A duality approach in the optimization of beams and plates

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Abstract

We introduce a class of nonlinear transformations called "resizing rules" which associate to optimal shape design problems certain equivalent distributed control problems, while preserving the state of the system. This puts into evidence the duality principle that the class of system states that can be achieved, under a prescribed force, via modifications of the structure (shape) of the system can be as well obtained via the modifications of the force action, under a prescribed structure.

We apply such transformations to the optimization of beams and plates and, in the simply supported or in the cantilevered cases, the obtained control problems are even convex. In all cases, we establish existence theorems for optimal pairs, by assuming only boundedness conditions. Moreover, in the simply supported case, we also prove the uniqueness of the global minimizer. A general algorithm that iterates between the original problem and the transformed one is introduced and studied. The applications also include the case of variational inequalities.

1 Introduction

It is our aim to study a class of control into the coefficients problems. The state equation has the form

$$\Delta(b\,u^3\,\Delta y) = f \quad \text{in } \Omega\,, \tag{1.1}$$

where Ω is a smooth bounded domain in R^n, $n \geq 1$, $f \in L^2(\Omega)$, $u \in L^\infty(\Omega)$, $b > 0$ is a constant. If $n \leq 2$, such models are used in the literature for the deflection y of plates or beams of thickness $u > 0$ a.e. in Ω, and subject to the transverse load f. The coefficient b is a material constant, and we shall fix $b = 1$ in the sequel. We quote Hlavacek, Bock, Lovisek [9], [10], Haslinger and Neittaanmäki [8], Casas [2], Neto and Polak [17], or Langenbach [14], for related beams or plate equations.

To (1.1) we add various boundary conditions:

$$y = \Delta y = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

(simply supported plates),

$$y = \frac{\partial y}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{1.3}$$

(clamped plates; $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative to $\partial\Omega$).

In space dimension one, cantilevered beams or unilaterally supported beams (variational inequalities) will be discussed as well.

We associate to (1.1) various optimization problems:

$$\text{Min} \int_\Omega u(x)\,dx \tag{1.4}$$

(minimization of the weight or volume),

$$\text{Min} \int_\Omega \Big(y(x) - y_d(x) \Big)^2 dx \tag{1.5}$$

(identification-type problems: the function $y_d \in L^2(\Omega)$ is a "desired" or "observed" deflection).

Moreover, natural control and state constraints will be imposed on u, y:

$$0 \le m \le u(x) \le M \quad \text{a.e. in } \Omega\,, \tag{1.6}$$

$$y(x) \ge -\tau \quad \text{a.e. in } \Omega\,, \tag{1.7}$$

(m, M, τ are positive constants),

$$y \in A\,. \tag{1.8}$$

$A \subset L^2(\Omega)$ is a prescribed closed subset, not necessarily convex.

Problems of this type are well-known in the literature and their difficulty, both from a theoretical and a numerical point of view, was put into evidence in the works of Neto and Polak [17] (with an example of approximating local minimizers converging to a nonstationary point of the original problem), Murat [16] (indicating counter-examples to the existence of minimizers for control into coefficients problems governed by second order equations) and Cheng and Olhoff [3], Rozvany, Cheng, Olhoff and Taylor [18] where comprehensive numerical experiments are discussed.

In general, in nonconvex minimization problems one may just expect approximation of stationary points. In the case of optimal design of beams this is discussed by Polak and Neto [17] via the use of consistent approximations.

In this work, we introduce a class of nonlinear transformations which may be applied to any of the problems (1.1)–(1.8). We call them "resizing rules" with reference to a partial similarity that exists with the Fully Stressed Design method (FSD) appearing in the engineering literature, Haftka, Gürdal and Kamat [7, Ch. 9].

Via the resizing rules the control into coefficients problem is transformed into an equivalent distributed control problem. In this way, we see that some of the problems (1.1)–(1.8) are convex or even strictly convex (after transformation). This gives the uniqueness of the global minimum in the original problem. Moreover, this approach allows to relax the compactness assumptions on the set of admissible controls, needed to show the existence of the minimizers. The boundedness condition (1.6) is sufficient for our method to work.

In Sections 2 and 3 such results are proved for simply supported, respectively clamped, plates and beams. In Section 4, an algorithmic approach is used for the optimization of a unilaterally supported beam and a numerical example is discussed. This shows the multiple possibilities of the "resizing rule" method. Such algorithms were previously used by Tiba and Sprekels [20], for classical types of beams (simply supported, cantilevered, clamped).

Finally, we point out that our method is a duality-type method: to the original minimization problem another optimization problem is associated which is simpler and gives relevant information on the first problem. From a theoretical point of view, the equivalence results are essential in proving convexity, uniqueness or existence. From a numerical point of view, simple "dual" problems may be considered that provide efficient approximations in the examples. Let us also notice that this duality approach has a mechanical background and is not inspired by the convex duality theory or its nonconvex extensions. A detailed comparison (from this point of view) was performed by Tiba and Sprekels [20], Sections 1 and 2.

2 Simply supported plates and beams

We start with a general equivalence result which, roughly speaking, says that the set of deflections obtained under a given load and for various thicknesses is the same as the set of deflections obtained for a fixed thickness, but with variable load. Namely, we consider the following two "state" systems:

$$\Delta(u^3\Delta y) = f \quad \text{in } \Omega\,, \tag{2.1}$$

$$y = \Delta y = 0 \quad \text{on } \partial\Omega\,, \tag{2.2}$$

$$0 < m \le u(x) \le M \quad \text{a.e. in } \Omega\,, \tag{2.3}$$

$$y \in A\,; \tag{2.4}$$

and

$$\Delta\Delta y = h \quad \text{in } \Omega\,, \tag{2.5}$$

$$y = \Delta y = 0 \quad \text{on } \partial\Omega\,, \tag{2.6}$$

$$\min\left\{m^{-3}z(x),\, M^{-3}z(x)\right\} \le \Delta y(x) \le \max\left\{m^{-3}z(x),\, M^{-3}z(x)\right\}, \text{a.e. in } \Omega, \tag{2.7}$$

$$y \in A\,. \tag{2.8}$$

In (2.1)–(2.4), $f \in L^2(\Omega)$ is fixed, $u \in L^\infty(\Omega)$ is the optimization parameter, $A \subset L^2(\Omega)$ is closed, and m, M are positive real constants. No sign conditions are imposed on f, and the unique weak solution y satisfies $y \in V := H^2(\Omega) \cap H_0^1(\Omega)$, $u^3\Delta y \in V$. One (convex) example for the set A is obtained via the constraint

$$y \ge -\tau \quad \text{in } \Omega\,, \tag{2.9}$$

with $\tau > 0$ given.

In (2.5)–(2.8) we assume that $h \in V^*$, and we define $g \in L^2(\Omega)$ as the unique transposition solution to

$$\Delta g = h \quad \text{in } \Omega\,, \tag{2.10}$$

$$g = 0 \quad \text{on } \partial\Omega\,, \tag{2.11}$$

that is, we have

$$\int_\Omega g\,\Delta\rho\,dx = \int_\Omega h\,\rho\,dx\,, \quad \forall\,\rho \in V\,. \tag{2.12}$$

Then, $y \in V$ is the strong solution to

$$\Delta y = g \quad \text{in } \Omega\,, \tag{2.13}$$
$$y = 0 \quad \text{on } \partial\Omega\,. \tag{2.14}$$

The second boundary condition, $\Delta y = 0$ on $\partial\Omega$, is included in the choice of test mappings ρ in (2.12) and not explicit. The mapping $z \in V$ from (2.7) is the strong solution to (2.10), (2.11), corresponding to $h = f$. We also mention that (2.10) is valid in the sense of distributions although $C_0^\infty(\Omega)$ is not dense in V. The constraint (2.7) shows that, for admissible y, the boundary condition $\Delta y = 0$ on $\partial\Omega$ has an explicit meaning.

Theorem 2.1. *For any admissible pair $[y, u]$ for (2.1) – (2.4), there is some $h \in V^*$ such that the pair $[y, h]$ is admissible for (2.5) – (2.8). The converse is also true if meas $\{x \in \Omega;\, z(x) = 0\} = 0$.*

Proof. If $[y, u]$ is admissible for (2.1)–(2.4), then

$$\Delta y = \frac{1}{u^3}\, z \in L^2(\Omega)\,. \tag{2.15}$$

We denote by $\tilde{h} \in V^*$ the linear bounded functional on V defined by

$$< \tilde{h}, \rho >_{V^* \times V} = \int_0^1 \frac{1}{u^3}\, z\, \Delta\rho\, dx\,, \quad \forall\, \rho \in V\,. \tag{2.16}$$

Then (2.16), (2.12) show that $\tilde{g} = \dfrac{1}{u^3}\, z$ is the transposition solution to (2.10)–(2.11) associated with this $\tilde{h}$. By (2.15) and (2.2), it follows that y satisfies (2.5), (2.6) with $\tilde{h}$ given by (2.16). Then, (2.7) is a clear consequence of (2.15) and (2.3).

Conversely, taking $[\hat{y}, \hat{h}]$ admissible for (2.5)–(2.8), and $\hat{g}$ satisfying (2.10), (2.11) with $h = \hat{h}$, we see that

$$\Delta \hat{y} = \hat{g} \quad \text{a.e. in } \Omega\,. \tag{2.17}$$

We shall multiply (2.17) by $z(x) \cdot [\hat{g}(x)]^{-1}$ which we denote $v(x)$. By (2.7) and (2.17), we notice that $v \in L^\infty(\Omega)$, and $\hat{u} = v^{1/3}$ satisfies constraint (2.3). To see this, we analyse in (2.7) the situations $z(x) > 0$, $z(x) < 0$, to get

$$0 < M^{-3} \leq \frac{\hat{g}(x)}{z(x)} = [\hat{u}(x)]^{-3} \leq m^{-3}\,. \tag{2.18}$$

Under our hypothesis, (2.18) is valid a.e. in Ω, and we obtain (2.3). Moreover,

$$\hat{u}^3(x)\, \Delta\hat{y}(x) = z(x) \quad \text{a.e. in } \Omega\,. \tag{2.19}$$

The definition of z and (2.19) show that $\hat{y}$ is a weak solution of (2.1) as well, and the proof is finished. $\square$

Corollary 2.2. *For any admissible pair* $[y,u]$ *for* (2.1) – (2.4), *there is some* $l \in L^2(\Omega)$ *such that the pair* $[y,l]$ *is admissible for the system*

$$\begin{aligned} \Delta y &= z\,l \quad \text{in}\,\Omega\,, & (2.20)\\ y &= 0 \quad \text{on}\,\partial\Omega\,, & (2.21)\\ M^{-3} &\le l(x) \le m^{-3} \quad \text{a.e. in}\,\Omega\,, & (2.22)\\ y &\in A\,. & (2.23) \end{aligned}$$

The converse is also true.

The proof is just a variant of the proof of Theorem 2.1.

Remark. While Theorem 2.1 has a physical interpretation which we have stressed from the very beginning, Corollary 2.2 represents a mathematical equivalence trick. Its advantages are to transform the fourth order equation into a second order one and to replace the "state" constraint (2.7) by the "control" constraints (2.22). Notice that no special assumption on z is necessary.

Remark. Theorem 2.1 and Corollary 2.2 are controllability-type results. They say that the reachable set of states is the same in the systems (2.1)–(2.4) or (2.5)–(2.8) or (2.20)–(2.23).

Remark. One basic property for the above results is that the set of admissible pairs $[y,h]$ defined by (2.5)–(2.7), as well as the set of admissible $[y,l]$ given by (2.20)–(2.22), are convex. If A is convex (which is generally the case — see (2.9)), then the systems (2.5)–(2.8) or (2.20)–(2.23) define convex pair sets in the appropriate product spaces.

This fundamental property is not valid, in general, for the original set of admissible pairs $[y,u]$ since the transformation that we use is nonlinear. However, there is one example, due to Kawohl [12], Kawohl and Lang [13], where the system (2.1)–(2.3) and (2.9) define a convex set of admissible "control" mappings u in $L^2(\Omega)$.

Example 2.3. We assume that $f \le 0$ a.e. in Ω. Then, the maximum principle gives that $z > 0$ in Ω, and we have the representation formula

$$y(x) = -\int_\Omega \sigma(x,y)\,\frac{z(y)}{u^3(y)}\,dy\,, \tag{2.24}$$

with σ being the Green function, again positive. Let u_1, u_2 be two admissible thicknesses for the system (2.1)–(2.3), (2.9), and $u_\lambda(x) = \lambda\,u_1(x) + (1-\lambda)\,u_2(x)$, $\forall\, x \in [0,1]$, $\forall\, \lambda \in [0,1]$. We denote by y_1, y_2, y_λ the solutions of (2.1), (2.2) corresponding to u_1, u_2, u_λ, respectively.

Then, (2.24) and the positivity of σ, z give

$$y_\lambda(x) \ge \lambda\,y_1(x) + (1-\lambda)\,y_2(x) \ge -\tau\,, \tag{2.25}$$

since the function $-u^{-3}$ is concave. We conclude that u_λ is admissible for any $\lambda \in [0,1]$, that is, the admissible set of controls is convex. For the set of admissible states y this is also true since the function $\bar{y}_\lambda = \lambda\,y_1 + (1-\lambda)\,y_2$ corresponds to the thickness $(\bar{u}_\lambda)^3 = (\lambda\,u_1^{-3} + (1-\lambda)u_2^{-3})^{-1}$ which satisfies (2.3) and (2.9). However, since the operator $u \mapsto y$ is nonlinear, we cannot expect $\bar{y}_\lambda = y_\lambda$ in (2.25), and the set of admissible pairs $[y, u]$ is not convex. Moreover, such properties do not extend beyond the condition (2.9) to general convex state constraints expressed by (2.4) or to nonnegative f.

It is our aim now to apply this equivalence, especially in the form given by Corollary 2.2, which is the simplest one, to certain optimization problems. Let us associate to the system (2.1)–(2.4) one of the following cost functionals, to be minimized:

$$\mathrm{Min} \int_0^1 u(x)\, dx\,; \tag{2.26}$$

$$\mathrm{Min} \int_0^1 \left(-\, u^{-3}(x)\right)\, dx\,; \tag{2.27}$$

$$\mathrm{Min} \int_0^1 (y(x) \;-\; y_d(x))^2\, dx\,. \tag{2.28}$$

The minimization parameter is $u \in L^\infty(\Omega)$, and we denote by $(\mathbf{P}_i), i = \overline{1,3}$, the obtained minimization problems, in this order. Obviously, $(\mathbf{P}_1)$ is the minimization of weight (volume) problem, subject to the given constraints. $(\mathbf{P}_2)$ is related to this question, as will be explained later, and $(\mathbf{P}_3)$ is an identification-type problem ($y_d \in L^2(\Omega)$ is an "observed" or "desired" deflection of the plate).

To the system (2.20)–(2.22) we associate the following cost functionals:

$$\mathrm{Min} \int_0^1 l^{-\frac{1}{3}}(x)\, dx\,; \tag{2.29}$$

$$\mathrm{Min} \int_0^1 \left(-\, l(x)\right)\, dx\,; \tag{2.30}$$

$$\mathrm{Min} \int_0^1 \left(y(x) \;-\; y_d(x)\right)^2 dx\,. \tag{2.31}$$

The minimization distributed control is the mapping $l \in L^2(\Omega)$ and we denote by $(\mathbf{D}_i)$, $i = \overline{1,3}$, the obtained optimization problems, in this order.

Theorem 2.4. *The problems $(\mathbf{P}_i)$ are equivalent to the problems $(\mathbf{D}_i)$, $i = \overline{1,3}$, in the sense that if $[y, u]$ is admissible for $(\mathbf{P}_i)$, then $[y, l]$, $l = 1/u^3$, is admissible for $(\mathbf{D}_i)$ with the same cost, and conversely.*

This follows directly from Corollary 2.2 and the definitions (2.26)–(2.31).

Corollary 2.5. *Under admissibility assumptions, the problem $(\mathbf{P}_1)$ has a unique global minimum $u^* \in L^\infty(\Omega)$.*

Proof. The existence of u^* can be established from standard estimates in (2.1), (2.2) and the boundedness of minimizing sequences, given by (2.3). The passage to the limit is a simplified variant of the one performed in Theorem 3.2. By Theorem 2.4, $l^* = (u^*)^{-3}$ is the (global) minimizer for $(\mathbf{D}_1)$. Since this latter problem is strictly convex, the uniqueness of l^*, u^* follows.

Remark. Instead of solving the nonconvex problems $(\mathbf{P}_i)$, $i = \overline{1,3}$, we suggest to solve the equivalent convex problems $(\mathbf{D}_i)$, $i = \overline{1,3}$. In numerical experiments, this avoids the "trap" of local minimum points, and the uniqueness of the global optimum enhances the numerical stability.

Remark. It is known that, in discussing weight minimization problems, any increasing function $\mu(u)$ may be relevant as an integrand in the cost functional. The problem $(\mathbf{P}_2)$ uses the increasing mapping

$$\mu(u) = -\frac{1}{u^3}, \quad u > 0,$$

which has the advantage that the equivalent problem $(\mathbf{D}_2)$ is a linear optimization problem.

Remark. Similar results may be obtained in dimension one for the simply supported beam and for the cantilevered beam, i.e. for the boundary conditions

$$y(0) = y'(0) = 0,$$

$$y''(1) = (u^3 y'')'(1) = 0.$$

One basic property which is important for the above analysis is that the state system can be decoupled into two independent second order differential equations. In the next sections, this property is no longer true; however, the results can be extended.

3 Clamped plates and beams

We investigate first the classical optimal shape design problem:

$$\text{Min} \int_\Omega u(x)\,dx, \tag{3.1}$$

subject to

$$\Delta(u^3\,\Delta y) = f \quad \text{in } \Omega, \tag{3.2}$$

$$y = \frac{\partial y}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{3.3}$$

$$0 < m \le u(x) \le M \quad \text{a.e. in } \Omega, \tag{3.4}$$

$$y \in A. \tag{3.5}$$

As usual, $f \in L^2(\Omega)$, $A \subset L^2(\Omega)$ are given, and $u \in L^\infty(\Omega)$ is the thickness of the plate, the minimization parameter.

The existence of a unique weak solution $y \in H_0^2(\Omega)$ to (3.2), (3.3) is obvious since the bilinear form

$$a(u, y, v) = \int_\Omega u^3\,\Delta y\,\Delta v\,dx$$

is coercive on $H_0^2(\Omega) \times H_0^2(\Omega)$.

Fix the mapping $g \in H^2(\Omega) \cap H_0^1(\Omega)$ by:

$$\Delta g = f \quad \text{in } \Omega\,, \tag{3.6}$$
$$g = 0 \quad \text{on } \partial\Omega\,. \tag{3.7}$$

Theorem 3.1. *a) The equation* (3.2), (3.3) *is equivalent to*

$$\Delta y = g\,l + h\,l \quad \textit{in } \Omega \tag{3.8}$$

and (3.3), *where* $h \in L^2(\Omega)$ *is a harmonic mapping in* Ω *and* $l = u^{-3} \in L^\infty(\Omega)$.
b) The optimization problem (3.1)–(3.5) *is equivalent to*

$$\text{Min} \int_\Omega l^{-\frac{1}{3}}(x)\,dx \tag{3.9}$$

subject to (3.8), (3.3), (3.5) *and*

$$M^{-3} \le l(x) \le m^{-3} \quad \text{a.e. in } \Omega\,. \tag{3.10}$$

Proof. a) By (3.2), (3.3) and the definition of $a(u\,,\cdot\,,\cdot)$, we see that

$$\int_\Omega (u^3\,\Delta y - g)\,\Delta v\,dx = 0\,, \quad \forall\ v \in H_0^2(\Omega)\,. \tag{3.11}$$

We denote $h = u^3\,\Delta y - g \in L^2(\Omega)$, and (3.11) gives $\Delta h = 0$ in the sense of distributions. The converse is obvious.
b) This is a clear consequence of a) and of $l^{-1/3} = u$. □

Remark. The above transformation shows that the obtained problem remains nonconvex. The harmonic mapping h may be determined from the "supplementary" boundary condition $\dfrac{\partial y}{\partial n} = 0$ on $\partial\Omega$. One such situation is explained in Corollary 3.4.

In general, we may interpret h as an extra control variable and $\dfrac{\partial y}{\partial n} = 0$ on $\partial\Omega$ as a new state constraint.

Theorem 3.2. *Under admissibility assumptions, the problem* $(\mathbf{P}_4)$ *given by* (3.1)–(3.5) *has at least one solution* $\tilde{u} \in L^\infty(\Omega)$.

Proof. By admissibility, there exists a minimizing sequence $\{u_n\} \subset L^\infty(\Omega)$ such that

$$\int_\Omega u_n(x)\,dx \to \inf(\mathbf{P_4}) \tag{3.12}$$

for $n \to \infty$. We denote by $l_n = u_n^{-3}$ and by $y_n \in H_0^2(\Omega)$ the corresponding weak solution of (3.2), (3.3). Conditions (3.4), (3.10) show that $\{u_n\}$, $\{l_n\}$ are bounded in $L^\infty(\Omega)$, and hence we may assume that $u_n \rightharpoonup \hat{u}$, $l_n \rightharpoonup \hat{l}$ weakly* in $L^\infty(\Omega)$. In general, $\hat{l} \ne \hat{u}^{-3}$!

We also notice that $\{y_n\}$ is bounded in $H_0^2(\Omega)$:

$$m \int_\Omega \big[\Delta y_n(x)\big]^2\,dx \le \int_\Omega u_n^3 (\Delta y_n)^2\,dx = \int_\Omega f\,y_n\,dx \le \big|f\big|_{L^2(\Omega)}\,\big|y_n\big|_{L^2(\Omega)}\,.$$

We may, as well, assume that $y_n \rightharpoonup \tilde{y}$ weakly in $H_0^2(\Omega)$, where $\tilde{y} \in A$ since A is closed in $L^2(\Omega)$. Moreover, by (3.8), we see that $h_n = u_n^3 \Delta y_n - g$ is bounded in $L^2(\Omega)$, and we may write $h_n \rightharpoonup \tilde{h}$ weakly in $L^2(\Omega)$. We now use

Lemma 3.3. *If a sequence of harmonic mappings is weakly convergent in $L^1(\Omega)$, then it is pointwisely convergent.*

We remark that the right-hand side in (3.8) is bounded in $L^2(\Omega)$, and hence we may assume that, with some $z \in L^2(\Omega)$,

$$g\, l_n + h_n\, l_n \rightharpoonup z \quad \text{weakly in } L^2(\Omega). \tag{3.13}$$

The difficulty is just to identify z, that is the limit of the product $h_n\, l_n$. By Lemma 3.3 and the Egorov theorem, for any $\varepsilon > 0$, there is $\Omega_\varepsilon \subset \Omega$ measurable, such that meas $(\Omega \backslash \Omega_\varepsilon) < \varepsilon$ and $h_n \to \tilde{h}$ uniformly in Ω_ε. Then, we can pass to the limit in (3.13) on Ω_ε, and we get $z = g\,\hat{l} + \tilde{h}\,\hat{l}$ in Ω_ε. Since ε is arbitrarily small, we obtain that $z(x) = g(x)\,\hat{l}(x) + \tilde{h}(x)\,\hat{l}(x)$ a.e. in Ω. Hence we can pass to the limit in (3.8) to obtain

$$\Delta \tilde{y} = g\,\hat{l} + \tilde{h}\,\hat{l} \quad \text{in } \Omega. \tag{3.14}$$

Using Theorem 3.1 in (3.14), we see that $\tilde{u} = \hat{l}^{-1/3}$ is the thickness in (3.2) which generates the deflection $\tilde{y}$. Obviously, the pair $[\tilde{y}, \tilde{u}]$ is admissible for the problem $(\mathbf{P}_4)$, and (3.12) yields:

$$\begin{aligned} \inf (\mathbf{P}_4) &= \lim \int_\Omega u_n(x) dx = \lim \int_\Omega l_n^{-\frac{1}{3}}(x) dx \geq \liminf \int_\Omega l_n^{-\frac{1}{3}}(x)\, dx \geq \int_\Omega \hat{l}^{-\frac{1}{3}}(x)\, dx \\ &= \int_\Omega \tilde{u}(x)\, dx \geq \inf (\mathbf{P}_4). \end{aligned}$$

This ends the proof. $\square$

Proof of Lemma 3.3. Since h_n, $\tilde{h}$ are harmonic in Ω, the Weyl lemma, Hörmander [11], shows that they belong to $C^\infty(\Omega)$. For any $x \in \Omega$ and any ball centered in x and of radius ρ, $B_\rho(x) \subset \Omega$, we have the solid mean property:

$$h_n(x) = \frac{m}{w_m\, \rho^m} \int_{B_\rho(x)} h_n(y)\, dy \to \frac{m}{w_m\, \rho^m} \int_{B_\rho(x)} \tilde{h}(y)\, dy = \tilde{h}(x).$$

Here, m is the dimension of Ω, and w_m denotes the area of the unit ball in R^m. $\square$

Remark. The passage to the limit in Theorem 3.2 is based on the following general property:

If $\{w_n\}$ is bounded in $L^p(\Omega)$, $p > 1$ and $w_n(x) \to w(x)$ a.e. in Ω, then $w_n \to w$ strongly in $L^s(\Omega)$, for any s such that $1 < s < p$.

Proof. Let $\varepsilon > 0$ be fixed and let $\Omega_\varepsilon \subset \Omega$ measurable, with meas $(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ be such that $w_n \to w$ uniformly in Ω_ε (by Egorov's theorem). We have

$$\begin{aligned} \int_\Omega |w_n - w|^s\, dx &= \int_{\Omega_\varepsilon} |w_n - w|^s\, dx + \int_{\Omega \backslash \Omega_\varepsilon} |w_n - w|^s\, dx \leq \int_{\Omega_\varepsilon} |w_n - w|^s\, dx \\ &+ \Big(\int_{\Omega \setminus \Omega_\varepsilon} |w_n - w|^p\Big)^{\frac{s}{p}} \text{meas } (\Omega \setminus \Omega_\varepsilon)^{\frac{p-s}{p}} \leq \int_{\Omega_\varepsilon} |w_n - w|^s\, dx + C\, \varepsilon^{\frac{p-s}{p}}. \end{aligned}$$

If $n \geq N(\varepsilon)$, we get $\int_\Omega |w_n - w|^s \, dx \leq c(\varepsilon)$, where $c(\varepsilon) \to 0$ for $\varepsilon \to 0$. This is a slight extension of Lemma 1.3, Lions [13]. □

Remark. By Theorem 3.2, we see that the "optimal" thickness $\tilde{u}$ is obtained by twice inverting the minimizing sequence $\{u_n\}$. If u_n is pointwisely convergent, then $\tilde{u} = \hat{u} = \lim u_n$. This is the case used in the existing literature, Haslinger and Neittaanmäki [8], Casas [2], Hlavacek, Bock and Lovisek [9], [10], Neto and Polak [17], Bendsoe [1]. Our result just shows that the strong compactness assumption (the boundedness of $\{\nabla u\}$) is not necessary to get existence in the optimal shape design problem. The numerical experiments from [3], [18] put into evidence the so-called "stiffeners" into the process of optimization of beams and plates, which correspond to unbounded gradients.

Remark. Obviously, the same argument applies to the cost functionals (2.27) or (2.28).

Corollary 3.4. *In the case of beams, the equation*

$$(u^3 \, y'')'' = f \quad in \;]0,1[\,,$$

$$y(0) = y(1) = y'(0) = y'(1) = 0\,,$$

is equivalent to

$$y'' = g\,l + (a_l\, x + b_l)\, l \quad in \;]0,1[\,,$$

with the same boundary conditions and with a_l, $b_l \in R$, g satisfying (3.6), (3.7) *and $l = u^{-3}$.*

Remark. It is clear, by direct calculus, that the harmonic mapping $h_l = a_l\, x + b_l$ can be uniquely determined from the "supplementary" boundary conditions $y'(0) = y'(1) = 0$. In general, by a finite element approximation, h will introduce a finite number of new entries into the state system (3.8) that can be determined from the discretization of $\dfrac{\partial y}{\partial n} = 0$, which will generate the same finite number of conditions.

Remark. If $f \leq 0$ in $[0,1]$, Tiba and Sprekels [20] proved that y'' has exactly two distinct roots in $[0,1]$ and that $y \leq 0$ in $[0,1]$ (see also Theorem 4.5). For general $f \in L^2(0,1)$, it is easy to see that y'' has at least two distinct roots in $[0,1]$. Otherwise $u^3\, y''$ (which is continuous) has at most one change of sign in $[0,1]$, and the maximum principle together with the Hopf maximum principle will contradict the boundary conditions.

Then, denoting by $\xi < \zeta$ two such roots, one can find a_l, b_l and h_l from the simple relations

$$g(\xi) + a_l \cdot \xi + b_l = 0\,,$$

$$g(\zeta) + a_l \cdot \zeta + b_l = 0.$$

In general, the determination of h is related to the zeros of Δy in Ω. This is an extension to the case of the clamped plate of the relation (2.7) which ensures (in the case of simply supported plates) that the zeros and the sign of Δy remain unmodified via the resizing transformation. The roots distribution is connected to the famous conjecture of Hadamard [6] on the positivity of the Green function for the biharmonic operator. While Duffin [4] provided a first counter-example, he also noticed that the sign of Δy in a neighbourhood of $\partial\Omega$ is the same as that of y. Later, Garabedian [5] and Tegmark and Shapiro [19] obtained counter-examples in eccentric ellipses. By reworking this last one, which has an elementary character, we see that Δy may change sign on an interior subdomain, but also in the neighbourhood of $\partial\Omega$ (even with f of constant sign). Therefore, the properties of Δy in dimension two are essentially different from Theorem 3.1, in Tiba and Sprekels [20] in the one-dimensional case.

4 Variational inequalities

We consider the elastic beam with a unilateral obstacle at the right end:

$$\left(u^3\, y''\,,\, y'' - z''\right)_{L^2(0,1)} \leq (f\,,\, y - z)_{V^* \times V}\,, \quad \forall\, z \in \mathcal{K}\,, \tag{4.1}$$

$$y \in \mathcal{K} = \{w \in V\,;\, y(1) \geq \alpha\}\,, \quad (\alpha \in R \text{ given})\,, \tag{4.2}$$

$$V = \{y \in H^2(0,1)\,;\, y(0) = y'(0) = 0\}\,. \tag{4.3}$$

The beam is clamped at the left end.

To any $u \in L^\infty(0,1)$ we associate the linear bounded operator $A(u) : V \to V^*$ via the bilinear form on V

$$a\,(u\,,\, y\,,\, z) = \int_0^1 u^3\, y''\, z''\, dx\,, \quad \forall\, y\,,\, z \in V\,. \tag{4.4}$$

Then the variational inequality (4.1), (4.2) may be rewritten in the abstract form

$$\left(A\,(u)\, y\,,\, y - z\right) = a\,(u\,, y\,,\, y - z) \leq (f\,,\, y - z)_{V^* \times V} \tag{4.5}$$

for any $z \in \mathcal{K}$ and with $y \in \mathcal{K}$.

If $u \in L^\infty(0,1)$ is positive, $A(u)$ is strictly maximal monotone, and if $u(x) \geq m > 0$ in $[0,1]$, then $A(u)$ is strongly monotone and coercive. This gives a unique weak solution $y \in V$ to the variational inequality (4.5), for any $f \in V^*$.

We define now two auxiliary problems. First, we consider a cantilevered beam (without unilateral conditions):

$$\begin{aligned} \left(u^3\, y_1''\right)'' &= f \quad \text{in }]0,1[\,, \\ y_1(0) &= y_1'(0) = 0\,, \\ y_1''(1) &= 0\,, \quad \left(u^3\, y_1''\right)'(1) = 0\,. \end{aligned} \tag{4.6}$$

Second, we introduce a clamped – simply supported beam:

$$\begin{aligned} \left(u^3\, y_2''\right)'' &= f \quad \text{in }]0,1[\,, \\ y_2(0) &= y_2'(0) = 0\,, \\ y_2''(1) &= 0\,, \quad y_2(1) = \alpha\,. \end{aligned} \tag{4.7}$$

It is simple to check by direct integration that both y_1, y_2 are in $H^2(0,1)$ and $u^3 y_1''$, $u^3 y_2'' \in H^2(0,1)$.

Theorem 4.1. *If $f \in L^2(0,1)$, then the solution y of the variational inequality (4.1) is either the solution of (4.6) or the solution of (4.7). It satisfies $u^3 y'' \in H^2(0,1)$.*

Proof. Assume first that $y_1(1) \geq \alpha$ (that is, $y_1 \in \mathcal{K}$). We multiply (4.6) by $y_1 - z$, for any $z \in \mathcal{K}$, and we see (by partial integration) that y_1 is also a solution of (4.1), $y = y_1$, and the claimed regularity is clear.

Assume now that $y \notin \mathcal{K}$. By (4.7), it is obvious that $y_2 \in \mathcal{K}$. We multiply (4.7) by $y_2 - z$, $z \in \mathcal{K}$, and integrate by parts:

$$(f\,,\, y_2 \,-\, z)_{L^2(0,1)} \;=\; \left(u^3\, y_2''\right)'(1)\,\left(\alpha \,-\, z(1)\right) \,+\, \int_0^1 u^3\, y_2''\,(y_2'' \,-\, z)\,dx\,. \tag{4.8}$$

Assume that

$$\gamma \,=\, \left(u^3\, y_2''\right)'(1) \,>\, 0\,, \tag{4.9}$$

and denote $w = y_2 - y_1$. By (4.6), (4.7), (4.9), we see that w satisfies

$$\left(u^3\, w''\right)'' \,=\, 0 \quad \text{in }]0,1[\,,$$

$$w(0) \,=\, w'(0) \,=\, 0\,,$$

$$w''(1) \,=\, 0\,, \quad \left(u^3\, w''\right)'(1) \,=\, \gamma \,>\, 0\,.$$

Then, $u^3(x)\, w''(x) = \gamma\, x - \gamma \leq 0$ in $[0,1]$. That is, w is a concave function, and $w(0) = w'(0) = 0$ gives $w \leq 0$ in $[0,1]$. Therefore, $y_2(1) \leq y_1(1) < \alpha$, according to the assumption $y_1 \notin \mathcal{K}$. But this is a contradiction to $y_2(1) = \alpha$, and it follows that (4.9) is false. Then (4.8) gives that y_2 is now the solution of (4.1), i.e. $y = y_2$ has again the claimed regularity. □

Remark. The boundary conditions in $x = 1$, associated to (4.5), are

$$y''(1) \,=\, 0\,, \quad y(1) \,\geq\, \alpha\,, \quad \left(u^3\, y''\right)'(1) \,\leq\, 0\,,$$

$$\left(y(1) \,-\, \alpha\right)\left(u^3\, y''\right)'(1) \,=\, 0\,.$$

We formulate the optimization problem $(\mathbf{P}_5)$:

$$\text{Min} \int_0^1 u(x)\,dx\,, \tag{4.10}$$

subject to (4.1) and to

$$m \,\leq\, u(x) \,\leq\, M \quad \text{a.e. in } [0,1]\,, \tag{4.11}$$

$$y(x) \,\geq\, -\,\tau \quad \text{in } [0,1]\,. \tag{4.12}$$

Without loss of generality, we may assume

$$\alpha > -\tau . \tag{4.13}$$

Otherwise, all the admissible pairs of $(\mathbf{P}_5)$ correspond to an inactive variational inequality (the case $y = y_1$), that is, to a cantilevered beam (by Theorem 4.1), and we can refer to Section 2. We call *extremal* for $(\mathbf{P}_5)$ any admissible "thickness" $u \in L^\infty(0,1)$ such that the associated state is active with respect to the constraint (4.12).

Proposition 4.2. *If $\alpha \geq 0$ and $m = 0$, any local minimum of $(\mathbf{P}_5)$ is an extremal of $(\mathbf{P}_5)$.*

Proof. If $[u, y]$ is local optimum for $(\mathbf{P}_5)$, but not extremal, there is some $\lambda > 1$ such that the pair $[\lambda^{-1/3} u, \lambda y]$ is admissible for $(\mathbf{P}_5)$ — it clearly satisfies the constraints and the variational inequality since $\lambda y \in \mathcal{K}$ by $\alpha \geq 0$.

Obviously $\lambda^{-1/3} u$ gives a lower cost which contradicts the local optimality of u when $\lambda \to 1+$. □

Remark. The case $\alpha = 0$ was considered by Hlavacek, Bock and Lovisek [9].

Proposition 4.3. *Assume that $f < 0$ in $[0,1]$. Then any extremal pair has exactly one active point in $]0,1[$.*

Proof. The existence of at least one point $x_u \in [0,1]$ such that $y(x_u) = -\tau$ is obvious by the definition. Assume that there are at least two such points $x_u \neq \bar{x}_u$, i.e. $y(x_u) = y(\bar{x}_u) = -\tau$. Again by definition, x_u and $\bar{x}_u$ are minimum points for y, different from 0 and 1, that is, $y'(x_u) = y'(\bar{x}_u) = 0$. Then $y + \tau$ satisfies the clamped beam conditions on $[x_u, \bar{x}_u]$. By Theorem 3.1, Tiba and Sprekels [20], we see that $y \leq -\tau$ on $[x_u, \bar{x}_u]$, therefore $y \equiv -\tau$ on $[x_u, \bar{x}_u]$. This contradicts $f < 0$ a.e. in $[0,1]$. □

Remark. Notice that by $y(0) = 0$ and $y(1) \geq \alpha > -\tau$ (by (4.13)) the end points cannot be active with respect to the state constraint.

Corollary 4.4. *If $f \leq 0$, any extremal of $(\mathbf{P}_5)$ satisfies (4.7) and $(u^3 y'')'(1) \leq 0$.*

This is a direct consequence of Theorem 4.1 and its subsequent Remark.

Remark. By Corollary 4.4 and Proposition 4.2, the shape optimization problem $(\mathbf{P}_5)$, governed by variational inequalities, is reduced to the linear state system (4.7). Some cases of control problems governed by variational inequalities of obstacle type which can be equivalently reformulated as convex control problems with state constraints are discussed in Tiba [21, Ch. III.5], by different approaches.

We formulate the "dual" problem:

$$(\mathbf{D}_5) \qquad \operatorname{Min} \int_0^1 f(x)\, dx ,$$

$$
\begin{aligned}
(\bar{u}^3\, y'')'' &= f \quad \text{in }]0,1[\,, && (4.14)\\
y(0) &= y'(0) = 0\,,\\
y(1) &= \alpha\,, \quad y''(1) = 0\,,\\
f &\le 0 \quad \text{a.e. in } [0,1]\,,\\
y &\ge -\tau \quad \text{in } [0,1]\,. && (4.15)
\end{aligned}
$$

Notice that this is again a linear optimization problem ($\bar{u}$ is a prescribed thickness).

Remark. The control constraint $f \le 0$ is a simplified stronger variant of (2.7), due to the maximum principle. Then, the equivalence results from Theorem 2.1 and Theorem 2.4 are not valid in this setting. However, we put into evidence that between the problems $(\mathbf{P}_5)$ and $(\mathbf{D}_5)$, there still exists a very useful relationship. In the cases discussed in Section 2 and Section 3 (only for beams), this weaker relationship was studied in Tiba and Sprekels [20]. The problem $(\mathbf{D}_5)$ is, in fact, a slightly simplified variant of the problem $(\mathbf{D}_2)$, Section 2. Moreover, this approach allows to consider $m = 0$ and gives another form for the resizing transformation.

Theorem 4.5. *Assume that $\bar{u}$ is continuous and let $[y, f]$ be extremal for $(\mathbf{D}_5)$. Then y'' has exactly one root in $[0,1[$. Moreover, $y \le \max\{0;\alpha\}$ in $[0,1]$.*

Proof. We have $\bar{u}^3\, y'' = g$ in $[0,1]$, where, moreover, $g'' = f$ in $[0,1]$ and $g(1) = 0$. Since $f \le 0$, then g is concave in $[0,1]$ and it may have at most one root in $[0,1[$, unless it is identically 0 in some subinterval.

In the last subcase, by concavity and $g(1) = 0$, there is some $\xi \in]0,1[$ such that $g(x) \equiv 0$, $x \in [\xi, 1]$, and $g(x) < 0$ in $[0,\xi[$. Then $y'' < 0$ in $[0,\xi[$ and, since $y(0) = y'(0) = 0$, we see that $y(\xi) < 0$, $y'(\xi) < 0$, and $y(x) = y'(\xi)\,(x-\xi) + y(\xi)$ for $x \in [\xi, 1]$. We obtain that

$$
\alpha = y(1) < y(x)\,, \quad \forall\, x \in [0,1]\,,
$$

which contradicts the extremality of $[y, f]$ and (4.10).

Therefore g has at most one root in $[0,1[$. Since $[y,f]$ is extremal, there is some $\bar{\xi}$ in $]0,1[$ such that $y(\bar{\xi}) = -\tau$, and this is a minimum point for y on $]0,1[$. Then $y'(\bar{\xi}) = 0$, and there is some $\eta \in]0,\bar{\xi}[$ such that $y''(\eta) = 0$, since $y'(0) = 0$ and y'' is continuous by the assumption on $\bar{u}$.

We conclude that y'' has exactly one root in $]0,\bar{\xi}[$. Let η be this root. Then $y'' \le 0$ in $[0,\eta]$ and $y'' \ge 0$ in $[\eta, 1]$. By $y(0) = y'(0) = 0$ and the concavity of y, we get $y \le 0$ in $[0,\eta]$. By $y(\eta) \le 0$, $y(1) = \alpha$ and the convexity of y, we get $y \le \max\{0;\alpha\}$ in $[\eta, 1]$, as well. This ends the proof. $\square$

Based on Theorem 4.5, we can formulate the following algorithm:

Algorithm 4.6. $(m = 0\,,\ M = +\infty)$

1. $n = 0\,,\quad u_0$ admissible for $(\mathbf{P}_5)$, continuous
2. $\mathrm{Min}(\tilde{D}_n)$ gives $[y_n\,,\, f_n]$, where
   $(\tilde{D}_n)$ is given by (4.15) with $\bar{u}$ replaced by u_n.

3. If $f_n - f$ "small", then STOP! Otherwise

4. ("resizing step")

   α) compute the unique root ξ_n in $[0,1[$ of y_n''

   β) denote $g_n = u_n^3\, y_n''$ and define $\tilde{g}_n$ by

   i) $$\begin{cases} \tilde{g}_n'' = f & \text{in }]\xi_n, 1[\,, \\ \tilde{g}_n(\xi_n) = 0\,, & \tilde{g}_n(1) = 0\,, \end{cases}$$

   ii) $$\begin{cases} \tilde{g}_n'' = f & \text{in }]0, \xi_n[\,, \\ \tilde{g}_n(\xi_n) = 0\,, & \tilde{g}_n'(\xi_n -) = \tilde{g}_n'(\xi_n +)\,. \end{cases}$$

   γ) resize u_n by $u_{n+1}^3 = u_n^3\,\dfrac{\tilde{g}_n}{g_n}$, and set $n := n+1$, GO TO step 2.

Remark. The resizing rule γ) is well defined even in ξ_n and in 1 by the Hopf maximum principle and l'Hospital rule. The sequence $\{u_n\}$ remains continuous in all iterations and $1/u_n^3 \in L^2(0,1)$ if $1/u_0^3 \in L^2(0,1)$.

Theorem 4.7. *The Algorithm* 4.6 *generates extremals* u_n *for* $(\mathbf{P}_5)$ *in each step* $n \geq 1$.

Proof. If f_n is a minimum for $(\tilde{\mathbf{D}}_n)$, then it is extremal for $(\tilde{\mathbf{D}}_n)$. Otherwise, $y_n(x) \geq -\tau + \varepsilon$ in $[0,1]$ for some positive ε. Consider $f_\delta = f - \delta$, δ positive constant and y_δ the corresponding solution of (4.11).

Clearly $y_\delta \to y_n$ uniformly in $[0,1]$, for $\delta \to 0$. Then, for some small δ, $[y_\delta\,,\, f_\delta]$ is an admissible pair for $(\tilde{\mathbf{D}}_n)$ with a lower cost. This is a contradiction to the optimality of f_n. Extremality is obviously preserved by the resizing rule, since γ) gives

$$\left(u_{n+1}^3\, y_n''\right)'' = (\tilde{g}_n)'' = f \quad \text{in }]0,1[\,,$$

i.e. y_n is the state associated to u_{n+1} in (4.7), or equivalently (4.1). $\square$

Remark. The algorithm has a global character since it iterates between extremals of $(\mathbf{P}_5)$. If the cost functional (4.10) is replaced by (2.28), then Algorithm 4.6 has the descent property as well (again by the resizing rule).

We close this section with a numerical example.

Example 4.8. We have made several experiments with the Algorithm 4.6 applied to the minimum weight problem $(\mathbf{P}_5)$. The state equation was discretized by usual finite difference approximations for the derivatives, using the grid $x_i = i\,h$, $i = \overline{0,m}$, $h = 1/m$. By the discretization process, the problem $(\mathbf{D}_5)$ is approximated by a linear programming problem, LPP (this is one of the advantages of an algorithm). The variables of the LPP are given by the discrete values of the pair $[y,f]$. The cost functional is evaluated using Simpson's approximation rule.

The numerical tests have been made with $m = 50$ which allows the LPP to be accurately solved via the simplex algorithm. The root ξ_n in Step 4 α) of the Algorithm 4.6 was found using a cubic spline approximation of y_n. The differential equations corresponding to $\tilde{g}_n$ were solved by integrating first mathematically, using convolution formulae, and approximating next the definite integrals by a sharp numerical integration routine.

Generally, the algorithm stopped by failing to solve the problem $(\tilde{\mathbf{D}}_n)$ when it cannot further decrease the thickness u. The numerical tests have been made on a PC Pentium with floating point arithmetic accuracy of order 10^{-20}. We have fixed the load $\bar{f} \equiv -50$ in $]0,1[$ and $\bar{f} \equiv -1$ in $x = 0$, $x = 1$ or $\hat{f} \equiv -1$ in $[0,0.5]$ and $\hat{f} \equiv -50$ in $]0.5,1]$. The obstacle α had the values 0.1 or 0 or -0.1, the state constraint was $\tau = 0.6$ or $\tau = 0.5$, and the initial iteration (thickness) was $\bar{u}_0 = 2 + x(x-1)$ or $\hat{u}_0 = 2 - x$. In all these variants a sharp decrease of the thickness was obtained in at maximum seven iterations, but usually only in three iterations. This information is collected in the following table (each column gives an experiment and $v(u_i)$ is the L^1 norm of u_i):

f	$\bar{f}$	$\bar{f}$	$\bar{f}$	$\hat{f}$	$\hat{f}$
α	0.0	- 0.1	0.1	- 0.1	0.1
τ	0.6	0.6	0.6	0.6	0.5
u_0	$\hat{u}_0$	$\hat{u}_0$	$\bar{u}_0$	$\bar{u}_0$	$\hat{u}_0$
$v(u_1)$	1.1369	1.2987	1.5064	1.6727	1.2513
$v(u_2)$	0.9194	1.1146	1.2586	1.5916	1.1598
$v(u_3)$	0.7583	0.9823	1.2682	1.5693	1.1100
$v(u_4)$			0.9280	1.4288	1.0740
$v(u_5)$			0.7596	1.3296	1.0468
$v(u_6)$			0.6540	1.3760	
$v(u_7)$			0.5781	1.2778	

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