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# Maximal monotonicity and convex programming


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**Abstract**

We introduce an explicit constraint qualification condition which is necessary and sufficient for the nondegenerate Lagrange multipliers rule to hold. We compare it with metric regularity conditions and we show that it is strictly weaker than the Slater assumption. Under certain weak smoothness hypotheses, our condition, the Slater condition and the existence of nondegenerate Lagrange multipliers are equivalent. The basic ingredient in the proof of the main result is the theory of maximal monotone operators (Minty's theorem). Another approach using a direct exact penalization argument yields a modified nondegenerate Lagrange multipliers rule involving the positive part of the constraint mapping.

Examples and applications to abstract optimal control problems are also indicated.

# 1 Introduction

Let $H$ be a Banach space, $g : H \to ]-\infty, +\infty]$ be a convex, lower semicontinuous proper function and $h_i : H \to ]-\infty, +\infty]$, $i = \overline{1,n}$ be convex mappings. Precise hypotheses will be stated in the sequel, for each result. We consider the standard convex programming problem with inequality constraints:

$$Minimize\ \{g(x)\} \tag{1.1}$$

subject to

$$h_i(x) \leq 0, \quad i = \overline{1,n}. \tag{1.2}$$

We define the feasible (convex) set

$$C = \{x \in H;\ h_i(x) \leq 0, \quad i = \overline{1,n}\} \tag{1.3}$$

assumed to be nonvoid (admissibility) and we denote by $\bar{x} \in C$ a solution to (1.1), (1.2), supposed to exist. We also introduce the convex mapping $h : H \to ]-\infty, +\infty]$, $h(x) = \max\ \{h_i(x);\ i = \overline{1,n}\}$ and the problem (1.1), (1.2) may be equivalently reformulated as (1.1) and

$$h(x) \leq 0. \tag{1.4}$$

This work discusses the classical question of the caracterization of $\bar{x}$ via the Karush-Kuhn-Tucker [10, 9] conditions. We introduce a new explicit constraint qualification which may be mainly compared with the metric regularity condition, Robinson [12], Jourani and Thibault [8] and we show that it is necessary and sufficient for the nondegenerate Lagrange multipliers rule (involving g essentially) to hold. In this sense, the condition that we introduce here is the weakest possible one. Let us also notice that by allowing g to be only lower semicontinuous, we can include in the problem (1.1), (1.2) other types of constraints, expressed in the abstract form

$$x \in A, \tag{1.5}$$

where $A \subset H$ is a closed convex nonvoid subset. This can be simply done by adding to g the indicator function $I_A$ of $A$ in $H$. The general form of our hypothesis is:

$$\begin{aligned} &\forall M \subset H \textit{ bounded}, \ \exists c_M > 0 : \\ &h(x) \geq c_M \textit{ dist } (x, C), \forall x \in M \setminus C. \end{aligned} \tag{1.6}$$

The relation makes sense only for sets $M$ such that $M \setminus C \neq \phi$, otherwise it has no object. If $C$ is bounded, then $c_M$ may be chosen independent of $M$. Comparing (1.6) to the general condition (0) of Jourani and Thibault [8] or with conditions (4), (5) of Robinson [12], we see that (1.6) is imposed exclusively on the explicit constraints (1.2) or (1.4) and does not involve $A$ (or $g$). Moreover, if some $\varepsilon > 0$ is added to any of the $h_i$, then the corresponding set $C_\varepsilon$ may be empty. That is, condition (1.6) is not related with the stability with respect to perturbations of the feasible set, as it is generally the case with metric regularity assumptions. Finally, under very weak differentiability suppositions, we also show that the hypothesis (1.6) is equivalent with the classical Slater [14] condition:

$$\exists x_0 \in C : h_i(x_0) < 0, \quad i = \overline{1, n} \tag{1.7}$$

and it follows, in this setting, that (1.7) is as well equivalent with the existence of nondegenerate Lagrange multipliers. In the nonsmooth case, we prove that (1.6) is strictly weaker than (1.7) and this expresses a general geometric property of convex functions. Our approach is based on the theory of maximal monotone operators (the Minty's theorem in Banach spaces), Brezis [5], Barbu [2]. In Section 2, we prove a formula giving the normal cone to the level set $C$ in terms of the original function $h$. The basic character of such a formula is wellknown, Lemarechal and Hiriart-Urruty [7]. In the book of Clarke [6], Theorem 2.4.7 proves a similar result under differentiability assumptions. The nondegenerate Lagrange multipliers rule, for general lower semicontinuous objective mappings $g$, is then an immediate consequence via subdifferential calculus and the Dubovitskij-Miljutin theorem, Tikhomirov [16]. In Section 3, we prove that the hypothesis (1.6) is as well necessary, under general hypotheses. Section 4 discusses the same problem in Banach spaces via an approximating projection idea. We prove a modified nondegenerate Lagrange multipliers rule, involving $h_+$, for $g$ continuous and $h$ lower semicontinuous. The last section is devoted to variants of the condition (1.6) and to an application to optimal control problems.

Throughout the text, we denote by $|\cdot|_H$ the norm in the Banach space $H$, by $S(z,r)$ the closed ball centered at $z \in H$ and with radius $r > 0$ and by $(\cdot,\cdot)_{H \times H^*}$ or $(\cdot,\cdot)_H$ the duality pairing between $H$ and its dual space $H^*$ or the scalar product in the Hilbert space $H$.

# 2 Maximal monotonicity

In this section, we assume that $H$ is a reflexive Banach space and, eventually by renorming, that $H$ and $H^*$ are strictly convex. The mapping $h$ is convex, proper, lower semicontinuous, $0 \in \partial h(0)$ and satisfies (1.6).

Let $F : H \to H^*$ be the duality mapping and $x_\lambda$ be the solution of

$$F(x_\lambda - y) + \lambda \partial h(x_\lambda) \ni 0 \tag{2.1}$$

where $\lambda \geq 0$ and $y \in H$ are arbitrarily fixed.

**Proposition 2.1** *If $\lambda$ is "big enough", then $x_\lambda \in C$.*

**Proof** We multiply (2.1) by $x_\lambda$ and, by $0 \in \partial h(0)$, we have

$$(x_\lambda - y, F(x_\lambda - y))_{H \times H^*} \leq -(y, F(x_\lambda - y))_{H \times H^*}, \tag{2.2}$$

that is $|x_\lambda|_H \leq 2|y|_H$.

We denote by $c_y > 0$ the constant associated by hypothesis (1.6) to the set $M = S(0, 2|y|_H)$ in $H$. If $x_\lambda \notin C$, we get

$$\begin{aligned} c_y|x_\lambda - proj_C(x_\lambda)|_H \leq h(x_\lambda) \leq h(x_\lambda)- \\ -h(proj_C(x_\lambda)) \leq (\partial h(x_\lambda), x_\lambda - proj_C(x_\lambda))_{H^* \times H} \end{aligned} \tag{2.3}$$

Here $proj_C(x_\lambda)$ is the projection of $x_\lambda$ on the closed convex set $C$ which exists and is unique under our assumptions.

It is known that $\partial h_\lambda(y) \in \partial h(x_\lambda)$, Barbu and Precupanu [3, Ch. II], where $h_\lambda(y) = \lambda^{-1}F(x_\lambda - y)$ is the Yosida approximation of $h$. Then, (2.3) gives

$$0 < c_y \leq \lambda^{-1}|x_\lambda - y|_H \leq 3\lambda^{-1}|y|_H.$$

This is a contradiction for $\lambda$ "big enough" since $c_y$ is independent of $\lambda$.

□

**Theorem 2.1** *If $h$ is continuous and the hypotheses of Proposition 2.1 are fulfilled, then the multivalued operator $N \subset H \times H^*$ given by the cone:*

$$N(x) = \begin{cases} 0 & if\ h(x) < 0, \\ \lambda w,\ \lambda \geq 0,\ w \in \partial h(x), & if\ h(x) = 0, \\ \phi & if\ h(x) > 0, \end{cases} \tag{2.4}$$

*is maximal monotone and $N(x) = \partial I_C(x)$ the normal cone to $C$ in $x$.*

**Proof** By the definition of the subdifferential, it is clear that $N \subset \partial I_C$ and, therefore, that it is monotone. We show that $N$ is maximal monotone in $H \times H^*$ and this will give the desired equality. Assume first that $h \geq 0$ in $H$. Then the first line in (2.4) disappears. Let $\bar{\lambda} > 0$ be such that $x_{\bar{\lambda}} \in C$, according to Proposition 2.1. Equation (2.1) may be, by (2.4), rewritten as

$$F(x_{\bar{\lambda}} - y) + N(x_{\bar{\lambda}}) \ni 0.$$

The converse part in Minty theorem shows that $N$ is maximal monotone and $N = \partial I_C$ follows. If $h$ has negative values in $H$, we consider $h_+ = \max\{0, h\}$ and we denote by $N_+ \subset H \times H^*$ the cone associated to $h_+$ via (2.4). Then, the first part of the proof gives that $N_+ = \partial I_C$. But, the Dubovitskij-Miljutin theorem, Tikhomirov [16], shows that $N = N_+$ if $h$ is continuous and the proof is finished.

□

**Corollary 2.1** *If $h$ is convex, proper, lower semicontinuous with $h \geq 0$ in $H$ and $0 \in \partial h(0)$, then $N = \partial I_C$.*

**Remark** The inclusion $N \subset N_C$ is wellknown and the equality $N = N_C$ is an abstract regularity condition, necessary and sufficient for the nondegenerate Lagrange multipliers rule to hold, Lemarechal and Hiriart-Urruty [7, Ch. VII.2]. Another "basic constraint qualification" may be formulated via tangent cones as well, Rockafellar [13]. In the book by Clarke [6], the regularity is defined as the equality between tangent and contingent cones and Theorem 2.4.7 and its corollaries give a description of the same type of $\partial I_C$, under differentiability assumptions.

**Corollary 2.2** *Let $g : H \to ]-\infty, +\infty]$ be convex lower semicontinuous proper, continuous in some point of $C$ and let $\bar{x}$ be a solution of (1.1), (1.2). If $h$ satisfies the assumptions of Theorem 2.1, then there are $\lambda_i \geq 0$ such that*

$$0 \in \partial g(\bar{x}) + \sum_{i=1}^{n} \lambda_i \partial h_i(\bar{x}),$$
$$\lambda_i h_i(\bar{x}) = 0, \ i = \overline{1, n}.$$

**Proof** Since $g$ is continuous in a point of $C$, we have the additivity rule $0 \in \partial g(\bar{x}) + \partial I_C(\bar{x})$. The proof is finished by Theorem 2.1 and the Dubovitskij-Miljutin theorem applied to $h$.

□

**Remark** The hypothesis $0 \in \partial h(0)$ has a technical character and it is not restrictive. If $h \geq 0$ in $H$, then any point in $C$ is a minimum point for $h$ and a simple shifting gives $0 \in \partial h(0)$. If $h$ has negative values as well then the Slater condition is fulfilled and Corollary 2.2 is wellknown.

# 3 Necessity

In this section, we denote by $(P_g)$ the problem (1.1), (1.2) associated to the performance index $g$ and by $x_g$ a (local) minimum of $(Pg)$ whose existence is supposed. The hypotheses on $g$ are as in Section 1, but now $h : H \to R$ is not necessarily convex and that is why $x_g$ may be only a local minimum.

We assume that $h : H \to R$ is continuous with bilateral derivative in every point of the Banach space $H$. This is a weaker supposition than the Gâteaux differentiability and if $h'(x, d)$ is the directional derivative of $h$ at $x$ in the direction $d$ (assumed to exist), we have

$$h'(x,d) = -h'(x,-d). \tag{3.1}$$

The set $C$ is closed, $C \neq \phi$, $C \neq H$, but it is not necessarily convex.

**Theorem 3.1** *If for any $g : H \rightarrow ]-\infty, +\infty]$ convex lower semicontinuous proper, there is $\mu = \mu(g) \geq 0$ such that $x_g$ is a (local) minimum on $H$ for:*

$$L_{\mu(g)} = g + \mu(g)h \tag{3.2}$$

*then $h$ has strictly negative values on $C$, that is the Slater condition is satisfied.*

**Proof** Assume that $h(x) = 0$, $\forall x \in C$. Then, any $x \in C$ is a global minimum of $h$ on $H$ and the bilateral differentiability gives

$$h'(x,d) = 0, \ \forall x \in C, \ \forall d \in H. \tag{3.3}$$

Let $\bar{x} \in \partial C$ and $\hat{x} \in H \setminus C \neq \phi$, such that the open-closed segment $]\bar{x}\hat{x}] \subset H \setminus C$, that is the closed segment satisfies $[\bar{x}\hat{x}] \cap C = \{\bar{x}\}$,. The existence of $\bar{x}$ with these properties follows by solving the problem:

$$\sup \{|\tilde{x} - x|_H; \ x \in C \cap [\tilde{x}\hat{x}]\}, \tag{3.4}$$

where $\tilde{x} \in C$ is given. The solution $\bar{x}$ of (3.4) exists by the Weierstrass theorem and $\bar{x} \neq \hat{x}$ since $\hat{x} \in H \setminus C$ open.

We define now $g : H \rightarrow ]-\infty, +\infty]$ convex, lower semicontinuous proper

$$g(x) = \begin{cases} \lambda - 1, & x = \lambda\bar{x} + (1-\lambda)\hat{x}, \quad \lambda \in [0,1], \\ +\infty & \text{otherwise.} \end{cases}$$

We have dom $g \cap C = \{\bar{x}\}$, $g(\bar{x}) = 0$ and $\bar{x}$ is the solution of $(Pg)$. However, for any $\mu \geq 0$, the nondegenerate Lagrange function $L\mu = g + \mu h$ satisfies (by (3.3)):

$$L'_\mu(\bar{x}, \hat{x} - \bar{x}) = g'(\bar{x}, \hat{x} - \bar{x}) + \mu h'(\bar{x}, \hat{x} - \bar{x}) = -1.$$

Clearly $L_\mu(\bar{x}) = 0$ and $L_\mu(x) < 0$ if $x \in [\bar{x}\hat{x}] \cap S(\bar{x}, \varepsilon_\mu)$, $\varepsilon_\mu > 0$ small. Then $\bar{x}$ cannot be a minimum for $L_\mu$, any $\mu \geq 0$. This provides the contradiction which ends the proof.

$\square$

**Remark** In the next section, we show that the Slater condition is stronger then our hypothesis (1.6). Then, Theorem 3.1 and Theorem 2.1 prove that (1.6) is necessary and sufficient for the nondegenerate Lagrange multipliers rule to hold, under bilateral differentiability conditions on $h$. And the same is valid for the Slater condition.

In the sequel, we indicate another case when (1.6) is necessary, without differentiability assumptions. We suppose that $H$ is a finite dimensional space and that $h : H \rightarrow R$ is convex, continuous. The directional derivative $h'(x,d)$ exists for any $x, d \in H$ and we assume a partial continuity property for it:

$$h'(y_n, v_n) \to h'(y, v) \tag{3.5}$$

if $y_n \to y,\ v_n \to v,\ y_n + \lambda v_n \in \overline{H \setminus C},\ \forall \lambda \geq 0.$

The set $C$ is convex closed, $C \neq \phi,\ C \neq H$ and $x_g$ denotes the global solution of $(Pg)$.

**Theorem 3.2** *If for any $g : H \to R$ convex continuous, there is $\mu = \mu(g) \geq 0$ such that $x_g$ is a minimum point for $L_{\mu(g)}$ on $H$, then $h$ satisfies (1.6).*

**Proof** Assume that $h \geq 0$ on $H$ and that $h$ does not satisfy (1.6), that is we can find $M \subset H$ bounded such that

$$\forall c > 0,\ \exists x_c \in M \setminus C : 0 < h(x_c) < c\ dist\ (x_c, C). \tag{3.6}$$

Clearly $\{x_c\}$ and $\{proj_C x_c\}$ are bounded in $H$ and we can take subsequences such that $x_c \to \bar{x},\ proj_C x_c \to \bar{x}$ and $h(\bar{x}) = 0$ by (3.6). Then $\bar{x} \in C$ and $\bar{x} \neq x_c,\ \forall c > 0$.

We can also write, as $h(x) = 0,\ \forall x \in C$, that:

$$0 < h(x_c) = h(x_c) - h(proj_C x_c) < c|x_c - proj_C x_c|_H.$$

If $\lambda \in [0,1]$, the convexity of $h$ gives:

$$\begin{aligned} 0 &\leq h(proj_C x_c + \lambda(x_c - \ proj_C x_c)) - h(proj_C x_c) \leq \\ &\leq \lambda[h(proj_C x_c + x_c - \ proj_C x_c) - h(proj_C x_c)] \leq \\ &\leq \lambda c|x_c - proj_C x_c|_H. \end{aligned}$$

Denote $d_c = (x_c - proj_C x_c)|x_c - proj_C x_c|_H^{-1}$. Then $|d_c|_H = 1$ and $h'(proj_C x_c, d_c) \leq c$. Let $d_c \to d$ on a subsequence. Then $|d|_H = 1$ since $H$ is finite dimensional. By the characterization of the projection we have $x_c - proj_C x_c \in \partial I_C(proj_C x_c)$, that is $d_c \in \partial I_C(proj_C x_c)$ and, consequently, $d \in \partial I_C(\bar{x})$. We can use hypothesis (3.5) to pass to limit and to get $h'(\bar{x}, d) \leq 0$. Since $\bar{x} \in C$ is as well a minimum point of $h$ on $H$, then $h'(\bar{x}, d) = 0$.

The contradiction follows by considering the mapping $g(x) = (d, \bar{x} - x)_H$ and arguing as in the previous proof.

If $h$ has as well negative values, then the Slater condition is fulfilled and the conclusion follows by Theorem 4.1.

□

**Remark** If $f : H \to R$ is convex continuous and Gâteaux differentiable and $h(x) = f(x)$ on $D = \{x \in H; f(x) \geq 0\}$, then $h'(y,v) = f'(y,v)$ if $y + \lambda v \in D$, for $\lambda \geq 0$, and the continuity property (3.5) follows. Even in this situation, the Slater condition is not necessarily valid (for instance, for $H = R$, take $f(x) = x$ and $h(x) = x_+$).

**Remark** We use that $H$ is finite dimensional only to obtain $|d|_H = 1$. The other arguments remain valid for $dim\ H = +\infty$.

# 4 Slater condition and exact penalization

In this section we compare (1.6) and the classical Slater condition via a general geometric property of convex functions and we give a modified nondegenerate Lagrange rule, under very weak hypotheses and using direct exact penalization arguments.

**Theorem 4.1** *Let $H$ be a Banach space and $h : H \to ]-\infty, +\infty]$ be convex lower semicontinuous proper such that (Slater):*

$$\exists \hat{x} \in C : h(\hat{x}) < 0. \tag{4.1}$$

*Then, $\forall r > 0$, $\forall x \in S(\hat{x}, r) \setminus C$, we have*

$$h(x) \geq -\frac{h(\hat{x})}{r} \; dist\ (x, C). \tag{4.2}$$

(If (4.1) is not valid, then (4.2) is satisfied in the trivial form $h(x) \geq 0$ in $H$.)

**Proof** Take $a = -h(\hat{x}) > 0$ and denote by $S$ the line passing through $\hat{x}$ and $x \in H$. If $h(x) = +\infty$, then (4.2) is clear. Therefore, we may assume that $h$ is finite on the closed segment $[\hat{x}x] \subset S$ since it is convex proper. Then $h|_S$ is continuous on the open segment $]\hat{x}x[ \subset S$. As $x \in H \setminus C$ and $C$ is closed, there is $\varepsilon > 0$ such that $h > 0$ on $S(x, \varepsilon) \cap [\hat{x}x]$, due to (1.3). Since $h$ is convex and $h(\hat{x}) < 0$, Proposition 3.1.2, Lemarechal and Hiriart-Urruty [7, Ch. I] gives that

$$\lim_{y \to \hat{x}+} h(y)|_S \leq h(\hat{x}) < 0.$$

Suppose that a unit vector $u$, $|u|_H = 1$ is chosen parallel to $S$ and a parametrization of $S$ with respect to $u$ and some origin is given. The above discussion shows the existence of $\tilde{x} \in ]\hat{x}x[$ such that $h(\tilde{x}) = 0$, $\tilde{x} \in C$. Let $\hat{\lambda}, \tilde{\lambda}, \lambda$ be the "coordinates" of $\hat{x}, \tilde{x}, x$ on $S$, respectively and put $\hat{\lambda} < \tilde{\lambda} < \lambda$ to fix the ideas. Take $y \in S$, with the representation $y = \mu u$, $\mu \in R$ and define the mapping

$$f(y) = \frac{a}{r}(\mu - \tilde{\lambda}) \tag{4.3}$$

which is affine on $S$. We notice, by (4.3), that

$$f(\tilde{x}) \;=\; 0 \;=\; h(\tilde{x}), \tag{4.4}$$

$$f(\hat{x}) \;=\; -\frac{h(\hat{x})}{r}\,(\hat{\lambda} - \tilde{\lambda}) \;=\; h(\hat{x})\frac{\tilde{\lambda} - \hat{\lambda}}{r} \geq h(\hat{x}) \tag{4.5}$$

since $h(\hat{x}) < 0$ and $0 \leq \dfrac{\tilde{\lambda} - \hat{\lambda}}{r} = \dfrac{|\tilde{x} - \hat{x}|_H}{r} \leq 1$.

Due to the convexity of $h|_S$ and the affine character of $f$, (4.4), and (4.5) yield

$$h(x) \geq f(x) = \frac{a}{r}(\lambda - \tilde{\lambda}) = \frac{a}{r}|x - \tilde{x}|_H \geq -\frac{h(\hat{x})}{r} \; dist\ (x, C)$$

since $\tilde{x} \in C$. This ends the proof.

□

**Remark** *A* special case of this result (bounded level sets) was studied by Azé and Rahmouni [1], while in the work of Lemaire [11], Lemma 4.1 gives a similar statement with different constants and another proof. Several properties of this type and more references are collected in the forthcoming book by Zălinescu [17]. If $h$ is affine, then (4.2) becomes an equality for certain $x$. This show that (4.2) has optimal constants.

**Remark** If $h$ satisfies (4.1), then $h_+$ will satisfy as well (4.2), but not (4.1). This example shows that the Slater condition is strictly stronger than (1.6).

**Theorem 4.2** *Let $h : H \to ]-\infty, +\infty]$ be convex lower semicontinuous proper and $g : H \to R$ be convex continuous. Then, if $\bar{x}$ is a solution to (1.1), (1.4) and (1.6) is fulfilled, there is $\lambda \geq 0$ such that $\bar{x}$ is a minimum point of $g + \lambda h_+$ over $H$.*

**Proof** Let $S(\bar{x}, 3\varepsilon)$ be a fixed "small" ball around $\bar{x}$. We show the minimum property of $g + \lambda h_+$ (for some $\lambda \geq 0$) on $S(\bar{x}, \varepsilon)$ and it will follow on $H$, by convexity.

For any $y \in S(\bar{x}, \varepsilon)$ and any $\delta > 0$, there is $y_\delta \in C$ such that (approximate projection):

$$|y - y_\delta|_H \leq \; dist\ (y, C) + \delta.$$

If $\delta < \varepsilon$, we have $|y_\delta - \bar{x}|_H \leq |y - \bar{x}|_H + |y - y_\delta| < 3\varepsilon$ and $y_\delta \in S(\bar{x}, 3\varepsilon)$. Denote by $m$ the Lipschitz constant of the continuous convex mapping $g$ on $S(\bar{x}, 3\varepsilon)$. We have

$$g(y) - g(y_\delta) \geq -m|y - y_\delta|_H \geq -m\ dist\ (y, C) - m\delta.$$

Then

$$g(y) + m\ dist\ (y, C) + m\delta \geq g(y_\delta) \geq g(\bar{x}). \tag{4.6}$$

By (1.6), we obtain

$$h_+(x) \geq c_\varepsilon\ dist\ (x, C),\ \forall x \in S(\bar{x}, \varepsilon) \setminus C \tag{4.7}$$

where $c_\varepsilon > 0$ is the constant associated to the bounded set $S(\bar{x}, \varepsilon)$ by (1.6).

Let $\delta \to 0$ in (4.6) and combine with (4.7) to get

$$\begin{aligned} g(y) + \frac{m}{c_\varepsilon} h_+(y) &\geq g(y) + m\ dist\ (y, C) \geq \\ &\geq g(\bar{x}) = g(\bar{x}) + \frac{m}{c_\varepsilon} h_+(\bar{x}),\ y \in S(\bar{x}, \varepsilon) \setminus C \end{aligned} \tag{4.8}$$

Relation (4.8) remains valid for $y \in S(\bar{x}, \varepsilon) \cap C$ since $h_+|_C = 0$ and this finishes the proof with $\lambda = \frac{m}{c_\varepsilon} \geq 0$.

□

**Remark** The above proof is based on direct exact penalization arguments as in Lemarechal and Hiriart-Urruty [7, Ch. VII 1.2] or in Bonnans [4], but avoids the use of the projection operator which allows us to work in general Banach spaces. We remark that only $g$ locally Lipschitz is sufficient in this setting, therefore a nonconvex variant of Theorem 4.2 is also valid (for local minimum points).

**Remark** Although $g$ is continuous, it is possible to include in (1.1), (1.4) abstract constraints (1.5). This can be done by adding $I_A$ directly to $h$ instead of $g$ as usual. Therefore Theorem 4.2 addresses to general mathematical programming problems as well.

**Remark** As in Corollary 2.2 we can reobtain the classical nondegenerate Lagrange multipliers rule via subdifferential calculus, from Theorem 4.2. However, it is necessary to impose $h_i$ continuous in order to apply the Dubovitskij-Miljutin theorem and this seems no more to allow abstract constraints (1.5) to be included in the problem (1.1), (1.2).

**Example** As a byproduct of Theorem 4.1, we put into evidence a class of functionals on $H$ which have the exact penalization property for convex closed sets $C \subset H$, such that $0 \in C$. We denote by $p_C : H \to ]-\infty, +\infty]$ the Minkowsky (gauge) functional associated with $C$ and we define for $\varepsilon > 0$:

$$h^{\varepsilon}(x) = \varepsilon(p_C(x) - 1)_+. \tag{4.9}$$

The following properties are obvious:

$$C = \{x \in H;\ h^{\varepsilon}(x) = 0\},\ \forall \varepsilon > 0,$$

$$h^{\varepsilon}(x) \to +\infty\ for\ \varepsilon \to \infty,\ \forall x \in H \setminus C,$$

$$h^{\varepsilon}(x) \to 0\ for\ \varepsilon \to 0,\ \forall x \in\ aff\ (C).$$

Moreover, if $\tilde{h}^{\varepsilon}(x) = \varepsilon(p_C(x) - 1)$, then it satisfies the Slater condition $\tilde{h}^{\varepsilon}(0) = -\varepsilon$, $\forall \varepsilon > 0$, and Theorem 4.1 shows that $h^{\varepsilon}$ satisfies (1.6). If $g$ is a continuous convex mapping with a minimum at $\bar{x} \in C$, Then (by the proof of Theorem 4.2) there is $\varepsilon > 0$ such that $\bar{x}$ is a minimum point of $g + h^{\varepsilon}$ on $H$.

It is wellknown that the distance function, *dist* $(x, C)$ has the exact penalization property, Lemarechal and Hiriart-Urruty [7, Ch. VII 1.2], while (4.9) is an example of a different nature.

# 5 Examples and applications

We consider first a sufficient condition for hypotheses (1.6) to hold. We assume that $H$ is a Hilbert space identified with its dual and $h : H \to ]-\infty, +\infty]$ is convex lower semicontinuous proper.

**Proposition 5.1** *Suppose that for any $M \subset H$ bounded there is $c_M > 0$ such that for any $\bar{x} \in M$ point of support for $C$ and for any normal vector $d \in \partial I_C(\bar{x})$, $|d|_H = 1$ we have*

$$h'(\bar{x}, d) \geq c_M. \tag{5.1}$$

*Then, condition (1.6) is fulfilled.*

**Proof** Let $\bar{x}$, point of support of $C$ be fixed and $M = S(\bar{x}, 1)$. We denote by $Z_{\bar{x}}$ the set

$$Z_{\bar{x}} = \{x \in H; x - y \in \partial I_C(y) \text{ for some } y \in M, y \text{ point of support of } C\}.$$

Since $h'$ is positively homogeneous, (5.1) yields

$$h'(y, x-y) \geq c_M |x - y|_H, \quad \forall x \in Z_{\bar{x}} \tag{5.2}$$

and $y \in M, y$ point of support for $C$. As $h$ is convex and $h(y) \leq 0$, we get

$$h(x) \geq c_M |x - y|_H \geq c_M \textit{ dist } (x, C), \quad \forall x \in Z_{\bar{x}}. \tag{5.3}$$

Here, we also use that if $h(y) < 0$, we get $h(x) = +\infty$ for $x \in Z_{\bar{x}}$ since $y$ is point of support of $C$. Finally, we show that $S(\bar{x}, 1)\setminus C \subset Z_{\bar{x}}$. We take any $z \in S(\bar{x}, 1)\setminus C$ and we have:

$$(z - proj_C z,\ proj_C z - v)_H \geq 0, \quad \forall v \in C \tag{5.4}$$

by the definition of the projection. Moreover, $proj_C z \in C \cap S(\bar{x}, 1)$ since the projection is nonexpansive and $proj_C \bar{x} = \bar{x}$. Since $proj_C z$ is a point of support for $C$, then (5.4) shows that $z - proj_C z \in \partial I_C(proj_C z)$, i.e. $z \in Z_{\bar{x}}$.

This argument shows that (5.3) yields (1.6) and the proof is finished. $\square$

**Remark** If $C$ is a smooth convex domain in $R^n$ (not necessarily bounded) and $\frac{\partial h}{\partial n}$, the outward normal derivative of $h$ to $\partial C$ (assumed to exist), is continuous and strictly positive, then (5.1) follows.

**Example** We show here that if (5.1) or (1.6) are not satisfied, then the nondegenerate Lagrange multipliers rule may be not valid.

Let $\quad x = (x_1, \ldots, x_n) \in R^n = H \quad$ and

$$h(x) = \sum_{i=1}^{n} h^i(x)$$

with $\quad h^i(x) = |x_i|, \quad i = \overline{2, n} \quad$ and

$$h^1(x_1) = \begin{cases} 0, & x_1 \in [0, +\infty[, \\ x_1^2, & x_1 \leq 0. \end{cases}$$

Then, $C = \{x \in R^n;\ x_1 \geq 0,\ x_i = 0,\ i = \overline{2, n}\}$ and conditions (5.1) or (1.6) are not fulfilled since $\frac{\partial h}{\partial x_1}(0) = 0$.

Let us take the objective function $g(x) = x_1$. The solution to the corresponding problem (1.1), (1.4) is $\bar{x} = 0 \in C$, clearly. For any $\lambda \geq 0$, we define the nondegenerate Lagrange function $g_\lambda(x) = g(x) + \lambda h(x)$. For $\lambda = 0, g_\lambda$ daes not attain its infimum on $R^n$ and for $\lambda > 0$, the global minimum point of $g_\lambda$ is $x_\lambda = (-\frac{1}{2\lambda}, 0 \ldots, 0)$ and $x_\lambda \neq \bar{x}$, $\forall \lambda > 0$.

**Example** In the end of this section, we briefly comment on the abstract control problem:

$$Min\ L(y,u), \tag{5.5}$$

subject to

$$Ay = Bu + f, \tag{5.6}$$

$$h(y,u) \leq 0. \tag{5.7}$$

Here $U$ and $V \subset H \subset V^*$ are Hilbert spaces, $L : V \times U \to R$, $h : V \times U \to R$ are convex continuous mappings, $A : V \to V^*$, $B : U \to V^*$ are linear bounded operators and $f \in V^*$.

The set $C = \{[y,u] \in V \times U;\ h(y,u) \leq 0\}$ is convex closed and we assume that there is an admissible pair $[\bar{y}, \bar{u}]$ such that $[\bar{y}, \bar{u}] \in\ int\, C$ in the topology of $V \times U$. Typical situations of problems (5.5) - (5.7) are obtained when $V, V^*$ are Sobolev spaces, $H, U$ are Lebesgue spaces and $A$ is an (elliptic) partial differential operator, while $B$ is some distributed or boundary control action.

If $(Ay, y)_{V^* x V} \geq w|y|_V^2$, $w > 0$, then the equation (5.6) has a unique solution for any $u \in U$. By shifting the domains of $L, h$ and redenoting the obtained mappings again by $L, h$, we may assume $f = 0$. We also notice the generality of the mixed constraint (5.7) which includes both state and control constraints.

We shall apply the results from Section 2. We consider the closed subspace $K = \{(y,u) \in V \times U;\ Ay = Bu\}$ and we replace $L$ by $L + I_K$ in (5.5). If $h$ satisfies (5.1) or (1.6) and $[y^*, u^*]$ is an optimal pair for (5.5) - (5.7), then Section 2 shows that there is $\lambda \geq 0$ such that

$$0 \in \partial L(y^*, u^*) + \partial I_K(y^*, u^*) + \lambda \partial h(y^*, u^*) \tag{5.8}$$

and $\lambda h(y^*, u^*) = 0$. Here, we also use that $int\ C \cap K \neq \phi$ in order to apply the additivity rule for the subdifferential. It is known that $\partial I_K(y^*, y^*) = K^\perp$ and a simple calculus shows that

$$K^\perp = \{[A^*p, -B^*p];\ p \in V\} \subset V^* \times U. \tag{5.9}$$

By (5.8), (5.9) we infer the optimality conditions for the problem (5.5) -(5.7):

$$\begin{aligned}
&-A^*p^* \in \partial_1 L(y^*, u^*) + \lambda \partial_1 h(y^*, u^*),\\
&B^*p^* \in \partial_2 L(y^*, u^*) + \lambda \partial_2 h(y^*, u^*),\\
&\lambda h(y^*, u^*) = 0, \quad \lambda \geq 0
\end{aligned}$$

where $\partial_i L, \partial_i h$, $i = 1,2$ denote the $i$-th component of the ordered pairs $\partial L, \partial h$ and not a partial subdifferential.

In the recent work of Tiba and Bergounioux [15], the interiority assumption on $C$ is removed, but the optimality system has a weaker form.

# References


[1] D. Azé, A. Rahmouni: *On primal-dual stability in convex optimization*, Preprint, University of Perpignan (1993).

[2] V Barbu: *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff, Leyden (1976 ).

[3] V. Barbu, Th. Precupanu: *Convexity and optimization in Banach spaces*, Noordhoff, Leyden (1978).

[4] J. F. Bonnans: *Théorie de la penalization exacte*, $M^2AN$, vol. 24, nr. 2 (1990), pp. 197–210.

[5] H. Brezis: *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North Holland, Amsterdam (1973).

[6] F. H. Clarke: *Optimization and nonsmooth analysis*, Wiley, New York (1983).

[7] J.-B. Hiriart-Urruty, Cl. Lemarechal: *Convex analysis and minimization algorithms*, Springer Verlag, Berlin (1993).

[8] A. Jourani, L. Thibault: *Approximation and metric regularity in mathematical programming in Banach spaces*, Math. Oper. Res., vol. 18, nr. 2 (1993), pp. 390–301.

[9] W. Karush: *Minima of functions of several variables with inequalities as side conditions*, Master's Thesis, Dept. of Mathematics, Chicago University (1939).

[10] H.W. Kuhn, A. W. Tucker: *Nonlinear programming*, in J. Neuman (ed.), Proseedings of second Berkeley symposium on mathematical statistics and probability Berkeley; University of California (1950).

[11] B. Lemaire: *Bounded diagonally stationary sequences in convex optimization*, J. Convex Analysis, vol. 1, nr. 1 (1994), pp. 75–86.

[12] St. M. Robinson: *Regularity and stability for convex multivalued functions*, Math. Oper. Res., vol. 1, nr. 2 (1976), pp. 130–143.

[13] R. T. Rockafellar: *Lagrange multipliers in optimization*, SIAM-AMS Proceedings 9 (1976).

[14] M. Slater: *Lagrange multipliers revisited*, Cowles comission discussion paper no. 403 (1950).

[15] D. Tiba, M. Bergounioux: *General optimality conditions for constrained convex control problems*, SIAM J. Control Optimiz, vol. 34, nr. 2 (1996), pp. 698–711.

[16] V. M. Tikhomirov: *Convex analysis*, Encyclopaedia of mathematical sciences vol. 14, Springer Verlag, Berlin (1990).

[17] C. Zălinescu: Book in preparation.

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