Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations [1]

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Abstract

In this article, dissipative perturbations of the nonlinear Schrödinger equation (NLS) are considered. For dissipative equations, when determining the stability of a solitary wave, one must locate both the point spectrum and the continuous spectrum. If the wave is to be stable, all the spectrum must reside in the left-half plane, except for the translational eigenvalue(s) at the origin. However, for the NLS the continuous spectrum is located on the imaginary axis, as the NLS can be thought of as an infinite-dimensional Hamiltonian system. Since dissipative perturbations will destroy this feature, it is then possible for eigenvalues to bifurcate out of the continuous spectrum and into the right-half plane, leading to an unstable wave. Here we show that the Evans function can be extended across the continuous spectrum, and hence it can be used to track these bifurcating eigenvalues. The extension is done for a general class of equations, and the result should therefore be useful for a larger class of problems than that presented here. Using the extended Evans function, we are then able to locate the spectrum for bright solitary-wave solutions to various perturbed nonlinear Schrödinger equations, and discuss their stability. In addition, we discuss the existence and stability of multi-bump solitary waves for a particular perturbation, the parametrically forced NLS equation.

1 Introduction

Compensating for the attenuation of pulses in nonlinear optical fibers is an important issue for the efficacy of optical communication systems. The standard model for the propagation of pulses in an ideal nonlinear fiber without loss is the cubic nonlinear Schrödinger equation (NLS)

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi = 0, \tag{1.1}$$

where $\omega > 0$. It is known to support stable pulses. If loss is present in the fiber, these pulses will cease to exist. Thus, amplifiers have to be used to compensate for the loss. The effects of linear loss in the fiber as well as other perturbations which account for amplifiers located along the fiber will then have to be incorporated into the model. The issue is whether pulses persist under the perturbation and what their stability might be. In this article, we shall concentrate on the stability of pulses for two different perturbations of (1.1).

The first equation is the cubic-quintic Schrödinger equation (CQNLS)

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + 3\alpha|\phi|^4\phi = 0, \tag{1.2}$$

where $\alpha < 0$ is real. The CQNLS is the correct model to describe the propagation of pulses in dispersive materials with either a saturable or higher-order refraction index ([5], [6]). An optical fiber which satisfies this condition can be constructed, for example, by doping with two appropriate materials ([3], [26], [27]). A physically realistic value for α is $3\alpha \approx -0.1$ ([7], [10], [38]), so the CQNLS cannot really be thought of as a small perturbation of the NLS. Equation (1.2) describes an ideal fiber; therefore, it is natural to consider the perturbed CQNLS (PCQNLS)

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + 3\alpha|\phi|^4\phi = i\epsilon(d_1\phi_{xx} + d_2\phi + d_3|\phi|^2\phi + d_4|\phi|^4\phi), \tag{1.3}$$

where $\epsilon > 0$ is small and the other parameters are real and of $O(1)$. The nonnegative parameter d_1 describes spectral filtering, d_2 describes the linear gain ($d_2 > 0$) or loss ($d_2 < 0$) due to the fiber, and d_3 and d_4 describe the nonlinear gain or loss due to the fiber.

The second equation is the parametrically-forced Schrödinger equation (PFNLS)

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^*) = 0, \tag{1.4}$$

where $\epsilon > 0$ is not necessarily small, $\gamma > 0$ is the dissipation factor (linear loss), and $\mu > 0$ is the parametric gain. It models the effect of linear loss and its compensation by phase-sensitive amplification ([8], [23], [25], [28], [30]). The PFNLS equation is valid when discussing optical fiber rings in which the length of the fiber loop is much less than the dispersion and loss lengths ([30]).

Existence of solitary waves is known for these equations; in fact, there is an analytic expression for the wave for each of the above equations ([29], [30], [35], [39]). We shall be interested in their stability. The nonlinear Schrödinger equations (1.1) and (1.2) are both infinite-dimensional Hamiltonian systems. Their linearization around a solitary-wave solution therefore has essential spectrum on the imaginary axis. In addition, the spectrum will contain several isolated eigenvalues of finite multiplicity. In particular, zero is such an eigenvalue by translation invariance. The major tool for tracking these eigenvalues upon adding perturbations is the Evans function (see [1]). However, the essential spectrum is more difficult to handle. While the essential spectrum itself is readily computed upon perturbations ([9, appendix to Section 5]), it is possible that eigenvalues may bifurcate from the essential spectrum. It is the problem of detecting such eigenvalues which is the primary issue of the present paper. Note that the perturbations mentioned above are in general not bounded and do not preserve the Hamiltonian structure of the system.

The issue is the detection of eigenvalues which are either embedded in the essential spectrum or which bifurcate from the essential spectrum upon adding perturbations. Investigating the FitzHugh-Nagumo equation, Jones [11] accomplished this task by extending the Evans function through the essential spectrum in an analytic fashion. He then showed that the extended Evans function has no zeros and therefore no eigenvalues can bifurcate from the essential spectrum. Pego and Weinstein [31] generalized this idea to a large class of equations. The interested reader should also consult Jones *et al.* [12], Kapitula ([15], [16], [17]), and Rubin [34] for other problems in which an extended Evans function has been used in stability calculations.

It is instructive to take a moment to understand the manner in which the Evans function has been extended across the continuous spectrum. Writing the eigenvalue equation under consideration as a first-order system, one obtains

$$Y' = M(\lambda, x)Y, \qquad Y \in \mathbb{R}^n,$$

where the matrix $M(\lambda, x)$ is analytic in λ. Since the solitary wave converges to a constant state as $|x| \to \infty$, the matrix $M_0(\lambda) = \lim_{|x|\to\infty} M(\lambda, x)$ exists and is also analytic in λ. By Henry's result [9, appendix to Section 5], λ is in the essential spectrum if, and only if, $M_0(\lambda)$ has eigenvalues on the imaginary axis. The Evans function is a priori only defined if the eigenvalues of $M_0(\lambda)$ have nonzero real part. Generalizing an idea of Jones [11], Pego and Weinstein [31] were able to extend the Evans function across the essential spectrum provided $M_0(\lambda)$ has precisely one eigenvalue with positive real part when λ is to the right of the essential spectrum. The class of equation considered in their work as well as in subsequent articles contains the KdV equation and other related systems. If there are several eigenvalues of $M_0(\lambda)$ on the imaginary axis for λ in the essential spectrum, and if these eigenvalues do not move all into the same half plane when λ moves off the essential spectrum, their method fails. In particular, the method is not applicable to equations of Schrödinger-type.

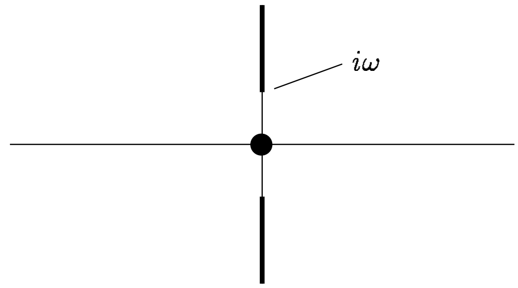

Figure 1: The spectrum for the NLS. The point $\lambda = 0$ is an isolated eigenvalue with algebraic multiplicity four. The rest of the spectrum is continuous spectrum, which is the curves $|\operatorname{Im}\lambda| \geq \omega$.

Consider the generalized perturbed NLS equation

$$i\partial_t\phi + (\partial_x^2 - \omega)\phi + f(|\phi|^2)\phi = i\epsilon d_1\partial_x^2\phi + i\epsilon R(\phi, \phi^*),$$

where $f(\eta)$ is real-valued and smooth with $f(0) = 0$, $\epsilon \geq 0$, and $R(\mu, \eta)$ is real-valued and smooth. Suppose that there exists a bright solitary-wave solution, $\Phi(x, \epsilon)$, which exists for $0 \leq \epsilon < \epsilon_0$. When $\epsilon = 0$, the continuous spectrum is given by the curves $|\operatorname{Im}\lambda| \geq \omega$. The Evans function is defined for $\operatorname{Re}\lambda > 0$ and on the strip $|\operatorname{Im}\lambda| < \omega$. Assuming that the spectral structure is understood when $\epsilon = 0$, in order to understand the spectrum for ϵ nonzero we must have a way of locating the possible points for which point spectrum can bifurcate from the continuous spectrum. As mentioned above, one such method is to extend the Evans function across the continuous spectrum, and then locate its zeros. However, as noted in the previous paragraph, it is not immediately clear that such an extension is possible. This leads us to the following theorem.

Theorem 1.1 *Consider the generalized perturbed NLS when $\epsilon = 0$. There exists an $M > 0$ such that the Evans function can be extended onto the strip*

$$|\operatorname{Im}\lambda| > \omega, \quad -M \leq \operatorname{Re}\lambda \leq 0$$

in an analytic fashion. Furthermore, it has a continuous limit at $\lambda = \pm i\omega$.

Remark 1.2 *The Evans function can also be extended for $\epsilon \neq 0$. The interested reader should consult Theorem 2.27 for an exact statement.*

Remark 1.3 *The extension is valid for a large class of problems, in which the generalized perturbed NLS is a subset (see Section 2).*

Another important consideration regarding the eigenvalue problem is that one must determine if it is possible for large eigenvalues to bifurcate out of the continuous spectrum. This is an important consideration for numerical calculations of the spectrum. Consider the linear operator

$$L = D(\mu)\partial_x^2 + N(\mu, x),$$

where the matrices are smooth in the parameters and the matrix $N(\mu, x)$ decays exponentially fast to constant matrices $N_{\pm}(\mu)$ as $x \to \pm\infty$. Suppose that the diffusivity matrix

$D(\mu)$ is diagonalizable with eigenvalues $\gamma_1(\mu), \ldots, \gamma_n(\mu)$. If $\mathrm{Re}\,\gamma_i(\mu) > 0$ for $i = 1, \ldots, n$, then by following the proof presented in Alexander *et al.* [1] it can be shown that there is an $M > 0$ and a $0 < \delta \ll 1$ such that if $|\lambda| > M$ with $|\arg\lambda| < \pi/2 + \delta$, then λ is not an eigenvalue. However, if $\mathrm{Re}\,\gamma_i(\mu) = 0$ for some i, which is the case for the linear operators associated with the PCQNLS and PFNLS (for example), then the proof only works for $\delta = 0$. Thus, it may be possible for arbitrarily large eigenvalues to bifurcate out of the continuous spectrum for Schrödinger-type operators represented by the linear operator L.

Theorem 1.4 *Suppose that $D(\mu)$ is a diagonalizable matrix whose eigenvalues satisfy*

$$|\arg\gamma_i(\mu)| \leq \pi/2, \quad i = 1, \ldots n.$$

There then exists an $M_1 > 0$ and $M_2 > 0$ such that if $|\lambda| > M_1$ and $\mathrm{Re}\,\lambda \geq -M_2$, then λ cannot be an eigenvalue for L whose corresponding eigenfunction is localized.

With the above theorems in hand, we now know that it is sufficient to look in bounded regions of the complex plane when looking for eigenvalues of perturbed NLS equations. The spectrum for the NLS is completely understood. The point $\lambda = 0$ is an isolated eigenvalue with geometric multiplicity two and algebraic multiplicity four, and the rest of the spectrum is continuous spectrum, which is the curves $|\mathrm{Im}\,\lambda| \geq \omega$ (see Figure 1). Furthermore, there are no eigenvalues embedded in the continuous spectrum ([20], [21]). When considering perturbations of the NLS, one must track the eigenvalues which are near zero in addition to locating any eigenvalues which may bifurcate out of the continuous spectrum. For the PCQNLS, two of the eigenvalues near zero will leave the origin and be real and of $O(\epsilon)$, while the other two will remain at the origin. Recently, Kapitula [14] was able to determine the location of the $O(\epsilon)$ eigenvalues, and showed that in a certain region of the (d_1, d_2, d_3, d_4) parameter space they both move into the left-half of the complex plane (see Lemma 5.1 for a complete statement). Therefore, assuming that the continuous spectrum moves into the left-half plane under perturbation, which will be the case if $d_1 > 0$ and $d_2 < 0$, then in order to determine the stability of the wave it is only necessary to locate any eigenvalues which happen to move out of the continuous spectrum. This problem was the original motivation for this paper.

Theorem 1.5 *Suppose that $0 < \epsilon \ll -\beta \ll 1$, where $\beta = \alpha\omega$. Assume that the parameter d_3 satisfies the existence condition specified in Lemma 5.1. Suppose that $d_1 > 0$ and that*

$$C_{\omega,\beta}(d_4 - \alpha d_1) < d_2 < 0,$$

where

$$C_{\omega,\beta} = \frac{2}{15}\omega^2\left(1 - \frac{22}{21}\beta + O(\beta^2)\right).$$

Then the solitary-wave solution Φ to the PCQNLS is orbitally exponentially stable, i.e., if $\|\phi_0 - \Phi\|$ is sufficiently small, then there exists a $b > 0$ and constants $\tau, \theta \in \mathbb{R}$ such that $\|\phi(t,\cdot) - \Phi(\cdot + \tau)e^{i\theta}\| \leq Ce^{-bt}$. Here $\|\cdot\|$ denotes the L^2-norm.

Remark 1.6 *It is shown in Lemma 4.4 that an eigenvalue bifurcates out of the continuous spectrum only if $\alpha > 0$. The wave will be stable for $\alpha > 0$ if it can be shown that this eigenvalue moves into the left-half plane for $\epsilon > 0$. The framework for this calculation is present in this paper, and we leave the actual calculation to the interested reader.*

Remark 1.7 *Since $\alpha < 0$, as a consequence of the above theorem a minimal condition on the term d_4 is that it must be negative for the wave to be stable. Furthermore, there must be a balance between the linear loss term d_2 and the nonlinear loss term d_4.*

Now consider the PFNLS

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^*) = 0.$$

The solitary-wave solution is given by

$$\Phi(x, \omega, \epsilon) = \sqrt{\frac{\beta}{2}}\operatorname{sech}(\sqrt{\beta}\, x), \tag{1.5}$$

where

$$\beta = \omega + \epsilon\mu \sin 2\theta, \quad \cos 2\theta = \frac{\gamma}{\mu}.$$

When considering the PFNLS, three of the eigenvalues will leave the origin and be of $O(\epsilon)$, and only one will remain. The reason that an extra eigenvalue leaves the origin is due to the fact that $\mu > 0$ breaks the rotational symmetry of the NLS. The location of the $O(\epsilon)$ eigenvalues is known for all $\epsilon > 0$ ([2], [24]). If $\mu \sin 2\theta < 0$, then there will be a positive real eigenvalue, while if $\mu \sin 2\theta > 0$, there will be an eigenvalue located at $\lambda = -2\epsilon\gamma$ and a complex conjugate pair located on the line $\operatorname{Re}\lambda = -\epsilon\gamma$. When $\mu \sin 2\theta > 0$, we will locate any eigenvalues which move out of the continuous spectrum, at least for $\epsilon > 0$ sufficiently small. In particular, we will show that only one complex conjugate pair leaves the continuous spectrum for $\epsilon > 0$ sufficiently small. Due to the symmetries associated with PFNLS, we will then be able to conclude that these eigenvalues are located on the line $\operatorname{Re}\lambda = -\epsilon\gamma$.

Theorem 1.8 *Consider the PFNLS. If $0 < \epsilon \ll 1$ and if $\mu \sin 2\theta > 0$, then the wave is orbitally exponentially stable, i.e., if $\|\phi_0 - \Phi\|$ is sufficiently small, then there exists a $b > 0$ and a constant $\tau \in \mathbb{R}$ such that $\|\phi(t, \cdot) - \Phi(\cdot + \tau)\| \le Ce^{-bt}$.*

Remark 1.9 *See Figure 2 for its spectrum.*

Now suppose that spectral filtering is added to the physical situation governed by the PFNLS, which means that one will consider the perturbed equation

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^*) = i\delta\phi_{xx}, \tag{1.6}$$

where $\delta > 0$. Note that equation (1.6) is reversible ($\phi(x)$ is a solution if and only if $\phi(-x)$ is) and admits the $\mathbb{Z}_2$-symmetry $\phi \to -\phi$ (ϕ is a solution if and only if $-\phi$ is). By exploiting this feature, we shall be interested in obtaining and proving the stability of multiple solitary-wave solutions. Multiple solitary waves are solutions of (1.6) which are formally constructed by concatenating N widely spaced copies of Φ or $-\Phi$, where $\Phi = \Phi_\delta$ is an $O(\delta)$ correction to the expression given in (1.5). Since Φ and $-\Phi$ are concatenated, N-pulses can be obtained in a variety of ways. Denoting Φ and $-\Phi$ by "up" and "down", respectively, we may then consider arbitrary sequences of ups and downs corresponding to whether Φ or $-\Phi$ is used. Based upon an application of the work of Sandstede *et al.* ([37, Theorems 1, 2, and 4]), we have the following theorem concerning existence and stability of multiple solitary waves of (1.4).

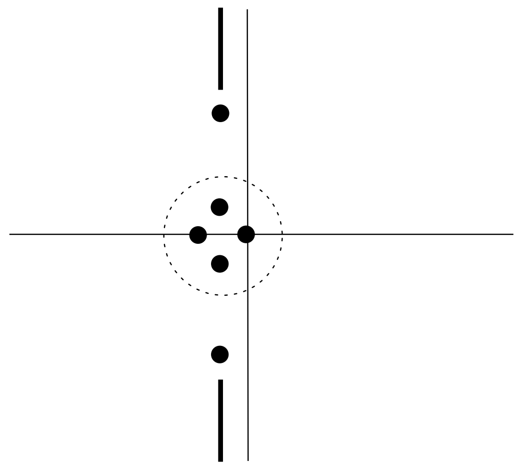

Figure 2: The spectrum for the PFNLS for $\mu \sin 2\theta > 0$. There are four eigenvalues ϵ-close to the origin and two eigenvalues which are ϵ^2-close to the points $-\epsilon\gamma \pm i\omega$ on the line $\operatorname{Re}\lambda = -\epsilon\gamma$.

Theorem 1.10 *Fix $\epsilon > 0$ small and $N > 1$. Suppose that $\mu \sin 2\theta > 0$. For any $0 < \delta < \delta(\epsilon, N) \ll 1$ small there exists a unique multiple solitary wave of up-down-up-down-... type. These pulses are orbitally exponentially stable with respect to equation (1.4). Any other N-pulse consisting of copies of Φ or $-\Phi$ is unstable.*

Remark 1.11 *By Theorem 1.8, the condition $\mu \sin 2\theta > 0$ means that the primary pulse is stable.*

Remark 1.12 *There exist many other N-pulses besides the ones of up-down-up-down-... type, and we refer to [37] for the details.*

Consider the PFNLS with an added quintic term, henceforth known as the parametrically forced cubic-quintic nonlinear Schrödinger equation (PFCQNLS):

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + 3\alpha|\phi|^4\phi + i\epsilon(\gamma\phi - \mu\phi^*) = 0. \tag{1.7}$$

This equation describes the periodic parametric (phase-sensitive) amplification of solitary waves for fibers with a saturable or higher-order refraction index. This equation can be thought of as encompassing the effects of both the CQNLS and the PFNLS. It turns out to be the case that a balancing of the quintic term α with the forcing amplitude ϵ will control the number of eigenvalues which move out of the continuous spectrum. Specifically, as a consequence of Lemma 4.8, if $0 < |\beta|, \epsilon \ll 1$ and

$$\alpha < \frac{8\mu \sin 2\theta}{\omega^2}\,\epsilon,$$

then no eigenvalues bifurcate out of the continuous spectrum. Otherwise, the picture is exactly as that given in Figure 2. As far as we know, this balancing effect between the parametric forcing and possibly destabilizing effect of a positive α has not yet been documented in the literature.

This paper is organized as follows. The Evans function is extended for Schrödinger-type equations in Section 2. Furthermore, some properties of the extended Evans function are derived. This section is of interest in its own right. In Section 3, we explicitly compute the extended Evans function for the cubic nonlinear Schrödinger equation. Eigenvalues bifurcating from the essential spectrum near its end points $\lambda = \pm i\omega$ are calculated in Section 4 for the cubic-quintic and the parametrically-forced Schrödinger equation. In Sections 5 and 6, these results are applied to equations (1.3) and (1.4), respectively. Finally, the existence of stable N-pulses is shown for (1.6).

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2 The extension of the Evans function

In this section, the Evans function is extended across the essential spectrum. The extension is first developed for λ in compact sets. We then consider the case of large λ.

2.1 Preliminaries

In this subsection, we consider a linear system

$$u' = (B(\lambda,\mu) + R(\mu,x))u, \tag{2.1}$$

where $u \in \mathbb{C}^n$, $(\lambda,\mu) \in \Omega \times \mathbb{R}^p$, and $x \in \mathbb{R}$. Here, $\Omega \subset \mathbb{C}$ is open. Assume that the following condition is satisfied.

Hypothesis 2.1 *There exists a vector $\eta(\lambda,\mu)$ such that $B(\lambda,\mu)\eta(\lambda,\mu) = 0$ for all (λ,μ) and $|\eta(\lambda,\mu)| \leq M$ for some M. Moreover, there are numbers $K_1 \geq 1$, $K_2 \geq 0$, $\beta \in \mathbb{R}$, and $\gamma > 0$ with $\gamma > \beta$ such that*

$$\begin{array}{rcll} \|e^{B(\lambda,\mu)x}\| & \leq & K_1 e^{\beta x} & \quad x \in \mathbb{R} \\ \|R(\mu,x)\| & \leq & K_2 e^{\gamma x} & \quad x \leq 0. \end{array}$$

We then have the following result which characterizes solutions decaying with the exponential rate γ to zero as $x \to -\infty$.

Lemma 2.2 *Assume that Hypothesis 2.1 is true. There exists a unique solution $u(\lambda,\mu)(x)$ of (2.1) defined for $x \leq 0$ such that there exists a constant C with*

$$|u(\lambda,\mu)(x) - \eta(\lambda,\mu)| \leq Ce^{\gamma x}$$

as $x \to -\infty$. In addition, we have

$$|u(\lambda,\mu)(x) - \eta(\lambda,\mu)| \leq \frac{2K_1K_2M}{\gamma - \beta} \tag{2.2}$$

uniformly for $x \in (-\infty, x_0]$ with $x_0 \leq 0$ such that

$$\frac{K_1K_2M}{\gamma-\beta}e^{\gamma x_0} \leq \frac{1}{2}.$$

Furthermore, $u(\lambda,\mu)$ is analytic in λ if B and η are. Similarly, if B, η, and R are C^m in μ for some $m \geq 0$ and

$$\|\frac{d^j}{d\mu^j}R(\mu,x)\| \leq C_j e^{\gamma x}, \quad x \leq 0$$

for $j = 1, \ldots, m$, then $u(\lambda,\mu)$ is C^k in μ.

Proof: We seek the desired solution $u(\lambda,\mu)$ in the form $u(\lambda,\mu)(x) = \eta(\lambda,\mu) + v(x)$. The function v will be sought as a solution of the integral equation

$$v(x) = \int_{-\infty}^{x} e^{B(\mu,\lambda)(x-y)} R(\mu,y)(\eta(\lambda,\mu) + v(y))\, dy, \tag{2.3}$$

for $x \in (-\infty, x_0]$ with $x_0 \leq 0$, see also [31, Proposition 1.2]. Note that any solution v of (2.3) satisfies (2.1) by Hypothesis 2.1. We have

$$\begin{aligned} \left|\int_{-\infty}^{x} e^{B(\mu,\lambda)(x-y)} R(\mu,y)\eta(\lambda,\mu)\, dy\right| &\leq K_1 K_2 \int_{-\infty}^{x} e^{\beta(x-y)} e^{\gamma y} |\eta(\lambda,\mu)|\, dy \\ &\leq \frac{K_1 K_2 M}{\gamma - \beta} e^{\gamma x}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \left|\int_{-\infty}^{x} e^{B(\lambda,\mu)(x-y)} R(\mu,y) v(y)\, dy\right| &\leq K_1 K_2 \int_{-\infty}^{x} e^{\beta(x-y)} e^{\gamma y} |v(y)|\, dy \\ &\leq \frac{K_1 K_2}{\gamma - \beta} e^{\gamma x} \|v\|, \end{aligned} \tag{2.4}$$

where

$$\|v\| := \sup_{y \leq x_0} |v(y)|.$$

Set

$$V := C^0(-\infty, x_0).$$

The integral equation (2.3) can be written in the function space V as

$$v = F(\lambda,\mu)(\eta(\lambda,\mu) + v), \tag{2.5}$$

with

$$\begin{aligned} \|F(\lambda,\mu)v\| &\leq \frac{K_1 K_2}{\gamma - \beta} e^{\gamma x_0} \|v\| \\ \|F(\lambda,\mu)\eta(\lambda,\mu)\| &\leq \frac{K_1 K_2 M}{\gamma - \beta}. \end{aligned}$$

Choose $x_0 \leq 0$ such that

$$\frac{K_1 K_2}{\gamma - \beta} e^{\gamma x_0} \leq \frac{1}{2},$$

so that $\|F(\lambda,\mu)\| \leq \frac{1}{2}$ in the operator norm on V. Since $F(\lambda,\mu)$ is then a uniform contraction, we can solve (2.5) and obtain the fixed point v

$$v = (\mathrm{id} - F(\lambda,\mu))^{-1} F(\lambda,\mu)\eta(\lambda,\mu).$$

In particular, we have

$$\|v\| \leq 2\|F(\lambda,\mu)\eta(\lambda,\mu)\| \leq \frac{2K_1K_2M}{\gamma-\beta}.$$

The estimates appearing in the lemma follow now immediately using (2.4).

Finally, the statements about the dependence of the fixed point v on the parameters (λ, μ) are true since the operator $F(\lambda, \mu)$ is then analytic in λ and C^m in μ. ■

2.2 Extension for λ in bounded sets

Consider the linear system

$$\mathbf{Y}' = A(\lambda, x)\mathbf{Y}, \tag{2.6}$$

where $\mathbf{Y} \in \mathbb{C}^n$, and the matrix A is analytic in λ for each fixed x. Here, $\lambda \in \Omega$ where Ω will be specified later in (2.11).

Hypothesis 2.3 *Assume that there exists a constant $\kappa > 0$ and matrices $A_\pm(\lambda)$ such that $A(\lambda, x) - A_\pm(\lambda)$ is independent of λ and*

$$\lim_{x\to\pm\infty} |A(\lambda, x) - A_\pm(\lambda)|e^{\pm 5\kappa x} \leq C, \tag{2.7}$$

where $C > 0$ is a fixed constant.

We begin with some hypotheses on the asymptotic matrices $A_\pm(\lambda)$.

Hypothesis 2.4 *If $\mathrm{Re}\,\lambda > 0$, then for some $1 \leq k < n$ both $A_\pm(\lambda)$ have k eigenvalues of positive real part and $n-k$ eigenvalues with negative real part.*

For $\mathrm{Re}\,\lambda > 0$, define

$$\begin{array}{rcl} \sigma^u_\pm(\lambda) & = & \sigma(A_\pm(\lambda)) \cap \{\mu \in \mathbb{C};\, \mathrm{Re}\,\mu > 0\} \\ \sigma^s_\pm(\lambda) & = & \sigma(A_\pm(\lambda)) \cap \{\mu \in \mathbb{C};\, \mathrm{Re}\,\mu < 0\} \end{array} \tag{2.8}$$

to be the sets corresponding to the k $(n-k)$ eigenvalues of $A_\pm(\lambda)$ with positive (negative) real part.

Hypothesis 2.5 *Let*

$$\Gamma = \bigcup_{j=1}^{N} (ia_j, ib_j) \subset i\mathbb{R},$$

where $a_j \leq b_j \leq a_{j+1}$ for $j = 1, .., N$ are real numbers, be such that if $\lambda \in \Gamma$, then the spectrum of $A_\pm(\lambda)$ is the disjoint union of two sets which are again denoted by $\sigma^u_\pm(\lambda)$ and $\sigma^s_\pm(\lambda)$. Moreover, $\sigma^u_\pm(\lambda)$ and $\sigma^s_\pm(\lambda)$ are the limits of $\sigma^u_\pm(\tilde\lambda)$ and $\sigma^s_\pm(\tilde\lambda)$, respectively, as $\tilde\lambda \to \lambda$ with $\mathrm{Re}\,\tilde\lambda > 0$.

If $\lambda = i\tau \in \Gamma$, it therefore is required that the spectrum of $A_\pm(\lambda)$ is the disjoint union of $\sigma^u_\pm(i\tau)$ and $\sigma^s_\pm(i\tau)$. As a consequence, for fixed τ, there are neighborhoods $U^u_\pm$ and $U^s_\pm$ of $\sigma^u_\pm(i\tau)$ and $\sigma^s_\pm(i\tau)$, respectively, in $\mathbb{C}$ such that any eigenvalue of $A_\pm(\tilde\lambda)$ is contained in either U^u or U^s for any $\tilde\lambda$ close to λ. Indeed, eigenvalues depend continuously on parameters ([19]). Hypothesis 2.5 then states that for all $\tilde\lambda$ close to λ with $\mathrm{Re}\,\tilde\lambda > 0$ any eigenvalue of $A_\pm(\tilde{\lambda})$ which lies in U^u (U^s) has positive (negative) real part. In other words, the sets $\sigma^u_\pm(\lambda)$ and $\sigma^s_\pm(\lambda)$, which were originally defined for $\operatorname{Re}\lambda > 0$, can be continued as disjoint sets for λ in an open neighborhood of Γ, see Figure 3.

In particular, there are numbers $\delta_j(\lambda) \geq 0$, $j = 1, \ldots, n$, such that for any $\lambda \in \tilde{\Sigma}_j$ defined by

$$\tilde{\Sigma}_j := \{\lambda \,:\, a_j < \operatorname{Im}\lambda < b_j,\ -\delta_j(\lambda) < \operatorname{Re}\lambda \leq 0\} \tag{2.9}$$

the spectrum of $A_\pm(\lambda)$ is the disjoint union of two sets $\sigma^u_\pm(\lambda)$ and $\sigma^s_\pm(\lambda)$ which are the continuation of $\sigma^u_\pm(i\tau)$ and $\sigma^s_\pm(i\tau)$ for $i\tau \in (ia_j, ib_j)$.

Set $\Sigma_j \subset \tilde{\Sigma}_j$ to be such that if $\lambda \in \Sigma_j$, then

$$\min\{\operatorname{Re}\mu \,:\, \mu \in \sigma^u_\pm(\lambda)\} > -\frac{\kappa}{n}, \quad \max\{\operatorname{Re}\mu \,:\, \mu \in \sigma^s_\pm(\lambda)\} < \frac{\kappa}{n}. \tag{2.10}$$

Finally, set Ω to be

$$\Omega = \left(\bigcup_{j=1}^{N} \Sigma_j\right) \cup \{\lambda \,:\, \operatorname{Re}\lambda > 0\}. \tag{2.11}$$

Note that Ω is open, simply connected, and $\Gamma \subset \Omega$. Some of the eigenvalues in the sets $\sigma^s_\pm(i\tau)$ and $\sigma^u_\pm(i\tau)$ might be contained in the imaginary axis and we will refer to these eigenvalues as those with small real part. Note that their number may depend on the interval (a_j, b_j) in which $i\tau$ is contained.

The goal of this subsection is to construct an Evans function for $\lambda \in \Omega$ which is an analytic extension of that constructed by Alexander *et al.* [1]. Under the current setup, the Evans function is defined only for those λ with positive real part. The following discussion mirrors much of the presentation of Alexander *et al.* [1].

By setting

$$x = \frac{1}{2\kappa}\ln\left(\frac{1+\tau}{1-\tau}\right),$$

the equation (2.6) becomes the autonomous system

$$\begin{aligned} \mathbf{Y}' &= A(\lambda,\tau)\mathbf{Y} \\ \tau' &= \kappa(1-\tau^2), \end{aligned} \tag{2.12}$$

where $' = d/d\tau$. By Alexander *et al.* [1] we have the following.

Lemma 2.6 *Assuming that equation (2.7) holds true, equation (2.12) is C^1 on $\mathbb{C}^n \times [-1,+1]$.*

If $\mathbf{Y}_1, \ldots, \mathbf{Y}_k$ are solutions of (2.6), then $\mathbf{Y}_1 \wedge \cdots \wedge \mathbf{Y}_k$ is a solution of

$$\mathbf{Y}' = A^{(k)}(\lambda, x)\mathbf{Y},$$

where $A^{(k)}(\lambda,x)$ is the linear derivation on $\Lambda^k\mathbb{C}^n$ induced by $A(\lambda,x)$. As above, this equation can be compactified to

$$\begin{aligned} \mathbf{Y}' &= A^{(k)}(\lambda,\tau)\mathbf{Y} \\ \tau' &= \kappa(1-\tau^2), \end{aligned} \tag{2.13}$$

which is C^1 on $\Lambda^k\mathbb{C}^n \times [-1,+1]$.

Consider the asymptotic systems

$$\mathbf{Y}' = A^{(k)}_\pm(\lambda)\mathbf{Y}. \tag{2.14}$$

The eigenvalues of $A_{\pm}^{(k)}(\lambda)$ are the sums of any k-tuples of eigenvalues of $A_{\pm}(\lambda)$. For $\lambda \in \Omega$, the spectral sets $\sigma_{-}^{u}(\lambda)$ and $\sigma_{+}^{s}(\lambda)$ are well-defined. The spectral projection of $A_{-}(\lambda)$ associated with $\sigma_{-}^{u}(\lambda)$ is denoted by $P_{-}^{u}(\lambda)$. If $\operatorname{Re}\lambda > 0$, it is the spectral projection onto the sum of all generalized eigenspaces of eigenvalues with positive real part. Similarly, $P_{+}^{s}(\lambda)$ denotes the spectral projection of $A_{+}(\lambda)$ associated with $\sigma_{+}^{s}(\lambda)$. Both projections depend analytically on $\lambda \in \Omega$. Set

$$\alpha_{-}(\lambda) = \operatorname{trace}(A_{-}(\lambda)P_{-}^{u}(\lambda)), \quad \alpha_{+}(\lambda) = \operatorname{trace}(A_{+}(\lambda)P_{+}^{s}(\lambda)). \tag{2.15}$$

In particular, $\alpha_{-}(\lambda)$ and $\alpha_{+}(\lambda)$ are analytic in λ. Then $\alpha_{-}(\lambda)$ equals the sum of the eigenvalues (counted with multiplicity) contained in $\sigma_{-}^{u}(\lambda)$. Similarly, $\alpha_{+}(\lambda)$ is the sum of the eigenvalues which lie in $\sigma_{+}^{s}(\lambda)$. If $\operatorname{Re}\lambda > 0$, then $\alpha_{-}(\lambda)$ is the eigenvalue of $A_{-}^{(k)}(\lambda)$ with largest real part, and $\alpha_{+}(\lambda)$ is the eigenvalue of $A_{+}^{(k)}(\lambda)$ with least real part. In addition, if $\operatorname{Re}\lambda > 0$, then $\alpha_{\pm}(\lambda)$ are simple eigenvalues.

Set

$$\mathbf{Z}(\lambda, x) = e^{-\alpha_{-}(\lambda)x}\mathbf{Y}(\lambda, x). \tag{2.16}$$

Then $\mathbf{Z}(\lambda, x)$ satisfies the ODE

$$\mathbf{Z}' = [A^{(k)}(\lambda, x) - \alpha_{-}(\lambda)\,\mathrm{id}]\mathbf{Z},$$

which, as above, can be compactified to

$$\begin{aligned} \mathbf{Z}' &= [A^{(k)}(\lambda, \tau) - \alpha_{-}(\lambda)\,\mathrm{id}]\mathbf{Z} \\ \tau' &= \kappa(1 - \tau^2). \end{aligned} \tag{2.17}$$

This again is a C^1 system on $\Lambda^k\mathbb{C}^n \times [-1, +1]$. In the invariant plane $\{\tau = -1\}$ this reduces to the autonomous system

$$\mathbf{Z}' = [A_{-}^{(k)}(\lambda) - \alpha_{-}(\lambda)\,\mathrm{id}]\mathbf{Z}. \tag{2.18}$$

The critical points are the eigenvectors, $\eta_{-}(\lambda)$, associated with $\alpha_{-}(\lambda)$, that is,

$$[A_{-}^{(k)}(\lambda) - \alpha_{-}(\lambda)\,\mathrm{id}]\eta_{-}(\lambda) = 0.$$

Since $\alpha_{-}(\lambda)$ is a simple eigenvalue of $A_{-}^{(k)}(\lambda)$ for $\operatorname{Re}\lambda > 0$, the associated eigenvector $\eta_{-}(\lambda)$ depends analytically on λ. However, $\alpha_{-}(\lambda)$ is not necessarily simple if $\operatorname{Re}\lambda \leq 0$. Still, there is an analytic continuation of $\eta_{-}(\lambda)$ for $\lambda \in \Omega$. Indeed, we may choose $\eta_{-}(\lambda)$ as the $\Lambda^k\mathbb{C}^n$-representative of the generalized eigenspace $R(P_{-}^{u}(\lambda))$ associated with the eigenvalues in $\sigma_{-}^{u}(\lambda)$.

To be more precise, choose analytic functions $e_1(\lambda), \ldots, e_k(\lambda) \in R(P_{-}^{u}(\lambda))$ for $\lambda \in \Omega$ such that these vectors are linearly independent for any $\lambda \in \Omega$. This is clearly possible, since $P_{-}^{u}(\lambda)$ is analytic for $\lambda \in \Omega$ and Ω is simply connected. Then define

$$\eta_{-}(\lambda) := e_1(\lambda) \wedge \ldots \wedge e_k(\lambda) \in \Lambda^k\mathbb{C}^n,$$

and note that $\eta_{-}(\lambda)$ is analytic and an eigenvector of $A_{-}^{(k)}(\lambda)$ associated with the eigenvalue $\alpha_{-}(\lambda)$.

Now linearize (2.17) at the critical point $(\eta_{-}(\lambda), -1)$. If $\operatorname{Re}\lambda > 0$, then there is exactly one unstable eigenvalue, 2κ, and the associated eigenvector lies in the τ-direction. This is the key which has been used in [1] to define the Evans function. Suppose now that $\lambda \in \Sigma_j$

for some j. We claim that if $\lambda \in \Sigma_j$, any eigenvalue of $A_-^{(k)}(\lambda) - \alpha_-(\lambda)$ has real part strictly less than 2κ. Indeed, let β_- be the eigenvalue of $A_-^{(k)}(\lambda)$ with largest real part. Then β_- is the sum of the k eigenvalues of $A_-(\lambda)$ with largest real part. We number the eigenvalues of $A_-(\lambda)$ according to

$$\begin{aligned}\sigma_\pm^u(\lambda) &= \{\sigma_1^\pm(\lambda), \ldots, \sigma_k^\pm(\lambda)\} \\ \sigma_\pm^s(\lambda) &= \{\sigma_{k+1}^\pm(\lambda), \ldots, \sigma_n^\pm(\lambda)\}\end{aligned}$$

and counted with multiplicity. Then β_- can be estimated by

$$\operatorname{Re}\beta_- - \sum_{i \in J^-(\lambda)} \operatorname{Re}\sigma_i^-(\lambda) < \frac{k}{n}\kappa, \tag{2.19}$$

where $J^-(\lambda)$ denotes the set of indices $1 \leq i \leq k$ which correspond to eigenvalues with positive real part. Indeed, for $\lambda \in \Sigma_j$, some of the $\sigma_i^-(\lambda)$ with $i \leq k$ may have crossed the imaginary axis. They are then possibly replaced by eigenvalues $\sigma_i^-(\lambda)$ with $i > k$. However, the real part of each of these eigenvalues is less than κ/n by the choice of Σ_j, see (2.10). Therefore, their real parts adds up to at most $\frac{k}{n}\kappa$, and (2.19) is proved. Let

$$\beta_c^- = \beta_- - \alpha_-(\lambda). \tag{2.20}$$

For $\lambda \in \Sigma_j$, using the estimate (2.19) and (2.10), we obtain

$$\operatorname{Re}\beta_c^- = \operatorname{Re}\beta_- - \sum_{i=1}^{k} \operatorname{Re}\sigma_i^-(\lambda) < \frac{2k}{n}\kappa. \tag{2.21}$$

This proves our claim.

Therefore, if $\lambda \in \Sigma_j$, the unstable eigenvalue with largest real part is 2κ, with the eigenvector still pointing in the τ-direction. Thus, for $\lambda \in \Omega$ the point $(\eta_-(\lambda), -1)$ has a one-dimensional strong unstable manifold. Since the tangent vector to this manifold points in the τ-direction, the manifold can be written as a function of τ, say $\mathbf{Z}_-(\lambda, \tau)$, for $-1 \leq \tau \ll 0$. It follows from Lemma 2.2 that $\mathbf{Z}_-(\lambda, \tau)$ is analytic in λ for $\lambda \in \Omega$. By applying the flow associated with (2.17), the solution $\mathbf{Z}_-(\lambda, \tau)$ is well-defined and analytic in λ for $\tau \in [-1, +1)$. By equation (2.16), this then defines a solution

$$\mathbf{Y}_-(\lambda, x) = \mathbf{Z}_-(\lambda, x)e^{\alpha_-(\lambda)x}, \tag{2.22}$$

which has the property that if $\operatorname{Re}\lambda > 0$, then $|\mathbf{Y}_-(\lambda, x)| \to 0$ exponentially fast as $x \to -\infty$. Note that $\mathbf{Y}_-(\lambda, x)$ is analytic in λ for $\lambda \in \Omega$.

Now set

$$\mathbf{Z} = e^{-\alpha_+(\lambda)x}\mathbf{Y}(\lambda, x),$$

where $\mathbf{Y} \in \Lambda^{n-k}\mathbb{C}^n$. Then $\mathbf{Z}(\lambda, x)$ satisfies the ODE

$$\mathbf{Z}' = [A^{(n-k)}(\lambda, x) - \alpha_+(\lambda)\,\mathrm{id}]\mathbf{Z},$$

and in a manner similar to that described above a solution, $\mathbf{Z}_+(\lambda, \tau)$, can be constructed as the strong stable manifold of the point $(\eta_+(\lambda), +1)$, where $\eta_+(\lambda)$ is an analytic eigenvector of $A_+^{(n-k)}(\lambda) - \alpha_+(\lambda)\,\mathrm{id}$ constructed as before using $P_+^s(\lambda)$ instead of $P_-^u(\lambda)$. This in turn yields a solution

$$\mathbf{Y}_+(\lambda, x) = \mathbf{Z}_+(\lambda, x)e^{\alpha_+(\lambda)x}, \tag{2.23}$$

which has the property that if $\operatorname{Re}\lambda > 0$, then $|\mathbf{Y}_+(\lambda, x)| \to 0$ exponentially fast as $x \to +\infty$. Again, $\mathbf{Y}_+(\lambda, x)$ is analytic in λ for $\lambda \in \Omega$.

Define the Evans function to be

$$E(\lambda) = \exp\left(-\int^x \operatorname{trace} A(\lambda, s)\, ds\right) \mathbf{Y}_-(\lambda, x) \wedge \mathbf{Y}_+(\lambda, x), \tag{2.24}$$

which for $\lambda \in \Omega$ has values in $\Lambda^n \mathbb{C}^n \cong \mathbb{C}$. It follows that $E(\lambda)$ is analytic for $\lambda \in \Omega$. We close with the following proposition.

Proposition 2.7 *Suppose that Hypotheses 2.3, 2.4 and 2.5 are true. Then the Evans function as described by equation (2.24) is analytic for $\lambda \in \Omega$, where Ω is described by equation (2.11). If λ is such that* $\operatorname{Re}\lambda > 0$, *then $E(\lambda)$ is the Evans function as constructed by Alexander* et al. *[1].*

Corollary 2.8 *Assume that the matrix $A(\lambda, \mu, x)$ depends in addition on a parameter $\mu \in \mathbb{R}^p$. Suppose that Hypothesis 2.3 is met for any μ and that Hypotheses 2.4 and 2.5 are satisfied for $\mu = 0$. In addition, suppose that $A(\lambda, \mu, x)$ is C^m in μ for some $m \geq 0$ and*

$$\|\frac{d^j}{d\mu^j}(A(\lambda, \mu, x) - A_\pm(\lambda, \mu))\| e^{\pm 5\kappa x} \leq C_j, \quad x \to \pm\infty$$

for $j = 1, \ldots, m$. Take any open subset $\tilde{\Omega}$ of Ω with $\operatorname{clos}\tilde{\Omega} \subset \Omega$. *The Evans function $E(\lambda, \mu)$ exists then for μ close to zero and $\lambda \in \tilde{\Omega}$. Moreover, $E(\lambda, \mu)$ is analytic in λ and C^m in μ.*

Proof: The statements follow easily from the above discussion and Lemma 2.2. ∎

2.3 Extension through branch points

Thus far, we considered regions in the complex plane such that the spectrum of the matrices $A_\pm(\lambda)$ was the disjoint union of the sets $\sigma^u_\pm(\lambda)$ and $\sigma^s_\pm(\lambda)$. In this subsection, we consider the case that the decomposition ceases to exist at an *isolated* point $\lambda \in \mathbb{C}$. In other words, we study neighborhoods of the points ia_j and ib_j appearing in the definition of the set Γ in Hypothesis 2.5.

We do not strive for the most general result possible, but instead restrict ourselves to cases which will arise in the analysis of perturbations of the cubic nonlinear Schrödinger equation. Therefore, let $n = 4$. Consider the linear system

$$\mathbf{Y}' = A(\lambda, \mu, x)\mathbf{Y}, \tag{2.25}$$

where $\mathbf{Y} \in \mathbb{C}^4$, and the matrix A is analytic in λ and smooth in $\mu \in \mathbb{R}^p$ for each fixed x. We assume that Hypotheses 2.3 and 2.4 are met with $k = 2$ for any small μ. In addition, suppose that $A_\pm(\lambda, \mu) = A(\lambda, \mu)$.

We start with the following assumption on the asymptotic matrix $A(\lambda, \mu)$. Set

$$K := \{\lambda : |\lambda - i\omega| \leq \delta, \operatorname{Re}\lambda \geq 0\}, \quad \hat{K} := K \setminus \{i\omega\}.$$

The point $i\omega$ should be thought of as a point $a_j = b_j = \omega$ in Hypothesis 2.5.

Hypothesis 2.9 *For $\lambda \in \hat{K}$ and any μ close to zero, the eigenvalues of $A(\lambda,\mu)$ can be written as continuous functions such that*

$$\sigma_1(\lambda,\mu),\, \sigma_2(\lambda,\mu) \in \sigma^u(\lambda,\mu), \quad \sigma_3(\lambda,\mu),\, \sigma_4(\lambda,\mu) \in \sigma^s(\lambda,\mu)$$

are disjoint. Moreover,

$$\operatorname{Re}\sigma_2(\lambda,\mu) \geq \delta > 0, \quad -\operatorname{Re}\sigma_4(\lambda,\mu) \geq \delta > 0$$

uniformly in $\lambda \in K$ and μ. Suppose that $\sigma_1(\lambda,0), \sigma_3(\lambda,0) \to 0$ as $\lambda \to i\omega$ such that the kernel of $A(i\omega,0)$ is one-dimensional. Also, assume that

$$\operatorname{Re}\sigma_1(\lambda,\mu) > 0, \quad -\operatorname{Re}\sigma_3(\lambda,\mu) > 0$$

for $\mu \neq 0$ and $\lambda \in K$.

We can then extend the Evans function $E(\lambda,\mu)$ as a continuous function in $\lambda \in K$ and μ.

Lemma 2.10 *Assume that Hypothesis 2.9 is met. There exists then an extension of the Evans function $E(\lambda,\mu)$ defined for $\lambda \in K$ and any μ close to zero such that $E(\lambda,\mu)$ is continuous in $\lambda \in K$ and μ.*

Proof: The eigenvalues of the matrix $A(\lambda,\mu)$ are simple for $(\lambda,\mu) \neq (i\omega,0)$ by Hypothesis 2.9. For $(\lambda,\mu) \neq (i\omega,0)$, denote the normalized eigenvectors of the matrix $A(\lambda,\mu)$ associated with $\sigma_j(\lambda,\mu)$ by $v_j^u(\lambda,\mu)$, where $j = 1,2$. It is clear from Hypothesis 2.9 that the eigenvector $v_2^u(\lambda,\mu)$ is continuous in $(\lambda,\mu) \in K \times \mathbb{R}^p$.

The kernel of $A(i\omega,0)$ is one-dimensional by Hypothesis 2.9 and therefore spanned by the normalized vector $\hat{v}_1^u$. We have

$$(A(\lambda,\mu) - A(i\omega,0))v_1^u(\lambda,\mu) + A(i\omega,0)v_1^u(\lambda,\mu) = A(\lambda,\mu)v_1^u(\lambda,\mu) = \sigma_1(\lambda,\mu)v_1^u(\lambda,\mu).$$

Since $\sigma_1(\lambda,\mu) \to 0$ as $(\lambda,\mu) \to (i\omega,0)$, $|v_1^u(\lambda,\mu)| = 1$, and $A(\lambda,\mu)$ is smooth in (λ,μ), we see that $A(i\omega,0)v_1^u(\lambda,\mu) \to 0$ as $(\lambda,\mu) \to (i\omega,0)$. Therefore, possibly after multiplying $\hat{v}_1^u$ with -1, the limit

$$\lim_{(\lambda,\mu)\to(i\omega,0)} v_1^u(\lambda,\mu) = \hat{v}_1^u$$

exists. Indeed, without loss of generality, the restriction of $A(i\omega,0)$ to its generalized kernel is given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and so the sign of $\langle v_1^u(\lambda,\mu), \hat{v}_1^u \rangle$ is not zero for μ small.

Therefore, we can extend $v_1^u(\lambda,\mu)$ continuously to $(\lambda,\mu) = (i\omega,0)$ by setting $v_1^u(i\omega,0) = \hat{v}_1^u$. We can then proceed as in Section 2.2 upon defining

$$\eta_-(\lambda,\mu) = v_1^u(\lambda,\mu) \wedge v_2^u(\lambda,\mu).$$

Continuity of the resulting Evans function follows from Lemma 2.2. ■

Finally, we consider differentiable extensions of the Evans function. Set

$$U := \{\lambda;\ |\lambda - i\omega| \leq \delta\} \setminus \{\lambda;\ \operatorname{Im}\lambda = i\omega,\ \operatorname{Re}\lambda < 0\}, \quad \hat{U} := U \setminus \{i\omega\}.$$

Hypothesis 2.11 *For $\lambda \in U$ and any μ, the eigenvalues of $A(\lambda, \mu)$ are independent of μ. They can be written as continuous functions such that*

$$\sigma_1(\lambda),\, \sigma_2(\lambda) \in \sigma^u(\lambda), \quad \sigma_3(\lambda),\, \sigma_4(\lambda) \in \sigma^s(\lambda)$$

are disjoint for $\lambda \in \hat{U}$. Moreover,

$$\operatorname{Re} \sigma_2(\lambda) \geq \delta > 0, \quad -\operatorname{Re} \sigma_4(\lambda) \geq \delta > 0$$

uniformly in $\lambda \in U$. Suppose that $\sigma_1(\lambda), \sigma_3(\lambda) \to 0$ as $\lambda \to i\omega$ such that the kernel of $A(i\omega, 0)$ is one-dimensional and spanned by the nonzero vector $\hat{v}_1(\mu)$.

Lemma 2.12 *Assume that Hypothesis 2.11 is met. There exists then an extension of the Evans function $E(\lambda, \mu)$ defined for $\lambda \in U$ and μ close to zero such that $E(\lambda, \mu)$ is continuous in $\lambda \in U$ and μ. Moreover, $E(\lambda, \mu)$ is differentiable in μ, and its derivative is continuous in (λ, μ).*

Proof: Again, we want to extend the 2-form $\eta_-(\lambda, \mu) = v_1^u(\lambda, \mu) \wedge v_2^u(\lambda, \mu)$ in a smooth fashion to the point $\lambda = i\omega$. A priori, the above 2-form is defined for $\lambda \in \hat{U}$ and $\mu \in \mathbb{R}^p$, and it is C^1 in μ with its derivative being continuous in λ. We can extend $\eta_-(\lambda, \mu)$ to $\lambda = i\omega$ by

$$\eta_-(i\omega, \mu) := \hat{v}_1(\mu) \wedge v_2^u(i\omega, \mu).$$

Note that $\hat{v}_1(\mu)$ is smooth in μ. It suffices therefore to show that $\eta_-(\lambda, \mu)$ is C^1 in μ for any $\lambda \in U$ with its derivative being continuous in (λ, μ).

On account of Hypothesis 2.11, we may assume that

$$A(i\omega, \mu) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

for any small μ with $\hat{v}_1(\mu) = (1,0)^*$. Writing $v_1^u(\lambda, \mu) = (1,0)^* + w(\lambda, \mu)$, we shall show that $w(\lambda, \mu)$ can be chosen such that it is C^1 in μ and continuous in λ. Set

$$B(\lambda, \mu) := A(\lambda, \mu) - A(i\omega, \mu),$$

and consider the following system

$$\begin{aligned} \langle (1,0)^*, w \rangle &= 0 \\ \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + B(\lambda, \mu) - \sigma_1(\lambda)\, \mathrm{id} \right] w &= (B(\lambda, \mu) + \sigma_1(\lambda)\, \mathrm{id})(1,0)^*. \end{aligned}$$

Since $\sigma_1(\lambda)$ is a simple eigenvalue of $A(\lambda, \mu)$ for $\lambda \in \hat{U}$ and any μ, we know that the above system has a unique solution. This solution can be easily obtained using the implicit function theorem and the claim follows. We omit the details. ∎

2.4 No large eigenvalues

Consider the linear eigenvalue problem $LP = \lambda P$, where

$$L = D(\mu)\partial_x^2 + N(\mu, x). \tag{2.26}$$

The goal of this subsection is to show that if λ is large, then the Evans function can be constructed as in Section 2.2. Furthermore, it will be shown that the extended Evans function will be nonzero for λ large uniformly in μ. We assume that the $n \times n$ matrix $N(\mu, x)$ is smooth in x, and that there exist asymptotic matrices $N_{\pm}(\mu)$ and a $\kappa > 0$ such that

$$\lim_{x \to \pm\infty} |N(\mu, x) - N_{\pm}(\mu)| e^{\pm 5\kappa x} \leq C. \tag{2.27}$$

Assume that the matrices $N(\mu, x)$, $N_{\pm}(\mu)$, and $D(\mu)$ are continuous in μ.

Hypothesis 2.13 *The eigenvalues $\gamma_1(\mu), \ldots, \gamma_n(\mu)$ of $D(\mu)$ are nonzero and satisfy*

$$|\arg \gamma_i(\mu)| \leq \pi/2$$

for all μ. Furthermore, assume that $D(\mu)$ is diagonalizable for any μ.

If $\mathbf{Y} = [P, Q]^T$, where $Q = P'$, the eigenvalue problem can be rewritten as the system

$$\mathbf{Y}' = A(\lambda, \mu, x)\mathbf{Y}, \tag{2.28}$$

where

$$A(\mu, \lambda, x) = \begin{bmatrix} 0 & \mathrm{id}_n \\ D^{-1}(\mu)(\lambda\, \mathrm{id}_n - N(\mu, x)) & 0 \end{bmatrix}.$$

As a consequence of (2.27), the matrix $A(\mu, \lambda, x)$ satisfies equation (2.7); therefore, (2.28) can be compactified as

$$\begin{aligned} \mathbf{Y}' &= A(\mu, \lambda, \tau)\mathbf{Y} \\ \tau' &= \kappa(1 - \tau^2). \end{aligned} \tag{2.29}$$

Set

$$r = |\lambda|^{-1/2}, \quad z = \frac{x}{r}, \quad \tilde{Q} = rQ.$$

Upon setting $\tilde{\mathbf{Y}} = [P, \tilde{Q}]^T$, equation (2.29) becomes

$$\begin{aligned} \tilde{\mathbf{Y}}' &= A(\mu, \lambda, r, \tau)\tilde{\mathbf{Y}} \\ \tau' &= r\kappa(1 - \tau^2), \end{aligned} \tag{2.30}$$

where now $' = d/dz$ and

$$A(\mu, \lambda, r, \tau) = \begin{bmatrix} 0 & \mathrm{id}_n \\ D^{-1}(\mu)(e^{i \arg \lambda}\, \mathrm{id}_n - r^2 N(\mu, \tau)) & 0 \end{bmatrix}. \tag{2.31}$$

Note that $A(\mu, \lambda, r, \tau)$ is smooth in the last three parameters. Letting $\nu_i(\mu) = 1/\gamma_i(\mu)$, $i = 1, \ldots, n$, denote the eigenvalues of $D^{-1}(\mu)$, we have the following lemma. Note that $\arg \nu_i = -\arg \gamma_i$, and that $|\nu_i| = 1/|\gamma_i|$.

Lemma 2.14 *Set*

$$A_{\pm}(\mu, \lambda, r) = \lim_{\tau \to \pm 1} A(\mu, \lambda, r, \tau).$$

The eigenvalues of $A_{\pm}(\mu, \lambda, 0)$ are given by

$$\begin{aligned} \sigma_j^-(\mu, \lambda, 0) &= +|\nu_j(\mu)|^{1/2} \exp(i\,(\arg \nu_j(\mu) + \arg \lambda)/2), & j &= 1, \ldots, n \\ \sigma_j^-(\mu, \lambda, 0) &= -|\nu_j(\mu)|^{1/2} \exp(i\,(\arg \nu_j(\mu) + \arg \lambda)/2), & j &= n+1, \ldots, 2n \\ \sigma_j^+(\mu, \lambda, 0) &= \sigma_j^-(\mu, \lambda, 0), & j &= 1, \ldots, 2n. \end{aligned}$$

Furthermore, for $j = 1, \ldots, n$

$$\sigma_j^{\pm}(\mu, \lambda, r) = \sigma_j^{\pm}(\mu, \lambda, 0) + O(r^2).$$

Proof: The eigenvalues σ of $A_\pm(\lambda, 0)$ satisfy the characteristic equation

$$|(e^{\arg\lambda} D^{-1}(\mu) - r^2 N_\pm(\mu)) - \sigma^2 \, \mathrm{id}_n| = 0,$$

from which one immediately gets the first part of the proposition. The second part follows from [19, Theorem II.5.11], since by Hypothesis 2.13 the matrix $D^{-1}(\mu)$ is diagonalizable. ∎

As a consequence of Lemma 2.14, if $\mathrm{Re}\,\lambda > 0$, then the eigenvalues $\sigma_j^\pm(0, \lambda, 0)$ are ordered according to equation (2.8), that is, $\mathrm{Re}\,\sigma_j^\pm(0, \lambda, 0) > 0$ for $j = 1, \ldots, n$ and $\mathrm{Re}\,\sigma_j^\pm(0, \lambda, 0) < 0$ for $j = n+1, \ldots, 2n$. Following the previous argument, in order to extend the Evans function across the imaginary axis, we must have the following: there exists a smooth positive function $\theta(r)$, with $\theta(r) \to 0$ as $r \to 0^+$, such that if $|\arg\lambda| < \pi/2 + \theta(r)$, then

$$\min_{j=1,\ldots,n} \mathrm{Re}\,\sigma_j^\pm(\mu, \lambda, r) > -\frac{\kappa}{2n} r, \quad \max_{j=n+1,\ldots,2n} \mathrm{Re}\,\sigma_j^\pm(\mu\lambda, r) < \frac{\kappa}{2n} r \tag{2.32}$$

uniformly in μ.

Lemma 2.15 *There exists an $r_0 > 0$ such that for any λ with*

$$|\arg\lambda| < \pi/2 + \frac{\kappa}{4n\nu^*} r$$

and $r < r_0$, equation (2.32) is satisfied. Here,

$$\nu^* = \max_{j=1,\ldots n} |\nu_j|^{1/2} = \min_{j=1,\ldots n} |\gamma_j|^{1/2}. \tag{2.33}$$

In other words, we may take

$$\theta(r) = \frac{\kappa}{4n\nu^*} r.$$

Proof: Without loss of generality, assume that $1 \le j \le n$. As a consequence of Lemma 2.14,

$$\mathrm{Re}\,\sigma_j^\pm(\mu, \lambda, r) = |\nu_j|^{1/2} \cos(\frac{1}{2}(\arg\nu_j + \arg\lambda)) + O(r^2),$$

so that equation (2.32) will be satisfied if for $0 \le r \ll 1$,

$$\cos(\frac{1}{2}(\arg\nu_j + \arg\lambda)) > -\frac{\kappa}{4n|\nu_j|^{1/2}} r. \tag{2.34}$$

Equation (2.34) will in turn be satisfied if

$$\begin{aligned} |\arg\lambda| &< 2\,\mathrm{Cos}^{-1}\left(-\frac{\kappa}{4|\nu_j|^{1/2} n} r\right) - |\arg\nu_j| \\ &= \pi - |\arg\nu_j| + \frac{\kappa}{2|\nu_j|^{1/2} n} r + O(r^2). \end{aligned} \tag{2.35}$$

Using the definition

$$\nu^* = \max_{j=1,\ldots n} |\nu_j|^{1/2} = \min_{j=1,\ldots n} |\gamma_j|^{1/2},$$

one can immediately see that if

$$|\arg\lambda| < \frac{\pi}{2} + \frac{\kappa}{4n\nu^*}r, \tag{2.36}$$

then (2.35) is satisfied. Thus, the function $\theta(r)$ discussed previously can be written as

$$\theta(r) = \frac{\kappa}{4n\nu^*}r, \tag{2.37}$$

and the lemma is proved. ∎

Remark 2.16 *Note that the definition of* $\arg\lambda$ *yields that*

$$\left|\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\right| < \theta(r) \Longleftrightarrow |\arg\lambda| < \frac{\pi}{2} + \theta(r). \tag{2.38}$$

With Lemma 2.15 in hand, the n-form $\mathbf{Y}_-(\mu,\lambda,r,x)$ can now be constructed as in Section 2.2. In a similar manner, the n-form $\mathbf{Y}_+(\mu,\lambda,r,x)$ can be constructed. Thus, for $0 \le r < r_0$ and $|\arg\lambda| < \pi/2 + \theta(r)$ the Evans function

$$E(\mu,\lambda,r) = \mathbf{Y}_-(\mu,\lambda,r,x) \wedge \mathbf{Y}_+(\mu,\lambda,r,x)$$

is well-defined. Since $\tau' = 0$ when $r = 0$, the n-forms $\mathbf{Y}_\pm(\lambda,0,x)$ can be constructed for any λ. As another consequence of Lemma 2.14, it is not difficult to see that if $|\arg\lambda| \le \pi/2$, then $E(\mu,\lambda,0) \neq 0$. We claim that the Evans function is nonzero for all r sufficiently small and $|\arg\lambda| \le \pi/2 + \theta(r)$.

To prove this claim, we proceed as in Section 2.2 and consider the equation

$$\begin{aligned} \mathbf{Z}' &= [A^{(n)}(\mu,\lambda,r,\tau) - \alpha_-(\lambda)\,\mathrm{id}]\mathbf{Z} \\ \tau' &= r\kappa(1-\tau^2). \end{aligned} \tag{2.39}$$

Here $A^{(n)}(\mu,\lambda,r,\tau)$ is induced by the matrix $A(\mu,\lambda,r,\tau)$ given in (2.31). When $r = 0$ the vector field (2.39) is autonomous and a solution is given by $(\eta_-(\lambda),\tau)$ for $\tau \in [-1,1]$. As in Section 2.2, for $r \neq 0$ the eigenvector $\eta_-(\mu,\lambda,r)$ extends. We seek the strong unstable manifold of the point $(\eta_-(\mu,\lambda,r),-1)$ and claim that it is a small perturbation of $\{(\eta_-(\lambda),\tau);\ \tau \in [-1,0]\}$.

Going back to the time variable z, we obtain the system

$$\mathbf{Z}' = [\tilde{A}^{(n)}(\mu,\lambda,r,rz) - \alpha_-(\mu,\lambda,r)\,\mathrm{id}]\mathbf{Z} \tag{2.40}$$

on $\Lambda^{2n}\mathbb{C}^n$, where

$$\tilde{A}(\mu,\lambda,r,rz) = \begin{bmatrix} 0 & \mathrm{id}_n \\ D^{-1}(\mu)(e^{i\arg\lambda}\,\mathrm{id}_n - r^2N(\mu,rz)) & 0 \end{bmatrix}. \tag{2.41}$$

Let $\tilde{A}^{(n)}_-(\mu,\lambda,r)$ be the limit of $\tilde{A}^{(n)}(\mu,\lambda,r,rz)$ as $z \to -\infty$. It is a consequence of the definition of the derivation $A^{(n)}$ and equation (2.27) that

$$\|\tilde{A}^{(n)}(\mu,\lambda,r,rz) - \tilde{A}^{(n)}(\mu,\lambda,r)\| \le Cr^2e^{5r\kappa z} \tag{2.42}$$

as $z \to -\infty$, where the constant C can be chosen independently of (μ,λ,r). In other words, we may write (2.40) according to

$$\mathbf{Z}' = [B(\mu,\lambda,r) + R(\mu,\lambda,r,z)]\mathbf{Z},$$

with

$$\begin{aligned} B(\mu,\lambda,r) &= \tilde{A}^{(n)}(\mu,\lambda,r) - \alpha_-(\mu,\lambda,r)\,\mathrm{id} \\ \|R(\mu,\lambda,r,z)\| &\leq Cr^2 e^{5r\kappa z}. \end{aligned}$$

For $|\arg\lambda| < \pi/2 + \theta(r)$, any eigenvalue of the matrix $B(\mu,\lambda,r)$ has real part less than $r\kappa$; therefore,

$$|e^{B(\mu,\lambda,r)z}| \leq Ce^{r\kappa z}.$$

Also, zero is an eigenvalue of $B(\mu,\lambda,r)$ with eigenvector $\eta_-(\mu,\lambda,r)$.

We may therefore apply Lemma 2.2 with $K_1 = C$, $K_2 = Cr^2$, $\beta = r\kappa$, and $\gamma = 5r\kappa$. As the result, the strong unstable manifold of $\eta_-(\mu,\lambda,r)$ is given by

$$\eta_-(\mu,\lambda,r) + O(r)$$

on $(-\infty, 0]$, since with the above choices we have

$$\frac{K_1K_2}{\gamma-\beta} = C^2 r\frac{1}{4\kappa}$$

and $\eta_-(\mu,\lambda,r)$ is bounded uniformly in (μ,λ,r).

Thus, since $\mathbf{Y}_- = e^{\alpha_- z}\mathbf{Z}_-$, we have that

$$\mathbf{Y}_-(\mu,\lambda,r,0) = \eta_-(\mu,\lambda,r) + O(r).$$

In a similar manner, one can show that

$$\mathbf{Y}_+(\mu,\lambda,r,0) = \eta_+(\mu,\lambda,r) + O(r).$$

Therefore, from the definition of the Evans function we have that

$$\begin{aligned} E(\mu,\lambda,r) &= (\mathbf{Y}_- \wedge \mathbf{Y}_+)(\mu,\lambda,r,0) \\ &= (\eta_- \wedge \eta_+)(\mu,\lambda,0) + O(r) \\ &\neq 0 \end{aligned}$$

for r sufficiently small (a consequence of Lemma 2.14).

Note that the above approach is still valid if the initial estimate on R is weakened to

$$\|R(\mu,\lambda,r,z)\| \leq Cre^{5r\kappa z},$$

for in this case a unique solution is initially guaranteed for $z < z_0 = O((\ln r)/r) \ll 0$, and can be continued for $z > z_0$ by applying the flow. However, the error term in the above identity of $E(\mu,\lambda,r)$ is then $O(1)$ instead of $O(r)$.

Upon going back to the original variables, we can close the discussion in this subsection with the following proposition which is a consequence of Lemma 2.15, (2.38) and the above discussion.

Proposition 2.17 *Suppose that Hypothesis 2.13 and equation (2.27) are met. There then exists an $L > 0$ such that if*

$$|\lambda| > L, \qquad \left|\frac{\mathrm{Re}\,\lambda}{\mathrm{Im}\,\lambda}\right| < \frac{\kappa}{4n\nu^*}|\lambda|^{-1/2},$$

where ν^* is defined in (2.33), then the extended Evans function is well-defined and nonzero.

Remark 2.18 *In particular, the Evans function can then be extended in a nonzero fashion into the strip*

$$0 \geq \mathrm{Re}\,\lambda \geq -\frac{q\kappa}{4n\nu^*}, \qquad |\,\mathrm{Im}\,\lambda| \geq L$$

for some $q = q(L) < 1$.

2.5 Example: perturbed nonlinear Schrödinger equations

Consider the generalized perturbed nonlinear Schrödinger equation

$$i\partial_t \phi + (\partial_x^2 - \omega)\phi + f(|\phi|^2, \alpha)\phi = i\epsilon d_1 \partial_x^2 \phi + i\epsilon R(\phi, \phi^*), \tag{2.43}$$

where $f(\eta, \alpha)$ is real-valued and smooth function with $f(0, \alpha) = 0$, ϵ is nonnegative, and $R(\mu, \eta)$ is real-valued and smooth. Let $\mu = (\alpha, \epsilon)$. Note that this equation encompasses both the perturbed cubic-quintic NLS and the parametrically-forced NLS.

Hypothesis 2.19 *There exists a smooth function $\Phi(x, \mu)$ which is a steady-state solution to (2.43) and satisfies the condition that $|\Phi(x, \mu)| \to 0$ at rate $O(e^{-5\kappa|x|})$ as $|x| \to \infty$. The same estimate is true for the derivative of $\Phi(x, \mu)$ with respect to μ. Furthermore, $\Phi_0(x) = \Phi(x, 0)$ is real-valued.*

Remark 2.20 *In order for the wave to decay exponentially fast, it must be true that when ϵ is small, then $\omega > 0$.*

By setting $\phi = u + iv$, where u and v are real, equation (2.43) can be rewritten as the system

$$\begin{aligned} \partial_t u + (\partial_x^2 - \omega)v + f(u^2 + v^2, \alpha)v &= \epsilon d_1 \partial_x^2 u + \epsilon R_1(u, v) \\ \partial_t v - (\partial_x^2 - \omega)u - f(u^2 + v^2, \alpha)u &= \epsilon d_1 \partial_x^2 v + \epsilon R_2(u, v), \end{aligned} \tag{2.44}$$

where

$$R_1(u, v) = \operatorname{Re} R(u + iv, u - iv), \quad R_2(u, v) = \operatorname{Im} R(u + iv, u - iv).$$

It will be assumed that $d_1 \geq 0$, so that (2.44) will have a well-posed initial-value problem. Upon setting $P = [u, v]^T$ and linearizing, we get the eigenvalue problem

$$\lambda P = D(\epsilon)\partial_x^2 P + (N_0(x, \alpha) + \epsilon N_1(x))P, \tag{2.45}$$

where

$$D(\epsilon) = \begin{pmatrix} \epsilon d_1 & -1 \\ 1 & \epsilon d_1 \end{pmatrix}, \quad N_0(x, \alpha) = \begin{pmatrix} 0 & \omega - f(\Phi_0^2, \alpha) \\ -\omega + f(\Phi_0^2, \alpha) + 2\Phi_0^2 + f'(\Phi_0^2, \alpha) & 0 \end{pmatrix},$$

and $N_1(x)$ is uniformly bounded and approaches an asymptotic matrix N_1^0 exponentially fast as $|x| \to \infty$. When $\epsilon = 0$, the continuous spectrum is given by

$$\Sigma_{ess} = \{\lambda;\ \operatorname{Re}\lambda = 0,\ |\operatorname{Im}\lambda| > \omega\}. \tag{2.46}$$

Indeed, we have that

$$N_0(x, \alpha) \to \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \quad |x| \to \infty,$$

and the limiting matrix is independent of α. We are now ready to prove the following lemma.

Lemma 2.21 *Assume that $d_1 \geq 0$. There exist $\alpha_0 > 0$ and $\epsilon_0 > 0$ (not necessarily small) and positive constants L_1 and L_2 which are independent of α and ϵ such that in the region*

$$|\lambda| \geq L_1, \quad \operatorname{Re}\lambda \geq -L_2,\ 0 < \epsilon < \epsilon_0,\ |\alpha| < \alpha_0,$$

the Evans function $E(\lambda, \alpha, \epsilon)$ for equation (2.43) is defined and nonzero.

Proof: It is a simple matter to check that the eigenvalues of $D(\epsilon)$ satisfy Hypothesis 2.13. The extension of the Evans function and the fact that it will be nonzero for large λ then follows immediately from Proposition 2.17. ∎

Remark 2.22 *Since the zeros of the Evans function locate those eigenvalues with localized eigenfunctions , we know that there will be no large eigenvalues, even if there is no diffusion present.*

Following the procedure of the previous subsection, the matrix $A(\mu, \lambda, x)$ is given by

$$A(\mu, \lambda, x) = \begin{pmatrix} 0 & \mathrm{id}_2 \\ D^{-1}(\epsilon)(\lambda \,\mathrm{id}_2 - N_0(x, \alpha) - \epsilon N_1(x)) & 0 \end{pmatrix},$$

where $\mu = (\alpha, \epsilon)$, and as before set

$$A_0(\epsilon, \lambda) = \lim_{|x| \to \infty} A(\mu, \lambda, x).$$

Note that $A_0(\epsilon, \lambda)$ does not depend on α.

For the moment, assume that $\epsilon = 0$. A routine calculation shows that the eigenvalues of $A_0(0, \lambda)$ are given by

$$\begin{aligned} \sigma_1^{\pm}(0, \lambda) &= \pm\sqrt{|\omega - i\lambda|}\, e^{\frac{i}{2}\arg(\omega - i\lambda)}, \qquad \arg(\omega - i\lambda) \in [-\frac{3\pi}{2}, \frac{\pi}{2}) \\ \sigma_2^{\pm}(0, \lambda) &= \pm\sqrt{|\omega + i\lambda|}\, e^{\frac{i}{2}\arg(\omega + i\lambda)}, \qquad \arg(\omega + i\lambda) \in [-\frac{\pi}{2}, \frac{3\pi}{2}). \end{aligned} \tag{2.47}$$

A simple observation reveals that if $\mathrm{Re}\,\lambda > 0$, then for $i = 1, 2$

$$\mathrm{Re}\,\sigma_i^{+}(0, \lambda) > 0, \quad \mathrm{Re}\,\sigma_i^{-}(0, \lambda) < 0,$$

and $\sigma_i^{\pm}(0, \lambda)$ are analytic across Σ_{ess}. As a consequence of Proposition 2.7, we now have the following lemma.

Lemma 2.23 *Assume that $\epsilon = 0$. Then the Evans function $E(\lambda, \alpha)$ can be extended across Σ_{ess} onto the strip*

$$\omega < |\,\mathrm{Im}\,\lambda| \leq L_1, \quad -L_3 < \mathrm{Re}\,\lambda \leq 0,$$

for some $L_3 > 0$.

Corollary 2.24 *Assume that $\epsilon = 0$, and set*

$$L_4 = \min\{L_2, L_3\},$$

where L_2 is given in Lemma 2.21. Then the Evans function can be extended across Σ_{ess} onto the strip

$$\omega < |\,\mathrm{Im}\,\lambda|, \quad -L_4 < \mathrm{Re}\,\lambda \leq 0.$$

Furthermore, the extended Evans function will be nonzero for $|\lambda| > L_1$.

Remark 2.25 *As it will be seen in the next section, if $f(\eta, \alpha) = 4\eta$, i.e., if one looks at the cubic NLS, then $L_4 = \infty$.*

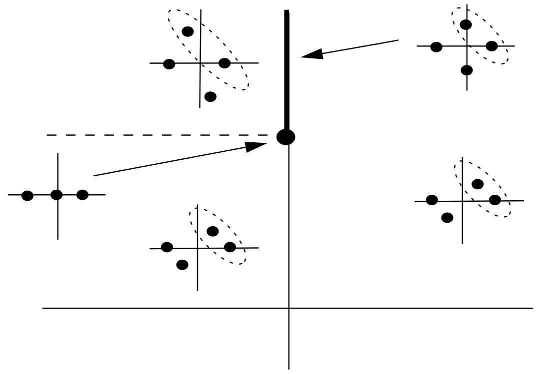

Figure 3: Here, the location of the eigenvalues $\sigma_j^{\pm}(0,\lambda)$ of $A_0(0,\lambda)$ with $j=1,2$ is indicated for λ in various regions of the complex plane. Eigenvalues inside the dotted ellipsoids belong to the unstable spectral set $\sigma^u(\lambda)$. The point $\lambda = i\omega$ corresponds to a branch point where the spectral decomposition ceases to exist. The dashed line emanating from the branch point indicates the cut defined in (2.47).

When $\epsilon = 0$, it is straightforward to prove that Hypotheses 2.9 and 2.11 are met with respect to the parameter α. Indeed, the limiting matrix does not depend on α at all. Applying Lemmata 2.10 and 2.12 then shows that the Evans function $E(\lambda,\alpha)$ is differentiable in α and can be extended to $\lambda = i\omega$. Combining the results obtained so far, we have the following theorem.

Theorem 2.26 *Assume that $\epsilon = 0$. Let*

$$\begin{aligned}\Sigma_1 &= \{\lambda : \operatorname{Re}\lambda > 0\}\\ \Sigma_2 &= \{\lambda : |\operatorname{Im}\lambda| < \omega\}\\ \Sigma_3 &= \{\lambda : |\operatorname{Im}\lambda| > \omega,\ -L_4 \le \operatorname{Re}\lambda \le 0\},\end{aligned}$$

and set

$$\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3. \tag{2.48}$$

The Evans function $E(\lambda,\alpha)$ is defined and analytic for $\lambda \in \Omega$, and is an analytic extension of that constructed by Alexander et al. *[1]. It is nonzero for sufficiently large $|\lambda|$, and has a continuous limit at $\lambda = \pm i\omega$. Finally, it is C^1 in α for $\lambda \in \Omega \cup \{i\omega\}$, and the derivative with respect to α is continuous in λ.*

Now suppose that $\epsilon > 0$ is small. As a consequence of Corollary 2.8 and Lemma 2.21, the following theorem is true.

Theorem 2.27 *Let $\delta > 0$ be given and small. Choose $\tilde\Omega \subset \Omega$ such that* $\operatorname{clos}\tilde\Omega \subset \Omega$ *where Ω is given in (2.48). There then exists an $\epsilon_0 > 0$ such that the Evans function $E(\lambda,\alpha,\epsilon)$ is defined for $0 < \epsilon < \epsilon_0$ and for $\lambda \in \tilde\Omega$. It is analytic for $\lambda \in \tilde\Omega$, smooth in ϵ, and is an extension of that constructed by Alexander* et al. *[1]. Furthermore, it is nonzero for sufficiently large $|\lambda|$.*

Now suppose that the Evans function can be shown to be nonzero if $\epsilon = 0$ and $|\operatorname{Im}\lambda| > \omega$. Then it will necessarily be true that for $0 < \epsilon < \epsilon_0$ there exists a $\delta = \delta(\epsilon) > 0$ such that the extended Evans function will be nonzero for $|\operatorname{Im}\lambda| > \omega + \delta$. Under this scenario it will only be possible for eigenvalues to bifurcate out of the continuous spectrum near $\lambda = \pm i\omega$. It turns out that the Evans function can be extended up to $\lambda = \pm i\omega$ such that it is differentiable in ϵ. A local bifurcation analysis near $\lambda = \pm i\omega$ will then reveal whether and how many eigenvalues bifurcate out of the essential spectrum. This idea will be exploited in the upcoming sections.

3 The Evans function for the cubic NLS

Instead of using the formulation in equation (2.44), we will write the cubic NLS as the system

$$\begin{aligned} i\phi_t + (\partial_x^2 - \omega)\phi + 4\phi^2\psi &= 0 \\ -i\psi_t + (\partial_x^2 - \omega)\psi + 4\phi\psi^2 &= 0, \end{aligned} \tag{3.1}$$

where ψ is defined by $\psi = \phi^*$. The system is written in this way so that the results of Kaup [20] and Kaup *et al.* [21] can be more easily exploited.

The bright solitary-wave solution is given by

$$\Phi(x,\omega) = \sqrt{\frac{\omega}{2}}\operatorname{sech}(\sqrt{\omega}\,x). \tag{3.2}$$

Linearization yields the system

$$iP_t + LP = 0,$$

where

$$L = (\partial_x^2 - \omega)\sigma_3 + 4\Phi^2(2\sigma_3 + i\sigma_2). \tag{3.3}$$

Here σ_2 and σ_3 are the Pauli spin matrices

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Setting $P(x,t) \to P(x)e^{\rho t}$, one then gets the linear eigenvalue problem

$$(L + i\rho)P = 0.$$

Upon setting

$$\lambda = -i\rho,$$

we then get the more conventional eigenvalue problem

$$(L - \lambda)P = 0. \tag{3.4}$$

It is important to note here that the wave will be unstable if there exists an eigenvalue with $\operatorname{Im}\lambda < 0$; hence, we will want to define the Evans function for $\operatorname{Im}\lambda < 0$, and extend it across $\operatorname{Im}\lambda = 0$.

Let $\mathbf{Y} = [P, Q]^T$, where $Q = P'$. Then $\mathbf{Y}$ satisfies the equation

$$\mathbf{Y}' = M(\lambda, x)\mathbf{Y}, \tag{3.5}$$

where

$$M(\lambda, x) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega + \lambda - g(x) & -h(x) & 0 & 0 \\ -h(x) & \omega - \lambda - g(x) & 0 & 0 \end{bmatrix}, \tag{3.6}$$

and

$$\begin{aligned} g(x) &= 8\Phi^2(x,\omega) \\ h(x) &= 4\Phi^2(x,\omega). \end{aligned}$$

Set

$$M_0(\lambda) = \lim_{|x|\to\infty} M(\lambda, x).$$

The eigenvalues of $M_0(\lambda)$ are given by $\pm\gamma_f(\lambda)$ and $\pm\gamma_s(\lambda)$, where

$$\begin{aligned} \gamma_s(\lambda) &= \sqrt{|\omega-\lambda|}\, e^{\frac{i}{2}\arg(\omega-\lambda)}, \qquad \arg(\omega-\lambda) \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \\ \gamma_f(\lambda) &= \sqrt{|\omega+\lambda|}\, e^{\frac{i}{2}\arg(\omega+\lambda)}, \qquad \arg(\omega+\lambda) \in [-\frac{3\pi}{2}, \frac{\pi}{2}), \end{aligned} \tag{3.7}$$

and the associated eigenvectors are $[1, 0, \pm\gamma_f(\lambda), 0]^T$ and $[0, 1, 0, \pm\gamma_s(\lambda)]^T$. The branch cuts of the above functions are being taken so that $\gamma_s(\lambda) > 0$ for $\lambda \in (-\infty, \omega)$, while $\gamma_f(\lambda) > 0$ for $\lambda \in (-\omega, \infty)$. Note that

$$\begin{aligned} \mathrm{Re}\,\lambda > 0 &\;\Rightarrow\; \mathrm{Re}\,\gamma_f(\lambda) > \mathrm{Re}\,\gamma_s(\lambda) \\ \mathrm{Re}\,\lambda < 0 &\;\Rightarrow\; \mathrm{Re}\,\gamma_f(\lambda) < \mathrm{Re}\,\gamma_s(\lambda), \end{aligned}$$

and that the functions are analytic if $\mathrm{Im}\,\lambda < 0$.

As a consequence of Theorem 2.26, we have the following lemma.

Lemma 3.1 *Let*

$$\begin{aligned} \Sigma_1 &= \{\lambda : \mathrm{Im}\,\lambda < 0\} \\ \Sigma_2 &= \{\lambda : |\mathrm{Re}\,\lambda| < \omega\} \\ \Sigma_3 &= \{\lambda : |\mathrm{Re}\,\lambda| > \omega,\ 0 \le \mathrm{Im}\,\lambda < L\}, \end{aligned}$$

and set

$$\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3.$$

There is an $L > 0$ such that the Evans function is defined and analytic for $\lambda \in \Omega$, and is an analytic extension of that constructed by Alexander et al. *[1]. Furthermore, it is nonzero for sufficiently large $|\lambda|$. Finally, it has a continuous limit at $\lambda = \pm\omega$.*

The goal in this section is to explicitly construct the extended Evans function. Once this is accomplished, we will then be able to locate its zeros, and hence be able to determine the location of the eigenvalues which may bifurcate out of the continuous spectrum. Before continuing, we need a couple of preliminary results.

Lemma 3.2 *Let $\mathbf{Y}(\lambda, x) = [P(\lambda, x), Q(\lambda, x)]^T$ be a solution to (3.5). Another solution to (3.5) is then $\mathbf{Y}(\lambda, x) = [P(\lambda, -x), -Q(\lambda, -x)]^T$. A solution to*

$$\mathbf{Y}' = M(\lambda^*, x)\mathbf{Y}$$

is given by $\mathbf{Y}^*(\lambda, x)$. Finally, if $\lambda \in \mathbb{R}$, then a solution to the adjoint problem

$$\mathbf{Z}' = -M^T(\lambda, x)\mathbf{Z}$$

is given by $\mathbf{Z}(\lambda, x) = [-Q(\lambda, x), P(\lambda, x)]^T$.

Proof: The first part follows immediately from the fact that both $g(x)$ and $h(x)$ are even functions. The second part follows as soon as one notices that

$$M(\lambda^*, x)^* = M(\lambda, x).$$

The third part is a simple calculation, and is left to the interested reader. ∎

Lemma 3.3 (Kaup [20], Kaup *et al.* **[21])** *When* $\operatorname{Re}\lambda > 0$*, a solution to (3.4) is given by*

$$\begin{aligned} P^+(\lambda, x) &= -\frac{e^{\gamma_s(\lambda)x}}{(\gamma_s(\lambda) - \sqrt{\omega})^2}\left\{(\lambda - 2\omega + 2\sqrt{\omega}\,\gamma_s(\lambda)\tanh(\sqrt{\omega}\,x))\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right. \\ &\qquad \left. +2\Phi^2(x,\omega,0)\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}. \end{aligned}$$

When $\operatorname{Re}\lambda < 0$*, a solution to (3.4) is given by*

$$\begin{aligned} P^-(\lambda, x) &= \frac{e^{-\gamma_f(\lambda)x}}{(\gamma_f(\lambda) + \sqrt{\omega})^2}\left\{(\lambda + 2\omega + 2\sqrt{\omega}\,\gamma_f(\lambda)\tanh(\sqrt{\omega}\,x))\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right. \\ &\qquad \left. -2\Phi^2(x,\omega,0)\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}. \end{aligned}$$

Furthermore, besides the functions $P^+(\omega + k^2, x)$ *and* $P^-(-(\omega + k^2), x)$*, where* $k \in \mathbb{R}^+$*, along with the eigenfunctions of* L *at* $\lambda = 0$*, there are no other bounded eigenfunctions of* L*.*

Since λ is be an eigenvalue if and only if $-\lambda$ is, it is sufficient to calculate the Evans function only for $\operatorname{Re}\lambda > 0$. For the rest of this discussion assume therefore that $\operatorname{Re}\lambda > 0$. The following arguments can easily be modified for the case $\operatorname{Re}\lambda < 0$.

There exists a unique solution $\mathbf{Y}_f^-$ to (3.5) such that

$$\lim_{x\to-\infty} \mathbf{Y}_f^-(\lambda, x)e^{-\gamma_f(\lambda)x} = \begin{bmatrix} 1 \\ 0 \\ \gamma_f(\lambda) \\ 0 \end{bmatrix}. \tag{3.8}$$

This is due to the fact that $\gamma_f(\lambda)$ is the positive eigenvalue of $M_0(\lambda)$ with largest real part. Similarly, there exists a unique solution $\mathbf{Z}_f^+(\lambda, z)$ to the adjoint problem with the asymptotics

$$\lim_{x\to\infty} \mathbf{Z}_f^+(\lambda, x)e^{\gamma_f(\lambda)x} = \begin{bmatrix} \gamma_f(\lambda) \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{3.9}$$

Define the reduced Evans function

$$E_f(\lambda) = \mathbf{Y}_f^-(\lambda, x)\cdot\mathbf{Z}_f^+(\lambda, x). \tag{3.10}$$

Before continuing, we need the following information regarding the reduced Evans function.

Lemma 3.4 *$E_f(\lambda)$ is analytic and nonzero for* $\operatorname{Re}\lambda > 0$.

Proof: The analyticity follows from the fact that the eigenvalue $\gamma_f(\lambda)$ is simple and thus analytic for $\operatorname{Re}\lambda > 0$ (see Lemma 2.2). In the following, it is important to note that if $E_f(\lambda) = 0$, then

$$\lim_{x\to\infty} |\mathbf{Y}_f^-(\lambda, x)e^{-\gamma_f(\lambda)x}| = 0.$$

First suppose that $\lambda \in (\omega, \infty)$. If $E_f(\lambda) = 0$, then $\mathbf{Y}_f^-$ is a uniformly bounded function which decays exponentially fast as $x \to -\infty$. However, Lemma 3.3 precludes the existence of such a solution.

Now suppose that $\lambda = \omega$. If $E_f(\omega) = 0$, then

$$\lim_{x\to\infty} |\mathbf{Y}_f^-(\omega, x)e^{-\gamma_f(\omega)x}| = 0.$$

Consider the 3-form $\mathbf{Y}_f^- \wedge \mathbf{Y}_s^- \wedge \mathbf{Y}_f^+$. This 3-form induces a solution to the adjoint equation, $\mathbf{Z}$. Since $\mathbf{Y}_f^+(\lambda, x) = \mu[P_f^-(\lambda, -x), -Q_f^-(\lambda, -x)]^T$ for some nonzero constant μ, where $\mathbf{Y}_f^-(\lambda, x) = [P_f^-(\lambda, x), Q_f^-(\lambda, x)]^T$, the adjoint solution then satisfies

$$\lim_{|x|\to\infty} |\mathbf{Z}(\omega, x)| = 0.$$

By Lemma 3.2, this then implies that there exists a solution to (3.5) which decays as $|x| \to \infty$. However, this contradicts Lemma 3.3.

Now suppose that $\lambda \in \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \geq 0,\ \lambda \notin [\omega, \infty)\}$. It is known that there are no eigenvalues to L, which implies by the result of Alexander *et al.* [1] that

$$\lim_{x\to\infty} \mathbf{Y}_f^-(\lambda, x) \wedge \mathbf{Y}_s^-(\lambda, x)e^{-(\gamma_f(\lambda)+\gamma_s(\lambda))x} = \mu[1, 0, \gamma_f(\lambda), 0]^T \wedge [0, 1, 0, \gamma_s(\lambda)]^T \qquad (3.11)$$

for some nonzero constant μ. By equation (3.16) we have

$$\lim_{x\to\infty} \mathbf{Y}_s^-(\lambda, x)e^{-\gamma_s(\lambda)x} = \frac{\lambda - 2\omega - 2\sqrt{\omega}\,\gamma_s(\lambda)}{\lambda - 2\omega + 2\sqrt{\omega}\,\gamma_s(\lambda)}[0, 1, 0, \gamma_s(\lambda)]^T.$$

If $E_f(\lambda) = 0$, then

$$\lim_{x\to\infty} |\mathbf{Y}_f^-(\omega, x)e^{-\gamma_f(\omega)x}| = 0.$$

Thus, in this case

$$\lim_{x\to\infty} |\mathbf{Y}_f^-(\lambda, x) \wedge \mathbf{Y}_s^-(\lambda, x)e^{-(\gamma_f(\lambda)+\gamma_s(\lambda))x}| = 0,$$

which violates (3.11).

It is now known that $E_f(\lambda) \neq 0$ for $\operatorname{Im}\lambda \geq 0$. By Lemma 3.2

$$\begin{aligned} E_f(\lambda^*) &= \mathbf{Y}_f^-(\lambda^*, x) \cdot \mathbf{Z}_f^+(\lambda^*, x) \\ &= (\mathbf{Y}_f^-(\lambda, x))^* \cdot (\mathbf{Z}_f^+(\lambda, x))^* \\ &= E_f(\lambda)^*. \end{aligned}$$

Thus, $E_f(\lambda) \neq 0$ for $\operatorname{Im}\lambda \geq 0$ necessarily implies that the same holds true for $\operatorname{Im}\lambda \geq 0$. ■

Remark 3.5 *The function $E_f(\lambda)$ can be extended to include the imaginary axis.*

Remark 3.6 *$E_f(\lambda) \to 0$ as $\lambda \to 0^+$.*

Using the definition of $E_f(\lambda)$ it is easy to check that

$$\lim_{x\to\infty} \mathbf{Y}_f^-(\lambda,x)e^{-\gamma_f(\lambda)x} = \frac{E_f(\lambda)}{2\gamma_f(\lambda)}[1,0,\gamma_f(\lambda),0]^T. \tag{3.12}$$

Since $E_f(\lambda) \neq 0$, the solution

$$\mathbf{Y}_f^+(\lambda,x) = 2\gamma_f(\lambda)[P_f^-(\lambda,-x), -Q_f^-(\lambda,-x)]^T, \tag{3.13}$$

where $\mathbf{Y}_f^-(\lambda,x) = [P_f^-(\lambda,x), Q_f^-(\lambda,x)]^T$, is well-defined for $\mathrm{Re}\,\lambda > 0$. Note that

$$\lim_{x\to-\infty} \mathbf{Y}_f^+(\lambda,x)e^{\gamma_f(\lambda)x} = E_f(\lambda)[1,0,-\gamma_f(\lambda),0]^T. \tag{3.14}$$

Set

$$\mathbf{Y}_s^-(\lambda,x) = \begin{bmatrix} P^+(\lambda,-x) \\ -Q^+(\lambda,-x) \end{bmatrix}, \quad \mathbf{Y}_s^+(\lambda,x) = \begin{bmatrix} P^+(\lambda,x) \\ Q^+(\lambda,x) \end{bmatrix}, \tag{3.15}$$

where $P^+(\lambda,x)$ is defined in Lemma 3.3. Note that

$$\begin{aligned} \lim_{x\to\mp\infty} \mathbf{Y}_s^\mp(\lambda,x)e^{\mp\gamma_s(\lambda)x} &= [1,0,\pm\gamma_s(\lambda),0]^T \\ \lim_{x\to\pm\infty} \mathbf{Y}_s^\mp(\lambda,x)e^{\mp\gamma_s(\lambda)x} &= \frac{\lambda-2\omega-2\sqrt{\omega}\,\gamma_s(\lambda)}{\lambda-2\omega+2\sqrt{\omega}\,\gamma_s(\lambda)}[1,0,\pm\gamma_s(\lambda),0]^T. \end{aligned} \tag{3.16}$$

For $\mathrm{Re}\,\lambda > 0$ the Evans function is given by

$$E(\lambda) = (\mathbf{Y}_f^- \wedge \mathbf{Y}_s^- \wedge \mathbf{Y}_f^+ \wedge \mathbf{Y}_s^+)(\lambda,x). \tag{3.17}$$

Based upon the above discussion, the Evans function can be explicitly calculated.

Proposition 3.7 *For $\mathrm{Re}\,\lambda > 0$ the Evans function is given by*

$$E(\lambda) = 4E_f(\lambda)\gamma_f(\lambda)\gamma_s(\lambda)\frac{\lambda-2\omega-2\sqrt{\omega}\,\gamma_s(\lambda)}{\lambda-2\omega+2\sqrt{\omega}\,\gamma_s(\lambda)}.$$

The analytic function $E_f(\lambda)$ is nonzero for $\mathrm{Re}\,\lambda > 0$, and can be scaled such that $E_f(\omega) = 1$.

Proof: By equations (3.8), (3.16), and (3.14), the behavior as $x \to -\infty$ is well-understood for all the functions comprising $E(\lambda)$. The result then follows immediately after evaluating

$$\lim_{x\to-\infty}(\mathbf{Y}_f^- \wedge \mathbf{Y}_s^- \wedge \mathbf{Y}_f^+ \wedge \mathbf{Y}_s^+)(\lambda,x),$$

and rescaling $\mathbf{Z}_f^+(\omega,x)$ such that $E_f(\omega) = 1$. ∎

4 Bifurcations from the essential spectrum near $\lambda = \omega$

As a consequence of Proposition 3.7, we now know that an eigenvalue may bifurcate out of the continuous spectrum only at $\lambda = \pm\omega$ for perturbations of the cubic NLS. Let $\tilde{\epsilon}$ represent the perturbation parameter for the cubic NLS, and let the perturbed extended Evans function be represented by $E(\lambda, \tilde{\epsilon})$.

In the following discussion, we assume the following, which will later be verified for specific perturbations using the results presented in Section 2.3.

Hypothesis 4.1 *The Evans function $E(\lambda, \tilde{\epsilon})$ can be defined for $\lambda \in U$ in a continuous fashion, where*

$$U := \{\lambda : |\lambda - \omega| \le \delta\} \setminus \{\lambda : \operatorname{Re}\lambda = \omega, \operatorname{Im}\lambda > 0\}.$$

It is C^1 in $\tilde{\epsilon}$, and its derivative with respect to $\tilde{\epsilon}$ is continuous in $\lambda \in U$.

By using a Taylor expansion, we can then write

$$E(\lambda, \tilde{\epsilon}) = E(\lambda, 0) + \partial_{\tilde{\epsilon}} E(\lambda, 0)\tilde{\epsilon} + o(\tilde{\epsilon}) = E(\lambda, 0) + (\partial_{\tilde{\epsilon}} E(\omega, 0) + g_1(\lambda, \tilde{\epsilon}))\tilde{\epsilon}$$

for $\lambda \in U$, where g_1 is continuous and $g_1(\omega, 0) = 0$. Using the expression for the Evans function for $\tilde{\epsilon} = 0$ given in Proposition 3.7, we then see that for $\lambda \in U$ the Evans function is given by

$$E(\lambda, \tilde{\epsilon}) = (\partial_{\tilde{\epsilon}} E(\omega, 0) + g_1(\lambda, \tilde{\epsilon}))\tilde{\epsilon} + 4\sqrt{2\omega}\,\gamma_s(\lambda)(1 + g_2(\lambda)), \tag{4.1}$$

where $g_2(\lambda)$ is continuous and $g_2(\omega) = 0$.

Due to the branch cut taken for $\gamma_s(\lambda)$, we then see that

$$\partial_{\tilde{\epsilon}} E(\omega, 0)\tilde{\epsilon} > 0 \Longrightarrow E(\lambda, \tilde{\epsilon}) \ne 0 \tag{4.2}$$

for $\lambda \in U$, and hence no eigenvalue bifurcates out of the continuous spectrum. Otherwise, a single eigenvalue bifurcates out of the continuous spectrum, and $E(\lambda^*(\tilde{\epsilon}), \tilde{\epsilon}) = 0$, where

$$\lambda^* = \omega\left(1 - \frac{(\partial_{\tilde{\epsilon}} E(\omega, 0))^2}{32\omega^2}\tilde{\epsilon}^2\right) + o(\tilde{\epsilon}^2) \Longleftarrow \partial_{\tilde{\epsilon}} E(\omega, 0)\tilde{\epsilon} < 0. \tag{4.3}$$

In order to perform the above calculation, we need an expression for $\partial_{\tilde{\epsilon}} E(\omega, 0)$. Write the perturbed eigenvalue equation as

$$\mathbf{Y}' = M(\lambda, x, \tilde{\epsilon}),$$

where $M(\lambda, x, 0)$ is the matrix given in equation (3.6). This equation induces the perturbed solutions $\mathbf{Y}_f^{\pm}(\lambda, x, \tilde{\epsilon})$ and $\mathbf{Y}_s^{\pm}(\lambda, x, \tilde{\epsilon})$, where $\mathbf{Y}_f^{\pm}(\lambda, x, 0)$ and $\mathbf{Y}_s^{\pm}(\lambda, x, 0)$ are those given in the previous section. Since $\mathbf{Y}_s^{-}(\omega, x, 0) = \mathbf{Y}_s^{+}(\omega, x, 0)$, a routine calculation shows that

$$\partial_{\tilde{\epsilon}} E(\omega, 0) = -\partial_{\tilde{\epsilon}}(\mathbf{Y}_s^{-} - \mathbf{Y}_s^{+})(\omega, x, 0) \wedge (\mathbf{Y}_f^{-} \wedge \mathbf{Y}_f^{+} \wedge \mathbf{Y}_s^{+})(\omega, x, 0).$$

The 3-form $(\mathbf{Y}_f^{-} \wedge \mathbf{Y}_f^{+} \wedge \mathbf{Y}_s^{+})(\omega, x, 0)$ is uniformly bounded as $|x| \to \infty$, with

$$\lim_{x \to -\infty} (\mathbf{Y}_f^{-} \wedge \mathbf{Y}_f^{+} \wedge \mathbf{Y}_s^{+})(\omega, x, 0) = 2\gamma_f(\omega)\,\mathbf{e}_{123}, \tag{4.4}$$

where $\mathbf{e}_{ijk} = \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k$. Writing

$$-(\mathbf{Y}_f^{-} \wedge \mathbf{Y}_f^{+} \wedge \mathbf{Y}_s^{+})(\omega, x, 0) = a_1(x)\mathbf{e}_{123} + a_2(x)\mathbf{e}_{124} + a_3(x)\mathbf{e}_{134} + a_4(x)\mathbf{e}_{234},$$

this 3-form induces a solution to the adjoint equation, $\mathbf{Z}_s(\omega, x, 0)$, which is given by

$$\mathbf{Z}_s(\omega, x, 0) = [a_4(x), -a_3(x), a_2(x), -a_1(x)]^T$$

(Kapitula [13]). In other words,

$$\partial_{\tilde{\epsilon}} E(\omega, 0) = \partial_{\tilde{\epsilon}}(\mathbf{Y}_s^- - \mathbf{Y}_s^+)(\omega, x, 0) \cdot \mathbf{Z}_s(\omega, x, 0). \tag{4.5}$$

Using (4.4) and Lemma 3.2, one can compute explicitly that

$$\mathbf{Z}_s(\omega, x, 0) = 2\gamma_f(\omega) \begin{bmatrix} -\partial_x P^+(\omega, x) \\ P^+(\omega, x) \end{bmatrix}, \tag{4.6}$$

where $P^+(\omega, x)$ is defined in Lemma 3.3. Unfortunately, the evaluation of $\partial_{\tilde{\epsilon}}(\mathbf{Y}_s^- - \mathbf{Y}_s^+)$ is not as straightforward. The following lemma gives us a computable quantity.

Lemma 4.2 *Assume that Hypothesis 4.1 is satisfied. The derivative of the Evans function is then given by*

$$\partial_{\tilde{\epsilon}} E(\omega, 0) = \langle \partial_{\tilde{\epsilon}} M(\omega, x, 0)\mathbf{Y}_s^+(\omega, x, 0), \mathbf{Z}_s(\omega, x, 0)\rangle,$$

where

$$\langle f(x), g(x)\rangle = \int_{-\infty}^{\infty} f(x) \cdot g(x)\, dx.$$

Proof: Let $\tilde{\mathbf{Y}}$ be any solution to (3.5) at $\lambda = \omega$ such that

$$D = \tilde{\mathbf{Y}} \cdot \mathbf{Z}_s \neq 0.$$

If $\partial_{\tilde{\epsilon}} E(\omega, 0) \neq 0$, then $\tilde{\mathbf{Y}} = \partial_{\tilde{\epsilon}}(\mathbf{Y}_s^- - \mathbf{Y}_s^+)$. Following the ideas in Kapitula [13], it can be shown that

$$\partial_{\tilde{\epsilon}}(\mathbf{Y}_s^- - \mathbf{Y}_s^+) = \frac{1}{D}\left(\langle \partial_{\tilde{\epsilon}} M \mathbf{Y}_s^+, \mathbf{Z}_s\rangle \tilde{\mathbf{Y}} + \langle \partial_{\tilde{\epsilon}} M \mathbf{Y}_s^+, \tilde{\mathbf{Z}}\rangle \mathbf{Y}_s^+\right), \tag{4.7}$$

where $\tilde{\mathbf{Z}}$ is a solution to the adjoint equation induced by the 3-form $\mathbf{Y}_f^- \wedge \tilde{\mathbf{Y}} \wedge \mathbf{Y}_f^+$. Since $\mathbf{Y}_s^+ \cdot \mathbf{Z}_s = 0$,

$$\begin{aligned}
\partial_{\tilde{\epsilon}}(\mathbf{Y}_s^- - \mathbf{Y}_s^+) \cdot \mathbf{Z}_s &= \left(\frac{\langle \partial_{\tilde{\epsilon}} M \mathbf{Y}_s^+, \mathbf{Z}_s\rangle}{D}\right) \tilde{\mathbf{Y}} \cdot \mathbf{Z}_s \\
&= \left(\frac{\langle \partial_{\tilde{\epsilon}} M \mathbf{Y}_s^+, \mathbf{Z}_s\rangle}{D}\right) D \\
&= \langle \partial_{\tilde{\epsilon}} M \mathbf{Y}_s^+, \mathbf{Z}_s\rangle.
\end{aligned}$$

Upon examination of (4.5), one gets the desired conclusion. ■

Remark 4.3 *A similar formulation of the derivative is given in the work of Rubin [34].*

4.1 Evaluation at $\lambda = \omega$: CQNLS

For the CQNLS,

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + 3\alpha|\phi|^4\phi = 0,$$

the solitary-wave solution is given by

$$\Phi^2(x, \omega, \alpha) = \frac{\omega}{1 + \sqrt{1 + \alpha\omega}\,\cosh(2\sqrt{\omega}\,x)} \tag{4.8}$$

([32]).

Following the formulation in Section 3, for the eigenvalue problem we get the matrix

$$M(\lambda, x, \alpha) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega + \lambda - g(x,\alpha) & -h(x,\alpha) & 0 & 0 \\ -h(x,\alpha) & \omega - \lambda - g(x,\alpha) & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} g(x,\alpha) &= 8\Phi^2 + 9\alpha\Phi^4 \\ h(x,\alpha) &= 4\Phi^2 + 6\alpha\Phi^4. \end{aligned}$$

Theorem 2.26 shows that Hypothesis 4.1 is met. By Lemma 4.2, we therefore know that

$$\partial_\alpha E(\omega, 0) = \langle \partial_\alpha M(\omega, x)\mathbf{Y}_s^+(\omega, x), \mathbf{Z}_s(\omega, x)\rangle. \tag{4.9}$$

Using the fact that

$$\partial_\alpha \Phi^2 = -\frac{1}{2}\Phi^2(\omega - \Phi^2),$$

which can be readily verified using the representation given in (4.8), it is easy to show that

$$\begin{aligned} \partial_\alpha g &= -4\omega\Phi^2 + 13\Phi^4 \\ \partial_\alpha h &= -2\omega\Phi^2 + 8\Phi^4. \end{aligned}$$

Since

$$\partial_\alpha M(\omega, x) = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \partial_\alpha g(x) & \partial_\alpha h(x) & 0 & 0 \\ \partial_\alpha h(x) & \partial_\alpha g(x) & 0 & 0 \end{bmatrix},$$

and

$$P^+(\omega, x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{2}{\omega}\Phi^2(x, \omega, 0)\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

a tedious calculation then shows that

$$\partial_\alpha M\mathbf{Y}_s^+ \cdot \mathbf{Z}_s = -2\gamma_f(\omega)\left(\frac{168}{\omega^2}\Phi^8 - \frac{132}{\omega}\Phi^6 + 37\Phi^4 - 4\omega\Phi^2\right).$$

Thus, upon using (4.9) and integrating,

$$\begin{aligned} \partial_\alpha E(\omega, 0) &= -\frac{2}{3}\gamma_f(\omega)\omega^{3/2} \\ &= -\frac{2\sqrt{2}}{3}\omega^2. \end{aligned} \tag{4.10}$$

Now set $\beta = \alpha\omega$. Equation (4.10) can then be rewritten as

$$\partial_\beta E(\omega, 0) = -\frac{2\sqrt{2}}{3}\omega.$$

As a consequence of equations (4.2) and (4.3), it can be seen that if $\beta < 0$, then $E(\lambda, \beta) \neq 0$ for $\lambda \in U$, while if $\beta > 0$, then $E(\lambda^*, \beta) = 0$, where

$$\lambda^* = \omega(1 - \frac{1}{36}\beta^2) + o(\beta^2) \in \mathbb{R}. \tag{4.11}$$

Thus, if $\beta > 0$ an eigenvalue moves out of the continuous spectrum. Note that $\lambda^* \in \mathbb{R}$ due to the symmetries of the eigenvalue problem. Indeed, λ is an eigenvalue if and only if $-\lambda$ is, see Section 3. Since we are in the region where the Evans function has not been extended artificially, any eigenvalue corresponds to a zero of $E(\lambda, \alpha)$. Thus, since there is precisely one eigenvalue bifurcating, it must be real. The following lemma has now been proved.

Lemma 4.4 *Let $\beta = \alpha\omega$. If $0 < \beta \ll 1$, then one and only one eigenvalue moves out of the continuous spectrum, with that eigenvalue being real and its location given by (4.11). Furthermore, λ^* is the only zero of the Evans function in the half-plane $Re\,\lambda > 0$. If $0 < -\beta \ll 1$, then the Evans function is nonzero for all λ such that $Re\,\lambda > 0$.*

Remark 4.5 *Equation (4.11) agrees with the result of Pelinovsky* et al. *[32] in the case that $\alpha = 1$.*

4.2 Evaluation at $\lambda = \omega$: PFNLS

The PFNLS is given by

$$i\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^*) = 0, \tag{4.12}$$

where $\epsilon \geq 0$ is not necessarily small. By setting $\phi \to \phi e^{-i\theta}$, where

$$\cos 2\theta = \frac{\gamma}{\mu}, \tag{4.13}$$

equation (4.12) can be rewritten as

$$i\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^* e^{-i2\theta}) = 0. \tag{4.14}$$

The solitary-wave solution is given by

$$\Phi(x, \omega, \epsilon) = \sqrt{\frac{\beta}{2}}\operatorname{sech}(\sqrt{\beta}\, x), \tag{4.15}$$

where

$$\beta = \omega + \epsilon\mu \sin 2\theta. \tag{4.16}$$

It is known that if $\mu \sin 2\theta < 0$, then the wave is unstable (Barashenkov *et al.* [4]).

As a system, equation (4.14) can be written as

$$\begin{aligned} i\phi_t + (\partial_x^2 - \omega)\phi + 4\phi^2\psi + i\epsilon(\gamma\phi - \mu\psi e^{-i2\theta}) &= 0 \\ -i\psi_t + (\partial_x^2 - \omega)\psi + 4\phi\psi^2 - i\epsilon(\gamma\psi - \mu\phi e^{+i2\theta}) &= 0, \end{aligned} \tag{4.17}$$

where $\psi = \phi^*$. Linearization yields the system

$$iP_t + LP + i\epsilon\gamma P = 0,$$

where

$$L = (\partial_x^2 - \omega)\sigma_3 + 4\Phi^2(2\sigma_3 + i\sigma_2) + \epsilon\mu\cos 2\theta\,\sigma_2 - \epsilon\mu\sin 2\theta\,\sigma_1. \tag{4.18}$$

Here the σ_i are the Pauli spin matrices, i.e.,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By setting $P(x,t) \to P(x)e^{\rho t}$, one then gets the linear eigenvalue problem

$$(L + i(\rho + \epsilon\gamma))P = 0.$$

Setting

$$\lambda = -i(\rho + \epsilon\gamma),$$

we then get the eigenvalue problem

$$(L - \lambda)P = 0. \tag{4.19}$$

Note that the eigenvalue problem again admits a symmetry: λ is an eigenvalue if and only if $-\alpha$ is.

Letting $\mathbf{Y} = [P, Q]^T$, where $Q = P'$, the eigenvalue equation can be rewritten as the first-order system

$$\mathbf{Y}' = M(\lambda, x, \epsilon)\mathbf{Y}, \tag{4.20}$$

where

$$M(\lambda, x, \epsilon) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega + \lambda - 8\Phi^2 & -4\Phi^2 + \epsilon\mu\sin 2\theta & 0 & 0 \\ -4\Phi^2 - \epsilon\mu\sin 2\theta & \omega - \lambda - 8\Phi^2 & 0 & 0 \end{pmatrix}. \tag{4.21}$$

We want to apply Lemma 4.2 and calculate the derivative of the Evans function $E(\lambda, \epsilon)$ with respect to ϵ. It suffices to verify Hypothesis 2.11 in Section 2.3, since Lemma 2.12 then shows that Hypothesis 4.1 is met. Thus, we have to show that the eigenvalues of the limiting matrix

$$M(\lambda, x, \epsilon) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega + \lambda & \epsilon\mu\sin 2\theta & 0 & 0 \\ -\epsilon\mu\sin 2\theta & \omega - \lambda & 0 & 0 \end{pmatrix}$$

are independent of ϵ. It is easy to check that they are independent of ϵ after replacing λ by

$$\lambda = \sqrt{\tilde{\lambda}^2 + (\epsilon\mu\sin 2\theta)^2} \tag{4.22}$$

for $\tilde{\lambda} \in U$. This transformations accounts for the fact that the essential spectrum, which is located on the real axis, moves towards zero as ϵ increases. Note that we have

$$\tilde{E}(\tilde{\lambda}, \epsilon) = E(\sqrt{\tilde{\lambda}^2 + (\epsilon\mu\sin 2\theta)^2}, \epsilon).$$

for the new Evans function $\tilde{E}(\tilde{\lambda}, \epsilon)$, and that $\tilde{E}(\tilde{\lambda}, \epsilon)$ satisfies Hypothesis 4.1.

By Lemma 4.2 we have that

$$\partial_\epsilon \tilde{E}(\omega, 0) = \langle \partial_\epsilon M(\omega, x) \mathbf{Y}_s^+(\omega, x), \mathbf{Z}_s(\omega, x) \rangle.$$

A routine, yet tedious, calculation shows that

$$\partial_\epsilon M \mathbf{Y}_s^+ \cdot \mathbf{Z}_s = 16\sqrt{2\omega}\,(-1 + \frac{6}{\omega}\Phi^2 - \frac{12}{\omega^2}\Phi^4)\partial_\epsilon(\Phi^2).$$

Since

$$\partial_\epsilon(\Phi^2) = \frac{\mu \sin 2\theta}{\omega}(\Phi^2 + \frac{1}{2} x \partial_x(\Phi^2)),$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} x \Phi^{2k}(x) \partial_x(\Phi^2(x))\, dx &= \frac{1}{k+1} \int_{-\infty}^{\infty} x \partial_x(\Phi^{2(1+k)}(x))\, dx \\ &= -\frac{1}{k+1} \int_{-\infty}^{\infty} \Phi^{2(1+k)}(x)\, dx,\end{aligned}$$

upon integrating we see that

$$\partial_\epsilon \tilde{E}(\omega, 0) = -\frac{16\sqrt{2}}{3} \mu \sin 2\theta. \tag{4.23}$$

As a consequence of equations (4.2) and (4.3), we see that if $\mu \sin 2\theta < 0$, then $\tilde{E}(\tilde{\lambda}, \epsilon) \neq 0$ for $\tilde{\lambda}$ near ω, while if $\mu \sin 2\theta > 0$, then $\tilde{E}(\tilde{\lambda}^*, \epsilon) = 0$, where

$$\tilde{\lambda}^* = \omega \left(1 - \frac{16}{9\omega^2}(\mu^2 - \gamma^2)\epsilon^2\right) + o(\epsilon^2).$$

In the above equation, the relation $\mu \sin 2\theta = \pm\sqrt{\mu^2 - \gamma^2}$ was used. Going back to the original variable λ given in (4.22), we have $E(\lambda^*, \epsilon) = 0$, where

$$\lambda^* = \omega \left(1 - \frac{23}{18\omega^2}(\mu^2 - \gamma^2)\epsilon^2\right) + o(\epsilon^2). \tag{4.24}$$

Note that $\lambda^* \in \mathbb{R}$ on account of the symmetries of (4.19) mentioned above. Summarizing the above discussion, we have the following lemma.

Lemma 4.6 *Let $0 < \epsilon \ll 1$. If $\mu \sin 2\theta < 0$, then the Evans function is nonzero for all λ such that $Re\,\lambda > O(\epsilon) > 0$. If $\mu \sin 2\theta > 0$, then one and only one eigenvalue moves out of the continuous spectrum. This eigenvalue is real and given by (4.24). Furthermore, λ^* is the only zero of the extended Evans function in the half-plane $Re\,\lambda > O(\epsilon) > 0$.*

Remark 4.7 *The Evans function will have four discrete zeros which are of $O(\epsilon)$ (see Section 6).*

4.3 Evaluation at $\lambda = \omega$: PFCQNLS

Consider the PFCQNLS

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + 3\alpha|\phi|^4\phi + i\epsilon(\gamma\phi - \mu\phi^*) = 0.$$

The Evans function will be given by $E(\lambda, \beta, \epsilon)$, where $\beta = \alpha\omega$. As a consequence of the results of the previous subsections, we know that after changing variables according to (4.22)

$$\begin{aligned} \partial_\beta E(\omega, 0, 0) &= -\frac{2\sqrt{2}}{3}\omega \\ \partial_\epsilon E(\omega, 0, 0) &= -\frac{16\sqrt{2}}{3}\mu \sin 2\theta. \end{aligned}$$

Therefore, as a result of equations (4.2) and (4.3), we get the following lemma.

Lemma 4.8 *Let* $0 < \epsilon, |\beta| \ll 1$. *If*

$$\alpha < \frac{8\mu \sin 2\theta}{\omega^2}\,\epsilon,$$

then the Evans function is nonzero for all λ *such that* $Re\,\lambda > O(\epsilon) > 0$, *and hence no eigenvalues bifurcate out of the continuous spectrum. Otherwise, one eigenvalue bifurcates out of the continuous spectrum.*

Remark 4.9 *From a physical viewpoint, this means that parametric forcing can overcome the possibly destabilizing effect that a positive* α *represents.*

5 The cubic-quintic nonlinear Schrödinger equation

The PCQNLS is given by

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + 3\alpha|\phi|^4\phi = i\epsilon(d_1\phi_{xx} + d_2\phi + d_3|\phi|^2\phi + d_4|\phi|^4\phi), \tag{5.1}$$

where $\epsilon > 0$ is small and the other parameters are real and of $O(1)$. In this section, we will investigate the stability of the solitary wave $\Phi(x, \omega, \epsilon)$, where

$$\Phi^2(x, \omega, 0) = \frac{\omega}{1 + \sqrt{1 + \alpha\omega}\,\cosh(2\sqrt{\omega}\,x)}.$$

The wave $\Phi(x, \omega, \epsilon)$ is a smooth perturbation of $\Phi(x, \omega, 0)$ ([39]). In Kapitula [14], it is shown that in order for the wave to persist, it must be true that $d_3 = d_3^*$, where

$$d_3^* = d_1 - C_{d_2}d_2 - C_{d_4}(d_4 - \alpha d_1) + O(\epsilon), \tag{5.2}$$

and the constants are given by

$$C_{d_2} = \frac{3}{\omega}\left(1 + \frac{4}{15}\beta + O(\beta^2)\right), \quad C_{d_4} = \frac{2}{5}\omega\left(1 - \frac{9}{35}\beta + O(\beta^2)\right).$$

The interested reader should consult [14] to get expressions for the constants when β is not small.

When locating the eigenvalues, it is first necessary to locate those eigenvalues near the origin. This study was undertaken in [14], and the following result was derived.

Lemma 5.1 *Consider the PCQNLS. Set $\beta = \alpha\omega$, and assume that $0 < \epsilon \ll 1$. Assume that $d_3 = d_3^*$, that $d_1 > 0$, and that $d_4 < \alpha d_1$. Now set*

$$d_2^* = C_{\omega,\beta}(d_4 - \alpha d_1),$$

where

$$C_{\omega,\beta} = \frac{2}{15}\omega^2\left(1 - \frac{22}{21}\beta + O(\beta^2)\right).$$

If $d_2 < d_2^* < 0$, then there is one positive real eigenvalue, and one negative real eigenvalue, both of which are $O(\epsilon)$. If $d_2^* < d_2 < 0$, then there are two negative real eigenvalues which are $O(\epsilon)$. Furthermore, except for the double eigenvalue at zero, there are no other eigenvalues of $O(\epsilon)$.

Remark 5.2 *The condition $d_1 > 0$ means that the PCQNLS is a well-posed PDE. The condition $d_2^* < 0$ means that the solution $\phi = 0$ is stable for the PCQNLS.*

Remark 5.3 *The constants given above are discussed in detail in Kapitula [14], in that an expression is given when $-1 < \beta \leq 0$ is not necessarily small.*

Remark 5.4 *One should consult Kodama* et al. *[22] for a formal calculation when $\alpha = O(\epsilon)$.*

For the rest of the discussion, assume that $d_2^* < d_2 < 0$, so that there are no unstable eigenvalues near $\lambda = 0$. In order to determine the stability of the wave, it is then necessary to locate all eigenvalues which are close to the curves $|\operatorname{Im}\lambda| \geq \omega$.

Since $d_1 > 0$ and $d_2 < 0$, when $\epsilon > 0$ the continuous spectrum is contained in the left-half plane and bounded away from the imaginary axis. It is then straightforward to verify Hypothesis 2.9 in Section 2.3. On account of Lemma 2.10, the Evans function can be extended continuously for $\epsilon \geq 0$ and all λ with $\operatorname{Re}\lambda \geq 0$. As a consequence of Lemma 4.4, if $0 < -\beta \ll 1$, then when $\epsilon = 0$ the Evans function is nonzero for $|\operatorname{Im}\lambda| > O(|\beta|) > 0$. Therefore, if $0 < \epsilon \ll -\beta$, the Evans function will continue to remain nonzero. Since the zeros of the Evans function locate eigenvalues if λ is to the right of the essential spectrum, this then means that there are no eigenvalues close to the imaginary axis except those near the origin. Observing that the linear operator is sectorial, we can now conclude that the wave is stable. Theorem 1.5 has now been proved.

Now that the primary pulse for the PCQNLS has been shown to be stable, it is natural to inquire as to the existence and stability of multiple-pulse solutions. The existence question has been partially answered in Kapitula *et al.* [18]. There the existence of N-pulses which are evenly spaced has been shown. However, other types of N-pulses certainly do exist, and the existence of these will be the topic of another paper. Sandstede [36] has developed a program to study the stability of the N-pulse solutions in the case that $\partial_\lambda E(0) \neq 0$. In order to determine the stability of the multiple-pulse solutions for the PCQNLS, these ideas must be extended to cover the case that $\partial_\lambda E(0) = 0$, but $\partial_\lambda^2 E(0) \neq 0$. This extension is also possible and will be the focus of a future paper.

6 The parametrically-forced nonlinear Schrödinger equation

The parametrically-forced nonlinear Schrödinger equation (PFNLS) is given by

$$i\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^*) = 0, \tag{6.1}$$

where $\omega > 0$ and $\epsilon \geq 0$. Initially, no size restriction on the size of ϵ will be made. By setting $\phi \to \phi e^{-i\theta}$, where

$$\cos 2\theta = \frac{\gamma}{\mu}, \tag{6.2}$$

equation (6.1) can be rewritten as

$$i\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^* e^{-i2\theta}) = 0. \tag{6.3}$$

The solitary-wave solution is given by

$$\Phi(x, \omega, \epsilon) = \sqrt{\frac{\beta}{2}}\mathrm{sech}(\sqrt{\beta}\, x), \tag{6.4}$$

where

$$\beta = \omega + \epsilon\mu \sin 2\theta. \tag{6.5}$$

Note that if θ satisfies (6.2), so does $\theta + \pi$. Thus the sign of the sine term in (6.5) can be chosen positive or negative as we wish.

It is known that if $\mu \sin 2\theta < 0$, then the wave Φ is unstable (Barashenkov *et al.* [4]). We will show that the wave is stable for all $\epsilon > 0$ sufficiently small if $\mu \sin 2\theta > 0$. Of interest is then the existence of multiple pulses resembling N copies of the stable primary wave Φ. Using results from [37], we prove that stable N-pulses exist provided a small dissipative term is added to the (6.1):

$$i\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^*) = i\delta\partial_x^2\phi, \tag{6.6}$$

$0 < \delta \ll \epsilon$. The dissipative term models spectral filtering of the signals in the optical fiber.

6.1 Stability of Φ

We consider equation (6.3)

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^* e^{-i2\theta}) = 0 \tag{6.7}$$

and investigate the stability of the primary solitary-wave

$$\Phi(x, \omega, \epsilon) = \sqrt{\frac{\beta}{2}}\mathrm{sech}(\sqrt{\beta}\, x)$$

with $\beta = \omega + \epsilon\mu \sin 2\theta$ and $\mu \sin 2\theta > 0$.

Theorem 6.1 *Let $\gamma > 0$, $\mu \neq 0$, and $\omega > 0$. Assume that θ is chosen such that $\mu \sin 2\theta > 0$. The solitary wave Φ given in (6.4) is then orbitally exponentially stable with respect to equation (6.7) for all $\epsilon > 0$ sufficiently small.*

Proof: First, we determine the spectrum of the linearization of (6.7) around the wave Φ for small $\epsilon > 0$. It is convenient to write equation (6.7) as a system by writing down the equations for the real and imaginary part of ϕ. Setting $\phi = u + iv$, we obtain

$$\begin{aligned} u_t &= -\left(v_{xx} - (2\omega - \beta)v + 4(u^2 + v^2)v\right) \\ v_t &= u_{xx} - \beta u + 4(u^2 + v^2)u - 2\epsilon\gamma v. \end{aligned} \tag{6.8}$$

The eigenvalue problem of the linearization of (6.8) about the wave Φ reads

$$LP = \lambda P,$$

where

$$L = \begin{pmatrix} 0 & -L_- \\ L_+ & -2\epsilon\gamma \end{pmatrix}, \tag{6.9}$$

and

$$L_- = \partial_x^2 + 4\Phi^2 - (2\omega - \beta), \quad L_+ = \partial_x^2 + 12\Phi^2 - \beta.$$

This eigenvalue problem has been considered in Section 4.2. By Lemma 4.6, the spectrum outside a small neighborhood of zero is contained in the line $\operatorname{Re}\lambda = -\epsilon\gamma$. Therefore, it suffices to consider eigenvalues near zero.

For that purpose, we rescale $y := \sqrt{\beta}x$ and denote the resulting operators again by $L_\pm$. We then have the equivalent eigenvalue problem

$$\begin{pmatrix} 0 & -L_- \\ L_+ & -\frac{2\epsilon\gamma}{\beta} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \frac{1}{\beta}\lambda \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \tag{6.10}$$

with

$$L_- = \partial_y^2 + 2\operatorname{sech}^2 y - q^2, \quad L_+ = \partial_y^2 + 6\operatorname{sech}^2 y - 1, \tag{6.11}$$

and

$$q^2 = \frac{2\omega - \beta}{\beta} = \frac{\omega - \epsilon\mu\sin 2\theta}{\omega + \epsilon\mu\sin 2\theta} < 1. \tag{6.12}$$

The eigenvalue problem (6.10) can be written as the fourth-order equation

$$L_- L_+ P_1 = -\frac{\lambda(\lambda + 2\epsilon\gamma)}{\beta^2} P_1. \tag{6.13}$$

In passing, we note that the spectrum is symmetric with respect to the axis $\operatorname{Re}\lambda = -\epsilon\gamma$, i.e., $\lambda - 2\epsilon\gamma$ is an eigenvalue whenever λ is.

It has been shown by Kutz and Kath [24] (see also [2]) that zero and $\nu_*(\epsilon) = O(\epsilon) > 0$ are all of the eigenvalues of the equation

$$L_- L_+ P_1 = \nu P_1 \tag{6.14}$$

inside a small neighborhood of zero for $\epsilon > 0$ small. Therefore, the eigenvalues of (6.13) near zero are simple and given by

$$\lambda_1 = 0, \quad \lambda_2 = -2\epsilon\gamma, \quad \lambda_{3,4} = -\epsilon\gamma \pm \sqrt{\epsilon^2\gamma^2 - \beta^2\nu_*(\epsilon)}$$

In particular, since $\nu_*(\epsilon) > c\epsilon$ for some $c > 0$, the eigenvalues $\lambda_{3,4}$ have nonzero imaginary part with $\operatorname{Re}\lambda_{3,4} < 0$ (see Figure 2).

Summarizing the above discussion, the spectrum of the operator L is contained in the left-half plane with the exception of a simple eigenvalue at zero. Unfortunately, however, L will generate only a C^0-semigroup. For these groups, the spectral theorem does not hold in general and therefore we cannot conclude asymptotic stability from the knowledge of the spectrum of L alone. However, it follows from a result by Prüß [33, Corollary 4] that if the resolvent $(L - \lambda)^{-1}$ is bounded uniformly in the right-half plane outside any small neighborhood of zero as an operator in $L^2(\mathbb{R})$, then the spectral theorem holds. In particular, the wave Φ and its translates form an exponentially attracting set in $L^2(\mathbb{R})$.

Let λ be such that $\mathrm{Re}\,\lambda \geq 0$. Set

$$\tilde{\lambda} = \frac{\lambda}{\beta}, \quad \tilde{\gamma} = \frac{\lambda}{\beta}, \quad \nu = \tilde{\lambda}(\tilde{\lambda} + 2\epsilon\tilde{\gamma}).$$

In the following, we will omit the tilde. In order to estimate the resolvent, we must solve

$$\begin{pmatrix} -\lambda & -L_- \\ L_+ & -(\lambda + 2\epsilon\gamma) \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \tag{6.15}$$

that is, $(L - \lambda)P = G$, where $G_i \in L^2(\mathbb{R})$. Since $0 < q^2 < 1$, the operator L_- is invertible ([2, Section 2]); therefore, we can solve the first equation for P_2 to get

$$P_2 = -L_-^{-1}(\lambda P_1 + G_1), \tag{6.16}$$

and substitute this result into the second equation to get

$$(L_+ + \nu L_-^{-1})P_1 = G_2 - (\lambda + 2\epsilon\gamma)L_-^{-1}G_1. \tag{6.17}$$

In solving equations (6.16) and (6.17) it is sufficient to consider the case that $|\lambda|$ is large, since the resolvent is bounded in bounded sets. Define the fourth-order operator

$$A = L_+ L_-,$$

and note that $A^* = L_- L_+$. We know from the results above that the fourth-order operators $A + \nu$ and $A^* + \nu$ are invertible for any large $|\lambda|$ with $\mathrm{Re}\,\lambda \geq 0$. Therefore, we can solve equations (6.16) and (6.17) to get

$$\begin{aligned} P_1 &= -(\lambda + 2\epsilon\gamma)(A^* + \nu)^{-1}G_1 + L_-(A + \nu)^{-1}G_2 \\ P_2 &= -L_+(A^* + \nu)^{-1}G_1 - \lambda(A + \nu)^{-1}G_2. \end{aligned} \tag{6.18}$$

We shall obtain estimates for $P = (P_1, P_2)$ in terms of $G = (G_1, G_2)$ when $|\lambda|$ is large. We claim that for $|\lambda|$ large

$$\begin{aligned} \|(A + \nu)^{-1}\| &\leq M/|\lambda| \\ \|L_-(A + \nu)^{-1}\| &\leq M, \end{aligned} \tag{6.19}$$

with analogous estimates for the adjoint operators. The constant $M > 0$ may depend on ϵ but not on λ. Assume for a moment that the claim is true. We then have from equations (6.18) and (6.19) that

$$(|P_1| + |P_2|) \leq (M + 1)(|G_1| + |G_2|)$$

for all λ with $\mathrm{Re}\,\lambda \geq 0$ and $|\lambda|$ large.

It remains therefore to prove the above claim, which means that we must estimate the norm of the operator $(A+\nu)^{-1}$. The operators A and A^* are sectorial, so that their resolvent can be estimated in a sector. However, $\nu = \lambda(\lambda + 2\epsilon\gamma)$ is not contained in any sector near the positive axis, but instead forms a parabola. A priori, it is then not obvious why the estimates (6.19) should be true.

The key is that the operator A is self-adjoint up to terms involving first-order derivatives. Indeed, it is easy to check that

$$\begin{aligned}Au &= (\partial_y^2 + 6\mathrm{sech}^2 y - 1)(\partial_y^2 + 2\mathrm{sech}^2 y - q^2)u \\ &= \partial_y^4 u + 4\mathrm{sech}^2 y \partial_y^2 u + \partial_y^2(4u\mathrm{sech}^2 y) - (1+q^2)\partial_y^2 u + \\ &\quad +2(\mathrm{sech}^2 y)_y u_y + (2(\mathrm{sech}^2 y)_{yy} + (2\mathrm{sech}^2 y - q^2)(6\mathrm{sech}^2 y - 1))u.\end{aligned}$$

In other words, we have

$$Au = Bu + Ru$$

where

$$\begin{aligned}Bu &= \partial_y^4 u + 4\mathrm{sech}^2 y \partial_y^2 u + \partial_y^2(4u\mathrm{sech}^2 y) + \\ &\quad (2(\mathrm{sech}^2 y)_{yy} + (2\mathrm{sech}^2 y - q^2)(6\mathrm{sech}^2 y - 1))u\end{aligned}$$

is self-adjoint and $RB^{-\frac{1}{4}}$ is a bounded operator.

Using the spectral family associated with B, we see that

$$\begin{aligned}\|(B+\nu)^{-1}\| &\le M/|\lambda| \\ \|B^{\frac{1}{4}}(B+\nu)^{-1}\| &\le M/|\lambda|^{1/2} \\ \|B^{\frac{1}{2}}(B+\nu)^{-1}\| &\le M\end{aligned} \tag{6.20}$$

uniformly for $\mathrm{Re}\,\lambda \ge 0$ and $|\lambda|$ large. We obtain

$$\begin{aligned}(A+\nu)^{-1}u &= (B+R+\nu)^{-1}u \\ &= (B+\nu)^{-1}(\mathrm{id} + R(B+\nu)^{-1})^{-1} \\ &= (B+\nu)^{-1}(\mathrm{id} + RB^{-\frac{1}{4}}B^{\frac{1}{4}}(B+\nu)^{-1})^{-1}.\end{aligned}$$

It follows from (6.20) and the boundedness of $RB^{-\frac{1}{4}}$ that the terms appearing in the above equation are well-defined for all $|\lambda|$ sufficiently large. Note that it is crucial that R is only of first order. Otherwise, it would not be clear whether the operator $(\mathrm{id} + R(B+\nu)^{-1})$ is invertible; for instance, for $R = B^{\frac{1}{2}}$ the operator $R(B+\nu)^{-1}$ can only be estimated by a constant. The estimates (6.19) are now an immediate consequence of (6.20), and the proof of Theorem 6.1 is complete. ∎

Remark 6.2 *Since $\nu = 0$ is a simple eigenvalue of (6.14) for all $\epsilon > 0$ (see [2]) and the eigenvalues λ of (6.10) satisfy $\nu = \lambda(\lambda + 2\epsilon\gamma)$, we know that if the wave is to become unstable as ϵ increases, it must do so through a Hopf bifurcation.*

Remark 6.3 *If $\mu \sin 2\theta < 0$, it follows from [24] that the eigenvalue $\nu_*(\epsilon)$ is negative and hence the pulse Φ is unstable for all small ϵ. Thus, one gets another proof of the local instability result presented in Barashenkov* et al. *[4]. From [2], one can conclude that the wave will never stabilize.*

6.2 Existence and stability of multiple pulses

Consider equation (6.6)

$$i\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^*) = i\delta\partial_x^2\phi$$

for $\delta > 0$ small. The associated steady-state equation reads

$$\phi_{xx} - \omega\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^*) = i\delta\phi_{xx}. \tag{6.21}$$

Note that (6.21) is reversible, that is, $\phi(x)$ satisfies (6.21) if and only if $\phi(-x)$ does. Since zero is simple eigenvalue of the linearization of (6.1) around Φ, it follows from the results of Vanderbauwhede *et al.* [40] that the pulse Φ persists for $\delta > 0$. Moreover, since the linearization of (6.6) around the perturbed wave is sectorial, the pulse will be stable for $\delta > 0$ small. Therefore, we have the following corollary of Theorem 6.1.

Corollary 6.4 *Equation (6.6) has a stable solitary-wave solution for all $\delta > 0$ sufficiently small which approaches Φ as $\delta \to 0$.*

Consider the steady-state equation

$$\phi_{xx} - \omega\phi + 4|\phi|^2\phi + i\epsilon(\gamma\phi - \mu\phi^*) = i\delta\phi_{xx} \tag{6.22}$$

of equation (6.6) for $\delta \geq 0$. By Theorem 6.1, equation (6.6) admits the stable solitary-wave solution Φ for $\epsilon > 0$, which by Corollary 6.4 persists for $0 \leq \delta \ll \epsilon$. Note that equation (6.22) is reversible ($\phi(x)$ is a solution if and only if $\phi(-x)$ is) and admits the $\mathbb{Z}_2$-symmetry $\phi \to -\phi$ (ϕ is a solution if and only if $-\phi$ is). We are interested in the existence and stability of multiple solitary waves. These are solutions of (6.22) resembling N widely spaced copies of Φ or $-\Phi$. There are several ways to obtain N-pulses of different shapes, since Φ and $-\Phi$ are concatenated. Denoting Φ and $-\Phi$ by "up" and "down", respectively, we may then consider arbitrary sequences of ups and downs corresponding to whether Φ or $-\Phi$ is used.

It has recently been proved in [37] that multiple pulses are expected to occur near so-called orbit-flip bifurcations. This bifurcation is characterized by the property that when $\delta = 0$, the wave Φ is contained in the strong stable manifold of the equilibrium $\phi = 0$, with this no longer being true for $\delta \neq 0$. Now, the eigenvalues of the linearization of (6.22) at $\phi = 0$ for $\delta = 0$ are given by $\sqrt{\omega + \epsilon\mu \sin 2\theta}$ with θ given by $\cos 2\theta = \gamma/\mu$. As mentioned previously, depending on the choice of θ, $\sin\theta$ may be positive or negative. The stable primary pulse $\Phi(x)$ satisfies (6.22) for $\delta = 0$ and converges to zero exponentially with rate $\sqrt{\omega + \epsilon\mu \sin 2\theta}$ for $\sin\theta > 0$ as $|x| \to \infty$. Thus, it converges with the largest exponential rate possible. Since Φ is contained in the strong stable manifold when $\delta = 0$, an orbit-flip bifurcation is possible.

We have the following theorem concerning existence and stability of multiple solitary waves of (6.22). It is based on an application of [37, Theorems 1, 2, and 4].

Theorem 6.5 *Fix $\epsilon > 0$ small and $N > 1$, then for any $0 < \delta < \delta(\epsilon, N)$ small, there exists a unique multiple solitary wave of up-down-up-down-... type. These pulses are stable with respect to equation (6.6). Any other N-pulse consisting of copies of Φ or $-\Phi$ is unstable.*

Remark 6.6 *There exist many other N-pulses besides the ones of up-down-up-down-... type, and we refer to [37] for the details.*

Proof: As mentioned above, the theorem is an application of results proved in [37]. In particular, we shall verify the hypotheses of Theorems 1 and 2 in that paper. Most of the hypotheses are concerned with the linearization of (6.22) for $\delta = 0$ around the wave Φ. However, this equation can be written as the fourth-order equation studied in [37, Section 4] (see (6.13) with $\lambda = 0$ and [37, (4.9)]). Thus, it turns out that most of these hypotheses have already been verified in [37, Theorem 4]. The only assumption which we have to consider here is Hypothesis (H4)(ii) in [37]. Assumption [37, (H4)(ii)] is used to compute the sign of a certain constant J_2 which determines the bifurcation direction. In fact, $J_2 > 0$ corresponds to the pulses bifurcating for $\delta > 0$.

The constant J_2 arises as follows. Recall that the steady-state equation of (6.6) written as a system for real and imaginary part is given by

$$\begin{aligned} \delta u_{xx} &= -(v_{xx} - (2\omega - \beta)v + 4(u^2+v^2)v) \\ \delta v_{xx} &= u_{xx} - \beta u + 4(u^2+v^2)u - 2\epsilon\gamma v. \end{aligned}$$

Let Φ_δ denote the stable primary pulse of (6.6), with $\Phi_0 = \Phi$. We need to calculate the first-order expansion of Φ_δ. Since Φ_δ is smooth, we can substitute Φ_δ into the above equation and take the derivative with respect to δ at $\delta = 0$. The function $(u,v) = \frac{d}{d\delta}\Phi_\delta|_{\delta=0}$ satisfies

$$\begin{aligned} \Phi_{xx} &= -(v_{xx} - (2\omega - \beta)v + 4\Phi^2 v) \\ 0 &= u_{xx} - \beta u + 12\Phi u - 2\epsilon\gamma v. \end{aligned}$$

Solving the second equation for v and substituting the resulting expression into the first equation, we get

$$(\partial_x^2 + 4\Phi^2 - (2\omega - \beta))(\partial_x^2 + 12\Phi^2 - \beta)v = -2\epsilon\gamma\Phi_{xx},$$

i.e., $L_- L_+ v = -2\epsilon\gamma\Phi_{xx}$. It is now clear that the fourth-order equation investigated in [37], that is, the left-hand side of the above equation, and the parametrically-forced NLS are related.

Substituting the expression for Φ and rescaling $y = \sqrt{\beta}x$, we obtain

$$(\partial_x^2 + 2\mathrm{sech}^2 y - q^2)(\partial_x^2 + 6\mathrm{sech}^2 y - 1)v = -\sqrt{\frac{2}{\beta}}\epsilon\gamma(\mathrm{sech} y - 2\mathrm{sech}^3 y) =: G(y),$$

where $q < 1$ has been defined in (6.12). The crucial point is that the constant J_2 is given by

$$\begin{aligned} J_2 &= \int_{-\infty}^{\infty} G(y)e^{qy}(q - \tanh y)\,dy \\ &= -\sqrt{\frac{2}{\beta}}\epsilon\gamma \int_{-\infty}^{\infty} (\mathrm{sech} y - 2\mathrm{sech}^3 y)e^{qy}(q - \tanh y)\,dy \end{aligned}$$

([37, Section 4.1]). A straightforward calculation following [37] yields

$$J_2 = 4\sqrt{\frac{2}{\beta}}\epsilon\gamma \int_{-\infty}^{\infty} e^{\frac{y}{q}}\mathrm{sech}^3 y \tanh y\,dy > 0,$$

which is positive since $q > 0$. This coincides with the sign computed in [37], and hence the multiple pulses bifurcate for $\delta > 0$. The conclusion of the theorem follows now from [37, Theorem 4]. ∎

Remark 6.7 *In fact, we have not used the assumption that $\epsilon > 0$ is small for the existence part of Theorem 6.5. It has only been used for concluding stability since then stability of the primary pulse Φ is required. This, however, has been shown in Theorem 6.1 only for $\epsilon > 0$ small. Therefore, for all $\epsilon > 0$, multiple solitary waves of up-down-up-... type exist for $\delta > 0$ small, and they are stable as long as Φ is stable for $\delta = 0$.*

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