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Balanced implicit methods for stiff stochastic systems: An introduction and numerical experiments

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Abstract

The paper introduces implicitness in stochastic terms of numerical methods for solving of stiff stochastic differential equations and especially a class of fully implicit methods, the balanced methods. Their order of strong convergence is proved. Systematic numerical experiments compare the numerical behaviour of these schemes with that of different other schemes. A wide class of model equations are also provided as one by-product in order to test numerical methods in the case of stochastic stiffness in the given system.

Contents

1 Introduction

During the last years several authors have proposed implicit numerical methods for stochastic differential equations with respect to strong and weak convergence criterions. We refer here to the papers of Talay ([14],1982), Klauder and Petersen ([5],1985), Milstein ([9],1988), Hernandez and Spigler ([4],1990), Mitsui and Saito ([11],1991), Drummond and Mortimer ([2],1991), Kloeden and Platen ([6],[7], 1992) just to mention few of them.
As in the deterministic case implicit methods are necessary to integrate stiff systems. However, the introduction of implicitness is restricted in the above mentioned papers to the deterministic terms, e.g. to the drift term in the Euler scheme. Let us call such methods deterministically implicit (or drift-implicit) and otherwise stochastically implicit. Deterministically (drift-) implicit methods are well adapted for stiff systems with small stochastic noise intensity or additive noise. But in those cases in which the stochastic part plays an essential role in the dynamics, e.g. as it is with large multiplicative noise, the application of fully implicit methods involving also implicit stochastic terms is unavoidable. A good illustration for such a situation is provided by the following one-dimensional Itô equation with multiplicative noise

$$dX_t \;=\; \sigma\, X_t\, dW_t \qquad , \quad t \geq 0 \tag{1.1}$$

starting at $X_0 = x_0$. Here $W = \{W_t, 0 \leq t\}$ is a standard Wiener process. The solution of (1.1) decreases rapidly to zero for $|\sigma| >> 1$ because its *Lyapunov* exponent $\lambda = -\sigma^2/2$ is negative.

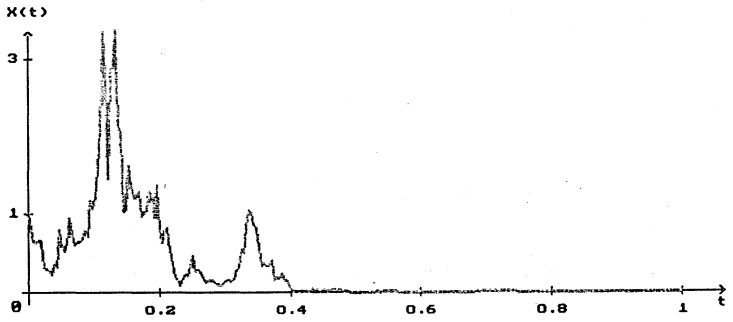

Figure 1.1 : A trajectory of the solution of equation (1.1)

The one-dimensional equation (1.1) cannot be simply called stiff (in the physical sense), but it has to be interpreted as an equation for one component in a multi-dimensional system, at least embedded in a two-dimensional system. From this view point we are going to consider the numerical solutions for the one-dimensional equation (1.1). For large parameters $|\sigma|$ in equation (1.1) one observes that explicit methods work unreliably and have large errors for not too small time step sizes. They even lead to computer overflows (for having an idea see figure 1.2). On the other hand using very small time step sizes may require too much computational

time. In stiff situations this is the crucial point where one has to look for other more suitable methods. For example, these difficulties occur in the estimation of the top Lyapunov exponent and in parametric estimation where the long-time behaviour of the numerical solution is decisive for the calculations. One way is to use the Itô formula to transform the solutions to reduce or remove the fastly varying property of both the exact and numerical solution. However this rarely works in more complicated multi-dimensional systems (see section 5 for an example). By the common transformation $Z_t = e^{\alpha t} X_t$ with $\alpha \in \mathbb{R}^+$ we obtain the linear Itô differential equation

$$dZ_t = \alpha Z_t \, dt + \sigma Z_t \, dW_t , \qquad t \geq 0, \; Z_0 = x \tag{1.2}$$

which, for α close to $\sigma^2/2$, can be integrated sufficiently accurate by explicit methods. Thus transforming is a useful idea when one can expect more stable behaviour of the explicit methods for the numerical solution of the original model. But the direct application of explicit methods for equation (1.1) yields as in the deterministic case for time step sizes larger than a critical value (more precisely, there even exists a critical random interval for $\sigma \neq 0$) suddenly to an explosion of the numerical solution and its global error. A natural task is to achieve control in numerical approximations for a larger range of integration step sizes and to overcome constructively this dilemma by introducing new numerical scheme techniques, such as the balanced methods. Obviously one cannot apply deterministically (drift-) implicit schemes to improve the stability of the numerical solution for the stochastic equation (1.1) which does not contain any drift component. Demonstrating the mentioned dilemma with a simple example the explicit Euler scheme provides poor results for time step sizes larger than 2^{-4} as it can be seen from figure 1.2, and later from figure 2.1 which show its poor global error behaviour.

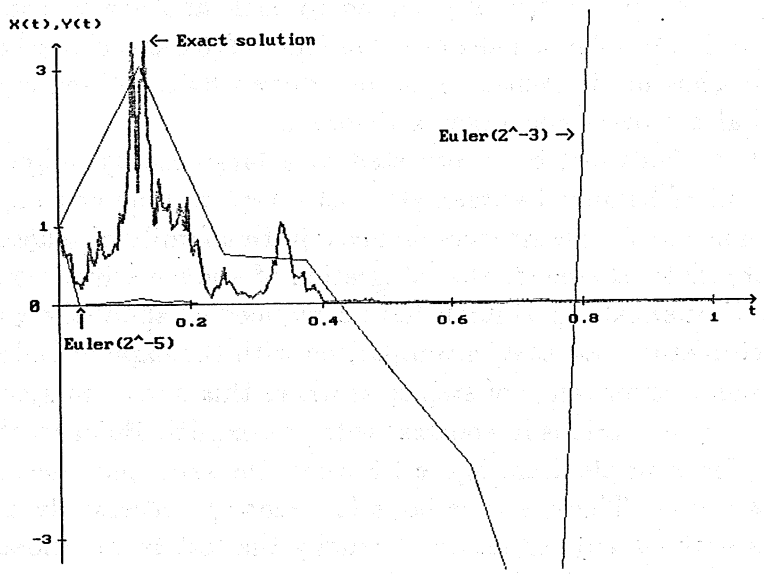

Figure 1.2 : Exact solution and Euler approximation for equation (1.1) with parameter $\sigma = 4$ and time step size $\Delta = 2^{-3}$ and 2^{-5}

Equation (1.1) is useful as a model equation that can be solved numerically by different methods with different features. With it we can discuss problems and typical properties encountered with explicit and implicit methods. Of course one could use smaller integration step sizes here, but the problem is still getting worse with larger noise intensities parameters σ because large σ requires extremely small step sizes for numerical integration. Which integration step size should be chosen to obtain reliable (explicit) approximations? This question can be decided experimentally. In our example (1.1) we have obtained experimentally that for integration step sizes smaller than 2^{-5} the explicit Euler method works reliably in the sense of having no explosions for $\sigma = 4$ and thereby the critical step size for the one Wiener path used for the figure 1.2 should be no larger than 2^{-5}.
We thus have to construct fully implicit methods which involve implicitness in the deterministic as well as in the stochastic terms, that is, stochastically implicit methods. In this paper we restrict ourselves to the construction of (fully) implicit methods with low order of strong convergence. In section 2 we will apply some numerical methods to equation (1.1). Among the methods of this section there is a new method which we call balanced method. Besides we will introduce an implicit method which has implicitness even in a stochastic term by analysing of the multiple Itô integral $\int\int dW_s dW_t$. It turns out that this method provides numerical solutions converge to the exact solution with strong order 1.0. Its pathwise behaviour seems to be similar to that of the balanced method in the range of large time step sizes, but in this range its accuracy is actually worse than that of the balanced method. Naturally, for sufficiently small time step sizes one obtains the strong order 1.0 and thereby the asymptotical error behaviour confirmed experimentally. This may be explained by the fact that the order of accuracy (asymptotical property) and the stability (property for 'real time step sizes') of the numerical solution are qualitative different assertions about the numerical method used, although both properties are related to each another to some extent. In section 3 we will state the structure of the class of balanced methods for multi-dimensional stochastic differential equations driven by multi-dimensional Wiener processes and also prove a convergence theorem.
The balanced methods can be interpreted as a large family of specific methods providing a kind of balance between stochastic terms. One can hope that by an appropriate choice of the parameters involved in this family of schemes one is able to find an acceptable scheme for the integration of a given stiff stochastic differential equation. Numerical experiments have convinced us about the good behaviour of the balanced method, at least in comparison with the explicit Euler method, for example, having a larger range of step sizes where this new numerical method still works without any explosions in contrast with the explicit Euler method. This can be easily seen, for example, from figure 1.3 where the same parameters and Wiener path have been used. There is thus hope for treating successfully a wide class of stiff equations without any explosion. Actually the last two sections 4 and 5 will provide some justification for this hope in more complicated situations.

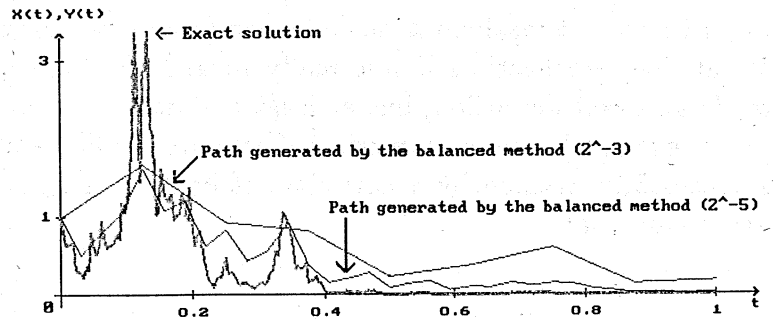

Figure 1.3 : Exact solution and numerical result of the balanced method simulating equation (1.1) with $\sigma = 4$ and $\Delta = 2^{-3}$ and 2^{-5}

Several numerical experiments in section 4 and 5 show the effects of stiffness in multi-dimensional stochastic systems, the failure of explicit methods in a certain range of time step sizes and the successful treatment by the balanced methods with suitable parameters for a very large range of time step sizes (e.g. the range where the roundoff error does not influence decisively the calculations). The formulation of specific recommendations for the choice of suitable parameters in these implicit methods for some given class of stochastic differential equations is a rather difficult task. It is closely related to the construction of appropriate classes of test equations and its parameters. In the deterministic case, by the Jordan theorem we have a rather simple linear complex valued equation which allows the characterization of stable numerical methods. The situation is similar for additive noise, see Milstein ([9],1988). But in the stochastic case with multiplicative or general noise so far no comparable result is known. Thus it still remains an open problem to find sufficiently simple, but also rich families of test equations which characterize numerical methods for stiff stochastic differential equations.
In this paper we understand stiffness in the physical sense involving the simultaneous appearence of relatively fast and slow velocities in the dynamics of the system, for example, as it occurs in this two-dimensional decoupled system (with its *Lyapunov* exponents $\lambda_1 = -50.0$ and $\lambda_2 = -0.28$)

$$\begin{aligned} dX_t^1 &= 10\, X_t^1\, dW_t^1 \\ dX_t^2 &= X_t^2\, dt \,+\, 1.6\, X_t^2\, dW_t^2 \end{aligned}$$

driven by two independent Gaussian noise sources W_t^1 and W_t^2 and having first component $X_t^1 = X_0^1 * \exp\{-50 * t + 10 * W_t^1\}$ which converges extremely rapidly to zero in contrast to its second component $X_t^2 = X_0^2 * \exp\{-0.28 * t + 1.6 * W_t^2\}$ which converges to zero slowly. We will put great emphasis in section 2, 4 and 5 on the systematic study of corresponding numerical experiments and on what stiffness could mean and imply in stochastic systems. However, it must be pointed out that all we have done is to provide some first steps for a class of model equations

analogous to the deterministic theory. As in the deterministic theory the intuitive meaning of stochastic stiffness is clear to all specialists, but there is still much controversy about its correct mathematical definition (compare Hairer and Wanner ([3],1991)). What does stochastic stiffness really mean? At the moment we are unable to provide an exact definition, but at least a class of model equations can be systematically developed, several numerical effects are studied and these model equations are successfully treated by a new class of implicit methods which have no deterministic counterpart.

2 Experiments for Multiplicative Noise

As already mentioned above there exists a well developed literature proposing and investigating different numerical methods using a time step size Δ in order to construct discrete time approximations converging to the solution X of the given stochastic differential equation as the step size Δ converges to zero. For simplicity in this paper numerical methods on a given time interval $[0,T]$ are fixed by schemes based on equidistant time discretization points $\tau_n = n\,\Delta,\ n = 0,1,\ldots,N$ with step size $\Delta = T/N, N = 1,2,\ldots$. Here we shall use the abbreviation Y_n instead of $Y_{n\Delta}$ to denote the value of the approximation at time $n\Delta$. To classify different methods with respect to the rate of strong convergence as in Kloeden and Platen ([6],1992) we say that a discrete time approximation Y^Δ converges with strong order $\gamma > 0$ if there exist constants $\Delta_0 \in (0,\infty)$ and $K < +\infty$, not depending on Δ, such that

$$I\!E|X_T - Y_N^\Delta| \leq K\Delta^\gamma \tag{2.1}$$

for all $\Delta \in (0,\Delta_0)$. The simplest useful method is the Euler method, which is sometimes called the Euler-Maruyama method, see Maruyama (1955). For equation (1.1) it has the form

$$Y_{n+1} = Y_n + \sigma Y_n \Delta W_n \tag{2.2}$$

where $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n},\ n = 0,1,\ldots,N-1$ and $Y_0 = X_0$. The Euler method is an explicit method. In fact there is no counterpart of the deterministic implicit Euler method because $I\!E|(1+\sigma\Delta W_n)^{-1}|$ does not exist. Nevertheless, to introduce implicitness in the numerical treatment for this special equation within the Itô calculus an outway could be to look at the next higher order method and try to introduce implicitness there. For this purpose we start from the scheme

$$Y_{n+1} = Y_n + \sigma Y_n \Delta W_n + \frac{1}{2}\sigma^2 Y_n((\Delta W_n)^2 - \Delta),\ Y_0 = X_0,\ n = 0,1,\ldots,N-1$$

which represents a numerical method of strong order 1.0 (see Milstein [9], Kloeden and Platen [6]). Again the introduction of implicitness in $\sigma Y_n \Delta W_n$ as for the explicit Euler method above fails, but one can analyse the stochastic term

$$\frac{1}{2}\sigma^2 Y_n((\Delta W_n)^2 - \Delta) \;=\; \frac{1}{2}\sigma^2 Y_n(\Delta W_n)^2 - \frac{1}{2}\sigma^2 Y_n \Delta$$

and introduce partial implicitness in it because of $\frac{1}{2}\sigma^2\Delta \geq 0$. This leads to the scheme

$$Y_{n+1} = Y_n + (\sigma\Delta W_n + \frac{\sigma^2}{2}(\Delta W_n)^2)Y_n - \frac{\sigma^2}{2}Y_{n+1}\Delta,\ n = 0,1,\ldots,N-1 \tag{2.3}$$

which is a stochastically implicit method in the Itô sense. For the special equation (1.1) it is identical with a scheme stated in Kloeden and Platen ([6],1992, p. 400), which was derived using the Stratonovich calculus, where it is a deterministically (drift-) implicit scheme for the corresponding Stratonovich equation. The numerical approximation discribed by the scheme (2.3) converges to the exact solution with

strong order $\gamma = 1.0$. This statement can be verified by the use of a convergence theorem (see Theorem 2 of section 3) proved in Milstein ([9], [10], 1988). We note that in both (2.2) and (2.3) no stochastic diffusion term is implicit.
In this paper the main emphasis is focused on a class of fully implicit schemes converging strongly and also allowing random terms to be made implicit. In a natural way the method we are proposing can be called balanced method. For equation (1.1) it takes the form

$$Y_{n+1} = Y_n + \sigma Y_n \Delta W_n + \sigma(Y_n - Y_{n+1})|\Delta W_n|, \; n = 0, 1, \ldots, N-1 \qquad (2.4)$$

starting in $Y_0 = X_0$ and using ΔW_n as in (2.2).
It makes it possible to introduce implicitness also in the stochastic terms. We are proving in section 3 that the balanced method converges with the same strong order $\gamma = 0.5$ as the Euler method does. There we will see the balanced method described by (2.4) is a special member of the class of balanced methods.
Let us perform several numerical experiments for the linear equation (1.1) which has the explicit solution

$$X_t \; = \; \exp\{\sigma W_t - \frac{\sigma^2}{2} t\} \, X_0 \quad , t \geq 0 \qquad (2.5)$$

In the previous section 1 the attention has been drawn to the pathwise behaviour of the numerical solution driven by different schemes for the integration of equation (1.1). Now we are investigating the dependence of the global error $|X_t - Y_t|$ of the above described numerical schemes on the time step size Δ at the discretization points $t = n\Delta, n = 0, 1, \ldots, N$. For this purpose estimated mean errors are plotted and corresponding error bars (90% confidence intervals) at the discretization points are added in the next three figures. To identify the errors corresponding to one and the same time step size we interpolate the estimated mean errors linearly. At first we look at the results of the Euler method.

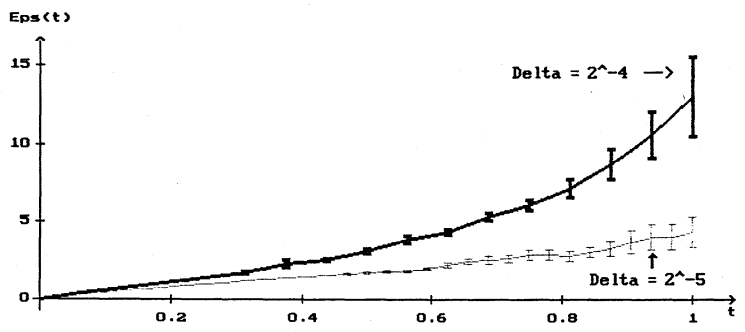

Figure 2.1 : Estimated mean global error Eps(t) of the Euler method at time t for different time step sizes using sample size 40 x 1000 and $\sigma = 4$

Both the estimate of the mean global error and its corresponding variance increase monotonically in t, e.g. we observe a worsening error behaviour of the Euler method for the time step sizes $\Delta = 2^{-4}$ and $= 2^{-5}$. On the base of figure 2.1 one suspects it exists a critical time step size Δ_0 such that for step sizes smaller than $\Delta_0 \leq 2^{-5}$ the error propagation in the Euler method remains under control. Beyond this random boundary for the time step size, e.g. for Δ larger Δ_0, the global error explodes in practice and the scheme becomes useless for such large time step sizes. We repeated the above simulations for the stochastically implicit scheme (2.3) which has a kind of implicitness only in the higher order term occuring by a purely stochastic multiple Itô integral.

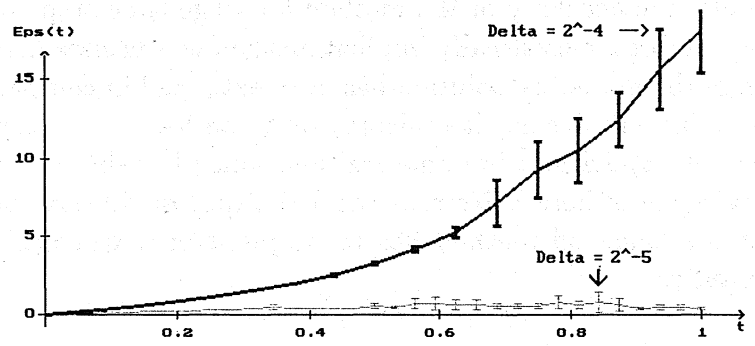

Figure 2.2 : Results of the numerical method (2.3)

Here we observe a slightly improved behaviour of the global error at the discretization points for time step size $\Delta = 2^{-5}$. This stochastically implicit method is able to reduce the variance of the global error estimates and keeps the mean global errors itself under control for time step sizes smaller than 2^{-5}. But for time step sizes larger than 2^{-4} it has no control on the mean global error.
Finally we apply the balanced method (2.4) representing a fully implicit method to the linear equation (1.1) and for the same step sizes. By the way, looking ahead to its general formulation in section 3, this method uses the functions $c^0(t,x) = 0$ and $c^1(t,x) = |\sigma|$.

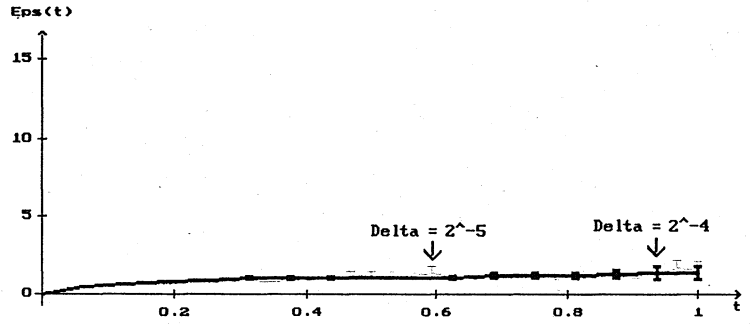

Figure 2.3 : Global error estimates Eps(t) of the balanced method (2.4) at time t

In the last two figures the same equation and parameters as well as sample size were used as in figure (2.1). Comparing the numerical experiments for the balanced method (2.4) with those where we applied the methods (2.2) and (2.3) it turns out that only the scheme (2.4) yields a significant improvement in limiting the error propagation. For the given stochastic differential equation (1.1) the balanced scheme is able to keep the propagation of errors under control if the time step size was increased to larger values and thereby for almost the whole range of possible time step sizes (except for too small Δ, where the roundoff error dominates). Besides this method causes a decisive variance reduction of the corresponding global error estimates. In additional we have seen that the global error could not be controlled successfully by one stochastically implicit method (see experiments for (2.3)) and thereby the accuracy of the method for large time step sizes. However, this does not mean this stochastically implicit method was useless here. Rather the stability region of the numerical solution has been extended in comparison with the 'explicit Euler' solution. Besides it indicates that one has to be very careful with the introduction of implicitness in numerical solutions (The discussion 'where' and 'which intensity' we omit here.). Now, as the first experiments have been done, the convergence of the balanced method has to be justified mathematically and not only experimentally.

3 Convergence of the Balanced Methods

To formulate the balanced methods in general we suppose the d-dimensional stochastic process $\{X_t : t \geq 0\}$ with $I\!E(X_0)^2 < \infty$ satisfies the d-dimensional stochastic differential equation

$$dX_t \;=\; a(t, X_t)\, dt \;+\; \sum_{j=1}^{m} b^j(t, X_t)\, dW_t^j \tag{3.1}$$

where $a, b^1, \ldots, b^m$ are d - dimensional Lipschitz continuous vector valued functions which fulfil also a linear growth condition. The processes $W^j = \{W_t^j;\ t \geq 0\}$, $j = 1, \ldots, m$ represent independent standard Wiener processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, I\!P)$ with its filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, see Kloeden and Platen ([6],1992). Usually one takes $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\}$ as the given filtration. Furthermore for $0 \leq s \leq t < \infty$ and $y \in I\!R^d$ $X_t^{s,y}$ denotes the value of a solution of (3.1) at time t which starts in $y \in I\!R^d$ at time s .
Now the balanced methods applied to (3.1) can be written in the form

$$Y_{n+1} = Y_n + a(\tau_n, Y_n)\Delta + \sum_{j=1}^{m} b^j(\tau_n, Y_n)\Delta W_n^j + C_n(Y_n - Y_{n+1}) \tag{3.2}$$

where

$$C_n = c^0(\tau_n, Y_n)\Delta + \sum_{j=1}^{m} c^j(\tau_n Y_n)|\Delta W_n^j| \quad , \tag{3.3}$$

$\Delta W_n^j = W_{\tau_{n+1}}^j - W_{\tau_n}^j$, $\Delta = \tau_{n+1} - \tau_n$, $n = 0, 1, \ldots, N-1$ and $c^0, c^1, \ldots, c^m$ represent d x d - matrix valued functions. We assume that for any sequence of real numbers (α_i) with $\alpha_0 \in [0, \bar{\alpha}], \alpha_1 \geq 0, \ldots, \alpha_m \geq 0, \bar{\alpha} < \infty$ and $(t, x) \in [0, \infty]$ x $I\!R^d$ the matrix

$$M(t, x) = I + \alpha_0\, c^0(t, x) + \sum_{j=1}^{m} \alpha_j\, c^j(t, x)$$

where I is unit matrix has only positive eigenvalues and it holds $M(t, x) = M^T(t, x)$, e.g. $M(t, x)$ is symmetric. Such matrices are invertible and fulfil the condition

$$\left|(M(t, x))^{-1}\right| \;\leq\; K \;< \infty. \tag{3.4}$$

Obviously (3.4) can be easily fulfilled in keeping $c^0, c^1, \ldots, c^m$ all positive definite. Thus under these conditions one obtains directly the one-step increment $Y_{n+1} - Y_n$ of the balanced method via the solution of a system of linear algebraic equations. Furthermore we suppose that the components of the matrices $c^0, c^1, \ldots, c^m$ are uniformly bounded (if even the components of a are uniformly bounded, then it is sufficient to demand that $c^0, c^1, \ldots, c^m$ have components satisfying a linear growth condition). The latter condition will be necessary to prove the convergence of the balanced methods via a theorem stated in Milstein ([9],1988).
We remark that method (3.2) - (3.3) turns out to be rather general. In the purely deterministic case it covers for instance the implicit Euler method with one or more Newton iteration steps. Now we are able to state the corresponding convergence theorem for the general balanced methods.

Theorem 1 :

Under the above assumptions the balanced methods (3.2) converge with strong order $\gamma = 0.5$, that is for all $k = 0, 1, \ldots, N$ and step size $\Delta = T/N$, $N = 1, 2, \ldots$.

$$\left(I\!E(|X_{\tau_k} - Y_k|^2 | \mathcal{F}_0)\right)^{1/2} \leq K' \, (1 + |X_0|^2)^{1/2} \, \Delta^{1/2} \tag{3.5}$$

where K' does not depend on Δ.

To prove Theorem 1 we recall a theorem concerning the order of strong convergence formulated in Milstein ([9],[10],1988) .

Theorem 2 (Milstein [9]) :

Assume for a one-step discrete time approximation Y that the local mean error and mean square error for all $N = 1, 2, \ldots,$ and $n = 0, 1, \ldots, N-1$ satisfy the estimates

$$\left| E(X^{\tau_n, Y_n}_{\tau_{n+1}} - Y_{n+1} | \mathcal{F}_{\tau_n}) \right| \leq K \, (1 + |Y_n|^2)^{1/2} \, \Delta^{p_1} \tag{3.6}$$

and

$$\left(E(|X^{\tau_n, Y_n}_{\tau_{n+1}} - Y_{n+1}|^2 | \mathcal{F}_{\tau_n}) \right)^{1/2} \leq K \, (1 + |Y_n|^2)^{1/2} \, \Delta^{p_2} \tag{3.7}$$

with $p_2 \geq \frac{1}{2}$ and $p_1 \geq p_2 + \frac{1}{2}$. Then

$$\left(E(|X^{0, X_0}_{\tau_k} - Y_k|^2 | \mathcal{F}_0) \right)^{1/2} \leq K' (1 + |X_0|^2)^{1/2} \, \Delta^{p_2 - \frac{1}{2}} \tag{3.8}$$

holds for each $k = 0, 1, ..., N$.

Pointing out the basic assertion for the proof of Theorem 1, we note that under the above assumptions on the local error Theorem 2 provides the global error bound and the strong order $\gamma = p_2 - \frac{1}{2}$ for the underlying one-step method. The proof of Theorem 1 consists of checking the validity of the assumptions of Theorem 2 for the balanced methods (3.2) assuming the boundary condition (3.4) for the matrix $M(t, x)$ and applying finally the Theorem 2.

Proof of Theorem 1 :

Firstly we show that the estimate (3.6) holds for the balanced methods (3.2) with $p_1 = \frac{3}{2}$. For this purpose the local Euler approximation step

$$Y^E_{k+1} = Y_k + a(\tau_k, Y_k)\Delta + \sum_{j=1}^{m} b^j(\tau_k, Y_k) \Delta W^j_k, \tag{3.9}$$

$k = 0, 1, \ldots, N-1$ is introduced and one can deduce for $n = 0, 1, \ldots, N-1$ that

$$
\begin{aligned}
H_1 &:= \left| I\!E(X^{\tau_n, Y_n}_{\tau_{n+1}} - Y_{n+1} | \mathcal{F}_{\tau_n}) \right| && (3.10)\\
&= \left| I\!E(X^{\tau_n, Y_n}_{\tau_{n+1}} - Y^E_{n+1} | \mathcal{F}_{\tau_n}) + I\!E(Y^E_{n+1} - Y_{n+1} | \mathcal{F}_{\tau_n}) \right| \\
&\leq K\,(1 + |Y_n|^2)^{1/2}\,\Delta^2 \;+\; H_2 && (3.11)
\end{aligned}
$$

with

$$
\begin{aligned}
H_2 &:= \left| I\!E(Y^E_{n+1} - Y_{n+1} | \mathcal{F}_{\tau_n}) \right| && (3.12)\\
&= \left| I\!E\left((I - (I + C_n)^{-1})(a(\tau_n, Y_n)\Delta + \sum_{j=1}^{m} b^j(\tau_n, Y_n)\Delta W^j_n) | \mathcal{F}_{\tau_n} \right) \right| && (3.13)\\
&= \left| I\!E\left((I + C_n)^{-1} C_n (a(\tau_n, Y_n)\Delta + \sum_{j=1}^{m} b^j(\tau_n, Y_n)\Delta W^j_n) | \mathcal{F}_{\tau_n} \right) \right| . && (3.14)
\end{aligned}
$$

Exploiting above the symmetry property of $\Delta W^j_n, j = 1, ..., m$ in those expressions involving this zero mean Gaussian variable we obtain

$$
H_2 = \left| I\!E((I + C_n)^{-1} C_n a(\tau_n, Y_n)\Delta | \mathcal{F}_{\tau_n}) \right| \qquad (3.15)
$$

and it follows with (3.4) that

$$
\begin{aligned}
H_2 &\leq K\; I\!E(|C_n a(\tau_n, Y_n)\Delta| | \mathcal{F}_{\tau_n}) && (3.16)\\
&\leq K\,(1 + |Y_n|^2)^{1/2}\,\Delta^{3/2} && (3.17)
\end{aligned}
$$

Thus the assumption (3.6) with $p_1 = 1.5$ in Theorem 2 is satisfied for the balanced methods.
Similarly we check assumption (3.7) for the local mean square error of the balanced methods (3.2) and obtain for $n = 0, 1, ..., N-1$ by standard arguments

$$
\begin{aligned}
H_3 &:= \left(I\!E(|X^{\tau_n, Y_n}_{\tau_{n+1}} - Y_{n+1}|^2 | \mathcal{F}_{\tau_n}) \right)^{1/2} && (3.18)\\
&\leq \left(I\!E(|X^{\tau_n, Y_n}_{\tau_{n+1}} - Y^E_{n+1}|^2 | \mathcal{F}_{\tau_n}) \right)^{1/2} + \left(I\!E(|Y^E_{n+1} - Y_{n+1}|^2 | \mathcal{F}_{\tau_n}) \right)^{1/2} && (3.19)\\
&\leq K\,(1 + |Y_n|^2)^{1/2}\,\Delta \quad . && (3.20)
\end{aligned}
$$

Thus we can choose in Theorem 2 the exponent $p_2 = 1.0$ together with $p_1 = 1.5$ and apply it to prove finally the strong order $\gamma = 0.5 (= p_2 - \frac{1}{2})$ of the balanced methods, as was claimed in Theorem 1 .

4 Experiments for a Simple System

In this and the next section some numerical experiments for multi-dimensional systems are reported. For simplicity we are going to only illustrate phase plane figures of the systems. For this we have chosen 'typical' trajectories driven by the stochastic system components. Further experiments such as the examination of first and second moments of the global error and their propagation for increasing time t are left to the reader. Already the phase plane figures will be sufficient to indicate graphically explosions in the numerical solution and better behaviour of the balanced methods. We start from the decoupled system of two equations

$$dX_t^1 = \sigma X_t^1 \, dW_t^1 \text{ and } dX_t^2 = \rho X_t^2 \, dW_t^2$$

and use the transformations

$$Y_t^1 = \frac{1}{2}(X_t^1 + X_t^2) \text{ and } Y_t^2 = \frac{1}{2}(X_t^1 - X_t^2)$$

to obtain the system

$$\begin{aligned} dY_t^1 &= \frac{1}{2}\sigma(Y_t^1 + Y_t^2)\, dW_t^1 + \frac{1}{2}\rho(Y_t^1 - Y_t^2)\, dW_t^2 \\ dY_t^2 &= \frac{1}{2}\sigma(Y_t^1 + Y_t^2)\, dW_t^1 - \frac{1}{2}\rho(Y_t^1 - Y_t^2)\, dW_t^2 \end{aligned} \tag{4.1}$$

It has already been seen in the introduction section that the Euler approximation explodes in contrast to that of the balanced method. For system (4.1) this is also clearly apparent from figure 4.1 where 'typical' trajectories for the same Wiener path have been plotted in the phase plane (Y_t^1, Y_t^2) on a time interval $0 \leq t \leq 2$.

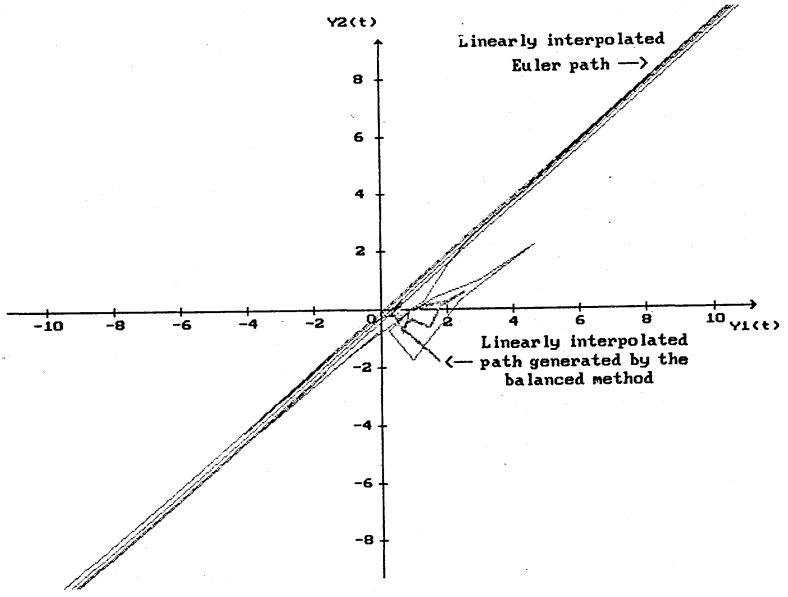

Figure 4.1 : Numerical solution of (4.1) of the Euler and balanced method for the parameter set $(\Delta, \sigma, \rho) = (0.04, 4, 1)$ starting at $(1, 0)$

The corresponding trajectory of the exact solution is not included here, but from multi-coloured computer experiments we know that the solution of the balanced method is close to the exact solution. For the balanced method with simply functions $c^0(t,x) = 0$, $c^1(t,x) = \sigma$ and $c^2(t,x) = \rho$ the trajectories (thick line) are attracted by a submanifold of the phase plane $(Y_t^1 = -Y_t^2)$ and stay closely to the zero point $(0,0)$, which replicates the behaviour of the exact solution. In contrast the Euler trajectories do not show this attraction and have large oscillating values, at least for time step sizes larger than 10^{-2} considered here. Of course the results need to be confirmed for other Wiener paths. In any case more reliable assertions can only be concluded from the error propagation graphs as in section 2. Nevertheless, figure 4.1 is confirmed in several experiments and one suspects that the global error of the Euler method behaves badly and has an unbounded error propagation for step sizes that are too large.
The same effects are obtained if there is a rotation in the system. For this purpose we add linear drift terms $\beta Y_t^1 dt$ and $-\beta Y_t^2 dt$ to system (4.1) to obtain the system

$$\begin{aligned} dY_t^1 &= \beta Y_t^2\, dt + \frac{1}{2}\sigma(Y_t^1 + Y_t^2)\, dW_t^1 + \frac{1}{2}\rho(Y_t^1 - Y_t^2)\, dW_t^2 \\ dY_t^2 &= -\beta Y_t^1\, dt + \frac{1}{2}\sigma(Y_t^1 + Y_t^2)\, dW_t^1 - \frac{1}{2}\rho(Y_t^1 - Y_t^2)\, dW_t^2 \end{aligned} \tag{4.2}$$

in which the solution is forced to rotate in the phase plane (Y_t^1, Y_t^2) depending on the magnitude of the drift parameter β. In this model the balanced method also stabilizes the numerical solution, as is shown in figure 4.2, compared with the 'Euler solution'. For time step sizes larger than 10^{-2} one can expect explosions in the Euler approximation as in this figure.

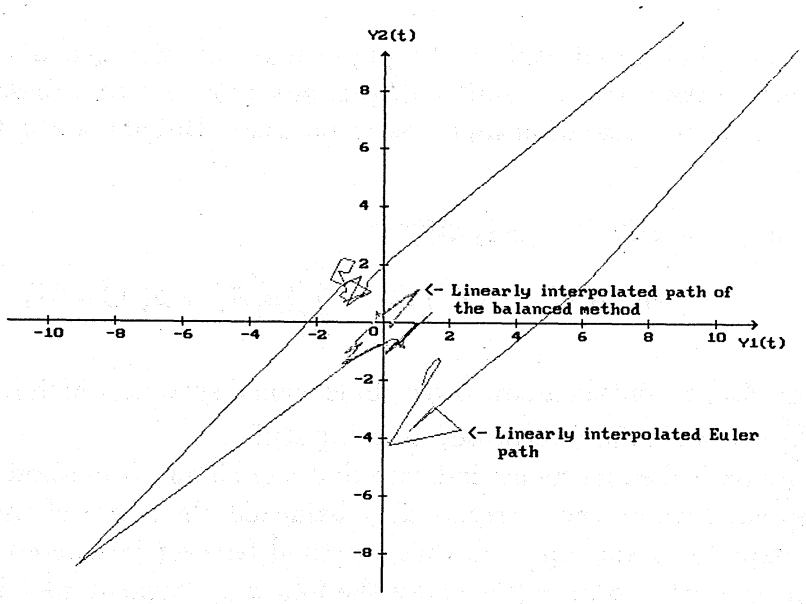

Figure 4.2 : Numerical solution of (4.2) of the Euler and balanced method using $\Delta = 0.02$ and $(\sigma, \beta, \rho) = (4, 5, 0.5)$ on $[0,1]$

Of course the figures depend strongly on the parameter choice $(\Delta, \sigma, \beta, \rho, T)$, but we are only demonstrating what can happen in numerical solutions, especially in stiff situations. Figure 4.2 displays the poor behaviour of the 'Euler solution' in comparison with that of the 'balanced solution' on the time interval $[0, 1]$. If one observes it for larger times T then very large fluctuations in the 'Euler solution' will occur and even result in overflow, but the exact solution and the 'balanced numerical solution' are moving closely to the zero point. That the 'balanced solution' better reflects the exact solution in fact can be easily checked by the use of very small step sizes Δ.

5 Experiments for a Linear Three-dimensional System

Let us consider the two–dimensional system

$$dX_t = AX_t\,dt + \sum_{j=1}^{m} \sigma^j X_t\,dW_t^j \tag{5.1}$$

$$\text{with} \quad A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{and} \quad \sigma^j = \begin{pmatrix} \sigma_{11}^j & \sigma_{12}^j \\ \sigma_{21}^j & \sigma_{22}^j \end{pmatrix}$$

where $\sigma_{11}^j = \sigma_{22}^j$ and $\sigma_{21}^j = -\sigma_{12}^j$, $j = 1, ..., m$. This system was studied in Auslender and Milstein ([1],1982) and has *Lyapunov* exponent

$$\lambda = a + \frac{1}{2}\sum_{j=1}^{m}[(\sigma_{12}^j)^2 - (\sigma_{11}^j)^2]. \tag{5.2}$$

But the system (5.1) is not stiff in the physical sense. We can always apply a transformation to the system to shift its *Lyapunov* exponent to a reasonable value which allows explicit integration methods to be used. But for $\lambda << 0$, if we add the equation

$$\begin{aligned} dY_t &= \alpha Y_t\,dt + \sigma Y_t\,dW_t^{m+1} \\ &\quad + (\gamma_1 X_t^1 + \gamma_2 X_t^2)\,dt + \sum_{j=1}^{m}(\mu_1^j X_t^1 + \mu_2^j X_t^2)\,dW_t^j \end{aligned} \tag{5.3}$$

to the system (5.1) we obtain a new three–dimensional system, which for $\alpha - \frac{\sigma^2}{2} >> \lambda$ or $\alpha - \frac{\sigma^2}{2} << \lambda$ can be interpreted as being stiff.
Once again numerical experiments indicate that the balanced method has a larger range of step sizes where the corresponding balanced algorithm of the form (3.2) provides no explosions and that the Euler method fails for large step sizes. For a special parameter set and $m = 2$ we have the following 'typical' plot in the phase plane (Y_t^1, Y_t^2) which describes graphically numerical solutions of the equations (5.1) for (X_t^1, X_t^2) components of the three-dimensional system (5.3) starting at $(1, 0, 0)$. Note that while these components effect the dynamics of the Y_t component, they are not themselves effected by the Y_t values.

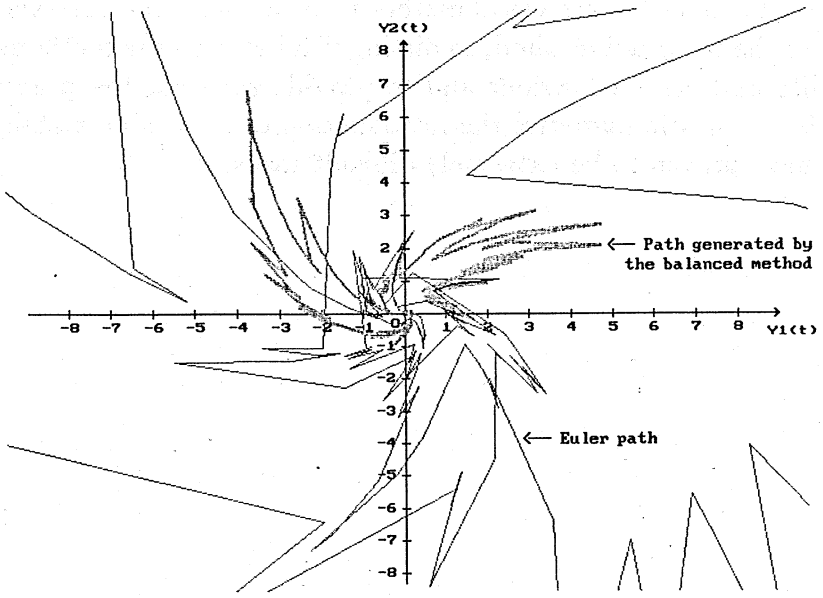

Figure 5.1 : Trajectories of the Euler and balanced method using $\Delta = 2^{-5}$

6 Some Final Remarks

The results of this paper show that new numerical methods also involving implicit stochastic terms can be implemented successfully. The class of balanced methods introduced here stabilizes the numerical solution for a larger range of step sizes than explicit methods. While a number of model equations (e.g. (1.1) (1.2), (4.1), (4.2) and (5.1) with (5.3)) stiffness in stochastic systems have been considered, a suitable class of test equations has not been determined.
Stochastically implicit methods are necessary for the successful numerical treatment of stiff systems. Our results show that one has to be very careful about how one introduces implicitness in numerical solutions. Often it is not sufficient simply to introduce implicitness just in one term of the scheme. One must also take into consideration the influence played in the dynamics by both the stochastic and deterministic terms. Implicit methods can be classified in fully implicit (such as the balanced methods (3.2)), deterministically implicit (drift-, see Kloeden and Platen ([6], 1992)) or pure-stochastically implicit (for example, a subclass of the balanced methods) methods. This interpretation depends on the calculus which one is working with, that is Itô or Stratonovich. In the balanced methods the type of implicitness can be chosen by appropriate weight functions $c^j(t,x), j = 1,2,...,m+1$. The appropriate choice of weights in a balanced method was not described constructively here and requires further investigation. This problem is closely connected with the problem of determining a suitable test equation for such methods.
Many questions remain open for the use of balanced methods for the numerical treatment of stiff stochastic systems. For instance, how can one generalize the concept of balanced methods to higher order methods? Do the higher order methods already have an inbuilt property of balancing stochastic terms?

Stability considerations will also be necessary to provide qualitive statements about the behaviour of stochastic numerical methods, to calculate the corresponding stability region of the balanced method, to distinguish between the methods regarding their reliability and error behaviour and to provide rules for the practical implementation. In stochastic numerics the introduction of a suitable stability concept and criteria have proven to be extremely difficult tasks.

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