Maximum Norm Wellposedness of Nonlinear Kinematic Hardening Models

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Abstract

We prove the wellposedness, with respect to the maximum norm, of stress-strain laws of nonlinear kinematic hardening type, in particular of the Chaboche model.

1 Introduction

In rate independent plasticity, the classical model of Prandtl and Reuß constitutes the basic and most well known form of the stress-strain law. It features an elastic region Z within stress space, bounded by the yield surface ∂Z. We will restrict our discussion to the yield condition due to v. Mises, that is, Z is a cylinder in stress space respectively a ball in deviatoric space.
Due to its obvious shortcomings in modelling real material behaviour, the Prandtl-Reuß-model has been refined in many ways. In the Melan-Prager model, which is also called linear kinematic hardening, the yield surface moves during plastic loading with a velocity proportional to the plastic strain rate $\dot{\varepsilon}^p$,

$$\dot{\sigma}^b = C\dot{\varepsilon}^p \,. \tag{1.1}$$

Here, the backstress σ^b describes the translation of the yield surface, that is, the yield surface at time t is located at $\sigma^b(t) + \partial Z$. Adding a nonlinear term in the right hand side of (1.1),

$$\dot{\sigma}^b = \gamma \left(R\dot{\varepsilon}^p - \sigma^b |\dot{\varepsilon}^p| \right) \,, \tag{1.2}$$

we obtain the model of Armstrong and Frederick [1], which is characterized by the constants γ and R. If we decompose the backstress σ^b into a sum

$$\sigma^b = \sum_{k \in I} \sigma^b_k \,, \tag{1.3}$$

where each constituent σ^b_k satisfies an equation of type (1.2), namely

$$\dot{\sigma}^b_k = \gamma(k) \left(R(k)\dot{\varepsilon}^p - \sigma^b_k |\dot{\varepsilon}^p| \right) \,, \quad k \in I \,, \tag{1.4}$$

we arrive at the model of Chaboche which enjoys a widespread popularity among engineers.
We refer to [11] and [6, 7, 8] for an in-depth discussion of those and other models.
We have proved in [3, 4] that the constitutive operators

$$\varepsilon = \mathcal{F}[\sigma] \,, \quad \sigma = \mathcal{G}[\varepsilon] \,, \tag{1.5}$$

generated by these models act in the space $W^{1,1}(t_0, t_1; \mathbb{T})$ of absolutely continuous functions, defined on a time interval $[t_0, t_1]$ with values in the space of symmetric $N \times N$-tensors, and are moreover locally Lipschitz continuous in that space. Our analysis was based on the study of the operator differential equation

$$\dot{u}(t) = \dot{\theta}(t) + \mathcal{M}(\theta, u, x^0, p)(t) |\dot{\xi}(t)| \,, \quad \xi = \mathcal{P}(u\,; x^0) \,, \quad u(t_0) = u^0 \,. \tag{1.6}$$

Here, θ represents a given input function, $\mathcal{P}$ denotes the play operator (see Section 2 below), $\mathcal{M}$ is a causal operator with certain additional properties, and p is a parameter.

In this paper, we discuss the wellposedness of the operators in (1.5) in the space of continuous functions, endowed with the maximum norm. Again, the analysis of (1.6) constitutes the main tool; it is carried out in Section 3. To prepare that discussion, in Section 2 we present some material concerning the play operator, including the new result of Proposition 2.5. Section 4 is devoted to the discussion of the Chaboche model, the main results being Theorem 4.2 and Theorem 4.3.

2 The Play Operator

Let $C([t_0,t_1];X)$ denote the space of continuous functions on $[t_0,t_1]$ with values in some normed space X, endowed with the maximum norm

$$\| u \|_\infty = \max_{t\in[t_0,t_1]} \| u(t) \|_X \,. \tag{2.7}$$

We will exclusively deal with evolutions which are completely determined by the past and present. Such evolutions are called causal.

Definition 2.1 *An operator $S : \mathcal{D}_S \subset C([t_0,t_1];Y) \to C([t_0,t_1];X)$ is said to be causal if the implication*

$$u = v \text{ on } [t_0,t] \quad \Rightarrow \quad Su = Sv \text{ on } [t_0,t]\,, \tag{2.8}$$

holds for all $u, v \in \mathcal{D}_S$ and all $t \in [t_0,t_1]$.

Obviously, we can consider a causal operator as acting on the whole family of spaces

$$S : \mathcal{D}_S(t) = \{u|_{[t_0,t]} : u \in \mathcal{D}_S\} \to C([t_0,t];X)\,, \quad t \in [t_0,t_1]\,. \tag{2.9}$$

In particular, the initial value $(Su)(t_0)$ depends only on $u(t_0)$ and thus can be expressed in terms of an initial value mapping from Y to X. In the sequel, we will freely use these properties and in particular not distinguish between the various domains $\mathcal{D}_S(t)$.

Definition 2.2 (Play and Stop Operator)
Let X be a separable Hilbert space and $Z \subset X$ a closed convex set. We define the operator $\mathcal{S}_Z : W^{1,1}(t_0,t_1;X) \times Z \to W^{1,1}(t_0,t_1;X)$ as the operator which maps any $u \in W^{1,1}(t_0,t_1;X), x^0 \in Z$ to the solution x of the variational inequality

$$\langle \dot{x}(t) - \dot{u}(t), x(t) - \tilde{x} \rangle \geq 0 \;\forall \tilde{x} \in Z\,, \quad a.e. \text{ in } (t_0,t_1)\,, \tag{2.10}$$

$$x(t) \in Z\,, \quad \forall t \in [t_0,t_1]\,, \tag{2.11}$$

$$x(t_0) = x^0\,, \tag{2.12}$$

where $\langle \cdot,\cdot \rangle$ denotes the scalar product in X. The operator $\mathcal{S}_Z$ is called the stop, and the operator $\mathcal{P}_Z$ defined by

$$\mathcal{P}_Z(u\,;x^0) = u - \mathcal{S}_Z(u\,;x^0) \tag{2.13}$$

is called the play. The set Z is called the characteristic of $\mathcal{S}_Z$ and $\mathcal{P}_Z$.

The definition of the operators $\mathcal{S}_Z$ and $\mathcal{P}_Z$ is meaningful since the variational inequality (2.10) - (2.12) is known to have a unique solution, see [2]. They have been studied in general in the monographs [9, 10, 12]. Here, we consider only the case where Z is a ball of radius $r > 0$, that is,

$$Z = B_r(0) = \{x : x \in X,\ |x| \le r\}\,, \tag{2.14}$$

and simply write $\mathcal{P}$ and $\mathcal{S}$ instead of $\mathcal{P}_Z$ and $\mathcal{S}_Z$. For ease of reference, we list some properties of $\mathcal{P}$ and $\mathcal{S}$ which are needed in the sequel. (They hold irrespectively of whether X is finite or infinite dimensional.) To further simplify the notation, we write

$$\xi = \mathcal{P}(u\,;x^0)\,,\quad x = \mathcal{S}(u\,;x^0)\,,\quad \eta = \mathcal{P}(v\,;x^0)\,,\quad y = \mathcal{S}(v\,;x^0)\,, \tag{2.15}$$

for any given pairs u, v of input functions and x^0, y^0 of initial values.

Proposition 2.3 *For every $u, v \in W^{1,1}(t_0, t_1; X)$ and every $x^0, y^0 \in B_r(0)$ there holds*

$$\frac{\mathrm{d}}{\mathrm{dt}}|x-y| \le |\dot u - \dot v|\,,\quad a.e.\ in\ (t_0,t_1)\,, \tag{2.16}$$

$$\Big||\dot\xi| - |\dot\eta|\Big| + \frac{1}{2r}\frac{\mathrm{d}}{\mathrm{dt}}\Big||x|^2 - |y|^2\Big| \le \frac{1}{r}\Big|\langle \dot u, x\rangle - \langle \dot v, y\rangle\Big|\,,\quad a.e.\ in\ (t_0,t_1)\,, \tag{2.17}$$

$$|\xi(t) - \eta(t)| \le \max\{|\xi(t_0) - \eta(t_0)|\,,\ \sqrt{(r+\Delta(t))^2 - r^2}\}\,,\quad \forall t \in [t_0,t_1]\,, \tag{2.18}$$

where $\Delta(t) = \max\{|u(\tau) - v(\tau)| : t \in [t_0, t]\}$.

Proof. (2.16) is well known, see e.g. [2] and [10]; (2.17) has been proved in [3] as inequality (A.32), and (2.18) can be found in Section 17 of [9] as well as in [10]. □

Another useful identity ist

$$|\dot\xi(t)| = \frac{1}{r}\langle \dot\xi(t), x(t)\rangle\,,\quad \dot\xi(t) = \frac{1}{r}|\dot\xi(t)|x(t)\,, \tag{2.19}$$

which follows directly from the variational inequality

$$\langle \dot\xi(t), x(t) - \tilde x\rangle \ge 0\ \forall \tilde x \in Z\,,\quad a.e.\ in\ (t_0,t_1)\,, \tag{2.20}$$

since (2.20) implies that $\dot\xi(t) = 0$ if $|x(t)| = r$ and that $\dot\xi(t)$ points in the direction of the outward normal if $|x(t)| = r$. Moreover, the inequalities (2.16) and (2.17) together with the elementary identities

$$\dot\xi - \dot\eta = \frac{1}{r}(|\dot\xi| - |\dot\eta|)x + \frac{1}{r}|\dot\eta|(x-y)\,, \tag{2.21}$$

$$\langle \dot u, x\rangle - \langle \dot v, y\rangle = \langle \dot u - \dot v, x\rangle - \langle \dot v, x-y\rangle\,, \tag{2.22}$$

imply the local Lipschitz continuity of $\mathcal{P}$ with respect to the norm of $W^{1,1}$, see [3].

From the inequality (2.18) we see that we can uniquely extend the operators $\mathcal{P}$ and $\mathcal{S}$ to operators

$$\mathcal{P}, \mathcal{S} : C([t_0,t_1];X) \times B_r(0) \to C([t_0,t_1];X)\,, \tag{2.23}$$

which are $\frac{1}{2}$-Hölder continuous. On $C([t_0,t_1];X)$, the play operator is known to possess the following smoothing property.

Proposition 2.4
The operator $\mathcal{P}$ maps $C([t_0,t_1];X)\times B_r(0)$ into $C([t_0,t_1];X)\cap BV([t_0,t_1];X)$ and satisfies

$$\lim_{n\to\infty} \operatorname{Var}\mathcal{P}(u_n\,;x_n^0) = \operatorname{Var}\mathcal{P}(u\,;x^0)\,, \tag{2.24}$$

if $\lim u_n = u$ *in* $C([t_0,t_1];X)$ *and* $\lim x_n^0 = x^0$ *in* $B_r(0)$.

Proof. The proof given in [10, Prop. 4.11] for the special case where x_n^0 and x^0 are the projections of $u_n(t_0)$ and $u(t_0)$ respectively also works without that restriction. □
The following result appears to be new. It improves the estimate (19.10) of [9].

Proposition 2.5 *Let $[a,b]\subset[t_0,t_1]$ be a given interval. Let $u\in C([t_0,t_1];X)$, $x^0\in B_r(0)$ and $\epsilon\geq 0$ be given with*

$$\max_{t\in[a,b]} |u(t)-u(a)| \leq \epsilon < r\,. \tag{2.25}$$

Then $\xi=\mathcal{P}(u\,;x^0)$ satisfies

$$\operatorname*{Var}_{[a,b]}\xi \leq \epsilon\left(1+\frac{\epsilon}{2(r-\epsilon)}\right)\,. \tag{2.26}$$

Proof. Assume first that $u\in W^{1,1}(t_0,t_1;X)$. For $t\in(a,b)$ with $\dot\xi(t)\neq 0$ we choose

$$\tilde x = u(t)-u(a)+(r-\epsilon)\chi(t)\,,\quad \chi(t)=\begin{cases}\frac{\dot\xi(t)}{|\dot\xi(t)|}\,, & \text{if } \dot\xi(t)\neq 0\,,\\ 0\,, & \text{if } \dot\xi(t)=0\,,\end{cases} \tag{2.27}$$

in the variational inequality (2.10) and obtain

$$(r-\epsilon)|\dot\xi(t)| \leq -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(|u(a)-\xi(t)|^2\right)\,. \tag{2.28}$$

If $|x(t)|<r$ for all $t\in[a,b]$, we have $\operatorname{Var}_{[a,b]}\xi=0$; otherwise, we define

$$t_* = \max\{t : t\in[a,b]\,,\ |x(t)|=r\}\,, \tag{2.29}$$

and conclude from (2.28) that

$$\begin{aligned}2(r-\epsilon)\int_a^b |\dot\xi(t)|\,dt &= 2(r-\epsilon)\int_a^{t_*}|\dot\xi(t)|\,dt \leq |x(a)|^2-|x(t_*)-u(t_*)+u(a)|^2\\ &\leq r^2-(r-\epsilon)^2=\epsilon(2r-\epsilon)\,,\end{aligned} \tag{2.30}$$

from which (2.26) readily follows. For arbitrary $u\in C([t_0,t_1];X)$, we choose a sequence $\{u_n\}\in W^{1,1}(t_0,t_1;X)$ with $u_n\to u$ in $C([t_0,t_1];X)$ and such that (2.25) holds for each u_n in place of u. With $\xi_n=\mathcal{P}(u_n\,;x^0)$, we obtain for any partition $\{\tau_i\}_{0\leq i\leq m}$ of $[a,b]$ that

$$\sum_{j=1}^m |\xi(\tau_j)-\xi(\tau_{j-1})| \leq 2(m+1)\|\,\xi-\xi_n\,\|_\infty + \int_a^b |\dot\xi_n|\,dt\,. \tag{2.31}$$

Since $\xi_n\to\xi$ uniformly by (2.18), (2.23), the assertion follows. □

We now extend the Lipschitz continuity result for the operator $\mathcal{P}$ from $W^{1,1}$ to $C \cap BV$. Let us first note that if a sequence of arbitrary functions $w_n : [a,b] \to X$ converges uniformly to some $w \in C([a,b];X) \cap BV([a,b];X)$, we have

$$\operatorname*{Var}_{[a,b]} w \le \liminf_{n\to\infty} \operatorname*{Var}_{[a,b]} w_n \,, \tag{2.32}$$

thus

$$\operatorname*{Var}_{[a,b]} w = \lim_{n\to\infty} \operatorname*{Var}_{[a,b]} w_n \,, \tag{2.33}$$

if in addition the w_n are piecewise linear interpolates of w.

Proposition 2.6
Let $u, v \in C([t_0,t_1];X) \cap BV([t_0,t_1];X)$ and $x^0, y^0 \in B_r(0)$ be given. Then there holds for every $[a,b] \subset [t_0,t_1]$

$$\operatorname*{Var}_{[a,b]}(V(u) - V(v)) \le \left(1 + \frac{1}{r} \operatorname*{Var}_{[a,b]} v\right) \left(|x(a) - y(a)| + \operatorname*{Var}_{[a,b]}(u - v)\right) , \tag{2.34}$$

where

$$V(u)(t) = \operatorname*{Var}_{[a,t]} \xi \,, \quad V(v)(t) = \operatorname*{Var}_{[a,t]} \eta \,. \tag{2.35}$$

Proof. If $u, v \in W^{1,1}(t_0,t_1;X)$, we have

$$\operatorname*{Var}_{[a,b]}(V(u) - V(v)) = \int_a^b \Big| |\dot\xi| - |\dot\eta| \Big| \, dt \,, \tag{2.36}$$

and (2.34) follows if we integrate (2.17) and use (2.22) as well as (2.16). Let now u, v belong to $C([t_0,t_1];X) \cap BV([t_0,t_1];X)$, let u_n, v_n denote their piecewise linear interpolates in $[a,b]$ on the grid $t_k = a + \frac{k}{n}(b-a)$, $0 \le k \le n$. By Proposition 2.4, the functions $V(u_n)$ and $V(v_n)$ converge pointwise to $V(u)$ and $V(v)$ respectively; since they are monotone, the convergence is actually uniform. We therefore obtain

$$\begin{aligned}
\operatorname*{Var}_{[a,b]}(V(u) - V(v)) &\le \liminf_{n\to\infty} \operatorname*{Var}_{[a,b]}(V(u_n) - V(v_n)) && \text{by (2.32)} \\
&\le \lim_{n\to\infty} \left(1 + \frac{1}{r} \operatorname*{Var}_{[a,b]} v_n\right) \left(|x(a) - y(a)| + \operatorname*{Var}_{[a,b]}(u_n - v_n)\right) \\
&= \left(1 + \frac{1}{r} \operatorname*{Var}_{[a,b]} v\right) \left(|x(a) - y(a)| + \operatorname*{Var}_{[a,b]}(u - v)\right) && \text{by (2.33)} \,,
\end{aligned}$$

which is what we wanted to prove. □

3 Main Results

This section is devoted to the study of the initial value problem

$$\dot u(t) = \dot\theta(t) + \mathcal{M}(\theta, u, x^0, p)(t) |\dot\xi(t)| \,, \quad \xi = \mathcal{P}(u\,; x^0) \,, \quad u(t_0) = u^0 \,. \tag{3.1}$$

We look for a solution $u : [t_0, t_1] \to X$, X being a finite dimensional Hilbert space, for a given input function $\theta : [t_0, t_1] \to X$, initial values $x^0 \in B_r(0)$, $u^0 \in X$ and a parameter value p belonging to some metric space P. In [3] we have proved the wellposedness of (3.1) in $W^{1,1}(t_0, t_1; X)$ through the study of its integrated form

$$u(t) = u^0 + \theta(t) - \theta(t_0) + \int_{t_0}^{t_1} \mathcal{M}(\theta, u, x^0, p)(s)\, dV(u)(s)\,, \quad V(u)(t) = \operatorname*{Var}_{[t_0,t]} \mathcal{P}(u\,; x^0)\,, \tag{3.2}$$

the integral being a Stieltjes integral with respect to the BV function $V(u)$. Because of Proposition 2.4, (3.2) makes sense also for $\theta, u \in C([t_0, t_1]; X)$. We thus investigate the wellposedness of (3.2) for a class of inputs

$$\Theta \subset C([t_0, t_1]; X) \tag{3.3}$$

being as large as possible. We assume that

$$\mathcal{M} : \Theta \times C([t_0, t_1]; X) \times B_r(0) \times P \to C([t_0, t_1]; \mathbb{R}) \tag{3.4}$$

is a causal operator with respect to its first two arguments. To obtain existence in the $W^{1,1}$-setting, we had to ensure that, for some $\kappa > 0$,

$$|\mathcal{M}(\theta, u, x^0, p)(t)| \le 1 - \kappa \tag{3.5}$$

is satisfied for all $t \in [t_0, t_1]$; indeed, this requirement is natural since we may have $\dot{\xi}(t) = \dot{u}(t)$ somewhere along the evolution. It turns out, however, that for some applications one cannot guarantee (3.5) to hold for arbitrary functions $u \in C([t_0, t_1]; X)$. On the other hand, we may restrict our attention to functions whose modulus of continuity

$$\mu_u(\delta) = \sup\{|u(t) - u(s)| : \ s, t \in [t_0, t_1],\ |t - s| \le \delta\} \tag{3.6}$$

is not too large. The following lemma furnishes the correct bound.

Lemma 3.1 *Let $u \in C([t_0, t_1]; X)$ be a solution of (3.2) for some given $\theta \in \Theta$, $u^0 \in X$, $x^0 \in B_r(0)$, $p \in P$, assume that (3.5) holds for all $t \in [t_0, t_1]$. Then we have*

$$\mu_u(\delta) \le \frac{2}{\kappa} \mu_\theta(\delta) \quad \forall\, \delta \in (0, \delta_\theta)\,, \tag{3.7}$$

where

$$\delta_\theta = \inf\{\eta : \eta > 0,\ \mu_\theta(\eta) \ge \frac{\kappa^2 r}{2}\}\,. \tag{3.8}$$

Proof. Let $\delta > 0$ be given such that $\epsilon = \mu_u(\delta) \le r\kappa$. If $|t - s| \le \delta$, we obtain from Proposition 2.5 that

$$\begin{aligned} |u(t) - u(s)| &\le |\theta(t) - \theta(s)| + (1-\kappa) \operatorname*{Var}_{[s,t]} \mathcal{P}(u\,; x^0) \\ &\le \mu_\theta(\delta) + (1-\kappa)\epsilon \left(1 + \frac{\epsilon}{2(r-\epsilon)}\right)\,, \end{aligned} \tag{3.9}$$

hence

$$\mu_\theta(\delta) \geq \epsilon - (1-\kappa)\epsilon\left(1 + \frac{\epsilon}{2(r-\epsilon)}\right) = \frac{\kappa}{2}\epsilon + \frac{\epsilon(\kappa r - \epsilon)}{2(r-\epsilon)} \geq \frac{\kappa}{2}\epsilon = \frac{\kappa}{2}\mu_u(\delta)\,. \tag{3.10}$$

Put $\delta_* = \sup\{\delta : \delta > 0, \mu_u(\delta) \leq r\kappa\}$. Then by (3.10) we have

$$\mu_\theta(\delta_*) \geq \frac{\kappa^2 r}{2}\,, \tag{3.11}$$

therefore $\delta_* \geq \delta_\theta$ and (3.7) follows from (3.10). □

For a given $u^0 \in X$, let us denote

$$U_0 = \{u : u \in C([t_0, t_1]; X),\ u(t_0) = u^0\}\,. \tag{3.12}$$

Theorem 3.2 (Existence) *Let $\theta \in \Theta$, $u^0 \in X$, $x^0 \in B_r(0)$, $p \in P$ be given. Assume that $\mathcal{M}(\theta, \cdot, x^0, p) : U_0 \to C([t_0, t_1]; \mathbb{R})$ is continuous, and that there exists a $\kappa > 0$ such that*

$$\| \mathcal{M}(\theta, u, x^0, p) \|_\infty \leq 1 - \kappa \tag{3.13}$$

holds for all $u \in U_0$ which satisfy (3.7). Then there exists a solution $u \in U_0$ of (3.2) which satisfies (3.13).

Proof. We use the retarded argument method. For $\rho > 0$ we introduce the shift operator $\tau^\rho : C([t_0, t_1]; X) \to C([t_0, t_1]; X)$ by the formula

$$(\tau^\rho u)(t) = \begin{cases} u(t-\rho)\,, & t \in [t_0 + \rho, t_1]\,, \\ u(t_0)\,, & t \in [t_0, t_0 + \rho]\,. \end{cases} \tag{3.14}$$

We consider the integral equation for the unknown function u_ρ

$$u_\rho(t) = u^0 + \theta(t) - \theta(t_0) + \int_{t_0}^{t_1} \mathcal{M}(\theta, \tau^\rho u_\rho, x^0, p)(s)\, dV(\tau^\rho u_\rho)(s)\,. \tag{3.15}$$

Since $V(\tau^\rho u_\rho) = 0$ on $I_1 = [t_0, t_0 + \rho]$, we have $u_\rho = u^0 + \theta - \theta(t_0)$ on I_1 and, because the constant $u \equiv u^0$ satisfies (3.7),

$$|\mathcal{M}(\theta, \tau^\rho u_\rho, x^0, p)(t)| \leq 1 - \kappa \tag{3.16}$$

on I_1. Assume now that u_ρ is defined by (3.15) on $I_k = [t_0, t_0 + k\rho]$ and that (3.16) holds on I_k as well as

$$\mu_{u_\rho|I_k}(\delta) \leq \frac{2}{\kappa}\mu_\theta(\delta)\,, \quad \forall\, \delta \in (0, \delta_\theta)\,. \tag{3.17}$$

We extend u_ρ to I_{k+1} by (3.15). The assumption on $\mathcal{M}$ together with (3.17) implies that (3.16) also holds on I_{k+1}. We now prove that (3.17) remains valid if we replace I_k by I_{k+1}. Indeed, let $\delta \in (0, \delta_\theta)$ be given, set $\epsilon = \frac{2}{\kappa}\mu_\theta(\delta) < \kappa r$. For any $s, t \in I_{k+1}$ with $|t - s| \leq \delta$ it follows from (3.15) that

$$|u_\rho(t) - u_\rho(s)| \leq |\theta(t) - \theta(s)| + (1-\kappa) \operatorname*{Var}_{[s,t]} \mathcal{P}(\tau^\rho u_\rho\,; x^0)\,, \tag{3.18}$$

whence, by (3.17) and Proposition 2.5,

$$\operatorname*{Var}_{[s,t]} \mathcal{P}(\tau^\rho u_\rho\,; x^0) \leq \epsilon \left(1 + \frac{\epsilon}{2(r-\epsilon)}\right), \tag{3.19}$$

hence

$$|u_\rho(t) - u_\rho(s)| \leq \epsilon \left(1 - \frac{\kappa}{2} + \frac{\epsilon(1-\kappa)}{2(r-\epsilon)}\right) = \epsilon \left(1 - \frac{\kappa r - \epsilon}{2(r-\epsilon)}\right) < \epsilon\,. \tag{3.20}$$

Consequently, (3.17) holds with I_k replaced by I_{k+1}, and by induction we conclude that the solution u_ρ of (3.15) on $[t_0, t_1]$ satisfies (3.16) on $[t_0, t_1]$ as well as

$$\mu_{u_\rho}(\delta) \leq \frac{2}{\kappa} \mu_\theta(\delta)\,, \quad \forall\, \delta \in (0, \delta_\theta)\,. \tag{3.21}$$

This implies that the family $\{u_\rho : \rho > 0\}$ is an equicontinuous and bounded subset of U_0. By virtue of the Arzelà-Ascoli theorem, there exists a sequence $\rho_n \to 0$ and a function $u \in U_0$ such that $u_{\rho_n} \to u$ uniformly. Since this implies that $\tau^{\rho_n} u_{\rho_n} \to u$ uniformly, we may use the continuity of $\mathcal{M}$ and Proposition 2.4 to pass to the limit in (3.15) and to conclude that u is a solution of (3.2) which satisfies (3.13). $\square$

Corollary 3.3
Let the assumptions of Theorem 3.2 hold, let $\theta \in C([t_0, t_1]; X) \cap BV([t_0, t_1]; X)$. Then each solution u of (3.2) which satisfies (3.13) belongs to $C([t_0, t_1]; X) \cap BV([t_0, t_1]; X)$, and there holds

$$\operatorname*{Var}_{[a,b]} u \leq \frac{1}{\kappa} \operatorname*{Var}_{[a,b]} \theta \tag{3.22}$$

on every subinterval $[a, b]$ of $[t_0, t_1]$.

Proof. We immediately obtain from (3.2) and (3.13) that

$$\operatorname*{Var}_{[a,b]} u \leq \operatorname*{Var}_{[a,b]} \theta + (1-\kappa) \operatorname*{Var}_{[a,b]} \mathcal{P}(u\,; x^0)\,, \tag{3.23}$$

and (3.22) easily follows since $\operatorname{Var}_{[a,b]} \mathcal{P}(u\,; x^0) \leq \operatorname{Var}_{[a,b]} u$, the latter being a consequence e.g. of (2.34), setting there $x(a) = y(a)$ and v constant. $\square$
We have uniqueness for the case where $\theta \in C([t_0, t_1]; X) \cap BV([t_0, t_1]; X)$. We do not know whether uniqueness holds in the general case $\theta \in C([t_0, t_1]; X)$.

Theorem 3.4 (Uniqueness) *Let $\theta \in \Theta$, $u^0 \in X$, $x^0 \in B_r(0)$, $p \in P$ be given. Assume that $\mathcal{M}$ satisfies, for some function $L : C([t_0, t_1]; X) \cap BV([t_0, t_1]; X) \to \mathbb{R}_+$,*

$$\sup_{s \in [t_0, t]} |\mathcal{M}(\theta, u, x^0, p)(s) - \mathcal{M}(\theta, v, x^0, p)(s)| \leq L(v) \cdot \operatorname*{Var}_{[t_0, t]}(u - v)\,, \tag{3.24}$$

for all $u, v \in U_0$ and all $t \in [t_0, t_1]$. Then any two solutions $u, v \in C([t_0, t_1]; X) \cap BV([t_0, t_1]; X)$ of (3.2) which satisfy (3.13) are identical.

Proof. Assume that $u \neq v$. Define

$$a = \inf\{t : t \in [t_0, t_1],\, u(t) \neq v(t)\}\,. \tag{3.25}$$

For $t > a$ we have

$$\begin{aligned} u(t) - v(t) &= \int_a^t \mathcal{M}(\theta, u, x^0, p)(s)\, d(V(u) - V(v))(s) \\ &\quad + \int_a^t \mathcal{M}(\theta, u, x^0, p)(s) - \mathcal{M}(\theta, v, x^0, p)(s)\, dV(v)(s)\,, \end{aligned} \tag{3.26}$$

hence

$$\operatorname*{Var}_{[a,t]}(u - v) \le (1-\kappa) \operatorname*{Var}_{[a,t]}(V(u) - V(v)) + L(v) \operatorname*{Var}_{[a,t]}(u-v) \operatorname*{Var}_{[a,t]} \mathcal{P}(v\,; x^0)\,. \tag{3.27}$$

By Proposition 2.6 we have, since $x(a) = y(a)$,

$$\operatorname*{Var}_{[a,t]}(V(u) - V(v)) \le \left(1 + \frac{1}{r} \operatorname*{Var}_{[a,t]} v\right) \operatorname*{Var}_{[a,t]}(u-v)\,. \tag{3.28}$$

Putting (3.27) and (3.28) together we obtain

$$\operatorname*{Var}_{[a,t]}(u - v) \le \left[1 - \kappa + \left(\frac{1-\kappa}{r} + L(v)\right) \operatorname*{Var}_{[a,t]} v\right] \operatorname*{Var}_{[a,t]}(u-v)\,. \tag{3.29}$$

Since $\mathrm{Var}_{[a,t]}(v) \to 0$ as $t \downarrow a$, we conclude from (3.29) that $\mathrm{Var}_{[a,t]}(u - v) = 0$ if t is sufficiently close to a, which contradicts the definition of a. □

We now discuss the question of continuous dependence on the data. For a given data vector $y = (\theta, u^0, x^0, p)$ and a given $\kappa > 0$ we define

$$U_{y,\kappa} = \{u : u \in C([t_0, t_1]; X),\ u \text{ solves (3.2) and satisfies (3.13) on } [t_0, t_1]\}\,. \tag{3.30}$$

Theorem 3.5 (Compactness of the Set of Solutions)
Let $Y \subset \Theta \times X \times B_r(0) \times P$ be compact, assume that $\mathcal{M}$ is continuous on its domain of definition given by (3.4). Then for any given $\kappa > 0$ the set

$$D_\kappa = \{(y, u) : y \in Y,\ u \in U_{y,\kappa}\} \tag{3.31}$$

is a compact subset of $Y \times C([t_0, t_1]; X)$.

Proof. Since Y is bounded and its projection to Θ is equicontinuous, Lemma 3.1 implies that D_κ is bounded and its projection to $\Theta \times C([t_0, t_1]; X)$ is equicontinuous. It remains to prove that D_κ is closed. Let (y_n, u_n) be a sequence in D_κ, so that

$$u_n(t) = u_n^0 + \theta_n(t) - \theta_n(t_0) + \int_{t_0}^{t_1} \mathcal{M}(\theta_n, u_n, x_n^0, p_n)(s)\, dV(u_n)(s)\,, \tag{3.32}$$

$$\|\, \mathcal{M}(\theta_n, u_n, x_n^0, p_n)\, \|_\infty \le 1 - \kappa \tag{3.33}$$

If (y_n, u_n) converges strongly to some $(y, u) \in Y \times C([t_0, t_1]; X)$, we can use the continuity of $\mathcal{M}$ and Proposition 2.4 as before to pass to the limit in (3.32) and (3.33) and to conclude that $u \in U_{y,\kappa}$. □

Corollary 3.6 (Continuous Dependence) *Assume that $\mathcal{M}$ is continuous on its domain of definition given by (3.4). Let a sequence of data $y_n = (\theta_n, u_n^0, x_n^0, p_n)$ be given with $y_n \to y = (\theta, u^0, x^0, p)$, assume that there exist unique solutions $u_n, u \in C([t_0, t_1]; X)$ of the corresponding initial value problem (3.1) such that (3.33) holds with a κ which does not depend on n. Then u_n converges to u uniformly.*

Proof. This is an immediate consequence of Theorem 3.5, applied to the set $Y = \{y_n : n \in \mathbb{N}\} \cup \{y\}$, and the fact that u is the only element of $U_{y,\kappa}$. □

4 Application to the Chaboche Model

The Chaboche model — which we will take for granted here, let us again refer to [11] and [6, 7, 8] — represents a particular rate independent stress-strain law in elastoplasticity. It defines the relation between the strain ε and the stress σ through a set of equations and inequalities which involve as internal variables the elastic strain ε^e, the plastic strain ε^p, the plastic stress σ^p and the backstress σ^b, the latter being decomposed into a weighted sum of individual backstresses σ^l and σ_k^b indexed by elements k from an index set I. The constitutive relations have the form

$$\sigma = \sigma^b + \sigma^p \,, \quad \varepsilon = \varepsilon^e + \varepsilon^p \,, \tag{4.1}$$

$$\sigma = A\varepsilon^e \,, \tag{4.2}$$

$$\varepsilon^p \in \mathbb{T}_d \,, \quad |\sigma_d^p| \le r \,, \quad \langle \dot\varepsilon^p, \sigma_d^p - \tilde\sigma \rangle \ge 0 \,, \quad \forall\, \tilde\sigma \in \mathbb{T}_d \,, |\tilde\sigma| \le r \,, \tag{4.3}$$

$$\sigma^b = \int_I \sigma_k^b \, d\nu(k) + \nu^l \sigma^l \,, \tag{4.4}$$

$$\dot\sigma_k^b = \gamma(k) \left(R(k)\dot\varepsilon^p - \sigma_k^b |\dot\varepsilon^p| \right) \,, \quad \text{for all } k \in I \,, \tag{4.5}$$

$$\sigma^l = C^l \varepsilon^p \,, \tag{4.6}$$

to be complemented below by initial conditions for those variables which carry the memory of the model.

We first explain the notation. By $\mathbb{T}$, we denote the space of symmetric $N \times N$ tensors endowed with the usual scalar product and the associated norm

$$\langle \tau, \eta \rangle = \sum_{i,j=1}^N \tau_{ij}\eta_{ij} \,, \quad |\tau| = \sqrt{\langle \tau, \tau \rangle} \,, \tag{4.7}$$

For $\tau \in \mathbb{T}$, we define its trace $\operatorname{Tr}\tau$ and its deviator τ_d by

$$\operatorname{Tr}\tau = \sum_{i=1}^N \tau_{ii} = \langle \tau, \delta \rangle \,, \quad \tau_d = \tau - \frac{\operatorname{Tr}\tau}{N}\delta \,, \tag{4.8}$$

where $\delta = (\delta_{ij})$ stands for the Kronecker symbol. We denote by

$$\mathbb{T}_d = \{\tau : \tau \in \mathbb{T}\,,\ \mathrm{Tr}\,\tau = 0\}\,, \quad \mathbb{T}_d^{\perp} = \{\tau : \tau = \lambda\delta\,,\ \lambda \in \mathbb{R}\}\,, \tag{4.9}$$

the space of all deviators respectively its orthogonal complement. The operator $A : \mathbb{T} \to \mathbb{T}$ represents the linear elastic law, and we assume A to be linear, symmetric and positive definite, a particular case being Hooke's law

$$A\varepsilon = 2\mu\varepsilon + \lambda \mathrm{Tr}\,(\varepsilon)\delta\,, \tag{4.10}$$

where $\lambda, \mu > 0$ denote the Lamé constants. The elastic domain has the form of a von Mises cylinder

$$K_r = \{\tau : \tau \in \mathbb{T},\ |\tau_d| \le r\}\,, \tag{4.11}$$

$r > 0$ being the yield stress. Due to (4.1) and (4.3), in the space of total stresses the elastic domain will be located at $\sigma^b(t) + \mathrm{int}(K_r)$ at time t. Concerning (4.4) - (4.6) we assume that I is a measure space, ν is a finite nonnegative measure on I, the numbers ν^l, C^l and the functions $R \in L^1_\nu(I)$, $\gamma \in L^\infty_\nu(I)$ satisfy $\nu^l, R \ge 0$, $C^l > 0$, $\int_I R(k)d\nu(k) > 0$ and

$$0 < \gamma_{min} \le \gamma(k) \le \gamma_{max}\,, \quad \text{a.e. in } I\,, \tag{4.12}$$

for some constants γ_{min} and γ_{max}. We also introduce the constants

$$\Gamma_m = \int_I \gamma(k)^m R(k)\,d\nu(k)\,, \quad m = 0, 1, 2\,. \tag{4.13}$$

We will prove that the model (4.1) - (4.6) is well posed in $C([t_0, t_1]; \mathbb{T}) \cap BV([t_0, t_1]; \mathbb{T})$, that is, that the corresponding operators

$$\varepsilon = \mathcal{F}[\sigma]\,, \quad \sigma = \mathcal{G}[\varepsilon]\,, \tag{4.14}$$

are well defined and continuous on a suitable subset of $C([t_0, t_1]; \mathbb{T}) \cap BV([t_0, t_1]; \mathbb{T})$, with respect to uniform convergence. In order to do this we have to pass to a weak formulation — in fact, the reduction to the initial value problem (3.1), which we have developed and used in [3] and [4] to prove wellposedness in the space $W^{1,1}$, works here too. We introduce the auxiliary variable

$$u = \alpha\varepsilon^p + \sigma^p_d\,, \tag{4.15}$$

with a certain constant $\alpha > 0$ to be fixed below. If we moreover fix an initial value

$$\sigma^p(t_0) = \sigma^p_0 \in K_r\,, \tag{4.16}$$

by virtue of (4.3) we can express (4.15) as

$$\sigma^p_d = x = \mathcal{S}(u\,; \sigma^p_{0d})\,, \quad \varepsilon^p = \frac{1}{\alpha}\xi\,, \quad \xi = \mathcal{P}(u\,; \sigma^p_{0d})\,. \tag{4.17}$$

Using the variations of constants formula and (2.19) which becomes $r\dot{\varepsilon}^p = \sigma_d^p|\dot{\varepsilon}^p|$, we can solve (4.5) for the backstresses σ_k^b and obtain

$$\sigma_k^b(t) = \exp\left(-\frac{\gamma(k)}{\alpha}V(t)\right)\left(\sigma_0^b(k) + \int_{t_0}^t \frac{R(k)}{r}x(s)\,dW_k(s)\right), \tag{4.18}$$

where

$$V(t) = \operatorname*{Var}_{[t_0,t]}\xi\,, \quad W_k(t) = \exp\left(\frac{\gamma(k)}{\alpha}V(t)\right), \tag{4.19}$$

and $\sigma_k^b(t_0) = \sigma_0^b(k)$ are given initial values for the backstresses σ_k^b. Indeed, if we define σ_k^b by (4.18), (4.19), then σ_k^b satisfies

$$\sigma_k^b(t) = \sigma_k^b(t_0) + \frac{\gamma(k)}{\alpha}\int_{t_0}^t \left(\frac{R(k)}{r}x(s) - \sigma_k^b(s)\right)\,dV(s)\,. \tag{4.20}$$

We express (4.18), (4.19) in operator form as

$$\sigma_k^b(t) = \mathcal{F}_k^\alpha(u, \sigma_{0d}^p, \sigma_0^b)(t)\,. \tag{4.21}$$

If we use (4.15) and the model equations, the choice

$$\alpha = \Gamma_1 + \nu^l C^l \tag{4.22}$$

leads, after formal differentiation, to

$$\dot{u} = \dot{\sigma}_d + \int_I \gamma(k)\sigma_k^b\,d\nu(k)\,|\dot{\varepsilon}^p|\,. \tag{4.23}$$

If we assume Hooke's law (4.10), the choice

$$\alpha = \Gamma_1 + \nu^l C^l + 2\mu \tag{4.24}$$

leads to

$$\dot{u} = 2\mu\dot{\varepsilon}_d + \int_I \gamma(k)\sigma_k^b\,d\nu(k)\,|\dot{\varepsilon}^p|\,. \tag{4.25}$$

The initial condition

$$u(t_0) = u^0 = \alpha\varepsilon^p(t_0) + \sigma^p(t_0)\,, \tag{4.26}$$

requires, in addition to (4.16), an initial value for $\varepsilon^p(t_0)$. In the strain controlled case with Hooke's law assumed, it is determined by $\varepsilon(t_0)$ and the other initial values through the equation

$$\sigma^b + \sigma_d^p = 2\mu\varepsilon^e = 2\mu(\varepsilon_d - \varepsilon^p)\,. \tag{4.27}$$

In the stress controlled case, we must have

$$\sigma(t_0) = \sigma_0^p + \int_I \sigma_0^b(k)\,d\nu(k) + \nu^l C^l \varepsilon^p(t_0)\,. \tag{4.28}$$

Thus, the initial data and $\sigma(t_0)$ determine $\varepsilon^p(t_0)$ in the case $\nu^l > 0$; if $\nu^l = 0$, we choose to fix

$$\varepsilon^p(t_0) = \varepsilon_0^p \,, \tag{4.29}$$

and treat (4.28) as a restriction on the stress inputs. Our weak formulation of the Chaboche model thus becomes

$$u(t) = u^0 + \theta(t) - \theta(t_0) + \int_{t_0}^t \mathcal{M}(u, \sigma_{0d}^p, \sigma_0^b)(s)\, dV(s) \,, \tag{4.30}$$

$$V(t) = \operatorname*{Var}_{[t_0,t]} \xi = \operatorname*{Var}_{[t_0,t]} \mathcal{P}(u\,; \sigma_{0d}^p) \,, \tag{4.31}$$

$$\mathcal{M}(u, \sigma_{0d}^p, \sigma_0^b)(t) = \frac{1}{\alpha} \int_I \gamma(k) \mathcal{F}_k^\alpha(u, \sigma_{0d}^p, \sigma_0^b)(t)\, d\nu(k) \,, \tag{4.32}$$

$$u^0 = \alpha \varepsilon^p(t_0) + \sigma_0^p \,, \tag{4.33}$$

where $\mathcal{F}_k^\alpha$ is defined in (4.18) - (4.21), and $\varepsilon^p(t_0)$ is fixed as discussed above. In the strain controlled case, we have

$$\alpha = \Gamma_1 + \nu^l C^l + 2\mu \,, \quad \theta = 2\mu \varepsilon_d \,, \tag{4.34}$$

and the stress σ is finally obtained from

$$\sigma = \mathcal{G}(\varepsilon\,; \sigma_0^p, \sigma_0^b) = A(\varepsilon - \varepsilon^p) = A\varepsilon - \frac{2\mu}{\alpha} \mathcal{P}(u\,; \sigma_{0d}^p) \,. \tag{4.35}$$

In the stress controlled case, we have

$$\alpha = \Gamma_1 + \nu^l C^l \,, \quad \theta = \sigma_d \,. \tag{4.36}$$

$$\varepsilon = \mathcal{F}(\sigma\,; \sigma_0^p, \sigma_0^b, \varepsilon_0^p) = \varepsilon^e + \varepsilon^p = A^{-1}\sigma + \frac{1}{\alpha} \mathcal{P}(u\,; \sigma_{0d}^p) \,. \tag{4.37}$$

According to (4.32), the initial values for the backstresses σ_k^b play the role of the parameter p for the operator $\mathcal{M}$. We set

$$P = \{\sigma_0^b : \sigma_0^b \in L_\nu^1(I; \mathbb{T}_d) \,,\, |\sigma_0^b(k)| \le R(k) \ a.e.\} \,. \tag{4.38}$$

The properties required of $\mathcal{M}$ follow from the corresponding properties of the operators $\mathcal{F}_k^\alpha$, which we now derive.

Lemma 4.1 *For every $\alpha > 0$, the family $(\mathcal{F}_k^\alpha)_{k \in I}$ of operators is defined in the domain $\mathcal{D}_F = C([t_0, t_1]; \mathbb{T}_d) \times B_r(0) \times P$ for almost every k. The individual operators $\mathcal{F}_k^\alpha$ are causal, map $\mathcal{D}_F$ into $C([t_0, t_1]; \mathbb{T}_d) \cap BV([t_0, t_1]; \mathbb{T}_d)$, are continuous if considered as operators $\mathcal{F}_k^\alpha : \mathcal{D}_F \to C([t_0, t_1]; \mathbb{T}_d)$, and satisfy*

$$\| \mathcal{F}_k^\alpha(u, \sigma_{0d}^p, \sigma_0^b) \|_\infty \le R(k) \tag{4.39}$$

on $\mathcal{D}_F$ for almost every $k \in I$. Moreover, for every $u, v \in C([t_0, t_1]; \mathbb{T}_d) \cap BV([t_0, t_1]; \mathbb{T}_d)$, $\sigma^p_{0d} \in B_r(0)$ and $\sigma^b_0 \in P$ we have

$$\begin{aligned} &\sup_{s\in[t_0,t]} |\mathcal{F}^\alpha_k(u, \sigma^p_{0d}, \sigma^b_0)(s) - \mathcal{F}^\alpha_k(v, \sigma^p_{0d}, \sigma^b_0)(s)| \leq \\ &\quad \leq \frac{R(k)\gamma(k)}{\alpha} \left[\frac{\gamma(k)}{\alpha r} \left(\operatorname*{Var}_{[t_0,t]} v \right)^2 + \left(\frac{\gamma(k)}{\alpha} + \frac{3}{r} \right) \operatorname*{Var}_{[t_0,t]} v + 2 \right] \operatorname*{Var}_{[t_0,t]} (u - v)\,. \end{aligned} \tag{4.40}$$

Proof. Except (4.40), all properties follow directly from (4.18), (4.19), Proposition 2.3 and Proposition 2.4, and for the same reasons it suffices to prove (4.40) for $u, v \in W^{1,1}(t_0, t_1; \mathbb{T}_d)$. In that case, we have, using (2.21), (2.16), (2.34) and (2.36),

$$\begin{aligned} |\mathcal{F}^\alpha_k(u, \sigma^p_{0d}, \sigma^b_0)(t) - \mathcal{F}^\alpha_k(v, \sigma^p_{0d}, \sigma^b_0)(t)| \quad &\leq \quad \frac{R(k)\gamma(k)}{\alpha} \left| \int_{t_0}^t |\dot\xi(\tau)| - |\dot\eta(\tau)| \, d\tau \right| \\ + \frac{R(k)\gamma(k)}{\alpha} \int_{t_0}^t & \left| \dot\xi(\tau) e^{-\frac{\gamma(k)}{\alpha} \int_\tau^t |\dot\xi(s)|\, ds} - \dot\eta(\tau) e^{-\frac{\gamma(k)}{\alpha} \int_\tau^t |\dot\eta(s)|\, ds} \right| d\tau \\ &\leq \frac{R(k)\gamma(k)}{\alpha} \int_{t_0}^t \Big| |\dot\xi(\tau)| - |\dot\eta(\tau)| \Big| + |\dot\xi(\tau) - \dot\eta(\tau)| \\ &\qquad + |\dot\eta(\tau)| \frac{\gamma(k)}{\alpha} \int_\tau^t \Big| |\dot\xi(s)| - |\dot\eta(s)| \Big| \, ds \, d\tau \\ &\leq \frac{R(k)\gamma(k)}{\alpha} \left(\left(2 + \frac{\gamma(k)}{\alpha} \int_{t_0}^t |\dot\eta(\tau)| \, d\tau \right) \int_{t_0}^t \Big| |\dot\xi(\tau)| - |\dot\eta(\tau)| \Big| \, d\tau \right. \\ &\qquad \left. + \frac{1}{r} \int_{t_0}^t |\dot\eta(\tau)| \, d\tau \| x - y \|_\infty \right) \\ &\leq \frac{R(k)\gamma(k)}{\alpha} \left(\left(2 + \frac{\gamma(k)}{\alpha} \int_{t_0}^t |\dot v(\tau)| \, d\tau \right) \left(1 + \frac{1}{r} \int_{t_0}^t |\dot v(\tau)| \, d\tau \right) \right. \\ &\qquad \left. + \frac{1}{r} \int_{t_0}^t |\dot v(\tau)| \, d\tau \right) \int_{t_0}^t |\dot u(\tau) - \dot v(\tau)| \, d\tau \,, \end{aligned} \tag{4.41}$$

from which (4.40) readily follows. □

Theorem 4.2 (Wellposedness, Strain Controlled Case)
Under the assumptions stated above (4.13), the weak formulation of the Chaboche model (4.30) - (4.35) together with Hooke's law (4.10) defines an operator

$$\sigma = \mathcal{G}(\varepsilon; \sigma^p_0, \sigma^b_0)\,, \quad \mathcal{G} : D \to C([t_0, t_1]; \mathbb{T}) \cap BV([t_0, t_1]; \mathbb{T})\,, \tag{4.42}$$

$$D := C([t_0, t_1]; \mathbb{T}) \cap BV([t_0, t_1]; \mathbb{T}) \times K_r \times P\,. \tag{4.43}$$

If $(\varepsilon_n, \sigma^p_{0,n}, \sigma^b_{0,n})$ is a sequence in D with $\varepsilon_n \to \varepsilon$ uniformly, $\sup_n \operatorname{Var}_{[t_0,t_1]} \varepsilon_n < \infty$, $\sigma^p_{0,n} \to \sigma^p_0$ in $\mathbb{T}$ and $\sigma^b_{0,n} \to \sigma^b_0$ in $L^1_\nu(I; \mathbb{T}_d)$, then the corresponding stresses satisfy $\sigma_n \to \sigma$ uniformly and $\sup_n \operatorname{Var}_{[t_0,t_1]} \sigma_n < \infty$.

Proof. Due to Lemma 4.1 and (4.34), we have

$$\| \mathcal{M}(u, \sigma_{0d}^p, \sigma_0^b) \|_\infty \le 1 - \kappa \,, \quad \kappa = \frac{2\mu + \nu^l C^l}{\Gamma_1 + 2\mu + \nu^l C^l} \,, \tag{4.44}$$

for all $(u, \sigma_{0d}^p, \sigma_0^b) \in C([t_0, t_1]; \mathbb{T}) \times B_r(0) \times P$, and moreover the assumptions of theorems 3.2 and 3.4 are satisfied, thus there exists a unique solution $u \in C([t_0, t_1]; \mathbb{T}_d)$ of (4.30) - (4.34) which is moreover an element of $BV([t_0, t_1]; \mathbb{T}_d)$ by virtue of Corollary 3.3. If $(\varepsilon_n, \sigma_{0,n}^p, \sigma_{0,n}^b)$ is a sequence with the properties stated above, then due to corollaries 3.3 and 3.6 we have $\sup_n \operatorname{Var}_{[t_0,t_1]} u_n < \infty$ and $u_n \to u$ uniformly for the corresponding solutions. From (4.35) we see that these properties carry over to σ_n and σ as well, taking into account Proposition 2.4. □

Theorem 4.3 (Wellposedness, Stress Controlled Case)
Let the assumptions stated above (4.13) hold.

(Case $\nu^l > 0$.) *The weak formulation of the Chaboche model (4.30) - (4.33), (4.36), (4.37) defines an operator*

$$\varepsilon = \mathcal{F}(\sigma; \sigma_0^p, \sigma_0^b) \,, \quad \mathcal{F} : D \to C([t_0, t_1]; \mathbb{T}) \cap BV([t_0, t_1]; \mathbb{T}) \,, \tag{4.45}$$

where D is given by (4.43). If $(\sigma_n, \sigma_{0,n}^p, \sigma_{0,n}^b)$ is a sequence in D with $\sigma_n \to \sigma$ uniformly, $\sup_n \operatorname{Var}_{[t_0,t_1]} \sigma_n < \infty$, $\sigma_{0,n}^p \to \sigma_0^p$ in $\mathbb{T}$ and $\sigma_{0,n}^b \to \sigma_0^b$ in $L_\nu^1(I; \mathbb{T}_d)$, then the corresponding strains satisfy $\varepsilon_n \to \varepsilon$ uniformly and $\sup_n \operatorname{Var}_{[t_0,t_1]} \varepsilon_n < \infty$.

(Case $\nu^l = 0$.) *The weak formulation of the Chaboche model (4.30) - (4.33), (4.36), (4.37), (4.29) defines an operator*

$$\varepsilon = \mathcal{F}(\sigma; \sigma_0^p, \sigma_0^b, \varepsilon_0^p) \,, \quad \mathcal{F} : D_0 \to C([t_0, t_1]; \mathbb{T}) \cap BV([t_0, t_1]; \mathbb{T}) \,, \tag{4.46}$$

where $D_0 \subset C([t_0, t_1]; \mathbb{T}) \cap BV([t_0, t_1]; \mathbb{T}) \times K_r \times P \times \mathbb{T}_d$ is the subset of quadruples $(\sigma, \sigma_0^p, \sigma_0^b, \varepsilon_0^p)$ which satisfy

$$\sigma(t_0) = \sigma_0^p + \int_I \sigma_0^b(k) d\nu(k) \,, \tag{4.47}$$

as well as

$$\left| \int_I \gamma(k) \sigma_0^b(k) d\nu(k) \right| < \Gamma_1 \,, \quad \| \sigma_d \|_\infty < \Gamma_0 + r \,. \tag{4.48}$$

If $(\sigma_n, \sigma_{0,n}^p, \sigma_{0,n}^b, \varepsilon_{0,n}^p)$ is a sequence in D_0 such that $\sigma_n, \sigma_{0,n}^p, \sigma_{0,n}^b$ are as in the previous case, $\varepsilon_{0,n}^p \to \varepsilon_0^p$ in $\mathbb{T}$, and which in addition satisfies

$$\sup_{n \in \mathbb{N}} \left| \int_I \gamma(k) \sigma_{0,n}^b(k) d\nu(k) \right| < \Gamma_1 \,, \quad \sup_{n \in \mathbb{N}} \| \sigma_{d,n} \|_\infty < \Gamma_0 + r \,, \tag{4.49}$$

then the corresponding strains satisfy $\varepsilon_n \to \varepsilon$ uniformly and $\sup_n \operatorname{Var}_{[t_0,t_1]} \varepsilon_n < \infty$.

Proof. Due to Lemma 4.1 and (4.36), we have

$$\| \mathcal{M}(u, \sigma_{0d}^p, \sigma_0^b) \|_\infty \leq 1 - \tilde{\kappa}\,, \quad \tilde{\kappa} = \frac{\nu^l C^l}{\Gamma_1 + \nu^l C^l}\,, \tag{4.50}$$

for all $(u, \sigma_{0d}^p, \sigma_0^b) \in C([t_0, t_1]; \mathbb{T}) \times B_r(0) \times P$. For the case $\nu^l > 0$, the arguments are completely analogous to those in the proof of Theorem 4.2 and will not be repeated. In the case $\nu^l = 0$ we have $\tilde{\kappa} = 0$. The required bound on $\mathcal{M}$ is then obtained through an a priori estimate involving the additional assumptions (4.48) and (4.49). The following proof extends the arguments of [4] to the present situation. Let $(\sigma, \sigma_0^p, \sigma_0^b, \varepsilon_0^p) \in D_0$ be given, let $u^0 = \alpha\varepsilon_0^p + \sigma_{0d}^p$, $\alpha = \Gamma_1$. Let

$$\beta = \frac{1}{\gamma_{min}} + \frac{r\gamma_{max}}{\Gamma_2}\,. \tag{4.51}$$

Choose $\kappa > 0$ small enough such that

$$\left| \int_I \gamma(k)\sigma_0^b(k)\,d\nu(k) \right| \leq \Gamma_1(1 - \kappa)\,, \tag{4.52}$$

$$\| \sigma_d \|_\infty \leq \Gamma_0 + r - \beta\Gamma_1\kappa\,, \tag{4.53}$$

Finally, we choose $\delta_0 > 0$ such that

$$\delta_0 < \delta_{\sigma_d} = \inf\{\eta : \eta > 0,\, \mu_{\sigma_d}(\eta) \geq \frac{\kappa^2 r}{2}\} \tag{4.54}$$

and

$$\epsilon := \frac{4}{\kappa}\mu_{\sigma_d}(\delta_0) \leq r + \frac{\kappa\Gamma_1^2}{4\Gamma_2} - \sqrt{r^2 + \left(\frac{\kappa\Gamma_1^2}{4\Gamma_2}\right)^2}\,. \tag{4.55}$$

We will prove that every solution $u \in C([t_0, \hat{t}_0]; \mathbb{T}_d)$ of

$$u(t) = u^0 + \sigma_d(t) - \sigma_d(t_0) + \int_{t_0}^t \mathcal{M}(u, \sigma_{0d}^p, \sigma_0^b)(s)\, dV(u)(s) \tag{4.56}$$

satisfying

$$\| \mathcal{M}(u, \sigma_{0d}^p, \sigma_0^b) \|_\infty \leq 1 - \kappa\,, \tag{4.57}$$

on $[t_0, \hat{t}_0]$, can be extended to a solution $u \in C([t_0, \hat{t}_1]; \mathbb{T}_d)$ of (4.56), where $\hat{t}_1 = \min\{\hat{t}_0 + \delta_0, t_1\}$, and that every such extension satisfies (4.57) on $[t_0, \hat{t}_1]$. To this end, we apply Theorem 3.2 on $[\hat{t}_0, \hat{t}_1]$ with $\hat{\kappa} = \frac{\kappa}{2}$ and

$$\hat{\Theta} = \{\hat{\theta}\}\,, \quad \hat{\theta} = \sigma_d|_{[\hat{t}_0, \hat{t}_1]}\,, \tag{4.58}$$

$$\hat{u}^0 = u(\hat{t}^0)\,, \quad \hat{U}_0 = \{\hat{u} : \hat{u} \in C([\hat{t}_0, \hat{t}_1]; \mathbb{T}_d),\, \hat{u}(\hat{t}^0) = \hat{u}^0\}\,, \tag{4.59}$$

and for $t \in [\hat{t}_0, \hat{t}_1]$ we define

$$\hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(t) = \mathcal{M}(u^*, \sigma_{0d}^p, \sigma_0^b)(t)\,, \tag{4.60}$$

where $\hat{x}^0 = \mathcal{S}(u\,; \sigma_{0d}^p)(\hat{t}^0)$, $\hat{p}(k) = \sigma_k^b(\hat{t}^0)$ and

$$u^*(t) = \begin{cases} u(t)\,, & \text{if } t \in [t_0, \hat{t}^0]\,, \\ \hat{u}(t)\,, & \text{if } t \in [\hat{t}_0, \hat{t}_1]\,. \end{cases} \tag{4.61}$$

Note that we have $\mathcal{S}(\hat{u}, \hat{x}^0) = \mathcal{S}(u^*, \sigma_{0d}^p)$ and $\mathcal{P}(\hat{u}, \hat{x}^0) = \mathcal{P}(u^*, \sigma_{0d}^p)$ on $[\hat{t}_0, \hat{t}_1]$. Causality and continuity of $\hat{\mathcal{M}}$ again follows from Lemma 4.1. To prove that the remaining assumption of Theorem 3.2 is satisfied, let us fix $\hat{u} \in \hat{U}_0$ with

$$\mu_{\hat{u}}(\delta) \le \frac{2}{\hat{\kappa}} \mu_{\hat{\theta}}(\delta)\,, \quad \delta < \delta_{\hat{\theta}}\,. \tag{4.62}$$

We then have for all $t \in [\hat{t}_0, \hat{t}_1]$

$$\begin{aligned} |F_k^{\Gamma_1}(u^*, \sigma_{0d}^p, \sigma_0^b)(t) - F_k^{\Gamma_1}(u^*, \sigma_{0d}^p, \sigma_0^b)(t)| &\le 2R(k)\left[1 - e^{-\frac{\gamma(k)}{\Gamma_1}(V(u^*)(t) - V(u^*)(\hat{t}^0))}\right] \\ &\le \frac{2R(k)\gamma(k)}{\Gamma_1} \operatorname*{Var}_{[\hat{t}_0, t]} \mathcal{P}(u^*\,; \sigma_{0d}^p)\,. \end{aligned} \tag{4.63}$$

From (4.55), (4.62) and Proposition 2.5 it follows that

$$\operatorname*{Var}_{[\hat{t}_0, t]} \mathcal{P}(u^*\,; \sigma_{0d}^p) \le \epsilon\left(1 + \frac{\epsilon}{2(r - \epsilon)}\right) \le \frac{\kappa \Gamma_1^2}{4\Gamma_2}\,, \quad t \in [\hat{t}_0, \hat{t}_1]\,. \tag{4.64}$$

From (4.63) and (4.64) we obtain

$$|\hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(t) - \hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(\hat{t}^0)| \le \frac{2\Gamma_2}{\Gamma_1^2} \operatorname*{Var}_{[\hat{t}_0, t]} \mathcal{P}(u^*\,; \sigma_{0d}^p) \le \frac{\kappa}{2}\,, \tag{4.65}$$

and consequently

$$|\hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(t)| \le 1 - \hat{\kappa}\,, \quad t \in [\hat{t}_0, \hat{t}_1]\,. \tag{4.66}$$

Thus, Theorem 3.2 can be applied and yields an extension of u to $[\hat{t}_0, \hat{t}_1]$. Let now $\hat{u} \in C([\hat{t}_0, \hat{t}_1]; \mathbb{T}_d)$ be any such extension. We want to prove that

$$|\hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(t)| \le 1 - \kappa\,, \quad t \in [\hat{t}_0, \hat{t}_1]\,. \tag{4.67}$$

If this is not the case, there exists a $t \in [\hat{t}_0, \hat{t}_1]$ such that

$$|\hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(t)| > 1 - \kappa\,, \tag{4.68}$$

$$|\hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(s)| < |\hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(t)|\,, \quad \forall\, s \in [\hat{t}^0, t)\,. \tag{4.69}$$

Put

$$e(t) = \frac{\hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(t)}{|\hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(t)|}\,. \tag{4.70}$$

By (4.69), we have for all $s \in [\hat{t}^0, t)$

$$\begin{aligned} 0 &< \langle \hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(t) - \hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(s), e(t)\rangle \\ &= \frac{1}{\Gamma_1^2} \int_s^t \int_I \gamma(k)^2 \langle \frac{R(k)}{r} x(\tau) - \sigma_k^b(\tau), e(t)\rangle \, d\nu(k)\, dV(u^*)(\tau)\,, \end{aligned} \tag{4.71}$$

because of (4.32), (4.20) and (4.21), where

$$x(t) = \mathcal{S}(u^*; \sigma_{0d}^p)(t)\,, \quad V(u^*)(t) = \operatorname*{Var}_{[t_0,t]} \mathcal{P}(u^*; \sigma_{0d}^p)\,, \quad \sigma_k^b = \mathcal{F}_k^{\Gamma_1}(u^*, \sigma_{0d}^p, \sigma_0^b)\,. \tag{4.72}$$

Since the functions x and σ_k^b are continuous and $V(u^*)$ is nondecreasing, it follows from (4.71) that

$$\int_I \gamma(k)^2 \langle \frac{R(k)}{r} x(t) - \sigma_k^b(t), e(t)\rangle \, d\nu(k) \geq 0\,. \tag{4.73}$$

Since $x(t) = \sigma_d^p(t) = \sigma_d(t) - \int_I \sigma_k^b(t)\, d\nu(k)$, (4.73) yields

$$\frac{\Gamma_2}{r} \langle \sigma_d(t), e\rangle \geq \int_I \left(\frac{\Gamma_2}{r} + \gamma(k)^2\right) \langle \sigma_k^b(t), e(t)\rangle \, d\nu(k)\,, \tag{4.74}$$

hence

$$\begin{aligned} \|\sigma_d\|_\infty &\geq \int_I \left(1 + \frac{r\gamma(k)^2}{\Gamma_2}\right) \langle \sigma_k^b(t), e(t)\rangle \, d\nu(k) \\ &= \Gamma_0 + r - \int_I \left(1 + \frac{r\gamma(k)^2}{\Gamma_2}\right) \langle R(k)e(t) - \sigma_k^b(t), e(t)\rangle \, d\nu(k)\,. \end{aligned} \tag{4.75}$$

For almost all $k \in I$ we have $\langle R(k)e(t) - \sigma_k^b(t), e(t)\rangle \geq 0$, hence by definition (4.51) of β we have

$$\begin{aligned} \int_I \left(1 + \frac{r\gamma(k)^2}{\Gamma_2}\right) \langle R(k)e - \sigma_k^b(t), e\rangle \, d\nu(k) &\leq \beta \int_I \gamma(k) \langle R(k)e - \sigma_k^b(t), e\rangle \, d\nu(k) \\ &= \beta\Gamma_1 (1 - |\hat{\mathcal{M}}(\hat{u}, \hat{x}^0, \hat{p})(t)|) \\ &< \beta\Gamma_1 \kappa\,. \end{aligned} \tag{4.76}$$

From (4.75) and (4.76) it follows that

$$\|\sigma_d\|_\infty > \Gamma_0 + r - \beta\Gamma_1\kappa\,, \tag{4.77}$$

which contradicts (4.53). Thus, no $t \in [\hat{t}_0, \hat{t}_1]$ can satisfy (4.68) and (4.69), and we have proved the statement concerning (4.56) and (4.57) completely. Uniqueness of the solution u in $[t_0, t_1]$ and its continuous dependence upon the data follows in the same manner as in the proof of Theorem 4.2. We only have to note that assumption (4.46) guarantees that the choice of κ to ensure (4.52) and (4.53) does not depend on n. □

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