Multiwavelet approximation methods for pseudodifferential equations on curves. Stability and convergence analysis

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Abstract. We develop a stability and convergence analysis of Galerkin-Petrov schemes based on a general setting of multiresolution generated by several refinable functions for the numerical solution of pseudodifferential equations on smooth closed curves. Particular realizations of such a multiresolution analysis are trial spaces generated by biorthogonal wavelets or by splines with multiple knots. The main result presents necessary and sufficient conditions for the stability of the numerical method in terms of the principal symbol of the pseudodifferential operator and the Fourier transforms of the generating multiscaling functions as well as of the test functionals. Moreover, optimal convergence rates for the approximate solutions in a range of Sobolev spaces are established.

1 Introduction

In the last two decades a significant number of papers has been devoted to Galerkin and collocation methods for the numerical solution of periodic boundary integral and pseudodifferential equations with various special choices of trial and test functions such as trigonometric polynomials, B-splines or biorthogonal wavelets. In particular, a stability and convergence analysis in Sobolev spaces has been developed in the papers [PS], [AW1], [AW2], [SW], [S], [PSr] for Galerkin and collocation methods using smoothest splines (see also [PSi] for the state of the art in this field) and in [DPS] for generalized Galerkin-Petrov methods in the framework of multiresolution, i.e. ascending sequences of nested spaces which are generated by translates and scaled versions of a single refinable function (interesting genuinely multivariate examples are given by various notions of multivariate splines).

However, until recently no rigorous convergence analysis was available for boundary element methods in which the trial functions are splines with multiple knots, e.g. Hermite quadratics or Hermite cubics that are often preferred to smoothest splines in engineering applications (cf. [MP], Section 6). Such an analysis has been provided in [MP] for the collocation of pseudodifferential equations on smooth closed curves and is based on a recurrence relation for the Fourier coefficients of the numerical solution. In particular, sufficient stability conditions and superconvergence results (with special choices of the collocation points) have been obtained [MP].

The results of the present paper constitute a natural generalization of the aforementioned results. They are concerned with a stability and convergence analysis of Galerkin-Petrov schemes based on a general setting of multiresolution generated by several refinable functions. Such a multiresolution analysis contains splines with multiple knots as well. The main result (Theorem 2.6) presents necessary and sufficient conditions for the stability of the numerical method in terms of the principal symbol of the pseudodifferential operator in consideration and the Fourier transforms of the generating multiscaling functions as well as of the test functionals. In the particular case of boundary element collocation methods using splines with multiple knots, Theorem 2.6 (together with Theorem 2.8) provides even the necessity of the stability conditions derived in [MP]. Moreover, the range of Sobolev spaces for which stability holds has been extended to $\beta \le s \le \beta + \frac{1}{2}$.

The proofs are essentially based on a thorough Fourier analysis of the corresponding stiffness matrices and on a new result concerning the equivalence between periodic Sobolev norms and certain discrete Sobolev norms.

The paper is organized as follows. In Sect. 2 we collect some important definitions on refinable functions and projection methods and formulate the main stability results which will be proved in Sect. 3.1 for the case of periodic pseudodifferential equations with constant symbols. In the remainder of Sect. 3 these results are applied to the biorthogonal Galerkin method and to the collocation. In Sect. 4 we give a characterization of the Strang-Fix condition which is a characterization of the approximation order of the trial spaces generated by a finite number of refinable functions. Then we prove the Jackson type approximation property as well as the Bernstein type inverse property for the orthogonal projectors onto the trial spaces and for the test projectors. In Sect. 5 we establish corresponding optimal convergence rates for the approximate solutions in a range of Sobolev spaces. Some of the theoretical results of Theorems 5.1 and 5.2 have been confirmed by numerical experiments in [MP].

2 Notation and the main stability results

Let us start with introducing a class of numerical methods for solving a periodic pseudodifferential equation of the form

$$L\,u = f\ . \tag{1}$$

Here $L : H^s(\mathbb{T}) \to H^{s-\beta}(\mathbb{T})$ is a periodic pseudodifferential operator (ΨDO) of order $\beta \in \mathbb{R}$ defined by

$$\begin{aligned}(Lu)(x) &:= \sum_{l\in\mathbb{Z}} e^{2\pi i l x}\,\sigma(l)\,\tilde{u}(l)\ ,\\ \tilde{u}(x) &:= \int_0^1 e^{-2\pi i x y}\,u(y)\,dy\ ,\end{aligned}$$

for $u \in \mathcal{C}^\infty(\mathbb{T})$, and $f \in H^{s-\beta}(\mathbb{T})$ is given. The symbol σ of the operator L can be written as

$$\sigma(x) := \begin{cases} (a_+ + a_-\ sign(x))\,|x|^\beta & for\quad x \neq 0\\ 1 & for\quad x = 0\end{cases}$$

where $a_+, a_- \in \mathbb{C}$. As usually $H^s(\mathbb{T})$ is the periodic Sobolev space of order $s \in \mathbb{R}$ which coincides with the completion of $\mathcal{C}^\infty(\mathbb{T})$ in the norm

$$\|u\|_s^2 := \sum_{l\in\mathbb{Z}} \langle l\rangle^{2s}\,|\tilde{u}(l)|^2 \tag{2}$$

where

$$\langle x\rangle \quad := \quad \begin{cases} |x| & for\ \ x \in \mathbb{R}\setminus\{0\}\\ 1 & for\ \ x = 0\end{cases}\ .$$

In what follows we require the ellipticity of the operator L , which means $a_+ \pm a_- \neq 0$.

In order to setup on numerical methods for (1), we have to introduce a finite dimensional trial space of approximating functions and a set of test functionals. For $M \in \mathbb{N}$ let

$$\Lambda_M := \mathbb{Z} \cap [-\tfrac{M}{2}, \tfrac{M}{2}) \, .$$

We choose a sequence $\phi := (\phi^j)_{j \in \Lambda_M}$ of generators for the space of approximation functions with

$$\phi^j \in \mathcal{L}_2 := \left\{ f \in L^2(\mathbb{R}) : \sum_{k \in \mathbb{Z}} |f(\cdot + k)| \in L^2([0,1]) \right\}$$

for $j \in \Lambda_M$. It is easy to see that $\mathcal{L}_2 \subseteq L^1(\mathbb{R})$ and therefore $\hat{f} \in \mathcal{C}(\mathbb{R})$ for $f \in \mathcal{L}_2$, where

$$\hat{f}(x) := \int_{\mathbb{R}} e^{-2\pi i x y} \, f(y) \, dy$$

is the Fourier transform on $L^2(\mathbb{R})$. For the stepsize $h := \frac{1}{N}$, $N := 2^m$ with $m \in \mathbb{N}_0$, we introduce the trial spaces

$$S_m(\phi) := Lin \left\{ \phi^j_{k,m} := 2^{\frac{m}{2}} [\phi^j(2^m \cdot - k)] : \ k \in \Lambda_N, \ j \in \Lambda_M \right\} \, .$$

Hereby the periodization operator $[\,\cdot\,]$ is defined by

$$[f] := \sum_{l \in \mathbb{Z}} f(\cdot + l)$$

for $f \in \mathcal{L}_2$.

Now we turn to the test functionals. Choose a family $\eta := (\eta^j)_{j \in \Lambda_M} \in \left(H^{-s'}(\mathbb{R})\right)^M$, $s' \geq 0$, with compact support to define the test functionals

$$\eta^l_{k,m}(f) := 2^{-\frac{m}{2}} \, \eta^l(f(2^{-m}(\cdot + k))) \ , \quad l \in \Lambda_M, \ k \in \Lambda_N \ , \tag{3}$$

for $f \in H^{s'}(\mathbb{T})$. The numerical method which we are going to investigate is the Galerkin-Petrov method corresponding to the just introduced trial spaces and test functionals. This method reads as follows:

Find an approximate solution $u_m \in S_m(\phi)$ such that

$$\eta^l_{k,m}(Lu_m) = \eta^l_{k,m}(f) \ , \quad l \in \Lambda_M, \ k \in \Lambda_N \tag{4}$$

for any sufficiently large $m \in \mathbb{N}_0$. The scheme (4) corresponding to the trial and test spaces generated by ϕ and η, respectively, is called **numerical method** $\{\eta, \phi\}$ for the operator L. The following two examples are special realizations of the scheme (4).

Example 2.1 *Collocation method: Choose a strictly increasing sequence* $(\epsilon_j)_{j\in\Lambda_M} \in [0,1)^M$ *and define the test functionals by*

$$\eta^j(f) := f(\epsilon_j)$$

for $j \in \Lambda_M$.

Example 2.2 *Biorthogonal Galerkin method: Let* $(\tilde{\eta}^j)_{j\in\Lambda_M} \in \mathcal{L}_2^M$ *be a family of functions biorthogonal to* $(\phi^j)_{j\in\Lambda_M}$*, e.g.*

$$\langle \phi^r, \tilde{\eta}^s(\cdot - k)\rangle_{L^2(\mathbb{R})} = \delta_{r,s}\delta_{0,k}$$

for $r, s \in \Lambda_M$ *and* $k \in \mathbb{Z}$. *Then*

$$\eta^j(f) := \langle f, \tilde{\eta}^j\rangle_{L^2(\mathbb{R})}\ , \quad j \in \Lambda_M,$$

for $f \in L^2(\mathbb{T})$.

It turns out that the convergence analysis of the numerical method (4) essentially depends on the behavior of the matrix valued function $[\eta\sigma\phi]$ defined by

$$[\eta\sigma\phi](x) := \sum_{l\in\mathbb{Z}} \left(\eta^r\left(e^{2\pi i(l+x)\cdot}\right)\ \sigma(l+x)\ \hat{\phi}^s(l+x)\right)_{(r,s)\in\Lambda_M^2}\ , \quad x \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

This function $[\eta\sigma\phi]$ will be called **numerical symbol** of the numerical method $\{\eta, \phi\}$ for the operator L with symbol σ. Using the notation

$$\begin{aligned} \hat{\eta^r}(x) &:= \overline{\eta^r(e^{2\pi i x\cdot})}\ , \\ \hat{\eta}_p(x) &:= \left(\hat{\eta^r}(pM+l+x)\right)_{(l,r)\in\Lambda_M^2}\ , \\ \hat{\phi}_p(x) &:= \left(\hat{\phi}^r(pM+l+x)\right)_{(l,r)\in\Lambda_M^2}\ , \\ f_p(x) &:= diag(f(pM+l+x))_{l\in\Lambda_M}\ , \end{aligned} \qquad (5)$$

for $f : \mathbb{R} \to \mathbb{R}$, the numerical symbol gets the simple form

$$[\eta\sigma\phi](x) = \sum_{p\in\mathbb{Z}} \hat{\eta}_p(x)^*\ \sigma_p(x)\ \hat{\phi}_p(x)\ . \qquad (6)$$

Now we are ready to define a class of admissible numerical methods.

Definition 2.3 *The numerical method* $\{\eta,\ \phi\}$ *is called* $s-$*admissible for* ΨDO*'s of order* $\beta \in \mathbb{R}$, $s \in \mathbb{R}$, *if the following is satisfied:*

i) the matrices $\hat{\phi}_0$ *and* $\hat{\eta}_0$ *are invertible on* $[-\frac{1}{2}, \frac{1}{2}]$;

ii) $\sum_{p\neq 0} \|\langle x\rangle_p^s \hat{\phi}_p(x)\hat{\phi}_0(x)^{-1}\langle x\rangle_0^{-s}\|^2$ *is uniformly bounded on* $[-\frac{1}{2},\frac{1}{2}]$;

iii) $\sum_{p\neq 0} \|\hat{\eta}_p(x)^* |x|_p^\beta \hat{\phi}_p(x)\|$ *is uniformly convergent on* $[-\frac{1}{2},\frac{1}{2}]$.

Here the matrices $\langle\cdot\rangle_p$, $|\cdot|_p^\beta$ arising in ii) and iii) are defined by (5) and $\|\cdot\|$ means any matrix norm. The letter s denotes the Sobolev index of the space $H^s(\mathbb{T})$.

Remark 2.4 *Properties i) and ii) are sufficient conditions for a certain discrete Sobolev norm to be equivalent to the continuous Sobolev norm* (2) (*see Section* 3). *Condition i) ensures the linear independence of the integer translates and is stronger than the Riesz stability* (*cf.* [JM], *Theorem* 5.1). *Property ii) is a uniform Strang-Fix condition combined with a growth condition for the* $(\hat{\phi}^j)_{j\in\Lambda_M}$ (*see Section* 4). *The last condition ensures the continuity of the numerical symbol for* $x \neq 0$.

We assume

Hypothesis H: There exist functions $\psi := (\psi^j)_{j\in\Lambda_M} \in \mathcal{L}_2^M$ with

- conditions i) and ii) of Definition 2.3 are fulfilled with replacing s and ϕ by $s-\beta$ and ψ, respectively;
- ψ satisfies the duality conditions $\eta^r(\psi^j(\cdot - k)) = \delta_{r,j}\delta_{0,k}$ for $r,j \in \Lambda_M$, $k \in \mathbb{Z}$;
- $\left\| |x|_0^{s-\beta}\hat{\psi}_0(x)\hat{\eta}_0(x)^*|x|_0^{\beta-s} \right\| \leq c$ and $\left\| |x|_0^{s-\beta}(\hat{\psi}_0(x)\hat{\eta}_0(x)^*)^{-1}|x|_0^{\beta-s} \right\| \leq c$

  for $x \in [-\frac{1}{2},\frac{1}{2}] \setminus \{0\}$ where c is a positive constant[1] independent of x.

Sufficient conditions for the above hypothesis are formulated in Section 3. At the end of Section 4 we will give some hints how to construct such functions ψ in a general situation. Note that the last condition of Hypothesis H is a uniform Strang-Fix condition, too (see Section 3). Moreover, the second property implies that the operators

$$Q_m(f) := \sum_{\substack{k\in\Lambda_N \\ l\in\Lambda_M}} \eta_{k,m}^l(f)\ \psi_{k,m}^l \tag{7}$$

are projectors defined for sufficiently smooth functions f. Using this notation of Q_m and representing $u_m \in S_m(\phi)$ as

$$u *' \phi := \sum_{j\in\Lambda_M}\sum_{k\in\Lambda_N} u_k^j\ \phi_{k,m}^j \tag{8}$$

with the coefficient vector $u := ((u_k^j)_{j\in\Lambda_M})_{k\in\Lambda_N} \in \mathbb{C}^{MN}$, the numerical scheme (4) is equivalent to the projection equation

$$Q_m(L(u *' \phi)) = Q_m(f)\ .$$

[1] From now on we use the letter c to denote a general positive constant the value of which varies from instance to instance

Definition 2.5 *The numerical method $\{\eta,\phi\}$ is called stable for $L: H^s(\mathbb{T}) \to H^{s-\beta}(\mathbb{T})$ if*

$$\|Q_m\, Lu_m\|_{s-\beta} \geq c\|u_m\|_s$$

for any $u_m \in S_m(\phi)$ and $m \in \mathbb{N}_0$ sufficiently large.

Theorem 2.6 *Let $\{\eta,\phi\}$ be $s-$admissible for ΨDO's of order β and let η fulfil Hypothesis H. Then the numerical method $\{\eta,\phi\}$ is stable for $L: H^s(\mathbb{T}) \to H^{s-\beta}(\mathbb{T})$ if and only if*

$$\left\| \left(\sigma_0(x)\langle x\rangle_0^{-\beta} + \sum_{p\neq 0} \langle x\rangle_0^{s-\beta} \left(\hat{\eta}_p(x)\hat{\eta}_0(x)^{-1}\right)^* \sigma_p(x)\hat{\phi}_p(x)\hat{\phi}_0(x)^{-1}\langle x\rangle_0^{-s} \right)^{-1} \right\| \leq c \qquad (9)$$

for any $x \in [-\frac{1}{2},\frac{1}{2}]$.

Definition 2.7 *The numerical symbol $[\eta\sigma\phi]$ is called elliptic of order β for s if condition* (9) *is fulfilled.*

In the case $M = 1$, the Definition 2.7 coincides with the definition of ellipticity given in [DPS], Section 4. The next theorem claims that Theorem 2.6 applies to the collocation method when $M = 2$ and $0 \leq \epsilon_{-1} < \frac{1}{2}$, $\epsilon_0 := \epsilon_{-1} + \frac{1}{2}$.

Theorem 2.8 *Let η be defined by the above choice of $(\epsilon_j)_{j\in\Lambda_2}$ (cf. Example* 2.1*). Further let $\phi := (\phi^j)_{j\in\Lambda_2} \in \mathcal{L}_2^2 \cap (H^{\beta+1/2+\delta}(\mathbb{R}))^2$, $\delta > 0$, be functions with compact support. Suppose $\{\eta,\phi\}$ is $s-$admissible for ΨDO's of order β. Then the collocation method is stable for $L: H^s(\mathbb{T}) \to H^{s-\beta}(\mathbb{T})$, $s-\beta \geq 0$, if and only if the numerical symbol $[\eta\sigma\phi]$ is elliptic of order β for s.*

The proofs of Theorems 2.6 and 2.8 will be postponed to the next section.

3 Stability

3.1 General results

Our next concern is the proof of Theorem 2.6. Later on we will apply it to collocation and to the biorthogonal Galerkin method. First we examine the linear system (4). The stiffness matrix is of the form

$$\left(\eta_{k,m}^r(L\phi_{n,m}^s)\right)_{((k,r),(n,s))\in(\Lambda_N\times\Lambda_M)^2} \; .$$

Since $\phi_{n,m}^s$ is the shift of $\phi_{0,m}^s$, $\eta_{k,m}^r$ is the shift of $\eta_{0,m}^r$ and since L commutes with the shift operator, we conclude

$$\begin{aligned}\eta^r_{k,m}(L\phi^s_{n,m}) &= 2^{-\frac{m}{2}}\ \eta^r((L\phi^s_{n,m})(2^{-m}(\cdot+k))) \\ &= 2^{-\frac{m}{2}}\ \eta^r((L\phi^s_{0,m})(2^{-m}(\cdot+[k-n]))) \\ &= \eta^r_{[k-n],m}(L\phi^s_{0,m})\end{aligned}$$

with $[k] := k\ mod\ N$. Hence we see that

$$\begin{aligned}\left(\eta^r_{k,m}(L\phi^s_{n,m})\right)_{((k,r),(n,s))\in(\Lambda_N\times\Lambda_M)^2} &= \left((\eta^r_{[k-n],m}(L\phi^s_{0,m}))_{(r,s)\in\Lambda^2_M}\right)_{(k,n)\in\Lambda^2_N} \\ &= \left(A_{[k-n]}\right)_{(k,n)\in\Lambda^2_N} =: A\end{aligned}$$

is a block circulant with

$$A_k := \left(\eta^r_{k,m}(L\phi^s_{0,m})\right)_{(r,s)\in\Lambda^2_M}\ .$$

Such matrices can be diagonalized by the unitary matrix

$$U := 2^{-\frac{m}{2}}\left(e^{2\pi iknh}\ \mathbf{1}_M\right)_{(k,n)\in\Lambda^2_N}$$

where $\mathbf{1}_M$ denotes the M-dimensional unit matrix. We obtain

$$A = UDU^*$$

with a block diagonal $D = diag(D(kh))_{k\in\Lambda_N} \in (\mathbb{C}^{M\times M})^{N\times N}$. The diagonal entries are given by

$$D(kh) = \sum_{j=0}^{N-1} A_j e^{-2\pi ijkh}\ . \tag{10}$$

To compute these elements of the block diagonal we use

$$\tilde{\phi}^l_{k,m}(\xi) = 2^{-\frac{m}{2}}\ e^{-2\pi ik\xi h}\ \hat{\phi}^l(h\xi)\ , \quad \xi\in\mathbb{Z}, \tag{11}$$

for $k\in\Lambda_N,\ l\in\Lambda_M$ and

$$\sum_{j=0}^{N-1} e^{-2\pi ihjk} = \begin{cases} 2^m & if\ \ k=2^m\xi\ \ with\ \ \xi\in\mathbb{Z} \\ 0 & else \end{cases}\ .$$

Thus we obtain

$$\begin{aligned} D(kh) &= \sum_{j=0}^{N-1} \sum_{l\in\mathbb{Z}} 2^{-\frac{m}{2}} \left(\eta^r \left(e^{2\pi i h l(\cdot+j)}\right) e^{-2\pi i j k h}\, \sigma(l)\, \tilde{\phi}^s_{0,m}(l)\right)_{(r,s)\in\Lambda_M^2} \\ &= \sum_{l\in\mathbb{Z}} 2^{\frac{m}{2}} \left(\eta^r \left(e^{2\pi i h(Nl+k)\cdot}\right) \sigma(Nl+k)\, \tilde{\phi}^s_{0,m}(Nl+k)\right)_{(r,s)\in\Lambda_M^2} \\ &= \sum_{l\in\mathbb{Z}} 2^{m\beta} \left(\eta^r \left(e^{2\pi i(l+kh)\cdot}\right) \sigma(l+kh)\, \hat{\phi}^s(l+kh)\right)_{(r,s)\in\Lambda_M^2} \\ &= 2^{m\beta}\, [\eta\sigma\phi](kh)\ . \end{aligned} \tag{12}$$

The above computation reveals the fundamental role of the numerical symbol.

To investigate the stability we need discrete norms equivalent to the Sobolev norm over $S_m(\phi)$. Using (12) with σ replaced by $\langle\cdot\rangle^{2s}$ and $\eta^j := \langle\cdot,\phi^j\rangle_{L^2(\mathbb{R})}$, a straightforward computation shows that (cf. (8))

$$\|u *' \phi\|_s^2 = 2^{2ms} \left\langle diag([\phi\,\langle\cdot\rangle^{2s}\phi](kh))_{k\in\Lambda_N}\, U^*\, u\ ,\ U^*\, u\right\rangle_{\mathbb{C}^{MN}}\ . \tag{13}$$

Now we introduce the discrete Sobolev norm defined by (cf. (5))

$$\begin{aligned} \|u *' \phi\|_{s,h}^2 &:= 2^{2ms} \left\langle diag([\phi\,\langle\cdot\rangle^{2s}\phi]_0(kh))_{k\in\Lambda_N}\, U^*\, u\ ,\ U^*\, u\right\rangle_{\mathbb{C}^{MN}}\ , \\ \left[\phi\,\langle\cdot\rangle^{2s}\phi\right]_0(x) &:= \hat{\phi}_0(x)^*\, \langle x\rangle_0^{2s}\, \hat{\phi}_0(x)\ . \end{aligned} \tag{14}$$

To examine the stability of the numerical method it is important to know the conditions under which the discrete and continuous Sobolev norms are equivalent on $S_m(\phi)$.

Theorem 3.1 *Let $s\in\mathbb{R}$, and $\phi := (\phi^j)_{j\in\Lambda_M} \in \mathcal{L}_2^M$ with $\hat{\phi}_0(x)$ invertible for $x\in[-\frac{1}{2},\frac{1}{2}]$. Suppose that the sum*

$$\sum_{p\neq 0} \|\langle x\rangle_p^s\, \hat{\phi}_p(x)\, \hat{\phi}_0(x)^{-1}\, \langle x\rangle_0^{-s}\|^2 \tag{15}$$

is bounded on $[-\frac{1}{2},\frac{1}{2}]$. Then the norms $\|\cdot *'\phi\|_s$ and $\|\cdot *'\phi\|_{s,h}$ are equivalent with equivalence constants independent of h.

Remark 3.2 *The invertibility of $\hat{\phi}_0$ is even necessary if the translates of ϕ form a Riesz basis (for the definition cf.* [JM]*). For the meaning of* (15) *see Remark* 2.4.

Proof of Theorem 3.1*:* i) It is obvious that

$$\left[\phi\,\langle\cdot\rangle^{2s}\phi\right]_0(x) \leq \left[\phi\,\langle\cdot\rangle^{2s}\phi\right](x)$$

for $x\in[-\frac{1}{2},\frac{1}{2}]$, hence $\|\cdot *'\phi\|_{s,h} \leq \|\cdot *'\phi\|_s$.

ii) For $u := (u^j)_{j\in\Lambda_M} \in \mathbb{C}^M$ and $\hat{\psi}_p(x) := \hat{\phi}_p(x)\hat{\phi}_0(x)^{-1}$ we have

$$\begin{aligned}
\sum_{p\in\mathbb{Z}} \left\langle \hat\psi_p(x)^*\langle x\rangle_p^{2s}\hat\psi_p(x)u\ ,\ u\right\rangle_{\mathbb{C}^M} &= \sum_{l\in\mathbb{Z}} \left| \sum_{j\in\Lambda_M} u^j\ \hat\psi^j(l+x)\right|^2 \langle l+x\rangle^{2s} \\
&\le \sum_{l\in\mathbb{Z}} \left(\sum_{j\in\Lambda_M} |u^j\ \hat\psi^j(l+x)|\right)^2 \langle l+x\rangle^{2s} \\
&\le \sum_{l\in\mathbb{Z}} \left(\sum_{j\in\Lambda_M} \frac{|\hat\psi^j(l+x)|^2\langle l+x\rangle^{2s}}{\langle j+x\rangle^{2s}}\right) \left(\sum_{j\in\Lambda_M} |u^j|^2\langle j+x\rangle^{2s}\right) \\
&= \left(\sum_{l\in\mathbb{Z}\setminus\Lambda_M}\sum_{j\in\Lambda_M} \frac{|\hat\psi^j(l+x)|^2\langle l+x\rangle^{2s}}{\langle j+x\rangle^{2s}} + M\right) \left\langle \hat\psi_0(x)^*\langle x\rangle_0^{2s}\hat\psi_0(x)u\ ,\ u\right\rangle_{\mathbb{C}^M} \\
&= \left(\sum_{p\ne 0} \left\| \langle x\rangle_p^{s}\hat\psi_p(x)\langle x\rangle_0^{-s}\right\|^2 + M\right) \left\langle \hat\psi_0(x)^*\langle x\rangle_0^{2s}\hat\psi_0(x)u\ ,\ u\right\rangle_{\mathbb{C}^M} \\
&\le c\ \left\langle \hat\psi_0(x)^*\langle x\rangle_0^{2s}\hat\psi_0(x)u\ ,\ u\right\rangle_{\mathbb{C}^M} .
\end{aligned}$$

Therefore we obtain

$$\left[\phi\ \langle\cdot\rangle^{2s}\phi\right](x) \le c\ \left[\phi\ \langle\cdot\rangle^{2s}\phi\right]_0(x)$$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$. Hence $\| \cdot *'\phi\|_s \le c\ \| \cdot *'\phi\|_{s,h}$ for any $h > 0$. ■

The following example shows that the assumptions of Theorem 3.1 are fulfilled for splines with multiple knots.

Example 3.3 *Let $(\phi^j)_{j\in\Lambda_M} \in \mathcal{L}_2^M$ be a family which generates the periodic multiresolution analysis of splines with degree r and defect M, $1 \le M \le r$, which means*

$$S_m(\phi) := \left\{ f \in \mathcal{C}^{r-M-1}(\mathbb{T}) :\ f|_{(nh,(n+1)h)} \text{ is a polynomial of degree } \le r-1,\ n \in \Lambda_N \right\} .$$

In the case $M = 2$ and $r = 3$, the generators are given by

$$\phi^{-1}(x) := \begin{cases} x & for \quad x\in[0,1] \\ 2-x & for \quad x\in[1,2] \\ 0 & else \end{cases} \quad , \quad \phi^{0}(x) := \begin{cases} x^2 & for \quad x\in[0,1] \\ (2-x)^2 & for \quad x\in[1,2] \\ 0 & else \end{cases} \quad .$$

For Hermite cubic splines, i.e., $M = 2$ and $r = 4$, the generators are given by

$$\phi^{-1}(x) := \begin{cases} 3x^2-2x^3 & for \quad x\in[0,1] \\ 3(2-x)^2-2(2-x)^3 & for \quad x\in[1,2] \\ 0 & else \end{cases} \quad ,$$

$$\phi^0(x) := \begin{cases} x^2 - x^3 & for \quad x \in [0,1] \\ -(2-x)^2 + (2-x)^3 & for \quad x \in [1,2] \\ 0 & else \end{cases} .$$

Let us turn to the general case. From [MP], Theorem 3.2 and (11) we obtain

$$\hat{\phi}_p(x) = (x)_p^{-r} \; W_p \; (x)_0^r \; \hat{\phi}_0(x) \; , p \in \mathbb{Z} \setminus \{0\}, \tag{16}$$

where W_p is an $M \times M$ matrix such that $\|W_p\| \leq c|p|^{M-1}$ for $p \in \mathbb{Z} \setminus \{0\}$ and $W_0 = \mathbf{1}_M$ (see [MP]*). The equivalence of the norms $\| \cdot *' \phi\|_s$ and $\| \cdot *' \phi\|_{s,h}$ independent of h is shown in* [MP]*, Theorem* 3.2*. Hence $\hat{\phi}_0(x)$ is invertible for $x \in [-\frac{1}{2}, \frac{1}{2}]$ (see also Remark* 3.2*). Further we have for $s < r - M + \frac{1}{2}$, $s \in \mathbb{R}$,*

$$\begin{aligned} \sum_{p \neq 0} \|\langle x\rangle_p^s \; \hat{\phi}_p(x) \; \hat{\phi}_0(x)^{-1} \; \langle x\rangle_0^{-s}\|^2 \;\; &\leq \;\; c \sum_{p \neq 0} \|\langle x\rangle_p^{s-r}\|^2 \; |p|^{2(M-1)} \\ &\leq \;\; c \sum_{p \neq 0} |p|^{2(M-r+s-1)} < \infty \end{aligned}$$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$. Thus the functions $(\phi^j)_{j \in \Lambda_M}$ fulfil the assumptions of Theorem 3.1 *for any $s < r - M + \frac{1}{2}$.*

Now we are ready for the

Proof of Theorem 2.6*:* For any $m \in \mathbb{N}_0$ and $u := ((u_k^j)_{j \in \Lambda_M})_{k \in \Lambda_N} \in \mathbb{C}^{MN}$ we conclude from (12) that

$$Q_m(L(u *' \phi)) = 2^{m\beta} \; \left(U \; diag \left([\eta\sigma\phi](kh)\right)_{k \in \Lambda_N} \; U^* u\right) *' \; \psi \; .$$

Moreover, using Theorem 3.1 and (14), we obtain

$$\begin{aligned} &\|Q_m(L(u *' \phi))\|_{s-\beta}^2 \doteq^2 \|Q_m(L(u *' \phi))\|_{s-\beta,h}^2 \qquad (17) \\ &= \;\; 2^{2ms} \; \left\langle diag \left([\psi \; \langle\cdot\rangle^{2(s-\beta)}\psi]_0(kh) \; [\eta\sigma\phi](kh)\right)_{k \in \Lambda_N} U^* u \; , \; diag \left([\eta\sigma\phi](kh)\right)_{k \in \Lambda_N} U^* u \right\rangle_{\mathbb{C}^{MN}} \end{aligned}$$

and

$$\|u *' \phi\|_s^2 \doteq \|u *' \phi\|_{s,h}^2 = 2^{2ms} \; \left\langle diag \left([\phi \langle\cdot\rangle^{2s}\phi]_0(kh)\right)_{k \in \Lambda_N} U^* \; , \; U^* u\right\rangle_{\mathbb{C}^{MN}} \; . \tag{18}$$

From (17) and (18) we get that $\{\eta, \phi\}$ is stable for $L : H^s(\mathbb{T}) \to H^{s-\beta}(\mathbb{T})$ if and only if for any $x \in [-\frac{1}{2}, \frac{1}{2}]$

$$[\eta\sigma\phi](x)^* \; [\psi\langle\cdot\rangle^{2(s-\beta)}\psi]_0(x) \; [\eta\sigma\phi](x) \geq c \; [\phi \; \langle\cdot\rangle^{2s}\phi]_0(x) \; .$$

[2] We write $\| \cdot \|_t \doteq \| \cdot \|_{t,h}$ if and only if there exist positive general constants $c_1, \; c_2$ such that $c_1 \; \| \cdot \|_{t,h} \leq \| \cdot \|_t \leq c_2 \; \| \cdot \|_{t,h}$.

Hence the stability is equivalent to

$$\begin{aligned} c &\geq \left\| \left(\sum_{p\in\mathbb{Z}} \langle x\rangle_0^{s-\beta} \hat{\psi}_0(x)\, \hat{\eta}_p(x)^* \, \sigma_p(x)\, \hat{\phi}_p(x)\, \hat{\phi}_0(x)^{-1} \langle x\rangle_0^{-s} \right)^{-1} \right\| \\ &:= \left\| \left(\sigma_0(x) \langle x\rangle_0^{-\beta} + \sum_{p\neq 0} \langle x\rangle_0^{s-\beta} \left(\hat{\eta}_p(x) \hat{\eta}_0(x)^{-1} \right)^* \sigma_p(x)\, \hat{\phi}_p(x)\, \hat{\phi}_0(x)^{-1} \langle x\rangle_0^{-s} \right)^{-1} \right\| . \end{aligned}$$

■

In the remainder of this subsection we give some general remarks concerning the validity of Hypothesis H of Section 2. Let us start with a proposition on the Poisson summation formula.

Proposition 3.4 *Let one of the following conditions be fulfilled*

i) $f,\ \tilde{F} \in \mathcal{L}_2,\quad F(g) := \langle g, \tilde{F} \rangle_{L^2(\mathbb{R})}\quad for\ g \in \mathcal{L}_2$;

ii) $0 \leq \epsilon < 1,\ F(g) := g(\epsilon)\quad for\ g \in \mathcal{C}(\mathbb{R})$,
$f \in L^1(\mathbb{R})$ *such that there exists* $s > \frac{1}{2}$ *with*
$\sum_{l\neq 0} |\hat{f}(l+x)|^2 |l+x|^{2s} \leq c \quad for\ \ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ *and* $\left(F(f(\cdot - k))\right)_{k\in\mathbb{Z}} \in l^1(\mathbb{Z})$.

Then the Poisson summation formula

$$\sum_{l\in\mathbb{Z}} \overline{\hat{F}(l+x)} \hat{f}(l+x) = \sum_{k\in\mathbb{Z}} F\left(f(\cdot - k)\right)\ e^{2\pi i x k} \tag{19}$$

is valid for any $x \in [-\frac{1}{2}, \frac{1}{2}]$.

Proof. If condition i) is fulfilled then the assertion follows from [JM], Theorem 3.2.
Now let ii) be valid for F and f. Using $|\hat{F}(x)| \equiv 1$, we get

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{l\in\mathbb{Z}} \overline{\hat{F}(l+x)} \hat{f}(l+x) \right|^2 dx \ \leq\ c \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{l\neq 0} \left| (l+x)^{-s} \hat{f}(l+x)(l+x)^s \right| + \left| \hat{f}(x) \right| \right)^2 dx$$

$$\leq\ c \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{l\neq 0} |l+x|^{-2s} + 1 \right) \left(\sum_{l\neq 0} \left| \hat{f}(l+x)(l+x)^s \right|^2 + \left| \hat{f}(x) \right|^2 \right) dx\ \leq\ c\,.$$

Hence there exists $(a_l)_{l\in\mathbb{Z}} \in l^2(\mathbb{Z})$ such that

$$\sum_{l\in\mathbb{Z}} \overline{\hat{F}(l+x)}\hat{f}(l+x) = \sum_{k\in\mathbb{Z}} a_k\, e^{2\pi ixk}$$

in the L^2-sense. Further we get $f \in H^s(\mathbb{R})$ because of

$$\begin{aligned}\int_{\mathbb{R}} |\hat{f}(x)|^2(1+|x|^2)^s\, dx &= \sum_{l\in\mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{f}(l+x)|^2(1+|l+x|^2)^s\, dx \\ &\leq c \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{l\neq 0} |\hat{f}(l+x)|^2|l+x|^{2s} + |\hat{f}(x)|^2(1+|x|^2)^s \right) dx < \infty\,.\end{aligned}$$

Therefore $f \in \mathcal{C}(\mathbb{R})$ by Sobolev's embedding theorem.

Now we compute $a_k,\ k\in\mathbb{Z}$. We get

$$\begin{aligned}a_k &= \int_0^1 \sum_{l\in\mathbb{Z}} \overline{\hat{F}(l+x)}\hat{f}(l+x)\, e^{-2\pi ikx}\, dx \\ &= \int_0^1 \sum_{l\in\mathbb{Z}} \overline{\hat{F}(l+x)}\hat{f}(\cdot - k)(l+x)\, dx \\ &= \int_{\mathbb{R}} e^{2\pi i\epsilon y}\ \hat{f}(\cdot - k)(y)\, dy \\ &= f(\cdot - k)(\epsilon) = F(f(\cdot - k))\,,\end{aligned}$$

since $\hat{f} \in L^1(\mathbb{R})$. Hence (19) is true in the L^2-sense. However the right hand side of the identity is a continuous function. So we only have to show that the left hand side is continuous too. This follows from the estimate

$$\left|\overline{\hat{F}(l+x)}\hat{f}(l+x)\right| \leq c\,|l+x|^{-2s} \leq c\,|l|^{-2s}$$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and $l \neq 0$. ■

Remark 3.5 *Condition i) of Proposition* 3.4 *is fulfilled in the case of the biorthogonal Galerkin method and ii) in the case of collocation.*

Using Poisson's summation formula, the biorthogonality condition of Hypothesis H can be expressed in terms of the Fourier transforms.

Corollary 3.6 *Let the Poisson summation formula be valid for η and ψ in the L^2-sense, i.e.,*

$$\sum_{p\in\mathbb{Z}} \hat{\eta}_p(x)^* \hat{\psi}_p(x) = \sum_{k\in\mathbb{Z}} \left(\eta^r(\psi^j(\cdot - k))\right)_{(r,j)\in\Lambda_M^2} e^{2\pi i x k}$$

in the L^2-sense. Then the following conditions are equivalent

$$i) \quad \sum_{p\in\mathbb{Z}} \hat{\eta}_p(x)^* \hat{\psi}_p(x) \equiv \mathbf{1}_M \ \text{ in } L^2 , \tag{20}$$

$$ii) \quad \eta^l(\psi^j(\cdot - k)) = \delta_{l,j}\ \delta_{0,k} , \quad l,j \in \Lambda_M, \ k \in \mathbb{Z} .$$

∎

In particular, the assumption of Corollary 3.6 is valid in the case i) or ii) of Proposition 3.4. In the following we require

$$\left(\sup_{x\in[-\frac{1}{2},\frac{1}{2}]} \|\hat{\eta}_p(x)\| \right)_{p\in\mathbb{Z}} \in l^\infty(\mathbb{Z}) \tag{21}$$

as well as that $\hat{\eta}_0$ is invertible and the inverse is uniformly bounded on $[-\frac{1}{2},\frac{1}{2}]$. This is valid for the biorthogonal Galerkin method with $\eta \in \mathcal{L}_2^M$ and for collocation. Now it turns out that the last condition of Hypothesis H follows from the first two conditions and some additional assumptions.

Proposition 3.7 *Let $\hat{\psi}_0$, $\hat{\eta}_0$ be invertible on $[-\frac{1}{2},\frac{1}{2}]$, $t \geq 0$,*

$$\textstyle\sum_{p\neq 0} \|\hat{\psi}_p(x)\ \hat{\psi}_0^{-1}(x)\ |x|_0^{-t}\| \leq c \quad , \quad \sum_{p\in\mathbb{Z}} \hat{\eta}_p(x)^*\ \hat{\psi}_p(x) \equiv \mathbf{1}_M$$

for $x \in [-\frac{1}{2},\frac{1}{2}] \setminus \{0\}$. Then we have for any $x \in [-\frac{1}{2},\frac{1}{2}] \setminus \{0\}$

$$\left\| |x|_0^t\ \hat{\psi}_0(x)\ \hat{\eta}_0(x)^*\ |x|_0^{-t} \right\| \leq c \quad , \quad \left\| |x|_0^t\ \left(\hat{\psi}_0(x)\ \hat{\eta}_0(x)^* \right)^{-1}\ |x|_0^{-t} \right\| \leq c .$$

Proof. For $x \neq 0$ we find that

$$\begin{aligned}
\left\| \left(1 - \hat{\psi}_0(x)\hat{\eta}_0(x)^*\right)\ |x|_0^{-t} \right\| &= \left\| \hat{\psi}_0(x)\ \left(1 - \hat{\eta}_0(x)^*\hat{\psi}_0(x)\right)\ \hat{\psi}_0(x)^{-1}|x|_0^{-t} \right\| \\
&= \left\| \hat{\psi}_0(x) \sum_{p\neq 0} \hat{\eta}_p(x)^*\hat{\psi}_p(x)\ \hat{\psi}_0(x)^{-1}|x|_0^{-t} \right\| \\
&\leq c \sum_{p\neq 0} \left\| \hat{\psi}_p(x)\ \hat{\psi}_0(x)^{-1}|x|_0^{-t} \right\| \leq c .
\end{aligned} \tag{22}$$

Because of (22) we get

$$\left\| \left(1 - (\hat{\psi}_0(x)\hat{\eta}_0(x)^*)^{-1}\right) \ |x|_0^{-t} \right\| \le \left\| \left(\hat{\psi}_0(x)\hat{\eta}_0(x)^*\right)^{-1} \right\| \left\| \left(1 - \hat{\psi}_0(x)\hat{\eta}_0(x)^*\right) \ |x|_0^{-t} \right\| \le c \,. \tag{23}$$

From (22) it follows that

$$\left\| |x|_0^t \hat{\psi}_0(x)\hat{\eta}_0(x)^* |x|_0^{-t} \right\| \le 1 + c \left\| \left(1 - \hat{\psi}_0(x)\hat{\eta}_0(x)^*\right) \ |x|_0^{-t} \right\| \le c \,,$$

and from (23)

$$\left\| |x|_0^t \ \left(\hat{\psi}_0(x)\hat{\eta}_0(x)^*\right)^{-1} \ |x|_0^{-t} \right\| \le c \,.$$

■

By the same arguments we obtain

Proposition 3.8 *Let $\hat{\psi}_0$, $\hat{\eta}_0$ be invertible on $[-\frac{1}{2}, \frac{1}{2}]$, $t \le 0$,*

$$\textstyle\sum_{p \ne 0} \|\hat{\eta}_p(x) \ \hat{\eta}_0(x)^{-1} \ |x|_0^t\| \le c \quad , \quad \sum_{p \in \mathbb{Z}} \hat{\eta}_p(x)^* \ \hat{\psi}_p(x) \equiv \mathbf{1}_M$$

for $x \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$. Then we have for any $x \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$

$$\left\| |x|_0^t \ \hat{\psi}_0(x) \ \hat{\eta}_0(x)^* \ |x|_0^{-t} \right\| \le c \quad , \quad \left\| |x|_0^t \ \left(\hat{\psi}_0(x) \ \hat{\eta}_0(x)^*\right)^{-1} \ |x|_0^{-t} \right\| \le c \,.$$

■

In the next two subsections we apply the results of 3.1 to the particular special cases of the biorthogonal Galerkin method and of collocation with special choices of the collocation points. In particular, we consider the case $M = 2$ in some more detail.

3.2 The biorthogonal Galerkin method

Suppose $(\eta^j)_{j \in \Lambda_M}$ is as in Example 2.2 and $L : H^s(\mathbb{T}) \to H^{s-\beta}(\mathbb{T})$ with $s = 0$. In this case it is possible to apply Theorem 2.6, Corollary 3.6 and Proposition 3.7 or 3.8 with $\psi = \phi$. For other choices of s one can get analogous results.

Theorem 3.9 *Suppose that $\hat{\eta}_0$ and $\hat{\phi}_0$ are invertible on $[-\frac{1}{2}, \frac{1}{2}]$, that $\sum_{p \ne 0} \|\hat{\phi}_p(x)\|^2$ is uniformly bounded on $[-\frac{1}{2}, \frac{1}{2}]$ and let one of the following conditions be fulfilled:*

i) $\sum_{p \ne 0} \left\| \hat{\phi}_p(x) \ \hat{\phi}_0(x)^{-1} \ \langle x \rangle_0^{-\beta} \right\| \le c$ *and* $\sum_{p \ne 0} \left\| \hat{\eta}_p(x)^* \ |x|_p^\beta \ \hat{\phi}_p(x) \right\| \le c$

in case $\beta \ge 0$;

ii) $\sum_{p\neq 0} \left\| \langle x\rangle_p^{-\beta}\ \hat{\phi}_p(x)\ \hat{\phi}_0(x)^{-1}\ \langle x\rangle_0^{\beta} \right\|^2 \leq c$ *and* $\sum_{p\neq 0} \left\| \hat{\eta}_p(x)\ \hat{\eta}_0(x)^{-1}\ \langle x\rangle_0^{\beta} \right\| \leq c$;

in case $\beta < 0$.

Then the biorthogonal Galerkin method $\{\eta, \phi\}$ *is stable if and only if the numerical symbol is elliptic of order* β *for* $s = 0$.

Proof. For the admissibility of the numerical method we only have to show property iii) of Definition 2.3 in case $\beta < 0$. We obtain

$$\sum_{p\neq 0} \left\| \hat{\eta}_p(x)^*\ |x|_p^{\beta}\ \hat{\phi}_p(x) \right\| \leq c \sum_{p\neq 0} \left\| |x|_0^{\beta}\ \left(\hat{\eta}_p(x)\ \hat{\eta}_0(x)^{-1}\right)^* \right\| \leq c\,.$$

Now the assertion follows from Theorem 2.6 and from Proposition 3.7 in case $\beta < 0$ or Proposition 3.8 if $\beta \geq 0$. ■

3.3 Collocation method

In the first part of the present subsection we prove Theorem 2.8 and then we obtain the admissibility of collocation for splines with multiple knots. In the last part we show the equivalence between the stability in the sense of [MP] for splines with multiple knots and the stability defined in this paper.

Suppose $(\epsilon_j)_{j\in\Lambda_M}$ and $(\eta_j)_{j\in\Lambda_M}$ are as in Example 2.1. It is easy to check that $\hat{\eta}_0$ is invertible on $[-\frac{1}{2}, \frac{1}{2}]$.

Proof of Theorem 2.8*:* First we have to show the existence of functions ψ as required in the Hypothesis H. Then we apply Theorem 2.6.

i) We define the B-splines of order $r \in \mathbb{N}_0$ by

$$M_r := M_0 * M_{r-1}\,, \quad r \geq 1,$$
$$M_0 := \frac{1}{2}\left(\chi_{[-\frac{1}{2},\frac{1}{2})} + \chi_{(-\frac{1}{2},\frac{1}{2}]}\right),$$

where $*$ is the convolution. It is known that the functions

$$\Phi_r(u) := \sum_{l\in\mathbb{Z}} M_r(l)\ e^{2\pi i l u}$$

have no zeros on $\mathbb{R}$ (cf. [DeVL] Chapter 13, Theorem 6.2). Hence there exists the inverse

$$\Phi_r^{-1}(u) = \sum_{l\in\mathbb{Z}} \omega_l^r\ e^{2\pi i l u}\,,$$

where $(\omega_l^r)_{l\in\mathbb{Z}}$ is a sequence of exponential decay, because Φ_r is a polynomial.

For fixed $r \in \mathbb{N}$ we define ψ by

$$\psi^{-1} := L_r\left(2(\cdot + \tfrac{1}{2} - \epsilon_0)\right) \ , \quad \psi^{0} := L_r\left(2(\cdot - \tfrac{1}{2} - \epsilon_{-1})\right) \ , \tag{24}$$

where

$$L_r(x) := \sum_{l\in\mathbb{Z}} \omega_l^r \ M_r(x-l) \ .$$

Because of the exponential decay of L_r we get $\psi \in \mathcal{L}_2^2$. Moreover, we have

$$L_r(k) = \delta_{0,k} \ , \quad k \in \mathbb{Z} \ ,$$

and

$$\psi^{-1}(\epsilon_0 + k) = L_r(2k+1) = 0 \quad , \quad \psi^0(\epsilon_{-1}+k) = L_r(2k-1) = 0 \ ;$$
$$\psi^{-1}(\epsilon_{-1} + k) = L_r(2k) = \delta_{0,k} \quad , \quad \psi^0(\epsilon_{0}+k) = L_r(2k) = \delta_{0,k} \ ,$$

for any $k \in \mathbb{Z}$. Here we have used $\epsilon_0 = \epsilon_{-1} + \frac{1}{2}$. Therefore the second condition of Hypothesis H in Section 2 is valid.

ii) The Fourier transforms of ψ are given by

$$\hat{\psi}^{-1}(x) = \tfrac{1}{2}\ e^{2\pi i(-\frac{1}{2}+\epsilon_0)x}\ \Phi_r^{-1}(\tfrac{x}{2})\ \hat{M}_r(\tfrac{x}{2}) \ ,$$
$$\hat{\psi}^{0}(x) = \tfrac{1}{2}\ e^{2\pi i(\frac{1}{2}+\epsilon_{-1})x}\ \Phi_r^{-1}(\tfrac{x}{2})\ \hat{M}_r(\tfrac{x}{2}) \ .$$

For any $x \in [-\frac{1}{2}, \frac{1}{2}]$ we have det $\hat{\psi}_0(x) \neq 0$ if and only if

$$\phi_r^{-1}(\tfrac{x-1}{2})\ \hat{M}_r(\tfrac{x-1}{2})\ \phi_r^{-1}(\tfrac{x}{2})\ \hat{M}_r(\tfrac{x}{2})\ \left(e^{-2\pi i\epsilon_0} - e^{-2\pi i\epsilon_{-1}}\right) \neq 0 \ .$$

Therefore $\hat{\psi}_0(x)$ is invertible on $[-\frac{1}{2}, \frac{1}{2}]$, since

$$\hat{M}_r(x) = \left(\frac{sin(\pi x)}{\pi x}\right)^r$$

has no zeros in $(-1,1)$.

iii) All we have to show is a sufficiently strong uniform Strang-Fix condition (see Remark 2.4). Indeed, we claim

$$\hat{\psi}_p(x) = e^{4\pi i p\epsilon_{-1}}\ (x)_p^{-r}\ (x)_0^r\ \hat{\psi}_0(x) \ . \tag{25}$$

If this is true for sufficiently large r, we obtain the third condition of Hypothesis H by Corollary 3.6 and Proposition 3.7. The assertion of Theorem 2.8 follows by Theorem 2.6.

Now we show assertion (25). For any $p \neq 0$, $\ l \in \mathbb{Z}$, $\ x \in [-\frac{1}{2}, \frac{1}{2}]$ we have

$$\begin{aligned}\hat{\psi}^{-1}(2p+l+x) &= e^{2\pi i(-\frac{1}{2}+\epsilon_0)2p}\ \Phi_r^{-1}(\tfrac{2p+l+x}{2})\ \Phi_r(\tfrac{l+x}{2})\ \left(\tfrac{l+x}{2p+l+x}\right)^r \\ &\quad \tfrac{1}{2}\ e^{2\pi i(-\frac{1}{2}+\epsilon_0)(l+x)}\ \Phi_r^{-1}(\tfrac{l+x}{2})\ \hat{M}_r(\tfrac{l+x}{2}) \\ &= e^{4\pi i p\epsilon_0}\ \Phi_r^{-1}(\tfrac{2p+l+x}{2})\ \Phi_r(\tfrac{l+x}{2})\ \left(\tfrac{l+x}{2p+l+x}\right)^r\ \hat{\psi}^{-1}(l+x)\ .\end{aligned}$$

Hence

$$\hat{\psi}^{-1}(2p+l+x) = e^{4\pi i p\epsilon_0}\ \left(\tfrac{l+x}{2p+l+x}\right)^r\ \hat{\psi}^{-1}(l+x)\ ,$$

since Φ_r is $1-$periodic. A similar computation shows that

$$\hat{\psi}^{0}(2p+l+x) = e^{4\pi i p\epsilon_{-1}}\ \left(\tfrac{l+x}{2p+l+x}\right)^r\ \hat{\psi}^{0}(l+x)\ ,$$

and (25) is proved. ∎

For the following theorem we refer the reader to the definition of periodic splines with multiple knots in Example 3.3.

Theorem 3.10 *Let* $r-M-\beta>0\ ,\ \ r-M+\frac{1}{2}-s>0\ ,\ \ r\geq M\geq 1.$
Then the collocation method for periodic splines of defect M is $s-$admissible for ΨDO's of order β.

Remark 3.11 *The assumption $r-M+\frac{1}{2}-s>0$ ensures that $S_m(\phi)\subseteq H^s(\mathbb{T})$. Further one has the continuity of Lu_m for $u_m\in S_m(\phi)$ since $r-M-\beta>0$.*

Proof of Theorem 3.10*:* The invertibility of $\hat{\phi}_0$ and condition ii) of Definition 2.3 have been shown in Example 3.3. It remains to prove property iii) of 2.3. We choose $t,\delta>0$, such that $r-M-\beta-\delta>0$ *and* $\frac{1}{2}<t<\frac{1}{2}+\frac{\delta}{2}$. Then for any $x\in[-\frac{1}{2},\frac{1}{2}]$ we have (cf. (21), (16))

$$\begin{aligned}\sum_{p\neq 0}\left\|\hat{\eta}_p(x)^*\ |x|_p^\beta\ \hat{\phi}_p(x)\right\| &\leq c\sum_{p\neq 0}\left\||x|_p^\beta\ \hat{\phi}_p(x)\ \hat{\phi}_0(x)^{-1}\right\| \\ &\leq c\sum_{p\neq 0}\left\||x|_p^{-t}\ |x|_p^{t+\beta}\ (x)_p^{-r}\ W_p\ (x)_0^r\right\| \\ &\leq c\left(\sum_{p\neq 0}|p|^{-2t}\right)\left(\sum_{p\neq 0}|p|^{2(t+\beta-r+M-1)}\right)<\infty\ ,\end{aligned}$$

since $2(t+\beta-r+M-1)<-1-\delta$. ∎

In the last part of this section we show the connection with the paper [MP]. The spaces $S_m(\phi)$ are defined as in Example 3.3. For the definition of stability in the sense of [MP], we need the matrix valued function D defined by

$$D(x) \quad := \quad \sum_{p\in\mathbb{Z}} \Phi_p(\epsilon)\ \sigma_p(x)\ (x)_p^{-r}\ W_p\ (x)_0^r\ \sigma_0(x)^{-1}\ , \quad x \in [-\tfrac{1}{2}, \tfrac{1}{2}], \tag{26}$$

$$\Phi_p(\epsilon) \quad := \quad \sum_{j\in\Lambda_M} \left(e^{2\pi i(Mp+s-r)\epsilon_j}\right)_{(r,s)\in\Lambda_M^2}\ . \tag{27}$$

Recall that the $M \times M$ matrices W_p (cf. (16)) satisfy $\|W_p\| < c\ |p|^{M-1}$ for $p \neq 0$. Moreover we remark that D is defined for $\beta < r$. Hence D has a continuous extension to zero, since $W_0 = \mathbf{1}_M$. The collocation method for periodic splines of defect M is called **stable** in the sense of [MP] if D is invertible on $[-\frac{1}{2}, \frac{1}{2}]$ and the inverse is uniformly bounded.

Theorem 3.12 *Let $s - \beta \geq 0$, $r - M - \beta > 0$, $r - M + \frac{1}{2} - s > 0$, $r \geq M \geq 1$ and consider $L : H^s(\mathbb{T}) \to H^{s-\beta}(\mathbb{T})$. Then the collocation method for periodic splines of defect M is stable in the sense of [MP] if and only if the numerical symbol $[\eta\sigma\phi]$ is elliptic of order β for any s.*

Proof. Fix $\delta > 0$ such that $r - \beta - \delta > 0$ and $r - s - \delta > 0$. Let D be invertible and suppose the inverse is uniformly bounded. Using (26) and the invertibility of $\Phi_0(\epsilon)$, we obtain for $x \neq 0$

$$\Phi_0(\epsilon)^{-1}\ D(x) = 1 + \Phi_0(\epsilon)^{-1} \sum_{p\neq 0} \Phi_p(\epsilon)\ \sigma_p(x)\ (x)_p^{-r}\ W_p\ (x)_0^r\ \sigma_0(x)^{-1}\ .$$

Therefore

$$\left(\Phi_0(\epsilon)^{-1}\ D(x)\right)_{l,0} = o(|x|^{r-\beta-\delta}) \quad , \ l \in \Lambda_M \setminus \{0\}\ .$$

Now use the adjugate for representing the inverse of D to conclude that

$$\left(\left(\Phi_0(\epsilon)^{-1}\ D(x)\right)^{-1}\right)_{l,0} = o(|x|^{r-\beta-\delta}) \quad , l \in \Lambda_M \setminus \{0\}\ .$$

Remark that we have used the boundedness of $(\Phi_0(\epsilon)^{-1}\ D(x))^{-1}$. Hence

$$\left(\left(\Phi_0(\epsilon)^{-1}\ D(x)\right)^{-1}\ \langle x\rangle_0^{\beta-s}\right)_{l,0} = O(1) \quad , l \in \Lambda_M \setminus \{0\}\ ,$$

since $s - \beta < r - \beta - \delta$. Therefore we get

$$\left(\langle x\rangle_0^s\ \sigma_0(x)^{-1}\ \left(\Phi_0(\epsilon)^{-1}\ D(x)\right)^{-1}\ \langle x\rangle_0^{\beta-s}\right)_{l,0} = O(1)$$

for $l \in \Lambda_M$. Using once more the boundedness of $D(x)^{-1}$, we obtain

$$\left\|\left(\langle x\rangle_0^{s-\beta}\ \Phi_0(\epsilon)^{-1}\ D(x)\ \sigma_0(x)\ \langle x\rangle_0^{-s}\right)^{-1}\right\| < c\ . \tag{28}$$

Taking into account (26), (16) and (6), we have

$$D(x) = \hat{\eta}_0(x)\ [\eta\sigma\phi](x)\ \hat{\phi}_0(x)^{-1}\ \sigma_0(x)^{-1}\ . \tag{29}$$

Therefore, using

$$\Phi_0(\epsilon)^{-1}\ \hat{\eta}_0(x) = \left(\hat{\eta}_0(x)^*\right)^{-1}\ ,$$

we conclude from (28) that the numerical symbol is elliptic of order β for s.

The converse assertion follows from formula (29) and the definition of the ellipticity for $s = \beta$. ∎

4 Strang-Fix condition and approximation property

First we give a characterization of the Strang-Fix condition. Then we show the approximation properties for the orthogonal projectors onto the trial spaces and for the test projectors Q_m (cf. (7)).
The Strang-Fix condition (cf. [P] or [JL]) is a characterization of the approximation order of the spaces $S_m(\phi)$ generated by a family of functions $(\phi^j)_{j\in\Lambda_M}$. The integer translates of such functions reproduce algebraic polynomials up to a certain degree. Now we give the precise definition.

Definition 4.1 *The functions $\phi := (\phi^j)_{j\in\Lambda_M}$ satisfy the Strang-Fix condition of order $d \in \mathbb{N}_0$ if there exist a vector of trigonometric polynomials $h \in \left(\mathcal{C}^\infty(\mathbb{T})\right)^M$ such that for $\hat{f}_p(x) := \left(\hat{f}(pM + l + x)\right)_{l\in\Lambda_M} := \hat{\phi}_p(x)\ h(x)$ the following conditions are valid:*

$$\begin{aligned} &\hat{f}_0(0) = (\delta_{0,k})_{k\in\Lambda_M}\,, \quad \hat{f}_p(0) = 0 \quad, \ p \in \mathbb{Z}\setminus\{0\}; \\ &\left(\left(\tfrac{d}{dx}\right)^n \hat{f}_p\right)(0) = 0\ , \quad p \in \mathbb{Z}, \quad n = 1, \ldots, d\ . \end{aligned} \tag{30}$$

The following theorem gives a new explicit characterization of the Strang-Fix condition under some mild assumption. The theorem reveals the connection of the Strang-Fix condition and property ii) of Definition 2.3.

Theorem 4.2 *Suppose that $\hat{\phi}_0$ is invertible on $[-\frac{1}{2}, \frac{1}{2}]$ where $\phi := (\phi^j)_{j\in\Lambda_M} \in \mathcal{L}_2^M$ and $(\hat{\phi}^j)_{j\in\Lambda_M} \in \left(\mathcal{C}^{d+1}(\mathcal{U})\right)^M$ for a neighborhood $\mathcal{U}$ of zero. Then the Strang-Fix condition of order $d \in \mathbb{N}$ is valid for ϕ if and only if*

$$\lim_{x\to 0}\left(\hat{\phi}_p(x)\ \hat{\phi}_0(x)^{-1}\ (x)_0^{-d}\right)_{l\in\Lambda_M,0} = \mathbf{0} \tag{31}$$

for any $p \in \mathbb{Z}\setminus\{0\}$.

For the proof of Theorem 4.2 we need two auxiliary lemmas.

Lemma 4.3 *Let* $\phi := (\phi^j)_{j\in\Lambda_M} \in \mathcal{L}_2^M$ *be as in Theorem* 4.2, $h \in (\mathcal{C}^\infty(\mathbb{T}))^M$ *be a vector of trigonometric polynomials and* $\hat{f}_p(x) := \left(\hat{f}(pM+l+x)\right)_{l\in\Lambda_M} := \hat{\phi}_p(x)\ h(x)$. *If* f *fulfils condition* (30) *then*

$$\left(\frac{d}{dx}\right)^n h(0) = \left(\frac{d}{dx}\right)^n \left(\hat{\phi}_0(\cdot)^{-1}\right)_{l\in\Lambda_M,0}(0)$$

for $n = 0, \ldots, d$.

Proof. For $n=0$ we have $\hat{\phi}_0(0)\ h(0) = (\delta_{0,k})_{k\in\Lambda_M}$. Since $\hat{\phi}_0(0)$ is invertible it follows that

$$h(0) = \left(\hat{\phi}_0(0)^{-1}\right)_{l\in\Lambda_M,0} .$$

For $1 \le k \le d$ we have

$$\left(\frac{d}{dx}\right)^k \hat{\phi}_0\ h(0) = 0 = \left(\frac{d}{dx}\right)^k \hat{\phi}_0\ \left(\hat{\phi}_0(\cdot)^{-1}\right)_{l\in\Lambda_M,0}(0) .$$

Therefore we obtain for $0 \le n < d$ by the induction hypothesis

$$\begin{aligned} 0 &= \left(\frac{d}{dx}\right)^{n+1} \left(\hat{\phi}_0\ \left(h - \left(\hat{\phi}_0(\cdot)^{-1}\right)_{l\in\Lambda_M,0}\right)\right)(0) \\ &= \hat{\phi}_0(0)\ \left(\frac{d}{dx}\right)^{n+1} \left(h - \left(\hat{\phi}_0(\cdot)^{-1}\right)_{l\in\Lambda_M,0}\right)(0) . \end{aligned}$$

■

The following lemma is a consequence of Taylor's theorem.

Lemma 4.4 *Let* $f \in \mathcal{C}^d(\mathcal{U})$ *where* $\mathcal{U}$ *is a neighborhood of zero,* $d \in \mathbb{N}_0$. *Then we have* $\lim_{x\to 0} \frac{f(x)}{x^d} = 0$ *if and only if* $\left(\frac{d}{dx}\right)^k f(0) = 0$ *for* $k = 0, \ldots, d$. ■

Proof of Theorem 4.2*:* First we show that (31) is necessary for the Strang-Fix condition. For any $p \in \mathbb{Z}\setminus\{0\}$ we have

$$\begin{aligned} &\lim_{x\to 0} \left(\hat{\phi}_p(x)\ \hat{\phi}_0(x)^{-1}\ (x)_0^{-d}\right)_{l\in\Lambda_M,0} \\ &= \lim_{x\to 0} \hat{\phi}_p(x)\ \left(\left(\hat{\phi}_0(x)^{-1}\right)_{l\in\Lambda_M,0} - h(x)\right)\ x^{-d} + \lim_{x\to 0} \hat{\phi}_p(x)\ h(x)\ x^{-d} = 0 \end{aligned}$$

by the assumption and by Lemmas 4.3 and 4.4.

Now we show the converse. Choose h such that

$$\left(\frac{d}{dx}\right)^n h(0) = \left(\frac{d}{dx}\right)^n \left(\hat{\phi}_0(\cdot)^{-1}\right)_{l\in\Lambda_M,0}(0) \tag{32}$$

for $n = 0, \ldots, d$. Such choice is possible since $\left((2\pi i n)^k\right)_{k,n=0,\ldots,d}$ is a Vandermonde matrix. From (32) we obtain

$$\left(\frac{d}{dx}\right)^n \hat{\phi}_p h(0) = \left(\frac{d}{dx}\right)^n \hat{\phi}_p \left(\hat{\phi}_0(\cdot)^{-1}\right)_{l\in\Lambda_M,0}(0)$$

for $n = 0, \ldots, d$. In case $p \neq 0$ the assertion follows from the assumption and Lemma 4.4. For $p = 0$ the right hand side equals to zero for $n = 1, \ldots, d$ and to $(\delta_{0,k})_{k\in\Lambda_M}$ for $n = 0$. Hence the assertion follows. ∎

Now we turn to the approximation properties. For the remainder of this section we require

Hypothesis A: $(\phi^j)_{j\in\Lambda_M} \in \left(\mathcal{C}_0^{d+\rho}(\mathbb{R})\right)^M, \; d \in \mathbb{N}_0, \; 0 < \rho < 1$, fulfills

- the Strang-Fix condition of order d ;
- $S_m(\phi) \subseteq S_{m+1}(\phi) \;\; for \;\; m \in \mathbb{N}_0$;
- $\hat{\phi}_0$ is invertible on $[-\frac{1}{2}, \frac{1}{2}]$.

Therefore, for the orthogonal projectors $\tilde{P}_m, \;\; m \in \mathbb{N}_0$, onto

$$\tilde{S}_m(\phi) := cl_{L^2(\mathbb{R})} Lin \left\{\phi^j(2^m \cdot -k) \;:\; j \in \Lambda_M, \;\; k \in \mathbb{Z}\right\}$$

we obtain from [JL] or [P] the relation

$$\|(1 - \tilde{P}_m)f\|_{0,\mathbb{R}} \leq c\, 2^{-m(d+1)} \|f\|_{d+1,\mathbb{R}}$$

for $f \in H^{d+1}(\mathbb{R})$. Here $\|\cdot\|_{d+1,\mathbb{R}}$ denotes the Sobolev norm on $\mathbb{R}$. Using a partition of unity we get for the orthogonal projectors P_m onto $S_m(\phi)$

$$\|(1 - P_m)f\|_0 \leq c\, 2^{-m(d+1)} \|f\|_{d+1}, \quad f \in H^{d+1}(\mathbb{T}) . \tag{33}$$

Next we introduce the norm $|||\cdot|||_{d+1}$ defined by $|||f|||_{d+1} := \|f^{(d+1)}\|_0 + |f(0)|$ for $f \in H^{d+1}(\mathbb{T})$. Using the norm equivalence of $\|\cdot\|_{d+1}$ and $|||\cdot|||_{d+1}$ and the fact that the constants are contained in $S_m(\phi)$, we obtain from (33) the relation

$$\|(1 - P_m)f\|_0 \leq c\, 2^{-m(d+1)} \|f^{(d+1)}\|_0, \quad f \in H^{d+1}(\mathbb{T}) . \tag{34}$$

Now we extent the approximation property (34) to other orders of Sobolev spaces. To this end we need the lth forward differences of u defined by

$$(\Delta_h^l\, u)(x) := \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} u(x+jh)$$

for $h \in \mathbb{R}$. As usually $\|\cdot\|_0(\Omega)$ denotes the L^2 norm relative to $\Omega \subseteq \mathbb{R}$. The corresponding lth order modulus of continuity is given by

$$\omega_l(u,t,\Omega) := \sup_{|h|\le t} \|\Delta_h^l\, u\|_0(\Omega_{h,l}) \ ,$$

where

$$\Omega_{h,l} := \{x \in \Omega \ :\ x + jh \in \Omega \ ,\ \ j = 0, \ldots, l\} \ .$$

For $\Omega = \mathbb{T}$ we write $\omega_l(u,t)$ instead of $\omega_l(u,t,\Omega)$. From (34) we get similarly to the proof of Proposition 4.1 in [DK]

$$\|(1-P_m)f\|_0 \le c\, \omega_{d+1}(f, 2^{-m}) \tag{35}$$

for $f \in H^{d+1}(\mathbb{T})$. Repeating the proof of Lemma 5.1 in [DPS] we obtain from (35) that for any $0 \le t \le d+1$,

$$\|(1-P_m)f\|_0 \le c\, 2^{-mt}\, \|f\|_t \ , \quad f \in H^t(\mathbb{T}) \ . \tag{36}$$

Lemma 4.5 *For any $u \in S_m(\phi)$ and $t \ge 0$ we have*

$$\omega_{d+1}(u,t) \le c\ (min\{1, t2^m\})^{d+\rho}\, \|u\|_0 \ .$$

Proof. For $j \in \Lambda_M$ and $h \ge 0$ we get

$$\begin{aligned} \left|\Delta_h^{d+1}\, \phi_{k,m}^j(x)\right| &= \left|\Delta_h^d\, \phi_{k,m}^j(x+h) - \Delta_h^d\, \phi_{k,m}^j(x)\right| \\ &= (2^m\, h)^d\, \left|(\phi_{k,m}^j)^{(d)}(\xi) - (\phi_{k,m}^j)^{(d)}(\theta)\right| \\ &\le c\, (2^m\, h)^d\, |2^m\xi - 2^m\theta|^\rho \le c\, (2^m\, h)^{d+\rho} \ , \end{aligned}$$

where $\xi,\ \theta \in \mathbb{T}$ and $|\xi - \theta| \le c\, h$. The constant does not depend on $x \in \mathbb{T}$. Hence

$$\left\|\Delta_h^{d+1}\, \phi_{k,m}^j\right\|_0 \le c\, (2^m\, h)^{d+\rho}\, 2^{-\frac{m}{2}} \ ,$$

since *supp* ϕ^j is compact. Now let $\left((u_k^j)_{j\in\Lambda_M}\right)_{k\in\Lambda_N} \in \mathbb{C}^{MN}$. Then we have

$$\begin{aligned}\left\|\Delta_h^{d+1}\, u *' \phi\right\|_0 &\leq \sum_{\substack{j\in\Lambda_M\\ k\in\Lambda_N}} |u_k^j| \,\left\|\Delta_h^{d+1}\, \phi_{k,m}^j\right\|_0 \\ &\leq \left(\sum_{\substack{j\in\Lambda_M\\ k\in\Lambda_N}} |u_k^j|^2\right)^{\frac{1}{2}} \left(\sum_{\substack{j\in\Lambda_M\\ k\in\Lambda_N}} \left\|\Delta_h^{d+1}\, \phi_{k,m}^j\right\|_0^2\right)^{\frac{1}{2}} \\ &\leq c\, \|u *' \phi\|_0\, (2^m\, h)^{d+\rho}\end{aligned}$$

where we have used the Riesz stability in the last step. Hence the assertion follows. ∎

Now we introduce the norm $\|\cdot\|_{\phi,t}$ on $\cup_{m\in\mathbb{N}_0} S_m(\phi)$ by

$$\|u\|_{\phi,t}^2 := \|u\|_0^2 + \sum_{m\in\mathbb{N}} 2^{2mt} \|(P_m - P_{m-1})\, u\|_0^2 \,.$$

In view of (35) and Lemma 4.5, Theorem 4.1 of [DK] applies and yields

$$\|u\|_t \doteq \|u\|_{\phi,t}\;, \quad u \in \cup_{m\in\mathbb{N}_0} S_m(\phi)$$

for $0 \leq t < d+\rho$. By (36) the smooth functions are contained in $cl_{\|\cdot\|_{\phi,t}} \cup_{m\in\mathbb{N}_0} S_m(\phi)$. Hence the norm equivalence is valid even on $H^t(\mathbb{T})$. Arguing as in the proof of Theorem 5.1 of [DPS], we obtain

Theorem 4.6 *Let $-d-1 \leq s < d+\rho$, $-d-\rho < t \leq d+1$ and $s \leq t$. Then the Jackson estimate*

$$\|f - P_m f\|_s \leq c\, 2^{m(s-t)}\, \|f\|_t$$

holds for $f \in H^t(\mathbb{T})$ and $m \in \mathbb{N}_0$. Moreover, when $s \leq t < d+\rho$ we have, for any $u_m \in S_m(\phi)$ and $m \in \mathbb{N}_0$, the Bernstein estimate

$$\|u_m\|_t \leq c\, 2^{m(t-s)}\, \|u_m\|_s \,.$$

∎

To prove the approximation property for the projectors Q_m it is necessary to require

Hypothesis B: There exist functions $\gamma := (\gamma^j)_{j\in\Lambda_M} \in \left(\mathcal{C}_0^{d+\rho}(\mathbb{R})\right)^M$, $d \in \mathbb{N}_0$, $0 < \rho < 1$, which fulfil

- the Hypothesis A for ϕ replaced by γ;

- there exist a sequence of $M \times M$ matrices $\omega := (\omega_l)_{l\in\mathbb{Z}}$ with exponential decay such that both $\Pi(x) := \sum_{l\in\mathbb{Z}} \omega_l \, e^{2\pi i l x}$ is invertible on $[0,1]$ and

$$\left(\psi^j\right)_{j\in\Lambda_M} = \omega *'_M \left(\gamma^j\right)_{j\in\Lambda_M} := \sum_{l\in\mathbb{Z}} \omega_l \, \left(\gamma^j(\cdot - l)\right)_{j\in\Lambda_M} . \tag{37}$$

Remark 4.7 *1. If one can choose $\psi = \phi$ then Hypothesis B reduces to Hypothesis A for ϕ. This is the case for the biorthogonal Galerkin method. For the special choice of $(\epsilon_j)_{j\in\Lambda_M}$ in Theorem* 2.8, *ψ has been constructed in Section* 3.3 (*see formula* (24)). *In this case Hypothesis B is valid too, as we will see later.*

2. A simple computation shows that $\hat{\psi}_0$ is invertible on $[-\frac{1}{2}, \frac{1}{2}]$ if and only if this is valid for $\hat{\gamma}_0$ and Π.

Since Π^{-1} is of the form $\sum_{l\in\mathbb{Z}} a_l \, e^{2\pi i l x}$ where $(a_l)_{l\in\mathbb{Z}} \in l^1(\mathbb{Z})^{M\times M}$ we obtain $S_m(\gamma) = S_m(\psi)$ for any $m \in \mathbb{N}_0$. The linear independence of the integer translates of γ ensures (cf. [B-AR]) the existence of dual functionals with compact support as required in the proof of Theorem 5.2 in [DPS]. Therefore the same proof implies

Theorem 4.8 *Let $s,\ s' < d+\rho,\ \ 0 \le s \le t$ and $0 \le s' \le t \le d+1$ where s' is defined by η (cf.* (3)).

Then there exist a constant $c > 0$ independent of $m \in \mathbb{N}_0$ such that

$$\|f - Q_m f\|_s \le c \, 2^{m(s-t)} \, \|f\|_t$$

holds for any $f \in H^t(\mathbb{T})$. ■

Example 4.9 *A straightforward computation shows that, in case of Theorem* 2.8, *the functions ψ defined by formula* (24) *fulfil*

$$\begin{pmatrix} \psi^{-1} \\ \psi^0 \end{pmatrix} = \sum_{l\in\mathbb{Z}} \begin{pmatrix} \omega^r_{2l+1} & \omega^r_{2l} \\ \omega^r_{2l} & \omega^r_{2l-1} \end{pmatrix} \begin{pmatrix} M_r\,(2(\cdot - l - \epsilon_0)) \\ M_r\,(2(\cdot - l - \epsilon_0) + 1) \end{pmatrix},$$

where $(\omega^r_l)_{l\in\mathbb{Z}}$ is defined as in the proof of 2.8 *in Section* 3.3. *Therefore ψ fulfills Hypothesis B by Remark* 4.7.

Remark 4.10 *The assertion of Theorems* 4.6 *and* 4.8 *remains valid when replacing the third property of Hypothesis A by the weaker assumption of linear independence of the integer translates of ϕ resp. γ.*

Reduced Hypothesis H: Choose sufficiently smooth functions $(\gamma^j)_{j\in\Lambda_M}$ with compact support such that

- η and γ fulfil the Poisson summation formula;

- $\sum_{k\in\mathbb{Z}} (\eta^r(\gamma^j(\cdot - k)))_{(r,j)\in\Lambda_M^2}\ e^{2\pi ixk} \neq 0$ for $x \in [0,1]$;
- $\sum_{p\neq 0} \|\langle x\rangle_p^s \hat{\gamma}_p(x)\hat{\gamma}_0(x)^{-1}\langle x\rangle_0^{-s}\|^2 < c$ for a sufficiently large $s \in \mathbb{N}_0$.

The second condition ensures the existence of a sequence of $M \times M$ matrices $(\omega_l)_{l\in\mathbb{Z}}$ with exponential decay such that

$$([\eta\gamma](x))^{-1} = \sum_{l\in\mathbb{Z}} \omega_l\, e^{2\pi ixl} .$$

Now we define ψ by (cf. (37))

$$\left(\psi^j\right)_{j\in\Lambda_M} := \omega *'_M \left(\gamma^j\right)_{j\in\Lambda_M} .$$

Straightforward computations show that if γ satisfies condition ii) of Definition 2.3 and the assumption of Proposition 3.7 then so does ψ. Further we get the second and therewith the complete set of conditions of Hypothesis H in Section 2. Hence, all we have to do is to choose γ with the above properties and Hypothesis A for γ replaced by ϕ. Obviously, the aforementioned functions γ satisfy Hypothesis B.

5 Error estimates

In this section we derive Sobolev norm estimates of the error between the approximate solution u_m (cf. (4)) and the exact solution u^* of the pseudodifferential equation

$$Lu^* = f \tag{38}$$

where $L : H^s(\mathbb{T}) \to H^{s-\beta}(\mathbb{T})$ is as in Section 2, $s \in \mathbb{R}$ is fixed, and $f \in H^t(\mathbb{T})$ with $t \geq s - \beta$ is given. Let us start with the assumptions on ϕ and η. First we consider those ones on η. For η we require that there exist $(\gamma^j)_{j\in\Lambda_M} \in \left(C_0^{d'+\rho'}(\mathbb{R})\right)^M$, $d' \in \mathbb{N}_0$, $0 < \rho' < 1$, $s - \beta < d' + \rho$ satisfying Hypothesis A for γ replaced by ϕ and

- η and γ fulfil the Poisson summation formula;
- property ii) of Definition 2.3 for s and ϕ replaced by $s - \beta$ and γ, respectively;
- $\sum_{k\in\mathbb{Z}} (\eta^r(\gamma^j(\cdot - k)))_{(r,j)\in\Lambda_M^2}\ e^{2\pi ixk} \neq 0$ for $x \in [0,1]$;
- $\sum_{p\neq 0} \|\hat{\gamma}_p(x)\ \hat{\gamma}_0(x)^{-1}\ |x|_0^{-(s-\beta)}\| \leq c$ on $[-\frac{1}{2}, \frac{1}{2}]$ in case $s - \beta \geq 0$ or
  $\sum_{p\neq 0} \|\hat{\eta}_p(x)\ \hat{\eta}_0(x)^{-1}\ |x|_0^{s-\beta}\| \leq c$ on $[-\frac{1}{2}, \frac{1}{2}]$ in case $s - \beta < 0$.

In the previous sections we have stated examples fulfilling the above conditions. In particular, for the choice of collocation points mentioned in Theorem 2.8, these conditions are valid for any $s - \beta \geq 0$.

Now we turn to the assumptions on ϕ. For the trial spaces defined by $\phi := (\phi^j)_{j\in\Lambda_M} \in \left(\mathcal{C}_0^{d+\rho}(\mathbb{R})\right)^M$, $d \in \mathbb{N}_0$, $0 < \rho < 1$, we have to require that $s < d + \rho$, $S_m(\phi) \subseteq S_{m+1}(\phi)$ for $m \in \mathbb{N}_0$ and that ϕ fulfills the Strang-Fix condition of order d. If $d \leq s < d + \rho$ this is a consequence of property ii) of Definition 2.3 and of Theorem 4.2; in this case no additional assumption is needed because we require the numerical method $\{\eta, \phi\}$ to be $s-$admissible for ΨDO's of order β.

Suppose all the aforementioned are satisfied. Then the numerical method is stable for $L : H^s(\mathbb{T}) \to H^{s-\beta}(\mathbb{T})$ if and only if the numerical symbol $[\eta\sigma\phi]$ is elliptic of order β for s (Theorems 2.6 and 2.8). Now we are ready to prove

Theorem 5.1 *Let $-d - 1 \leq s < min\{d + \rho\ ,\ d' + \rho'\}$ and $0 \leq s - \beta < d' + \rho'$. In case of the classical Galerkin method, i.e., if $Q_m = P_m$, it is sufficient to require that $-d - 1 \leq s - \beta < d' + \rho'$. Suppose u^* is the solution of* (38) *and u_m is the approximate solution of the numerical method $\{\eta, \phi\}$ (cf.* (4)*), which is assumed to be stable. If s' is defined by η (cf.* (3)*) and $s' < d+\rho-\beta$, $s' \leq t-\beta \leq d'+1$, $-d-\rho < t \leq d+1$, $t \geq s$, then we have for any $f \in H^t(\mathbb{T})$*

$$\|u^* - u_m\|_s \leq c\, 2^{-m(t-s)}\, \|u^*\|_t\ .$$

Proof. From Theorems 4.6 and 4.8 we obtain

$$\begin{aligned} \|(Q_m L P_m - L)\, u^*\|_{s-\beta} &\leq \|(1 - Q_m) L P_m\, u^*\|_{s-\beta} + \|L(P_m - 1)\, u^*\|_{s-\beta} \\ &\leq c\, 2^{-m(t-s)}\, \|u^*\|_t\ . \end{aligned} \tag{39}$$

Using once more Theorem 4.8, we get

$$\begin{aligned} \|Q_m f - f\|_{s-\beta} &\leq c\, 2^{-m(t-s)}\, \|f\|_{t-\beta} = c\, 2^{-m(t-s)}\, \|L\, u^*\|_{t-\beta} \\ &\leq c\, 2^{-m(t-s)}\, \|u^*\|_t\ . \end{aligned} \tag{40}$$

Moreover, applying (39), (40) and the stability, we conclude that

$$\begin{aligned} \|P_m(u^* - u_m)\|_s &\leq c\, \|Q_m L P_m (u^* - u_m)\|_{s-\beta} \\ &\leq c\, \|Q_m L P_m u^* - L u^*\|_{s-\beta} + c\, \|L u^* - Q_m L P_m u_m\|_{s-\beta} \\ &= c\, \|Q_m L P_m u^* - L u^*\|_{s-\beta} + c\, \|f - Q_m f\|_{s-\beta} \\ &\leq c\, 2^{-m(t-s)}\, \|u^*\|_t\ . \end{aligned}$$

Now the assertion follows from

$$\|u^* - u_m\|_s \le \|(1 - P_m)u^*\|_s + \|P_m(u^* - u_m)\|_s$$

and Theorem 4.6. ∎

The next Theorem gives an error estimate with respect to the norm of $H^{\tilde{s}}(\mathbb{T})$ with $\tilde{s} \le s$ provided $s' \le s - \beta$ and the stability holds for s.

Theorem 5.2 *Let s be as in Theorem 5.1 and suppose, in addition, $s' \le s - \beta$ (cf. (3)). If the numerical method $\{\eta, \phi\}$ is stable for $L : H^s(\mathbb{T}) \to H^{s-\beta}(\mathbb{T})$, then for any $f \in H^t(\mathbb{T})$ with $s' \le t - \beta \le d' + 1$, $-d - \rho < t \le d + 1$, $t \ge s$ we have*

$$\|u^* - u_m\|_{\tilde{s}} \le c\, 2^{-m(t-\tilde{s})}\, \|u^*\|_t \,, \quad max\{-d-1\,,\ \beta\} \le \tilde{s} \le s \,,$$

where u_m is the approximate solution defined by (4) and u^* is the exact solution of (38). For the classical Galerkin method, i.e., if $Q_m = P_m$, one has to require $max\{-d-1\,,\ -d-1-\beta\} \le \tilde{s} \le t$ instead of $max\{-d-1\,,\ \beta\} \le \tilde{s} \le s$.

We skip the proof, because it is the same as the second part of the proof to Theorem 6.3 in [DPS]. ∎

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