trajectory $\mathbf{y} \in \mathrm{Tr}_{\boldsymbol{J}}(f, \mathcal{Y})$ for a given integer interval $\boldsymbol{J}$ was fixed rather than just its 0th point $y \in \mathcal{Y}$ as in the above definition.

Theorem 4. *Let $f : \mathcal{X} \mapsto \mathcal{X}$ be Lispchitz continuous with Lipschitz constant L, let $\mathcal{Y} \subset \mathcal{X}$ and let $\eta > 0$ be small enough so that $\mathcal{Y}_\eta \subset \mathcal{X}$. If f is continuously $\mathcal{M}$-shadowing on $\mathcal{Y}_\eta$ with parameters α and β, then f is inverse $\mathcal{M}$-shadowing on $\mathcal{Y}$ with parameters*

$$\alpha, \qquad \beta_1 = \min\left(\beta, \frac{\eta}{\alpha}, \frac{\eta}{\alpha L + 1}\right), \qquad \eta.$$

Proof: Let a continuous $(\beta_1, \mathcal{Y}_\eta)$-method φ and let $x \in \mathcal{Y}$ be fixed. Hence the ball $B = \{y \in \mathbb{R}^d \; : \; |x - y| \leq \eta\} \subset \mathcal{Y}_\eta$, so we can define a mapping $F : B \mapsto \mathbb{R}^d$ by

$$F(y) = x - W(y) + y,$$

where W is a continuous operator given by continuous $\mathcal{M}$-shadowing on $\mathcal{Y}_\eta$. The mapping F is thus obviously also continuous.

Now by shadowing, for $y \in B$ we have

$$|F(y) - x| = |W(y) - y| \leq \alpha \mathcal{D}(\varphi(y); f, \mathcal{Y}_\eta) \leq \alpha\beta_1 \leq \eta,$$

from which it follows that $F(B) \subset B$. By the Brouwer Fixed Point Theorem there exists a fixed point $x^* \in B$ of F and by the definition of F we have $x = W(x^*)$.

Suppose that the inclusions (5.1) hold and let n_* be the largest integer for which

$$\varphi(x_*)_i \in \mathcal{Y}_\eta, \qquad i = 0, 1, \ldots, n_*.$$

Suppose that $n_* < n$. Then by the shadowing property

$$|\varphi(x_*)_{n_*} - x_{n_*}| \leq \alpha\beta_1,$$

so $|f(\varphi(x_*)_{n_*}) - f(x_{n_*})| \leq \alpha\beta_1 L$ and consequently

$$|\varphi(x_*)_{n_*+1} - x_{n_*+1}| \leq \beta_1(\alpha L + 1) \leq \eta.$$

That is $\varphi(x_*)_{n_*+1} \in \mathcal{Y}_\eta$, which contradicts the definition of n_*. Thus we must have $n_* \geq n$ and the desired relation (5.2) then follows from the shadowing property which was established in the previous theorem. □

The theorems 3 and 4 imply

Corollary 1. *Let the mapping $f \in \mathcal{L}(\mathcal{X})$ be (s, h, δ)–semi-hyperbolic on a compact set $\mathcal{Y} \subset \mathcal{X}$. Then for any sufficiently small $\varepsilon > 0$, f is inverse $\mathcal{M}$-shadowing on $\mathcal{Y}$ with parameters*

$$\alpha > \alpha(\varepsilon), \quad \beta < \min\left\{\beta(\varepsilon), \frac{\eta(\varepsilon)}{\alpha(\varepsilon)}, \frac{\eta(\varepsilon)}{\alpha(\varepsilon)L + 1}\right\}, \quad \eta(\varepsilon),$$

where L is the Lipschitz constant of f, $\alpha(\varepsilon) = a(\mathrm{s}(\varepsilon), h - \varepsilon)$ and $\beta(\varepsilon) = \beta(\mathrm{s}(\varepsilon), h - \varepsilon, \delta - \varepsilon)$ while $\eta(\varepsilon)$ is the same as in Lemma 1.

6 Proof of the Continuous Shadowing Lemma

We will prove Theorem 2 using the Contraction Mapping Principle. For this we need some auxiliary facts.

6.1 Auxiliary results

For each x, $y \in \mathcal{Y}$ with $|f(x)-y| \leq \delta$ and for each $z \in \mathbb{R}^d$ satisfying $|P^s_x z| \leq \delta$ define the mapping $F_{x,y,z} : B^u_x(\delta) \mapsto E^u_y$ by

$$F_{x,y,z}(v) = P^u_y \left(f(x + P^s_x z + v) - f(x + P^s_x z) \right). \tag{6.1}$$

Lemma 2. *Let x, $y \in \mathcal{Y}$ with $|f(x) - y| \leq \delta$ and $z \in \mathbb{R}^d$ satisfy $|P^s_x z| \leq \delta$. Then $F_{x,y,z}\left(B^u_x(r)\right) \supseteq B^u_y(\lambda_u r)$ for $0 \leq r \leq \delta$.*

Proof: We need only consider the case $r > 0$. Denote the boundary and the interior of $B^u_x(r)$ by $\partial B^u_x(r)$ and $\operatorname{int} B^u_x(r)$, respectively. Clearly,

$$F_{x,y,z}(0) = P^u_y \left(f(x + P^s_x z) - f(x + P^s_x z) \right) = 0 \in \operatorname{int} B^u_y(\lambda_u r), \tag{6.2}$$

while by the inequality (2.4)

$$F_{x,y,z}(\partial B^u_x(r)) \bigcap \operatorname{int} B^u_y(\lambda_u r) = \emptyset. \tag{6.3}$$

By Property SH0 and the Invariance of Domain Principle (see, e.g., [1], p.396), the relation (6.3) implies that $\partial F_{x,y,z}(B^u_x(r)) \bigcap \operatorname{int} B^u_y(\lambda_u r) = \emptyset$. This combines with (6.2) to imply that $F_{x,y,z}\left(B^u_x(r)\right) \supseteq B^u_y(\lambda_u r)$, which proves the lemma. □

¿From this lemma and from inequality (2.4) we immediately obtain

Corollary 2. *Under conditions of Lemma 2 the operator $Q_{x,y,z} = F^{-1}_{x,y,z}$ is defined and continuous on $B^u_y(\lambda_u \delta)$ and satisfies the estimates*

$$|Q_{x,y,z}(v)| \leq \lambda_u^{-1} |v|, \quad v \in B^u_y(\lambda_u \delta)$$

and

$$|Q_{x,y,z}(v_1) - Q_{x,y,z}(v_2| \leq \lambda_u^{-1} |v_1 - v_2|, \quad v_1, v_2 \in B^u_y(\lambda_u \delta).$$

6.2 Operator H and its properties

Let $n \leq \infty$ be fixed. Denote by $J[n]$ the set of all finite integers satisfying $0 \leq i \leq n$ and let $J(n]$, $J[n)$ and $J(n)$ be the subsets of $J[n]$ obtained by excluding the first, the last and both the first and last elements, respectively. Let $\mathcal{Z}^n$ be the space of finite sequences

$$\mathbf{z} = \{z_0, z_1, \ldots, z_n\} \tag{6.4}$$

if $n < \infty$ and the corresponding space of infinite sequences otherwise. Choose a pseudo-trajectory $\mathbf{y} \in \operatorname{Tr}^n(f, \mathcal{Y}, \beta)$ and, when n if finite, also choose a point $z^u \in E^u_{y_n}$. Introduce an operator $H : \mathcal{Z}^n \mapsto \mathcal{Z}^n$ taking $\mathbf{z}$ to $\mathbf{w}$ which is defined by the boundary conditions

$$P^s_{y_0} w_0 \;=\; 0, \tag{6.5}$$

$$P^u_{y_n} w_n \;=\; z^u \quad \text{if} \quad n < \infty, \tag{6.6}$$

and the relations

$$P^s_{y_i} w_i \;=\; P^s_{y_i}(f(y_{i-1} + z_{i-1}) - y_i), \tag{6.7}$$

$$P^u_{y_{i-1}} w_{i-1} \;=\; Q_{y_{i-1},y_i,z_{i-1}}(P^u_{y_i}(y_i - f(y_{i-1} + P^s_{y_{i-1}} z_{i-1}) + z_i)) \tag{6.8}$$

for $i \in J(n]$ and $i \in J(n)$ in the first and second equations, respectively. A further restriction will be placed on the point z^u in the boundary condition (6.6) in Lemma 4 below.

Define a and b by

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1-\lambda_s & \mu_s \\ \mu_u/\lambda_u & 1-1/\lambda_u \end{pmatrix}^{-1} \begin{pmatrix} h \\ h/\lambda_u \end{pmatrix} = (I - M(\mathrm{s}))^{-1} \begin{pmatrix} h \\ h/\lambda_u \end{pmatrix}, \tag{6.9}$$

where $M(\mathrm{s})$ is the split matrix (2.2) which, as we saw earlier, has spectral radius $\sigma(\mathrm{s})$ strictly less than 1. Hence the matrix

$$(I - M(\mathrm{s}))^{-1} = \sum_{j=0}^{\infty} M(\mathrm{s})^j$$

exits and has nonnegative entries with strictly positive diagonal entries, from which it follows that a and b are strictly positive. The set

$$S = \left\{ \mathrm{z} \in \mathcal{Z}^n : \; |P^s_{y_i} z_i| \le a\beta \text{ and } |P^u_{y_i} z_i| \le b\beta, \; i \in J[n] \right\} \tag{6.10}$$

is thus nonempty.

Lemma 3. *Let* $\mathrm{z} = \{z_i\}_{i=0}^n \in S$. *Then*

$$|P^s_{y_i} z_i| \le a\beta(\mathrm{s}, h, \delta) \le \delta, \qquad |P^u_{y_i} z_i| \le b\beta(\mathrm{s}, h, \delta) \le \delta \tag{6.11}$$

and

$$y_i + z_i \in \mathcal{X} \tag{6.12}$$

hold for each $i \in J[n]$.

Proof: First note from (6.9) that

$$a + b = \alpha(\mathrm{s}, h) \quad \text{and} \quad \max\{a, b\} = \frac{\delta}{\beta(\mathrm{s}, h, \delta)}, \tag{6.13}$$

the second of which implying that

$$a\beta(\mathrm{s}, h, \delta) \le \delta, \quad b\beta(\mathrm{s}, h, \delta) \le \delta. \tag{6.14}$$

On the other hand, by the definition of the set S we have

$$|P^s_{y_n} z_n| \le a\beta(\mathrm{s}, h, \delta), \quad |P^u_{y_n} z_n| \le b\beta(\mathrm{s}, h, \delta). \tag{6.15}$$

Inequalities (6.14) and (6.15) lead to (6.11). The inclusion (6.12) then follows from (6.11) and the inclusion $x + u + v \in \mathcal{X}$ of property SH2. $\square$

Now, define an auxiliary norm $\|\cdot\|_*$ on Z^n by

$$\|\mathbf{z}\|_* = \max_{i \in J[n]} |z_i|_* \tag{6.16}$$

with $|z_i|_* = \max\{|P^s_{y_i} z|, \gamma |P^s_{y_i} z|\}$ where

$$\gamma = \frac{1}{\mu_s}\sigma(\mathrm{s}) - \frac{\lambda_s}{\mu_s}$$

and $\sigma(\mathrm{s})$ is the spectral radius (2.3) of the split matrix $M(\mathrm{s})$. Finally, the boundary condition (6.6) will be called *restricted* if z^u satisfies $|z^u| < a\beta(\mathrm{s}, h, \delta)$.

Lemma 4. *Let $n = \infty$ or let $n < \infty$ and the boundary condition (6.6) be restricted. Then the operator H exists and is $\sigma(\mathrm{s})$-contracting in the norm* (6.16) *on the set S.*

Proof: We consider only the more complicated case when n is finite. By inclusion (6.12) of Lemma 3, the right hand side of (6.7) is defined and depends continuously on $\mathbf{z} \in S$. We need thus only to prove that for any $i \in J(n]$ the right hand side of (6.8) is defined and continuous for $\mathbf{z} \in S$. By inclusion (6.12) of Lemma 3 again, the expression

$$P^u_{y_i}(y_i - f(y_{i-1} + P^s_{y_{i-1}} z_{i-1}) + z_i)$$

which is the argument of the operator $Q_{y_{i-1}, y_i, z_{i-1}}$ in (6.8), is defined and continuous for $\mathbf{z} \in S$. It remains then to verify that the conditions of Corollary 2 hold, that is that

$$y_{i-1}, y_i \in \mathcal{Y}, \qquad |f(y_{i-1}) - y_i| \le \delta, \qquad |P^s_{y_{i-1}} z_{i-1}| \le \delta \tag{6.17}$$

and

$$\left| P^u_{y_i}\left(y_i - f(y_{i-1} + P^s_{y_{i-1}} z_{i-1}) + z_i\right)\right| \le \lambda_u \delta\ . \tag{6.18}$$

are valid for all $i \in J(n]$. The inclusions in (6.17) follow from the assumptions of the theorem and the first inequality in (6.17) is valid by (3.3), while the second one follows from Lemma 3 and the first inequality of (6.11). To verify (6.18) we first rewrite it as

$$|J_1 + J_2 + J_3| \le \lambda_u \delta, \tag{6.19}$$

where

$$J_1 = P^u_{y_i}(y_i - f(y_{i-1})),\ J_2 = P^u_{y_i}(f(y_{i-1}) - f(y_{i-1} + P^s_{y_{i-1}} z_{i-1})),\ J_3 = P^u_{y_i} z_i, \tag{6.20}$$

and estimate each of $|J_1|$, $|J_2|$ and $|J_3|$. To estimate $|J_1|$ note that $|y_i - f(y_{i-1})| \le \gamma$ because $\mathbf{y}$ is a γ-pseudo trajectory of f. By property SH1 we thus obtain

$$|J_1| \le \gamma h. \tag{6.21}$$

By inequality (2.4) we also have

$$|J_2| \le \mu_u |P^s_{y_{i-1}} z_{i-1}|, \tag{6.22}$$

and obviously

$$|J_3| = |P^u_{y_i} z_i|. \tag{6.23}$$

On the other hand, $\mathbf{z} \in S$ implies that

$$|P^s_{y_{i-1}} z_{i-1}| \le a\gamma, \qquad |P^u_{y_i} z_i| \le b\gamma. \tag{6.24}$$

Hence by (6.22)–(6.24)

$$|J_1 + J_2 + J_3| \ \le\ |J_1| + |J_2| + |J_3| \ \le\ \gamma(h + a\mu_u + b),$$

so it remains to show that

$$\gamma(h + a\mu_u + b) \ \le\ \lambda_u \delta. \tag{6.25}$$

¿From (6.9) we see that

$$a = h\frac{\lambda_u - 1 + \mu_s}{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u}, \qquad b = h\frac{1 - \lambda_s + \mu_u}{(1-\lambda_s)(\lambda_u - 1) - \mu_s\mu_u},$$

so $h + a\mu_u + b = \lambda_u b$ and (6.25) can be rewritten as

$$\gamma\lambda_u b \ \le\ \lambda_u \delta. \tag{6.26}$$

Inequality (6.26) then follows from the second equality of (6.13).

Let us now prove the contracting property of the operator H. For this we consider the difference

$$\mathbf{w}^1 - \mathbf{w}^2 = H\mathbf{z}^1 - H\mathbf{z}^2$$

for arbitrary $\mathbf{z}^1$, $\mathbf{z}^2 \in S$. We will verify that the inequality

$$\|\mathbf{w}^1 - \mathbf{w}^2\|_* \le \sigma(\mathrm{s})\|\mathbf{z}^1 - \mathbf{z}^2\|_*$$

holds, or equivalently the pairs of inqualities

$$|P^s_{y_i}(w^1_i - w^2_i)| \le \sigma(\mathrm{s})\|\mathbf{z}^1 - \mathbf{z}^2\|_*, \tag{6.27}$$

$$\gamma|P^u_{y_i}(w^1_i - w^2_i)| \le \sigma(\mathrm{s})\|\mathbf{z}^1 - \mathbf{z}^2\|_*. \tag{6.28}$$

In view of the boundary conditions it suffices to establish the estimates

$$|P^s_{y_i}(w^1_i - w^2_i)| \le \sigma(\mathrm{s})|z^1_{i-1} - z^2_{i-1}|_*, \tag{6.29}$$

$$\gamma|P^u_{y_i}(w^1_{i-1} - w^2_{i-1})| \le \sigma(\mathrm{s})|z^1_i - z^2_i|_* \tag{6.30}$$

for $i \in J(n]$. Reasoning in the same way as above, we obtain

$$|P^s_{y_i}(w^1_i - w^2_i)| \le \lambda_s|P^s_{y_{i-1}}(z^1_{i-1} - z^2_{i-1})| + \mu_s|P^u_{y_{i-1}}(z^1_{i-1} - z^2_{i-1})| \tag{6.31}$$

and

$$|P^u_{y_{i-1}}(w^1_{i-1} - w^2_{i-1})| \le \frac{\mu_u}{\lambda_u}|P^s_{y_i}(z^1_i - z^2_i)| + \frac{1}{\lambda_u}|P^u_{y_i}(z^1_i - z^2_i)| \tag{6.32}$$

for $i \in J(n]$. To see that (6.29) and (6.30) follow from (6.31) and (6.32) we recall that the norm $\|M(\mathrm{s})\|_*$ is just $\sigma(\mathrm{s})$. □

Lemma 5. *Let the boundary condition (6.6) be restricted. Then for any fixed point* $\mathbf{z} \in S$ *of* H*, the sequence*

$$\mathbf{x} = \mathbf{y} + \mathbf{z} \tag{6.33}$$

is a trajectory of f*.*

Proof: It is sufficient to show for a given $\mathbf{z} \in \mathcal{Z}^n$ that

$$y_i + z_i = f(y_{i-1} + z_{i-1}), \qquad i \in J(n]. \tag{6.34}$$

are equivalent to the following system of pairs of equalities

$$P^s_{y_i} z_i = P^s_{y_i}(f(y_{i-1} + z_{i-1}) - y_i), \tag{6.35}$$

$$P^u_{y_{i-1}} z_{i-1} = Q_{y_{i-1},y_i,z_{i-1}}(P^u_{y_i} y_i - f(y_{i-1} + P^s_{y_{i-1}} z_i) + z_i)) \,. \tag{6.36}$$

Clearly (6.35) is equivalent to

$$P^s_{y_i}(y_i + z_i) = P^s_{y_i} f(y_{i-1} + z_{i-1}). \tag{6.37}$$

On the other hand, applying the (nonlinear) operator $F_{y_{i-1},y_i,z_{i-1}} = Q^{-1}_{y_{i-1},y_i,z_{i-1}}$ to both sides of (6.36), by (6.1) with $x = y_{i-1}$, $y = y_i$ and $z = z_{i-1}$ we get

$$P^u_{y_i}\left(f(y_{i-1} + z_{i-1}) - f(y_{i-1} + P^s_{y_{i-1}} z_{i-1})\right) =$$

$$P^u_{y_i}\left((y_i + z_i - f(y_{i-1} + z_{i-1})) + (f(y_{i-1} + z_{i-1}) - f(y_{i-1} + P^s_{y_{i-1}} z_{i-1}))\right),$$

or $0 = P^u_{y_i}(z_i - f(y_{i-1} + z_{i-1}) + y_i)$. That is, (6.36) is equivalent to

$$P^u_{y_i}(y_i + z_i) = P^u_{y_i} f(y_{i-1} + z_{i-1}). \tag{6.38}$$

Hence the system (6.35)–(6.36) is equivalent to the system (6.37)–(6.38), which in turn is equivalent to (6.34) and the lemma is proved. □

Lemma 6. *Let the boundary condition (6.6) be restricted. Then the set* S *is invariant under* H*.*

Proof: Choose and fix any $\mathbf{z} \in S$. We need to show that $\mathbf{w} = H(\mathbf{z}) \in S$ or, what is the same, to establish the estimates

$$|P^s_{y_i} w_i| \le a\gamma, \qquad |P^u_{y_i} w_i| \le b\gamma, \qquad i \in J[n]. \tag{6.39}$$

First we rewrite (6.7) in the form $P^s_{y_i} w_i = (I_1 + I_2)$, where

$$I_1 = P^s_{y_i}\left(f(y_{i-1}) - y_i\right), \qquad I_2 = P^s_{y_i}\left(f(y_{i-1} + z_{i-1}) - f(y_{i-1})\right) \,. \tag{6.40}$$

and then rewrite (6.8) as $P^u_{y_{i-1}} w_{i-1} = Q_{y_{i-1},y_i,z_{i-1}}(J_1 + J_2 + J_3)$, where J_1, J_2 and J_3 are defined in (6.20).

To estimate $|I_1|$ note that $|y_i - f(y_{i-1})| \leq \gamma$, so by property SH1 we have

$$|I_1| \leq h\gamma. \tag{6.41}$$

Also by (2.4) and (2.4) it follows that

$$|I_2| \leq \lambda_s |P^s_{y_{i-1}} z_{i-1}| + \mu_s |P^u_{y_{i-1}} z_{i-1}|. \tag{6.42}$$

¿From the estimates (6.41)–(6.42) and the definition (6.10) of S we obtain

$$|P^s_{y_i} w_i| \leq \gamma h + \gamma \lambda_s a + \gamma \mu_s b \qquad i \in J(n). \tag{6.43}$$

Analogously, from (6.21)–(6.23), the definition of S and Corollary 2 it follows that

$$|P^u_{y_{i-1}} w_{i-1}| \leq \lambda_u^{-1}(\gamma \mu_u a + \gamma b + \gamma h), \qquad i \in J(n]. \tag{6.44}$$

Inequalities (6.43) and (6.44) together with the boundary conditions (6.5)–(6.6) then lead to the component-wise estimate

$$\left(|P^s_{y_i} w_i|, |P^u_{y_i} w_i|\right)^T \leq \gamma M(\mathrm{s})(a,b)^T + \gamma \mathbf{h} \tag{6.45}$$

for all $i \in J[n]$. On the other hand, in view of (6.9) we have

$$\begin{aligned} \gamma M(\mathrm{s})(a,b)^T + \gamma \mathbf{h} &= \gamma(M(\mathrm{s})(I - M(\mathrm{s}))^{-1} + I)\mathbf{h} \\ &= \gamma(I - M(\mathrm{s}))^{-1}\mathbf{h} = \gamma(a,b)^T. \end{aligned}$$

Hence (6.45) implies that

$$\left(|P^s_{y_i} w_i|, |P^u_{y_i} w_i|\right)^T \leq \gamma(a,b)^T, \qquad \mathbf{z} \in S,$$

which is equivalent to (6.39). □

Lemma 7. *Let the boundary condition (6.6) be restricted. Let $0 \leq m \leq n < \infty$ and let $\mathbf{y}_*$ is the restriction of $\mathbf{y}$ to $J[m]$. Then*

$$\mathbf{w}_* = H_* \mathbf{z}_*$$

holds for each $\mathbf{z} \in \mathcal{Z}^n$, where $\mathbf{w}_*$ and $\mathbf{z}_*$ are the restrictions of $\mathbf{z}$ and $\mathbf{w}$ to $J[m]$ and $H_* = H_{\mathbf{y}_*, z^s_*, z^u_*}$ with

$$z^u_* = \begin{cases} P^u_{y_m} w_n, & \text{if} \quad m < n, \\ z^u, & \text{otherwise.} \end{cases}$$

Proof: The proof follows from the previous lemma and the definitions. □

6.3 Proof of Theorem 2

Let us now finish the proof of the Continuous Shadowing Lemma (Theorem 2). It is known that the unique fixed points of parametrized family of contraction mappings on a common complete metric space depend continuously on the parameter if the mappings do and if they have a common contraction constant. Clearly, the operators H depend continuously on y and $z^u \in B^u_{y_n}(r^u(\mathrm{s},\delta))$ if $n < \infty$ and on y if $n = \infty$. Moreover, these operators are uniformly contracting with the common contraction constant $\sigma(\mathrm{s})$ and each maps the same set S into itself. Hence their unique fixed points $\mathbf{z} = \mathrm{z}(\mathrm{y}, z^u)$ if $n < \infty$ or $\mathbf{z} = \mathrm{z}(\mathrm{y})$ if $n = \infty$ depend continuously on their variables. Define

$$W^n(\mathrm{y}, z^u) = \mathrm{y} + \mathrm{z}(\mathrm{y}, z^u), \quad \text{if} \quad n < \infty,$$

and

$$W^\infty(\mathrm{y}) = \mathrm{y} + \mathrm{z}(\mathrm{y}).$$

It is straightforward to verify that these mappings have the required properties. The theorem is thus proved. □

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