An application of the Implicit Function Theorem to an energy model of the semiconductor theory

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Abstract

In this paper we deal with a mathematical model for the description of heat conduction and carrier transport in semiconductor heterostructures. We solve a coupled system of nonlinear elliptic differential equations consisting of the heat equation with Joule heating as a source, the Poisson equation for the electric field and drift–diffusion equations with temperature dependent coefficients describing the charge and current conservation, subject to general thermal and electrical boundary conditions. We prove the existence and uniqueness of Hölder continuous weak solutions near thermodynamic equilibria points using the Implicit Function Theorem. To show the continuous differentiability of maps corresponding to the weak formulation of the problem we use regularity results from the theory of nonsmooth linear elliptic boundary value problems in Sobolev–Campanato spaces.

1 Introduction

We consider the following stationary drift–diffusion model (1), (2) with recombination and generation for self-heating semiconductors (cf. WACHUTKA [7]) consisting of continuity equations of electron and hole flow, Poisson's equation and a heat equation with Joule heating, but no thermoelectric effects. All functions are suitably scaled, especially we have set $q = 1$ for the elementary charge:

$$\left.\begin{array}{ll} -\nabla\cdot(\mu n\nabla u) = R & \text{on } \Omega, \\ -\nabla\cdot(\nu p\nabla v) = R & \text{on } \Omega, \\ -\nabla\cdot(\varepsilon\nabla\psi) = p - n + D & \text{on } \Omega, \\ -\nabla\cdot(\kappa\nabla\theta) = \mu n|\nabla u|^2 + \nu p|\nabla v|^2 - (u+v)R & \text{on } \Omega. \end{array}\right\} \tag{1}$$

Let us complete these equations with mixed boundary conditions for the electrical and thermal boundary behaviour:

$$\left.\begin{array}{llll} u = u_\mathrm{o} & \text{on } \partial\Omega\setminus\Gamma & \text{and} & \mathfrak{e}\cdot(\mu n\nabla u) = 0 \quad \text{on } \Gamma, \\ v = v_\mathrm{o} & \text{on } \partial\Omega\setminus\Gamma & \text{and} & \mathfrak{e}\cdot(\nu p\nabla v) = 0 \quad \text{on } \Gamma, \\ \psi = \psi_\mathrm{o} & \text{on } \partial\Omega\setminus\Gamma & \text{and} & \mathfrak{e}\cdot(\varepsilon\nabla\psi) = 0 \quad \text{on } \Gamma, \\ \theta = \theta_\mathrm{o} & \text{on } \partial\Omega\setminus\Sigma & \text{and} & \mathfrak{e}\cdot(\kappa\nabla\theta) = 0 \quad \text{on } \Sigma. \end{array}\right\} \tag{2}$$

Here and later on Ω is a bounded domain of the m-dimensional Euclidean space $\mathbb{R}^m$ for $m \geq 2$. We denote by $\mathfrak{e}$ the outward unit normal vector field on the boundary $\partial\Omega$, by ∇a the gradient of a function $a : \Omega \to \mathbb{R}$, by $\nabla\cdot\mathfrak{a}$ the divergence of a vector field $\mathfrak{a} : \Omega \to \mathbb{R}^m$ and for the scalar product in $\mathbb{R}^m$ we use a centered dot. Finally, Γ, $\Sigma \subset \partial\Omega$ are the possibly different Neumann parts of $\partial\Omega$ and $\partial\Omega\setminus\Gamma$, $\partial\Omega\setminus\Sigma$ the corresponding Dirichlet parts.

There occur several variables and related quantities in the model equations (1), namely,

$-u$	quasi–Fermi level of electrons,	n	concentration of electrons,
v	quasi–Fermi level of holes,	p	concentration of holes,
ψ	electrostatic potential,	R	recombination rate,
θ	thermal voltage,	D	concentration of dopants.

To get a self-consistent system of equations we have to formulate constitutive laws:

$$\left.\begin{aligned} n &= N\,F\left(\frac{u+\psi-E_n}{\theta}\right), \\ p &= P\,F\left(\frac{v-\psi+E_p}{\theta}\right), \\ R &= r\left\{1-\exp\left(\frac{u+v}{\theta}\right)\right\}. \end{aligned}\right\} \tag{3}$$

In order to involve the very important situation of heterogeneous materials we assume that the coefficients occuring in (1) and (3) depend on spatial variables and some other arguments

$\mu=\mu(x,\theta)$	mobility of electrons,	$E_n=E_n(x,\theta)$	band edge quantity,
$\nu=\nu(x,\theta)$	mobility of holes,	$E_p=E_p(x,\theta)$	band edge quantity,
$N=N(x,\theta)$	electron state density,	$\varepsilon=\varepsilon(x)$	dielectric permittivity,
$P=P(x,\theta)$	hole state density,	$\kappa=\kappa(x,\theta)$	thermal conductivity,
$F=F(t)$	distribution function,	$r=r(x,u,v,\psi,\theta)$	relaxation rate.

Our aim is to prove existence and uniqueness of Hölder continuous weak solutions to problem (1), (2) near thermodynamic equilibria points. The main tool in our investigations is a regularity result from the theory of nonsmooth linear elliptic problems with mixed boundary conditions in Sobolev–Campanato spaces (see Recke [4]), which works also in space dimensions $m>2$ in contrast to the $W^{1,p}$-theory (cf. Gröger [2]). Working in these spaces we are able to derive differentiability properties of operators corresponding to the weak formulation of our problem and to show that the Implicit Function Theorem can be applied to ensure the announced existence and uniqueness result.

The paper is organized as follows. In Section 2 we specify the assumptions on the data of our problem. Section 3 is devoted to the functional analytic background. Here we collect some properties of Sobolev–Campanato spaces and some results of the regularity theory for nonsmooth linear elliptic boundary value problems. It also contains differentiability properties for superposition operators connected with the weak formulation of problem (1), (2). In Section 4 we define the appropriate Banach spaces and the open set for the application of the Implicit Function Theorem. Furthermore, we formulate our problem in a weak sense and develop an equivalent formulation. Section 5 contains our main result. Here we show the validity of the assumptions for the application of the Implicit Function Theorem.

2 Assumptions on the data

To specify the assumptions on the coefficients we define a special class of Carathéodory functions (cf. Recke [5]):

Definition 1 Let $l\in\mathbb{N}$, $I\subset\mathbb{R}$ an open interval and $Q\subset\mathbb{R}^l$ be a domain. We call a function $a:\Omega\times Q\to I$ admissible if and only if it fulfils the following properties:

$$\left.\begin{array}{l} y\mapsto a(x,y) \text{ is continuously differentiable for almost all } x\in\Omega, \\ x\mapsto a(x,y) \text{ and } x\mapsto D_2a(x,y) \text{ are measurable for all } y\in Q. \\ \text{For every compact set } C\subset Q \text{ there exist a compact interval } I_C\subset I \\ \text{and a compact set } J_C\subset\mathbb{R}^l \text{ such that} \\ a(x,y)\in I_C \text{ for almost all } x\in\Omega \text{ and all } y\in C, \\ D_2a(x,y)\in J_C \text{ for almost all } x\in\Omega \text{ and all } y\in C. \\ \text{For every compact set } C\subset Q \text{ and } \tau>0 \text{ there exists a } \delta>0 \\ \text{such that for all } y,z\in C \text{ it holds} \\ |y-z|<\delta \Longrightarrow |a(x,y)-a(x,z)|<\tau \text{ for almost all } x\in\Omega, \\ |y-z|<\delta \Longrightarrow |D_2a(x,y)-D_2a(x,z)|<\tau \text{ for almost all } x\in\Omega. \end{array}\right\} \tag{4}$$

To work with regularity results for nonsmooth linear elliptic equations defined in nonsmooth domains we want to use the following very general concept of regular sets (see Gröger [2]).

Definition 2 A bounded set $H \subset \mathbb{R}^m$ is called regular if and only if

$$\left.\begin{array}{l}\text{for every point } x \in \partial H \text{ of the boundary there exist two open}\\ \text{neighborhoods } U \subset \mathbb{R}^m \text{ of } x \text{ and } V \text{ in } \mathbb{R}^m \text{ and a bijective}\\ \text{transformation } B \text{ from } U \text{ onto } V, \text{ such that } B \text{ and } B^{-1} \text{ are}\\ \text{Lipschitz continuous and } B(U \cap H) \text{ is one of the sets:}\\ E_1 = \big\{x \in \mathbb{R}^m \,\big|\, |x| < 1 \;,\; x_m < 0\big\},\\ E_2 = \big\{x \in \mathbb{R}^m \,\big|\, |x| < 1 \;,\; x_m \le 0\big\},\\ E_3 = \big\{x \in E_2 \,\big|\, x_m < 0 \text{ or } x_1 > 0\big\}.\end{array}\right\} \tag{5}$$

For all further considerations we will assume that

$$\left.\begin{array}{l}\Omega \subset \mathbb{R}^m \text{ is a bounded domain } (m \ge 2) \text{ and } \Gamma, \Sigma \subset \partial\Omega \text{ such that}\\ G = \Omega \cup \Gamma \text{ and } S = \Omega \cup \Sigma \text{ are regular,}\\ \partial\Omega \setminus \Gamma \text{ and } \partial\Omega \setminus \Sigma \text{ contain nonempty open subsets,}\\ \mu, \nu, \kappa, N, P : \Omega \times (0, +\infty) \to (0, +\infty) \text{ are admissible,}\\ E_n, E_p : \Omega \times (0, +\infty) \to \mathbb{R} \text{ are admissible,}\\ r : \Omega \times \mathbb{R}^3 \times (0, +\infty) \to (0, +\infty) \text{ is admissible,}\\ F : \mathbb{R} \to (0, +\infty) \text{ is monotonously increasing, continuously}\\ \text{differentiable on } \mathbb{R} \text{ and it holds } \lim\limits_{t \to +\infty} F(t) = +\infty,\\ D : \Omega \to \mathbb{R} \text{ is measurable and bounded on } \Omega,\\ \varepsilon : \Omega \to (0, +\infty) \text{ is measurable and bounded on } \Omega,\\ 0 < \varepsilon_\star \le \varepsilon \le \varepsilon^\star < +\infty \text{ almost everywhere on } \Omega.\end{array}\right\} \tag{6}$$

The above general assumption on F includes the most physically relevant Boltzmann and Fermi–Dirac distribution functions. Furthermore, the nonsmoothness of the coefficients in the spatial variables is devoted to the situation of heterogeneous semiconductor devices.

3 Functional analytic background

Now, for every regular set $H \subset \mathbb{R}^m$ and its interior Ω we define the following subspaces of the usual Sobolev space $W^{1,2}(\Omega)$:

$$\begin{array}{l} C_\circ^\infty(H) := \big\{w|_\Omega \,\big|\, w \in C_\circ^\infty(\mathbb{R}^m) \text{ with } \operatorname{supp}(w) \cap (\overline{H} \setminus H) = \emptyset\big\} \text{ and}\\ W_\circ^{1,2}(H) \text{ as the closure of } C_\circ^\infty(H) \text{ in } W^{1,2}(\Omega).\end{array}$$

Let λ^m the usual Lebesgue measure on the Lebesgue measurable sets of $\mathbb{R}^m$. We denote by $W^{-1,2}(H)$ the dual space to $W_\circ^{1,2}(H)$ and by $\langle\,,\rangle_H$ and $J_H : W_\circ^{1,2}(H) \longrightarrow W^{-1,2}(H)$ the dual pairing and the corresponding duality map

$$\langle J_H w, h\rangle_H = \int\limits_H (\nabla w \cdot \nabla h + wh)\, \mathrm{d}\lambda^m \qquad \text{for all } w, h \in W_\circ^{1,2}(H), \tag{7}$$

respectively. We introduce suitable Banach spaces connected with the Campanato spaces $\mathfrak{L}^{2,\omega}(\Omega)$ for real numbers $\omega \in (m-2, m)$:

$$\begin{array}{l} \mathfrak{L}^{2,\omega}(\Omega) := \big\{w \in L^2(\Omega) \,\big|\, [w]_{\mathfrak{L}^{2,\omega}(\Omega)} < \infty\big\},\\ W_\omega(\Omega) := \big\{w \in W^{1,2}(\Omega) \,\big|\, \nabla w \in \mathfrak{L}^{2,\omega}(\Omega, \mathbb{R}^m)\big\},\\ X_\omega(H) := \big\{w \in W_\circ^{1,2}(H) \,\big|\, \nabla w \in \mathfrak{L}^{2,\omega}(\Omega, \mathbb{R}^m)\big\},\\ Y_\omega(H) := \big\{h \in W^{-1,2}(H) \,\big|\, \exists w \in X_\omega(H) : J_H w = h\big\},\end{array}$$

$$[w]^2_{\mathfrak{L}^{2,\omega}(\Omega)} := \sup_{\substack{x \in \Omega\\ r > 0}} \Big(r^{-\omega} \int\limits_{\Omega \cap B(x,r)} \Big|w - \frac{1}{\lambda^m(\Omega \cap B(x,r))} \int\limits_{\Omega \cap B(x,r)} w\, \mathrm{d}\lambda^m\Big|^2 \mathrm{d}\lambda^m\Big),$$

$$\|w\|^2_{\mathfrak{L}^{2,\omega}(\Omega)} := \|w\|^2_{L^2(\Omega)} + [w]^2_{\mathfrak{L}^{2,\omega}(\Omega)},$$
$$\|w\|^2_{W_\omega(\Omega)} := \|w\|^2_{W^{1,2}(\Omega)} + \|w\|^2_{\mathfrak{L}^{2,\omega}(\Omega)},$$
$$\|w\|_{X_\omega(H)} := \|w\|_{W_\omega(\Omega)},$$
$$\|h\|_{Y_\omega(H)} := \|J_H^{-1}h\|_{X_\omega(H)}.$$

Now, we collect some properties of the Sobolev–Campanato spaces $W_\omega(\Omega)$ (see, for instance, Troianiello [6], Hong Xie [8]):

Theorem 1 *Let $\Omega \subset \mathbb{R}^m$ be the interior of a regular set $H \subset \mathbb{R}^m$ and $\omega \in (m-2, m)$, then*
(1) *there exists a constant $c_\infty = c_\infty(\omega) > 0$ such that for all $w \in \mathfrak{L}^{2,\omega}(\Omega)$ and $z \in L^\infty(\Omega)$ the product zw belongs to $\mathfrak{L}^{2,\omega}(\Omega)$ and can be estimated by*

$$\|zw\|_{\mathfrak{L}^{2,\omega}(\Omega)} \le c_\infty \|z\|_{L^\infty(\Omega)} \|w\|_{\mathfrak{L}^{2,\omega}(\Omega)},$$

(2) *there exists a $c_W = c_W(\omega) > 0$ such that for all $w \in W_\omega(\Omega)$ we have $w \in \mathfrak{L}^{2,\omega+2}(\Omega)$ and it holds the norm estimate*

$$\|w\|_{\mathfrak{L}^{2,\omega+2}(\Omega)} \le c_W \|w\|_{W_\omega(\Omega)},$$

(3) *the space $\mathfrak{L}^{2,\omega+2}(\Omega)$ is isomorphic to the space $C^{0,\eta}(\overline{\Omega})$ of Hölder continuous functions, where the Hölder exponent is $\eta = (\omega - m + 2)/2$.*

Now, we formulate a regularity result for diagonal systems of nonsmooth linear elliptic equations (for a proof see Recke [4], compare also with Troianiello [6] and Hong Xie [8]). Let $l \in \mathbb{N}, \gamma \in (0,1)$ and denote by $\mathfrak{M}_l$ and $\mathfrak{S}_l$ the spaces of all real $l \times l$-matrices and real symmetric $l \times l$-matrices, respectively and, finally, by $L^\infty_\gamma(\Omega, \mathfrak{S}_m)$ the set of all matrices $M \in L^\infty(\Omega, \mathfrak{S}_m)$ such that

$$\gamma|\xi|^2 < M(x)\xi \cdot \xi < \frac{1}{\gamma}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^m \setminus \{0\} \text{ and for almost all } x \in \Omega.$$

Theorem 2 *Let $k \in \mathbb{N}, \gamma \in (0,1)$ and $H_1 = \Omega \cup \Gamma_1, \dots, H_k = \Omega \cup \Gamma_k$ with $\Gamma_1, \dots, \Gamma_k \subset \partial\Omega$ be regular sets with the common interior $\Omega \subset \mathbb{R}^m$. For*

$$A = (A_i) \in L^\infty_\gamma(\Omega, (\mathfrak{S}_m)^k), \quad d = (d_{ij}) \in L^\infty(\Omega, \mathfrak{M}_k),$$
$$b = (b_{ij}) \in L^\infty(\Omega, (\mathfrak{M}_k)^m), \quad c = (c_{ij}) \in L^\infty(\Omega, (\mathfrak{M}_k)^m)$$

we define the linear bounded operator

$$L(A,b,c,d) : W^{1,2}_{\mathrm{o}}(H_1) \times \dots \times W^{1,2}_{\mathrm{o}}(H_k) \longrightarrow W^{-1,2}(H_1) \times \dots \times W^{-1,2}(H_k) \quad \textit{by}$$

$$\langle L(A,b,c,d)w, \varphi\rangle := \sum_{i,j=1}^k \int_\Omega ((A_i \nabla w_i + b_{ij} w_j) \cdot \nabla\varphi_i + (c_{ij} \cdot \nabla w_j + d_{ij} u_j)\varphi_i) \,\mathrm{d}\lambda^m$$

for all $w = (w_1, \dots, w_k), \varphi = (\varphi_1, \dots, \varphi_k) \in W^{1,2}_{\mathrm{o}}(H_1) \times \dots \times W^{1,2}_{\mathrm{o}}(H_k)$.

Then there exist constants $\omega_1 \in (m-2, m)$ and $\omega_{\mathrm{o}} = \omega_{\mathrm{o}}(\gamma) \in (m-2, \omega_1)$ such that
(1) *the operator $L(A,b,c,d)$ maps $X_\omega(H_1) \times \dots \times X_\omega(H_k)$ continuously into the space $Y_\omega(H_1) \times \dots \times Y_\omega(H_k)$ for all $\omega \in [0, \omega_1]$. Moreover, the following map is continuous:*

$$(A,b,c,d) \in L^\infty_\gamma(\Omega, (\mathfrak{S}_m)^k) \times L^\infty(\Omega, (\mathfrak{M}_k)^m) \times L^\infty(\Omega, (\mathfrak{M}_k)^m) \times L^\infty(\Omega, \mathfrak{M}_k) \longmapsto$$
$$L(A,b,c,d) \in \mathcal{L}(X_\omega(H_1) \times \dots \times X_\omega(H_k), Y_\omega(H_1) \times \dots \times Y_\omega(H_k)).$$

(2) *For all $j \in \{1, \dots, k\}$ and $\omega \in [0, \omega_1]$ the space $Y_\omega(H_j)$ equals the set of all functionals $h \in W^{-1,2}(H_j)$ such that there exist functions $z \in \mathfrak{L}^{2,\omega}(\Omega, \mathbb{R}^m)$ and $z_{\mathrm{o}} \in \mathfrak{L}^{2,\omega-2}(\Omega)$ with*

$$\langle h, \varphi_j\rangle = \int_\Omega (z \cdot \nabla\varphi_j + z_{\mathrm{o}}\varphi_j) \,\mathrm{d}\lambda^m \quad \textit{for all } \varphi_j \in W^{1,2}_{\mathrm{o}}(H_j).$$

Moreover, in that case there exists a constant $c_Y = c_Y(\omega) > 0$ such that

$$\|h\|_{Y_\omega(H_j)} \le c_Y \left(\|z\|_{\mathfrak{L}^{2,\omega}(\Omega,\mathbb{R}^m)} + \|z_{\mathrm{o}}\|_{\mathfrak{L}^{2,\omega-2}(\Omega)}\right),$$

where the constant c_Y does not depend on z and z_{o}.
(3) *$L(A,b,c,d)$ is a Fredholm operator (index zero) from $X_\omega(H_1) \times \dots \times X_\omega(H_k)$ into $Y_\omega(H_1) \times \dots \times Y_\omega(H_k)$ if $\omega \in (m-2, \omega_{\mathrm{o}}]$.*

For further considerations we have to ensure the continuous differentiability of several operators containing superposition operators. Here we will apply the following differentiability result of RECKE [5]:

Lemma 3 *Let $Q \subset \mathbb{R}^l$ be a domain and $l \in \mathbb{N}$. Furthermore, let $\Omega \subset \mathbb{R}^m$ be a bounded domain and $a : \Omega \times Q \to \mathbb{R}$ an admissible function. Then the following superposition operator is continuously differentiable:*

$$S_a : \{w \in C(\overline{\Omega}, \mathbb{R}^l) \mid \forall x \in \overline{\Omega} : w(x) \in Q\} \longrightarrow L^\infty(\Omega) \quad \textit{defined by}$$
$$(S_a(w))(x) := a(x, w(x)) \quad \textit{for almost all } x \in \overline{\Omega}.$$

4 Weak formulations

Now, we are in the situation to specify the functional analytic setting of the problem (1), (2). In order to apply the Implicit Function Theorem we have to consider suitable continuously differentiable maps resulting from a weak formulation of the problem (1), (2) and defined on an open subset of an appropriate Banach space.

Definition 3 We choose the Banach spaces for the problem (1), (2) as Cartesian products of function spaces defined above:

$$\begin{array}{ll} T_\infty := W_{\mathrm{o}}^{1,2}(G, \mathbb{R}^3) \times (W_{\mathrm{o}}^{1,2}(S) \cap L^\infty(\Omega)) & \text{space of test functions for problem (9),} \\ T := W_{\mathrm{o}}^{1,2}(G, \mathbb{R}^3) \times W_{\mathrm{o}}^{1,2}(S) & \text{space of test functions for problem (10),} \\ W_\omega := W^{1,\infty}(\Omega, \mathbb{R}^2) \times W_\omega(\Omega, \mathbb{R}^2) & \text{space of boundary value functions,} \\ X_\omega := X_\omega(G, \mathbb{R}^3) \times X_\omega(S) & \text{space of the homogeneous parts,} \\ Y_\omega := Y_\omega(G, \mathbb{R}^3) \times Y_\omega(S) & \text{space of functionals.} \end{array}$$

Definition 4 We will decompose solutions of the problem (1), (2) into a sum of given boundary value functions in W_ω and the homogeneous parts of solutions which we want to find in X_ω:

$$((u_{\mathrm{o}}, v_{\mathrm{o}}, \psi_{\mathrm{o}}, \theta_{\mathrm{o}}), (U, V, \Psi, \Theta)) \in W_\omega \times X_\omega \longmapsto$$
$$(u, v, \psi, \theta) = (u_{\mathrm{o}} + U, v_{\mathrm{o}} + V, \psi_{\mathrm{o}} + \Psi, \theta_{\mathrm{o}} + \Theta) \in W_\omega.$$

Definition 5 Assume, that $\omega \in (m-2, m)$, $c > b > 0$ and $0 < \theta_\star < \theta^\star$. Then we define the open subset $M_\omega = M_\omega(b, c, \theta_\star, \theta^\star) \subset W_\omega \times X_\omega$ by the following rule:

$$\left.\begin{array}{l} ((u_{\mathrm{o}}, v_{\mathrm{o}}, \psi_{\mathrm{o}}, \theta_{\mathrm{o}}), (U, V, \Psi, \Theta)) \in M_\omega \quad \text{if and only if} \\ ((u_{\mathrm{o}}, v_{\mathrm{o}}, \psi_{\mathrm{o}}, \theta_{\mathrm{o}}), (U, V, \Psi, \Theta)) \in W_\omega \times X_\omega, \\ |u_{\mathrm{o}} + U| < b, |v_{\mathrm{o}} + V| < b, |\psi_{\mathrm{o}} + \Psi| < c \quad \text{and} \\ \theta_\star < \theta_{\mathrm{o}} + \Theta < \theta^\star \text{ on } \Omega. \end{array}\right\} \tag{8}$$

Definition 6 For $((u_{\mathrm{o}}, v_{\mathrm{o}}, \psi_{\mathrm{o}}, \theta_{\mathrm{o}}), (U, V, \Psi, \Theta)) \in M_\omega$ and all $\phi = (\phi_u, \phi_v, \phi_\psi, \phi_\theta) \in T_\infty$ we define by

$$\langle g((u_{\mathrm{o}}, v_{\mathrm{o}}, \psi_{\mathrm{o}}, \theta_{\mathrm{o}}), (U, V, \Psi, \Theta)), \phi \rangle :=$$
$$= \int_\Omega (\mu n \nabla(U + u_{\mathrm{o}}) \cdot \nabla\phi_u - R\phi_u)\, \mathrm{d}\lambda^m + \int_\Omega (\nu p \nabla(V + v_{\mathrm{o}}) \cdot \nabla\phi_v - R\phi_v)\, \mathrm{d}\lambda^m +$$
$$+ \int_\Omega (\varepsilon \nabla(\Psi + \psi_{\mathrm{o}}) \cdot \nabla\phi_\psi - (p - n + D)\phi_\psi)\, \mathrm{d}\lambda^m +$$
$$+ \int_\Omega (\kappa \nabla(\Theta + \theta_{\mathrm{o}}) \cdot \nabla\phi_\theta + (U + u_{\mathrm{o}} + V + v_{\mathrm{o}}) R\phi_\theta)\, \mathrm{d}\lambda^m -$$
$$- \int_\Omega \mu n |\nabla(U + u_{\mathrm{o}})|^2 \phi_\theta\, \mathrm{d}\lambda^m - \int_\Omega \nu p |\nabla(V + v_{\mathrm{o}})|^2 \phi_\theta\, \mathrm{d}\lambda^m$$

a functional $g((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta))$ on T_∞. We call (u, v, ψ, θ) a weak solution of the problem (1), (2) to the boundary value functions $(u_o, v_o, \psi_o, \theta_o)$ if and only if

$$\left.\begin{array}{l} ((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)) \in M_\omega \text{ and} \\ \langle g((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)), \phi\rangle = 0 \text{ for all } \phi \in T_\infty. \end{array}\right\} \tag{9}$$

Analogously to the thermistor problem (see Howison, Rodrigues, Shillor [3]), it is now possible to find an equivalent formulation to (9) replacing the quadratic terms $|\nabla(U + u_o)|^2$ and $|\nabla(V + v_o)|^2$ by $\nabla(U + u_o) \cdot \nabla u_o$ and $\nabla(V + v_o) \cdot \nabla v_o$ and further lower order terms:

Theorem 4 *For $((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)) \in M_\omega$ and all $\varphi = (\varphi_u, \varphi_v, \varphi_\psi, \varphi_\theta) \in T$ we define the map $f : M_\omega \to W^{-1,2}(G, \mathbb{R}^3) \times W^{-1,2}(S)$ in the following way:*

$$\begin{aligned} &\langle f((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)), \varphi\rangle := \\ &= \int_\Omega (\mu n \nabla(U + u_o) \cdot \nabla\varphi_u - R\varphi_u)\, d\lambda^m + \int_\Omega (\nu p \nabla(V + v_o) \cdot \nabla\varphi_v - R\varphi_v)\, d\lambda^m + \\ &+ \int_\Omega (\varepsilon\nabla(\Psi + \psi_o) \cdot \nabla\varphi_\psi - (p - n + D)\varphi_\psi)\, d\lambda^m + \\ &+ \int_\Omega (\kappa\nabla(\Theta + \theta_o) \cdot \nabla\varphi_\theta + (u_o + v_o)R\varphi_\theta)\, d\lambda^m - \\ &- \int_\Omega \mu n \nabla(U + u_o) \cdot \nabla u_o \cdot \varphi_\theta\, d\lambda^m + \int_\Omega \mu n U \nabla(U + u_o) \cdot \nabla\varphi_\theta\, d\lambda^m - \\ &- \int_\Omega \nu p \nabla(V + v_o) \cdot \nabla v_o \cdot \varphi_\theta\, d\lambda^m + \int_\Omega \nu p V \nabla(V + v_o) \cdot \nabla\varphi_\theta\, d\lambda^m. \end{aligned}$$

Then (u, v, ψ, θ) is a weak solution of problem (1), (2) to the boundary value functions $(u_o, v_o, \psi_o, \theta_o)$ if and only if

$$\left.\begin{array}{l} ((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)) \in M_\omega \textit{ and} \\ \langle f((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)), \varphi\rangle = 0 \textit{ for all } \varphi \in T. \end{array}\right\} \tag{10}$$

Proof: Let (u, v, ψ, θ) be a weak solution of the problem (1), (2) to the given boundary value functions $(u_o, v_o, \psi_o, \theta_o)$, that means by definition:

$$\begin{array}{l} ((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)) \in M_\omega \text{ and} \\ \langle g((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)), \phi\rangle = 0 \text{ for all } \phi \in T_\infty. \end{array}$$

Now, we consider a test function $\varphi = (\varphi_u, \varphi_v, \varphi_\psi, \varphi_\theta) \in T$. Since $W_o^{1,2}(S)$ is the closure of $C_o^\infty(S)$ in $W^{1,2}(\Omega)$ we can choose a convergent sequence $\{\varphi_\theta^i\}_{i\in\mathbb{N}} \subset C_o^\infty(S)$ with $\varphi_\theta^i \to \varphi_\theta$ in $W^{1,2}(\Omega)$. Obviously, then

$$\phi^i = (\phi_u^i, \phi_v^i, \phi_\psi, \phi_\theta^i) := (\varphi_u + U\varphi_\theta^i, \varphi_v + V\varphi_\theta^i, \varphi_\psi, \varphi_\theta^i) \in T_\infty$$

is a sequence of test functions in T_∞ and for $\varphi^i = (\varphi_u, \varphi_v, \varphi_\psi, \varphi_\theta^i)$ we easily obtain:

$$\langle f((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)), \varphi^i\rangle = \langle g((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)), \phi^i\rangle = 0.$$

Finally, the limiting process $i \to \infty$ yields $\langle f((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)), \varphi\rangle = 0$.
To proof the opposite direction let us now assume, that

$$\begin{array}{l} ((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)) \in M_\omega \text{ and} \\ \langle f((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)), \varphi\rangle = 0 \text{ for all } \varphi \in T. \end{array}$$

Fixing a test function $\phi = (\phi_u, \phi_v, \phi_\psi, \phi_\theta) \in T_\infty$ we can find a sequence $\{\phi_\theta^i\}_{i\in\mathbb{N}} \subset C_\mathrm{o}^\infty(S)$ which converges to ϕ_θ in $W^{1,2}(\Omega)$ and, furthermore, $\|\phi_\theta^i\|_{L^\infty(\Omega)} \le 2\|\phi_\theta\|_{L^\infty(\Omega)}$ for all $i \in \mathbb{N}$. Then $\phi^i = (\phi_u, \phi_v, \phi_\psi, \phi_\theta^i) \in T_\infty$ and

$$\varphi^i = (\varphi_u^i, \varphi_j^i, \varphi_\psi, \varphi_\theta^i) := (\phi_u - U\phi_\theta^i, \phi_v - V\phi_\theta^i, \phi_\psi, \phi_\theta^i) \in T$$

are test functions in T and it holds

$$\langle g((u_\mathrm{o}, v_\mathrm{o}, \psi_\mathrm{o}, \theta_\mathrm{o}), (U, V, \Psi, \Theta)), \phi^i\rangle = \langle f((u_\mathrm{o}, v_\mathrm{o}, \psi_\mathrm{o}, \theta_\mathrm{o}), (U, V, \Psi, \Theta)), \varphi^i\rangle = 0.$$

Again the limiting process $i \to \infty$ yields $\langle g((u_\mathrm{o}, v_\mathrm{o}, \psi_\mathrm{o}, \theta_\mathrm{o}), (U, V, \Psi, \Theta)), \phi\rangle = 0.$ ■

5 Local existence and uniqueness

Theorem 5 *There exists a constant $\omega_1 \in (m-2, m)$ such that for all parameters $c > b > 0$, $0 < \theta_\star < \theta^\star$ and exponents $\omega \in (m-2, \omega_1]$ the map $f : M_\omega(b, c, \theta_\star, \theta^\star) \longrightarrow Y_\omega$ is continuously differentiable.*

Proof: First of all, because of the assumptions (6) and Theorems 1 and 2 there exists a number $\omega_1 \in (m-2, m)$ such that the map $f : M_\omega(b, c, \theta_\star, \theta^\star) \longrightarrow Y_\omega$ is continuous for $\omega \in (m-2, \omega_1]$.

For the proof of existence and continuity of the partial Fréchet derivatives $D_1 f : M_\omega \longrightarrow \mathcal{L}(W_\omega, Y_\omega)$ and $D_2 f : M_\omega \longrightarrow \mathcal{L}(X_\omega, Y_\omega)$ we want to utilize an argument of Recke [5]. To do so, we consider admissible functions $a : \Omega \times M_\omega \to \mathbb{R}$ and introduce with the help of the associated superposition operators (see Lemma 3) the following operators

$$A_{ij\alpha\beta}, A^\mathrm{o}_{ij\alpha\beta}, A_{i\alpha\beta}, A^\mathrm{o}_{i\alpha\beta}, A_{j\beta}, A_\beta : M_\omega \longrightarrow Y_\omega \text{ by}$$

$$\langle A_{ij\alpha\beta}(w_\mathrm{o}, W), \varphi\rangle := \int_\Omega S_a(w_\mathrm{o}, W) D_i W_\alpha D_j \varphi_\beta \, \mathrm{d}\lambda^m,$$

$$\langle A^\mathrm{o}_{ij\alpha\beta}(w_\mathrm{o}, W), \varphi\rangle := \int_\Omega S_a(w_\mathrm{o}, W) D_i w_{\mathrm{o}\alpha} D_j \varphi_\beta \, \mathrm{d}\lambda^m,$$

$$\langle A_{i\alpha\beta}(w_\mathrm{o}, W), \varphi\rangle := \int_\Omega S_a(w_\mathrm{o}, W) D_i W_\alpha \cdot \varphi_\beta \, \mathrm{d}\lambda^m,$$

$$\langle A^\mathrm{o}_{i\alpha\beta}(w_\mathrm{o}, W), \varphi\rangle := \int_\Omega S_a(w_\mathrm{o}, W) D_i w_{\mathrm{o}\alpha} \cdot \varphi_\beta \, \mathrm{d}\lambda^m,$$

$$\langle A_{j\beta}(w_\mathrm{o}, W), \varphi\rangle := \int_\Omega S_a(w_\mathrm{o}, W) D_j \varphi_\beta \, \mathrm{d}\lambda^m,$$

$$\langle A_\beta(w_\mathrm{o}, W), \varphi\rangle := \int_\Omega S_a(w_\mathrm{o}, W) \varphi_\beta \, \mathrm{d}\lambda^m$$

for all $\varphi \in T$, where $i, j \in \{1, \ldots, m\}$, $\alpha, \beta \in \{1, \ldots, 4\}$ and

$$(w_{\mathrm{o}1}, \ldots, w_{\mathrm{o}4}) = (u_\mathrm{o}, v_\mathrm{o}, \psi_\mathrm{o}, \theta_\mathrm{o}),\ (W_1, \ldots, W_4) = (U, V, \Psi, \Theta) \text{ and } (\varphi_1, \ldots, \varphi_4) = (\varphi_u, \varphi_v, \varphi_\psi, \varphi_\theta).$$

Except of the two operators $f^u, f^v : M_\omega \longrightarrow Y_\omega$ defined as

$$\langle f^u(w_\mathrm{o}, W), \varphi\rangle := -\int_\Omega S_{\mu n}(w_\mathrm{o}, W) \nabla(U + u_\mathrm{o}) \cdot \nabla u_\mathrm{o} \cdot \varphi_\theta \, \mathrm{d}\lambda^m,$$

$$\langle f^v(w_\mathrm{o}, W), \varphi\rangle := -\int_\Omega S_{\nu p}(w_\mathrm{o}, W) \nabla(V + v_\mathrm{o}) \cdot \nabla v_\mathrm{o} \cdot \varphi_\theta \, \mathrm{d}\lambda^m,$$

and to be considered separately, each part of the map f can be splitted into a sum of operators of the type $A_{ij\alpha\beta}, A^\mathrm{o}_{ij\alpha\beta}, A_{i\alpha\beta}, A^\mathrm{o}_{i\alpha\beta}, A_{j\beta}, A_\beta : M_\omega \longrightarrow Y_\omega$. Because of the continuous embedding of $W_\omega \times X_\omega$ into the space of bounded continuous functions we are able to prove the continuous differentiability of the map $f - f^u - f^v : M_\omega \longrightarrow Y_\omega$ for $\omega \in (m-2, \omega_1]$ in the same manner as in the announced paper of RECKE [5]. This argument also yields the existence and the continuity of the partial Fréchet derivatives $D_2 f^u, D_2 f^v : M_\omega \longrightarrow \mathcal{L}(X_\omega, Y_\omega)$.

It remains to consider the partial Fréchet derivatives $D_1 f^u, D_1 f^v$. Let (w_o, W) be an arbitrarily chosen point of M_ω. Now, we will prove that the linear map $D_1 f^u(w_\mathrm{o}, W) : W_\omega \longrightarrow Y_\omega$ defined as

$$\begin{aligned}
&\langle D_1 f^u(w_\mathrm{o}, W)\widehat{w}_\mathrm{o}, \varphi\rangle := \\
&= \int_\Omega S_{D_2(\mu n)}(w_\mathrm{o}, W)\widehat{w}_\mathrm{o}\, \nabla(U + u_\mathrm{o}) \cdot \nabla u_\mathrm{o} \cdot \varphi_\theta \,\mathrm{d}\lambda^m + \\
&+ \int_\Omega S_{\mu n}(w_\mathrm{o}, W)\,(\nabla U + 2\nabla u_\mathrm{o}) \cdot \nabla \widehat{u}_\mathrm{o} \cdot \varphi_\theta \,\mathrm{d}\lambda^m
\end{aligned}$$

for all $\varphi \in T$ and $\widehat{w}_\mathrm{o} = (\widehat{u}_\mathrm{o}, \widehat{v}_\mathrm{o}, \widehat{\psi}_\mathrm{o}, \widehat{\theta}_\mathrm{o}) \in W_\omega$ yields the sought-for partial Fréchet derivative of f^u. To do so, we define for $(w_\mathrm{o} + \widehat{w}_\mathrm{o}, W) \in M_\omega$ the functional $\Lambda^u \in Y_\omega$ by

$$\langle \Lambda^u, \varphi\rangle := \langle f^u(w_\mathrm{o} + \widehat{w}_\mathrm{o}, W) - f^u(w_\mathrm{o}, W) - D_1 f^u(w_\mathrm{o}, W)\widehat{w}_\mathrm{o}, \varphi\rangle \quad \text{for } \varphi \in T$$

and split it into the three parts $\Lambda^u = \Lambda_1^u + \Lambda_2^u + \Lambda_3^u$ as follows

$$\langle \Lambda_1^u, \varphi\rangle := \int_\Omega \big(S_{\mu n}(w_\mathrm{o} + \widehat{w}_\mathrm{o}, W) - S_{\mu n}(w_\mathrm{o}, W) - S_{D_2(\mu n)}(w_\mathrm{o}, W)\widehat{w}_\mathrm{o}\big)\, \nabla(U + u_\mathrm{o}) \cdot \nabla u_\mathrm{o} \cdot \varphi_\theta \,\mathrm{d}\lambda^m,$$

$$\langle \Lambda_2^u, \varphi\rangle := \int_\Omega \left(S_{\mu n}(w_\mathrm{o} + \widehat{w}_\mathrm{o}, W) - S_{\mu n}(w_\mathrm{o}, W)\right)(\nabla U + 2\nabla u_\mathrm{o}) \cdot \nabla \widehat{u}_\mathrm{o} \cdot \varphi_\theta \,\mathrm{d}\lambda^m,$$

$$\langle \Lambda_3^u, \varphi\rangle := \int_\Omega S_{\mu n}(w_\mathrm{o} + \widehat{w}_\mathrm{o}, W)\, |\nabla \widehat{u}_\mathrm{o}|^2\, \varphi_\theta \,\mathrm{d}\lambda^m.$$

Because of Theorem 1 and 2 there exists a constant $c_\Lambda = c_\Lambda(\omega) > 0$ such that the following norm estimates hold

$$\begin{aligned}
&\|\Lambda_1^u\|_{Y_\omega} \le c_\Lambda \|S_{\mu n}(w_\mathrm{o} + \widehat{w}_\mathrm{o}, W) - S_{\mu n}(w_\mathrm{o}, W) - S_{D_2(\mu n)}(w_\mathrm{o}, W)\widehat{w}_\mathrm{o}\|_{L^\infty(\Omega)} \|\nabla u_\mathrm{o}\|_{L^\infty(\Omega)}^2 \\
&+ c_\Lambda \|S_{\mu n}(w_\mathrm{o} + \widehat{w}_\mathrm{o}, W) - S_{\mu n}(w_\mathrm{o}, W) - S_{D_2(\mu n)}(w_\mathrm{o}, W)\widehat{w}_\mathrm{o}\|_{L^\infty(\Omega)} \|\nabla u_\mathrm{o}\|_{L^\infty(\Omega)} \|\nabla U\|_{\mathfrak{L}^{2,\omega}(\Omega)},
\end{aligned}$$

$$\begin{aligned}
&\|\Lambda_2^u\|_{Y_\omega} \le 2\, c_\Lambda \|S_{\mu n}(w_\mathrm{o} + \widehat{w}_\mathrm{o}, W) - S_{\mu n}(w_\mathrm{o}, W) - S_{D_2(\mu n)}(w_\mathrm{o}, W)\widehat{w}_\mathrm{o}\|_{L^\infty(\Omega)} \|\nabla u_\mathrm{o}\|_{L^\infty(\Omega)} \|\nabla \widehat{u}_\mathrm{o}\|_{L^\infty(\Omega)} \\
&+ c_\Lambda \|S_{\mu n}(w_\mathrm{o} + \widehat{w}_\mathrm{o}, W) - S_{\mu n}(w_\mathrm{o}, W) - S_{D_2(\mu n)}(w_\mathrm{o}, W)\widehat{w}_\mathrm{o}\|_{L^\infty(\Omega)} \|\nabla U\|_{\mathfrak{L}^{2,\omega}(\Omega)} \|\nabla \widehat{u}_\mathrm{o}\|_{L^\infty(\Omega)} \\
&+ c_\Lambda \|S_{D_2(\mu n)}(w_\mathrm{o}, W)\widehat{w}_\mathrm{o}\|_{L^\infty(\Omega)} \big(2\|\nabla u_\mathrm{o}\|_{L^\infty(\Omega)} + \|\nabla U\|_{\mathfrak{L}^{2,\omega}(\Omega)}\big) \|\nabla \widehat{u}_\mathrm{o}\|_{L^\infty(\Omega)},
\end{aligned}$$

$$\|\Lambda_3^u\|_{Y_\omega} \le c_\Lambda \|S_{\mu n}(w_\mathrm{o} + \widehat{w}_\mathrm{o}, W)\|_{L^\infty(\Omega)} \|\nabla \widehat{u}_\mathrm{o}\|_{L^\infty(\Omega)}^2.$$

Now, with Lemma 3 it follows that $D_1 f^u(w_\mathrm{o}, W) : W_\omega \longrightarrow Y_\omega$ is the partial Fréchet derivative of f^u in $(w_\mathrm{o}, W) \in M_\omega$. Analogously, we prove existence of the partial Fréchet derivative $D_1 f^v(w_\mathrm{o}, W) : W_\omega \longrightarrow Y_\omega$ for $\omega \in (m-2, \omega_1]$.

Let $\{(w_\mathrm{o}^i, W^i)\}_{i\in\mathbb{N}} \subset M_\omega$ be a convergent sequence with $(w_\mathrm{o}^i, W^i) \to (w_\mathrm{o}, W)$ in M_ω. From Theorem 2(2) it follows the norm estimate

$$\begin{aligned}
&\|\left(D_1 f^u(w_\mathrm{o}^i, W^i) - D_1 f^u(w_\mathrm{o}, W)\right)\widehat{w}_\mathrm{o}\|_{Y_\omega} \le \\
&\le c\, \|S_{D_2(\mu n)}(w_\mathrm{o}^i, W^i)\nabla(U^i + u_\mathrm{o}^i) \cdot \nabla u_\mathrm{o}^i - S_{D_2(\mu n)}(w_\mathrm{o}, W)\nabla(U + u_\mathrm{o}) \cdot \nabla u_\mathrm{o}\|_{\mathfrak{L}^{2,\omega}(\Omega)} \|\widehat{w}_\mathrm{o}\|_{L^\infty(\Omega)} \\
&+ c\, \|S_{\mu n}(w_\mathrm{o}^i, W^i)(\nabla U^i + 2\nabla u_\mathrm{o}^i) - S_{\mu n}(w_\mathrm{o}, W)(\nabla U + 2\nabla u_\mathrm{o})\|_{\mathfrak{L}^{2,\omega}(\Omega)} \|\nabla \widehat{u}_\mathrm{o}\|_{L^\infty(\Omega)}.
\end{aligned}$$

Again, Theorem 1 and 2 yield a constant $c_L = c_L(\omega) > 0$ such that we are able to estimate

$$\begin{aligned}
&\| \left(D_1 f^u(w_o^i, W^i) - D_1 f^u(w_o, W)\right) \|_{\mathcal{L}(W_\omega, Y_\omega)} \leq \\
&\leq c_L \|S_{D_2(\mu n)}(w_o^i, W^i)\|_{L^\infty(\Omega)} \left(\|\nabla(u_o^i - u_o)\|_{L^\infty(\Omega)} + \|\nabla(U^i - U)\|_{\mathfrak{L}^{2,\omega}(\Omega)}\right) \|\nabla u_o^i\|_{L^\infty(\Omega)} \\
&+ c_L \|S_{D_2(\mu n)}(w_o^i, W^i)\|_{L^\infty(\Omega)} \|\nabla(u_o^i - u_o)\|_{L^\infty(\Omega)} \|\nabla(U + u_o)\|_{\mathfrak{L}^{2,\omega}(\Omega)} \\
&+ c_L \|S_{D_2(\mu n)}(w_o^i, W^i) - S_{D_2(\mu n)}(w_o, W)\|_{L^\infty(\Omega)} \|\nabla(U + u_o)\|_{\mathfrak{L}^{2,\omega}(\Omega)} \|\nabla u_o\|_{L^\infty(\Omega)} \\
&+ c_L \|S_{\mu n}(w_o^i, W^i)\|_{L^\infty(\Omega)} \left(2\|\nabla(u_o^i - u_o)\|_{L^\infty(\Omega)} + \|\nabla(U^i - U)\|_{\mathfrak{L}^{2,\omega}(\Omega)}\right) \\
&+ c_L \|S_{\mu n}(w_o^i, W^i) - S_{\mu n}(w_o, W)\|_{L^\infty(\Omega)} \left(2\|\nabla u_o\|_{L^\infty(\Omega)} + \|\nabla U\|_{\mathfrak{L}^{2,\omega}(\Omega)}\right).
\end{aligned}$$

According to Lemma 3 the limiting process $i \to \infty$ yields $\| \left(D_1 f^u(w_o^i, W^i) - D_1 f^u(w_o, W)\right) \|_{\mathcal{L}(W_\omega, Y_\omega)} \to 0$. Analogously, we get the continuity of the partial Fréchet derivative $D_1 f^v : M_\omega \longrightarrow \mathcal{L}(W_\omega, Y_\omega)$. ■

Theorem 6 *Let $\sigma \in (m-2, m)$ and*

$$\left.\begin{aligned}
&(\overline{u}_o, \overline{v}_o, \overline{\psi}_o, \overline{\theta}_o) \in W_\sigma \text{ such that} \\
&\nabla \overline{u}_o = \nabla \overline{v}_o = \nabla \overline{\theta}_o = 0 \text{ and } \overline{u}_o + \overline{v}_o = 0 \text{ on } \Omega.
\end{aligned}\right\} \tag{11}$$

We take positive constants $\theta_\star, \theta^\star, b, \kappa_\star = \kappa_\star(\theta_\star, \theta^\star), \mu_\star = \mu_\star(\theta_\star, \theta^\star), N_\star = N_\star(\theta_\star, \theta^\star), e_\star = e_\star(\theta_\star, \theta^\star)$ fulfilling

$$|\overline{u}_o|, |\overline{v}_o| < b \text{ and } \theta_\star < \overline{\theta}_o < \theta^\star \text{ on } \Omega \text{ and } \kappa_\star < \kappa < \frac{1}{\kappa_\star} \text{ almost everywhere on } \Omega \times [\theta_\star, \theta^\star],$$

$$\mu_\star < \mu < \frac{1}{\mu_\star} \text{ and } \mu_\star < \nu < \frac{1}{\mu_\star} \text{ almost everywhere on } \Omega \times [\theta_\star, \theta^\star],$$

$$N_\star < N < \frac{1}{N_\star} \text{ and } N_\star < P < \frac{1}{N_\star} \text{ almost everywhere on } \Omega \times [\theta_\star, \theta^\star],$$

$$e_\star > |E_n| \text{ and } e_\star > |E_p| \text{ almost everywhere on } \Omega \times [\theta_\star, \theta^\star].$$

Finally, using the properties of F (see (6)) we choose a constant $c = c(b, \theta_\star, \theta^\star) > 0$ such that

$$c > b + e_\star \text{ and } c > |\overline{\psi}_o| \text{ on } \Omega,$$

$$N_\star F\left(\frac{c - b - e_\star}{\theta^\star}\right) - \frac{1}{N_\star} F\left(\frac{b + e_\star - c}{\theta^\star}\right) - \|D\|_{L^\infty(\Omega)} > 0.$$

Then, there exists a constant $\omega = \omega(b, c, \theta_\star, \theta^\star, \varepsilon_\star, \varepsilon^\star) \in (m-2, \sigma)$ such that

(1) $f((\overline{u}_o, \overline{v}_o, \overline{\psi}_o, \overline{\theta}_o), (\overline{U}, \overline{V}, \overline{\Psi}, \overline{\Theta})) = 0$ has a unique solution in $M_\omega(b, c, \theta_\star, \theta^\star)$ and it holds $\overline{U} = \overline{V} = \overline{\Theta} = 0$ (thermodynamic equilibrium point),

(2) $D_2 f((\overline{u}_o, \overline{v}_o, \overline{\psi}_o, \overline{\theta}_o), (\overline{U}, \overline{V}, \overline{\Psi}, \overline{\Theta}))$ is a linear isomorphism from X_ω onto Y_ω,

(3) there exist open sets K_o, K and a uniquely determined map $s \in C^1(K_o, X_\omega)$ such that

$K_o \subset W_\omega$ is an open neighborhood of $(\overline{u}_o, \overline{v}_o, \overline{\psi}_o, \overline{\theta}_o)$,
$K \subset X_\omega$ is an open neighborhood of $(0, 0, \overline{\Psi}, 0)$,
$K_o \times K \subset M_\omega(b, c, \theta_\star, \theta^\star)$ and

$(U, V, \Psi, \Theta) = s((u_o, v_o, \psi_o, \theta_o))$ if and only if
$(u_o, v_o, \psi_o, \theta_o) \in K_o$, $(U, V, \Psi, \Theta) \in K$ and $f((u_o, v_o, \psi_o, \theta_o), (U, V, \Psi, \Theta)) = 0$.

Proof: 1. At first we look for a solution $(0,0,\overline{\Psi},0)$ to the given boundary value functions $(\overline{u}_{\rm o},\overline{v}_{\rm o},\overline{\psi}_{\rm o},\overline{\theta}_{\rm o})$. That means, we have to deal with the remaining nonlinear Poisson equation. To do so, we follow closely the methods of GRÖGER [1]. Because of the properties of F (see (6)) we can find a constant $c_1 \in (0,c)$ such that

$$c_1 > b + e_\star \text{ and } c_1 > |\overline{\psi}_{\rm o}| \text{ on } \Omega,$$

$$N_\star F\left(\frac{c_1 - b - e_\star}{\theta^\star}\right) - \frac{1}{N_\star} F\left(\frac{b + e_\star - c_1}{\theta^\star}\right) - \|D\|_{L^\infty(\Omega)} > 0.$$

Let us define for this constant $c_1 > 0$ and a function $a : \Omega \to \mathbb{R}$ the cut-off operator Π as

$$(\Pi a)(x) := \begin{cases} c_1 & \text{for } x \in \Omega, \quad c_1 \le a(x), \\ a(x) & \text{for } x \in \Omega, \quad -c_1 \le a(x) \le c_1, \\ -c_1 & \text{for } x \in \Omega, \quad a(x) \le -c_1 \end{cases}$$

and an operator f_Π for a corresponding regularized problem

$$f_\Pi : W_{\rm o}^{1,2}(G) \longrightarrow W^{-1,2}(G) \quad \text{by}$$

$$\langle f_\Pi(\overline{\Psi}), \varphi_\psi \rangle := \int_\Omega \varepsilon \nabla(\overline{\Psi} + \overline{\psi}_{\rm o}) \cdot \nabla \varphi_\psi \, \mathrm{d}\lambda^m +$$

$$+ \int_\Omega \left(\overline{N} F\left(\frac{\overline{u}_{\rm o} + \Pi(\overline{\Psi} + \overline{\psi}_{\rm o}) - \overline{E}_n}{\overline{\theta}_{\rm o}} \right) - \overline{P} F\left(\frac{\overline{v}_{\rm o} - \Pi(\overline{\Psi} + \overline{\psi}_{\rm o}) + \overline{E}_p}{\overline{\theta}_{\rm o}} \right) - D \right) \varphi_\psi \, \mathrm{d}\lambda^m$$

for all $\varphi_\psi \in W_{\rm o}^{1,2}(G)$.

Note, that here and later on coefficients considered in a thermodynamic equilibrium point are overlined. Obviously, by (6) the so defined operator f_Π is strongly monotone and Lipschitz continuous. Therefore, the regularized problem

$$\langle f_\Pi(\overline{\Psi}), \varphi_\psi \rangle = 0 \quad \text{for all } \varphi_\psi \in W_{\rm o}^{1,2}(G) \tag{12}$$

admits a unique solution $\overline{\Psi} \in W_{\rm o}^{1,2}(G)$. To show the boundedness of $\overline{\Psi}$ we consider the test function

$$\varphi_\psi = \max\left\{\overline{\Psi} + \overline{\psi}_{\rm o} - c_1, 0\right\} \in W^{1,2}(\Omega).$$

Because of $\overline{\psi}_{\rm o} < c_1$ we have

$$\overline{\Psi} + \overline{\psi}_{\rm o} - c \le \overline{\Psi} \text{ and } 0 \le \varphi_\psi \le \max\left\{\overline{\Psi}, 0\right\} \text{ almost everywhere on } \Omega.$$

With $\overline{\Psi} \in W_{\rm o}^{1,2}(G)$ it follows that

$$\varphi_\psi \in W_{\rm o}^{1,2}(G) \text{ and } \nabla(\overline{\Psi} + \overline{\psi}_{\rm o}) \cdot \nabla\varphi_\psi = |\nabla\varphi_\psi|^2.$$

Inserting φ_ψ in (12) we get for $\Omega_\Pi = \left\{x \in \Omega \,\middle|\, \overline{\Psi} + \overline{\psi}_{\rm o} \ge c_1\right\}$

$$\int_\Omega \varepsilon |\nabla\varphi_\psi|^2 \, \mathrm{d}\lambda^m = \int_\Omega D\varphi_\psi \, \mathrm{d}\lambda^m + \int_{\Omega_\Pi} \left(\overline{P} F\left(\frac{\overline{v}_{\rm o} - c_1 + \overline{E}_p}{\overline{\theta}_{\rm o}}\right) - \overline{N} F\left(\frac{\overline{u}_{\rm o} + c_1 - \overline{E}_n}{\overline{\theta}_{\rm o}}\right)\right) \varphi_\psi \, \mathrm{d}\lambda^m \le$$

$$\le \int_{\Omega_\Pi} \left(\frac{1}{N_\star} F\left(\frac{b - c_1 + e_\star}{\theta_\star}\right) - N_\star F\left(\frac{c_1 - b - e_\star}{\theta_\star}\right) + \|D\|_{L^\infty(\Omega)} \right) \varphi_\psi \, \mathrm{d}\lambda^m \le 0.$$

Hence, we have proved the estimate $\overline{\Psi} + \overline{\psi}_{\rm o} \le c_1 < c$ almost everywhere on Ω. Analogously, testing with

$$\varphi_\psi = \min\{\overline{\Psi} + \overline{\psi}_{\rm o} + c_1, 0\} \in W_{\rm o}^{1,2}(G),$$

we see that $\overline{\Psi} + \overline{\psi}_{\rm o} \ge -c_1 > -c$ and finally, $|\overline{\Psi} + \overline{\psi}_{\rm o}| \le c_1 < c$ almost everywhere on Ω. Therefore, it holds

$$\int_\Omega \varepsilon \nabla\overline{\Psi} \cdot \nabla\varphi_\psi \, \mathrm{d}\lambda^m = -\int_\Omega \varepsilon \nabla\overline{\psi}_{\rm o} \cdot \nabla\varphi_\psi \, \mathrm{d}\lambda^m + \int_\Omega (\overline{p} - \overline{n} + D)\varphi_\psi \, \mathrm{d}\lambda^m$$

for all $\varphi_\psi \in W^{1,2}_{\mathrm{o}}(G)$.

Having in mind Theorem 1(1), Theorem 2(2) and assumption (6) it is easy to see, that there exists a constant $\omega_1 \in (m-2,\sigma)$ such that the integral terms on the right hand side define a functional in $Y_{\omega_1}(G)$. Furthermore, by Theorem 2(3) there exists a constant $\omega_{\mathrm{o}} = \omega_{\mathrm{o}}(\varepsilon) \in (m-2,\omega_1)$ such that the integral on the left hand side defines an injective Fredholm operator from $X_{\omega_{\mathrm{o}}}(G)$ into $Y_{\omega_{\mathrm{o}}}(G)$, which is then even a linear isomorphism from $X_{\omega_{\mathrm{o}}}(G)$ onto $Y_{\omega_{\mathrm{o}}}(G)$. Therefore, to the given boundary value functions it exists a thermodynamic equilibrium point

$$((\overline{u}_{\mathrm{o}},\overline{v}_{\mathrm{o}},\overline{\psi}_{\mathrm{o}},\overline{\theta}_{\mathrm{o}}),(0,0,\overline{\Psi},0)) \in M_{\omega_{\mathrm{o}}}(b,c,\theta_\star,\theta^\star), \quad f((\overline{u}_{\mathrm{o}},\overline{v}_{\mathrm{o}},\overline{\psi}_{\mathrm{o}},\overline{\theta}_{\mathrm{o}}),(0,0,\overline{\Psi},0)) = 0.$$

To show its uniqueness let $(\overline{U},\overline{V},\overline{\Psi},\overline{\Theta}) \in X_{\omega_{\mathrm{o}}}$ be any homogeneous solution to the given boundary value functions. Inserting at first the test function $(\overline{U},\overline{V},0,0) \in T$ and then $(0,0,0,\overline{\Theta}) \in T$ in (10) it follows immediately from (11) that

$$\overline{U} = \overline{V} = \overline{\Theta} = 0.$$

Finally, we consider two homogeneous solutions $(0,0,\overline{\Psi}_1,0) \in X_{\omega_{\mathrm{o}}}$ and $(0,0,\overline{\Psi}_2,0) \in X_{\omega_{\mathrm{o}}}$. Testing (10) with $(0,0,\overline{\Psi}_1-\overline{\Psi}_2,0) \in T$ and subtracting the remaining integral identities the monotonicity of F yields $\overline{\Psi}_1 = \overline{\Psi}_2$.

2. Now, Theorem 5 gives us a constant $\omega_2 \in (m-2,\omega_{\mathrm{o}}]$ such that the partial Fréchet derivative

$$D_2 f((\overline{u}_{\mathrm{o}},\overline{v}_{\mathrm{o}},\overline{\psi}_{\mathrm{o}},\overline{\theta}_{\mathrm{o}}),(0,0,\overline{\Psi},0)) \in \mathcal{L}(X_{\omega_2},Y_{\omega_2})$$

exists. After a little computation we obtain the following matrix formulation of this derivative:

$$\left\langle D_2 f((\overline{u}_{\mathrm{o}},\overline{v}_{\mathrm{o}},\overline{\psi}_{\mathrm{o}},\overline{\theta}_{\mathrm{o}}),(0,0,\overline{\Psi},0))(\widehat{U},\widehat{V},\widehat{\Psi},\widehat{\Theta}),(\varphi_u,\varphi_v,\varphi_\psi,\varphi_\theta)\right\rangle =$$

$$= \int_\Omega \begin{pmatrix} \nabla\widehat{U} \\ \nabla\widehat{V} \\ \nabla\widehat{\Psi} \\ \nabla\widehat{\Theta} \end{pmatrix} \begin{pmatrix} \overline{\mu}\,\overline{n} & 0 & 0 & 0 \\ 0 & \overline{\nu}\,\overline{p} & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & \overline{\kappa} \end{pmatrix} \begin{pmatrix} \nabla\varphi_u \\ \nabla\varphi_v \\ \nabla\varphi_\psi \\ \nabla\varphi_\theta \end{pmatrix} + \begin{pmatrix} \widehat{U} \\ \widehat{V} \\ \widehat{\Psi} \\ \widehat{\Theta} \end{pmatrix} \begin{pmatrix} \overline{r}/\overline{\theta}_{\mathrm{o}} & \overline{r}/\overline{\theta}_{\mathrm{o}} & a_{13} & 0 \\ \overline{r}/\overline{\theta}_{\mathrm{o}} & \overline{r}/\overline{\theta}_{\mathrm{o}} & a_{23} & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & a_{43} & 0 \end{pmatrix} \begin{pmatrix} \varphi_u \\ \varphi_v \\ \varphi_\psi \\ \varphi_\theta \end{pmatrix} \,\mathrm{d}\lambda^m$$

for all $(\widehat{U},\widehat{V},\widehat{\Psi},\widehat{\Theta}) \in X_{\omega_2}$ and $(\varphi_u,\varphi_v,\varphi_\psi,\varphi_\theta) \in T$.

Noting the diagonal structure of the first matrix under the integral we can apply the regularity theory (Theorem 2(3)) for weakly coupled systems to get a constant $\omega \in (m-2,\omega_2]$ such that the partial Fréchet derivative

$$D_2 f((\overline{u}_{\mathrm{o}},\overline{v}_{\mathrm{o}},\overline{\psi}_{\mathrm{o}},\overline{\theta}_{\mathrm{o}}),(0,0,\overline{\Psi},0))$$

is a Fredholm operator (index zero) from X_ω into Y_ω. Considering the equation

$$D_2 f((\overline{u}_{\mathrm{o}},\overline{v}_{\mathrm{o}},\overline{\psi}_{\mathrm{o}},\overline{\theta}_{\mathrm{o}}),(0,0,\overline{\Psi},0))(\widehat{U},\widehat{V},\widehat{\Psi},\widehat{\Theta}) = 0$$

from the structure of the second matrix under the integral it follows at first

$$\widehat{U} = \widehat{V} = \widehat{\Theta} = 0$$

because of $\overline{r}/\overline{\theta}_{\mathrm{o}} \geq 0$ on Ω. In the third column of the second matrix then only the element a_{33} is of interest. Indeed, with the monotonicity of F we get

$$a_{33} = \frac{\overline{N}}{\overline{\theta}_{\mathrm{o}}} F'\left(\frac{\overline{u}_{\mathrm{o}}+\overline{\psi}_{\mathrm{o}}+\overline{\Psi}-\overline{E}_n}{\overline{\theta}_{\mathrm{o}}}\right) + \frac{\overline{P}}{\overline{\theta}_{\mathrm{o}}} F'\left(\frac{\overline{v}_{\mathrm{o}}-\overline{\psi}_{\mathrm{o}}-\overline{\Psi}+\overline{E}_p}{\overline{\theta}_{\mathrm{o}}}\right) \geq 0 \text{ on } \Omega$$

and therefore $\widehat{\Psi} = 0$. Hence, $D_2 f((\overline{u}_{\mathrm{o}},\overline{v}_{\mathrm{o}},\overline{\psi}_{\mathrm{o}},\overline{\theta}_{\mathrm{o}}),(0,0,\overline{\Psi},0))$ is injective and thus a linear isomorphism from X_ω onto Y_ω.

3. The third assertion is an immediate consequence of the above results and the Implicit Function Theorem.

■

References

[1] GRÖGER, K.: On steady-state carrier distributions in semiconductor devices. Aplikace Matematiky **32** (1987) No.1, 49–56.

[2] GRÖGER, K.: A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations. Math. Ann. **283** (1989), 679–687.

[3] HOWISON, S., RODRIGUES, J., SHILLOR, M.: Stationary solutions to the thermistor problem. J. Math. Anal. Appl. **174** (1993), 573–588.

[4] RECKE, L.: Solvability properties of linear elliptic boundary value problems with nonsmooth data. Preprint Nr. 94 – 3, Fachbereich Mathematik, Humboldt–Universität zu Berlin, 1994.

[5] RECKE, L.: Applications of the Implicit Function Theorem to quasi-linear elliptic boundary value problems with non-smooth data. Comm. Part. Diff. Equations **20** (1995), 1457–1479.

[6] TROIANIELLO, G.M.: Elliptic Differential Equations and Obstacle Problems. Plenum Press, New York, London, 1987.

[7] WACHUTKA, G.: Rigorous thermodynamic treatment of heat generation and conduction in semiconductor device modelling. In BACCARANI, G., RUDAN, M. (eds.): Simulation of semiconductor devices and processes, Vol. 3, Bologna (Italy), September 26–28, 1988, pp. 83–95.

[8] HONG XIE: $\mathfrak{L}^{2,\mu}(\Omega)$ estimate to the mixed boundary value problem for second order elliptic equations and its application in the thermistor problem. Nonlinear Analysis **24** (1995), 9–27.