Martingale problem for (ξ, Φ, k)-superprocesses

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Abstract. The martingale problem for superprocesses with parameters (ξ, Φ, k) is studied where $k(ds)$ may not be absolutely continuous with respect to the Lebesgue measure. This requires a *generalization* of the concept of martingale problem: we show that for any process X which *partially* solves the martingale problem, an extended form of the liftings defined in [11] exists; these liftings are part of the statement of the *full martingale problem*, which is hence not defined for processes X who fail to solve the *partial martingale problem*. The existence of a solution to the martingale problem follows essentially from Itô's formula. The proof of uniqueness requires that we find a sequence of (ξ, Φ, k^n)-superprocesses "approximating" the (ξ, Φ, k)-superprocess, where $k^n(ds)$ has the form $\lambda^n(s, \xi_s)ds$. Using an argument in [12], applied to the (ξ, Φ, k^n)-superprocesses, we derive, passing to the limit, that the *full martingale problem* has a unique solution. This result is apply to construct superprocesses with interactions via a Dawson-Girsanov transformation.

1. Introduction

While the characterization of superprocesses by *evolution equations* has had a spectacular development in the recent years, the characterization of these processes by martingale problems was stopped by difficulties arising when considering (ξ, Φ, k)-superprocess with k a non-absolutely continuous additive functional. The purpose of this paper is to generalize [12] to the case where k cannot be written as $k(ds) = \lambda(s, \xi_s)ds$.

The difficulties first come from the fact that it is not possible to get, in the case of a general k, the classical form of the $(\mathrm{A}, \mathcal{D}(\mathrm{A}))$-martingale problems, where A is an operator with domain $\mathcal{D}(\mathrm{A})$. The statement of the martingale problem itself is problematic. It requires to identify (see Theorem 4) additive functionals K of X, corresponding to additive functionals k of the motion process ξ (the lifting K of k). But in our context, unlike in [10] or [11], X is not, in general, a Markov process and we must find new methods to determine and manipulate them.

Furthermore, the simplest way to prove uniqueness of the solution to martingale problems for superprocesses (cf. [12, p. 254] or [4, p. 112]), involves the derivation of the log-Laplace functional v of X. But this is unapplicable here since v solves an evolution equation of the form

$$v_{r,t}(f)(x) - \pi_{r,x} f(\xi_t) - \pi_{r,x} \int_r^t \Phi(s, \xi_s, v_{s,t}(f)(\xi_s)) k(ds)$$

where k cannot be written as $k(ds) = \lambda(s, \xi_s)ds$.

Finally martingale problems can used (see §6) to construct superprocesses with interactions. This is one of the advantages of the *martingale problem characterization* over the *evolution equation characterization*. Here the interaction is given by an additionnal term $\Re$, and the process is called the $(\xi, \Phi, k, \Re)$-*superprocess with interactions*. This process is characterized as the unique solution of a martingale problem obtained by a *Dawson-Girsanov transformation* of the (ξ, Φ, k)-full martingale problem (cf. [3] and [4, Th. 7.2.2]). Here, the difficulty is essentially to *properly state* the martingale problem: Dawson's argument completely extends to our more general context. This difficulty comes from the fact that the statement of this martingale problem involves additive functionals which cannot be explicitly described. But unlike in the case of (ξ, Φ, k)-superprocesses, a $(\xi, \Phi, k, \Re)$-superprocess with interactions *does not* solve a partial martingale problem (cf. §§1.2 and 1). The problem is overpassed by defining these additive functionals as the liftings of the (ξ, Φ, k)-superprocesses. This involves the choice of a version of these liftings, but it turns out that the (unique) solution $P_{r,\mu}^{(\xi,\Phi,k,\Re)}$ of the $(\xi, \Phi, k, \Re)$-martingale problem does not depend on this choice, since it is absolutely continuous with respect to the distribution $P_{r,\mu}^{(\xi,\Phi,k)}$ of the (ξ, Φ, k)-superprocess.

1.1. Historical background . A measure valued process X whose log-Laplace functional $v_{r,t}(f)(x)$ solves the integral equation

$$v_{r,t}(x) = \pi_{r,x} f(\xi_t) - \pi_{r,x} \int_r^t \psi(s, \xi_s, v_{s,t}) k(ds)$$

where $\xi = (\xi_t, \Im, \pi_{r,x})$ is a Markov process, $k(ds)$ is an additive functional of ξ and ψ is an operator, is called a (ξ, ψ, k)-superprocess. In view of the advantages and the intrinsic interest in characterizing superprocesses by a martingale problem, Roelly-Coppoletta [23] posed and solved in 1986 the martingale problem for the $(\xi, (.)^2, ds)$-superprocesses where ξ is a Feller process. In 1987, El-Karoui and Roelly-Coppoletta extended the result to a large class of (ξ, ψ, ds)-superprocesses where ξ is a Feller process. In 1988, Fitzsimmons obtained some results on the martingale problem for the (ξ, ψ, ds)-superprocesses (where ξ is a right process) and in particular he showed that interesting properties can be derived from a well posed martingale problem. Multitype superprocesses were characterized by martingale problems in 1990 by Gorostiza and Lopez-Mimbela [16]. In 1992, Fitzsimmons solved the martingale problem for the (ξ, Φ, ds)-superprocesses for ξ a right process and

$$\Phi(x, \lambda) = b(x)\lambda^2 + \int_0^\infty \left(e^{-\lambda u} - 1 + \lambda u\right) n(x, du)$$

where the measure $n(x, du)$ satisfies some properties. No results until recently were available when the branching additive functional rate $k(ds)$ is not the Lebesgue measure. Difficulties are inherent even in the statement of a martingale problem for superprocesses with branching rates $k(ds)$ which are not absolutely continuous with respect to the Lebesgue measure. In 1994, Dawson and Fleischmann showed in [5] that the one point catalytic super Brownian motion, that is the $(\xi, (.)^2, L^c)$-superprocesses (where L_t^c is the local time of the Brownian motion ξ at time t), solves a martingale problem related to the density of the occupation time process.

In fact, in order to be able to state the martingale problem for the (ξ, Φ, k)-superprocesses, we first need to extend the notion of lifting and projection introduced in [11] to the case where X may not be a Markov process. Let $(E, \mathcal{B})$ be a metrizable Luzin space. Given an $\mathcal{M}_{\mathrm{f}}(\mathcal{B})$-valued Hunt process $X = (X_t, \Im, P_{r,\mu})$ and a E-valued Hunt process ξ, Dynkin, Kuznetsov and Skorohod defined the lifting $A(ds)$ of an additive functional $a(ds)$ of ξ as an additive functional $A(ds)$ of X such that for every $r, t \in R_+$, for every $\mu \in \mathcal{M}_{\mathrm{f}}(\mathcal{B})$ and for every bounded non-negative measurable $\varphi(.)$

$$P_{r,\mu} A(r,t] = \int_E \mu(dx) P_{r,\delta_x} A(r,t] \tag{1}$$

and more generally

$$P_{r,\mu} \int_r^\infty \varphi(s) A(ds) = \pi_{r,\mu} \int_r^\infty \varphi(s) a(ds). \tag{2}$$

If $A(ds)$ is a lifting of $a(ds)$, then $a(ds)$ is said to be the projection of $A(ds)$. And in fact, given a *linear* additive functional $A(ds)$ of X, that is an additive functional such that (1) is verified, one can find an additive functional $a(ds)$ of ξ which is the projection of A. The authors proved that the lifting-projection relation establishes a one to one correspondence between the additive functionals of ξ and the linear additive functionals of X. Their proof makes use of the Markov property of X. For our purposes, it was necessary to reduce that condition to the assumption that a certain *partial-martingale problem* is verified.

In fact we also obtained a criteria for the convergence of the liftings of a convergent sequence of additive functionals which may have some independent interest. The lifting's existence allowed us to define a process, $t \mapsto M_t$, playing the role of the martingale problem statement that "for every $r > 0$ and every $\mu \in \mathcal{M}_{\mathrm{f}}(\mathcal{B})$ there exists only one distribution, $P_{r,\mu}$, on the space of càdlàg trajectories in $\mathcal{M}_{\mathrm{f}}(\mathcal{B})$, such that $P_{r,\mu}(X_r = \mu) = 1$ and M_t is a $P_{r,\mu}$-martingale for $t \geq r$ ".

The proof relies on a sequence of superprocesses that we can construct to "approximate" (in a sense specified below) our given superprocess. The approximating superprocesses, X^n, have the property that their branching additive functional rates $k^n(ds)$ are absolutely continuous with respect to the Lebesgue measure: $k^n(ds) = \lambda^n(s, \xi_s)ds$. But then, El Karoui and Roelly-Coppoletta's proof of uniqueness can be decomposed into different steps making sense in terms of the approximating superprocesses, and finally passing to the limit, their proof finds an expression in our more general context.

1.2. Partial and full martingale problem. In general, a martingale problem can be formulated in the following way: first, to any (canonical càdlàg) process X, a real valued process $t \mapsto (M_G^r)_t$, $t \geq r$, is defined up to $P_{r,\mu}$-indistinguishability, for every function G in a certain set S. A càdlàg process $X = (X_t, \Im, P_{r,\mu})$ is said to be a *solution to the martingale problem* if the processes $(M_G^r)_t$ are $P_{r,\mu}$-martingales for every G in S. The martingale problem $((M_G^r), S)$ is said to be *well posed* if there exists one and only one solution to the martingale problem.

We see a well posed martingale problem as a "test" which characterizes a process. Pick a process $X = (X_t, \Im, P_{r,\mu})$. The test goes like this:

- For every $G \in S$, check if the process $t \mapsto (M_G^r)_t$ is a $P_{r,\mu}$-martingale.

If the test is a success, X is the only solution to the $((M_G^r), S)$ martingale problem. In the test, the order in which the processes $t \mapsto (M_G^r)_t$ (for $G \in S$) are tested has no importance. We introduce now a slight modification to this procedure. Let $S = S_1 \cup S_2$ where S_1 and S_2 are two disjoint sets. Our new "test" is the following:

- Test whether or not X is a solution to the $((M_G^r), S_1)$-martingale problem
- If the test has not failed continue, otherwise, stop.
- Test whether or not X is a solution to the $((M_G^r), S_2)$-martingale problem

The non well posed martingale problem $((M_G^r), S_1)$ is called the partial martingale problem. A solution to the partial martingale problem is called a solution to the full martingale problem if it is a solution to the $((M_G^r), S_2)$-martingale problem.

In this paper, partial martingale problems are used to determine certain processes -in terms of the solutions to the partial martingale problem- which enter into the *statement* of the full martingale problem; the statement of the full martingale problem is not well defined for process $X = (X_t, \Im, P_{r,\mu})$ which are *not* solutions to the partial martingale problem.

1.3. Assumptions and basic elements . We fix a constant $T > 0$ and consider our processes only during the time interval $[0, T]$. Throughout this paper we assume, *unless specifically mentioned*, that

1.3A $\xi = (\xi_t, \mathcal{F}, \pi_{r,x})$, is a (time homogeneous) Feller processes living in a locally compact separable metric space (E, d).

We denote by $\mathcal{B}$ the σ-algebra generated by d; given a family F of measurable functions, we denote by bF the bounded members of F and by pF the non negative $f \in F$. $\hat{C}(E)$ denotes the set of continuous functions vanishing at infinity. We denote by S_t the semigroup of ξ.We often make use of time inhomogeneous *notation* and in particular:

$$S_t^r(f)(x) := \pi_{r,x} f(\xi_t) := \pi_x f(\xi_{t-r}) = S_{t-r}(f)(x).$$

We denote by $\mathcal{L}$ be the set of bounded measurable functions f such that $S_t(f)(x)$ is *strongly* continuous, that is $\|S_t(f)(.) - S_{t+h}(f)(.)\|_\infty \to 0$ as $h \to 0$. Obviously, for a Feller process ξ, $\hat{C}(E) \subset \mathcal{L}$. We denote by $(\mathrm{A}, \mathcal{D}(\mathrm{A}))$ the infinitesimal (strong) generator of ξ.

In addition to a motion process, we need a *branching mechanism*:

1.3B $b(x)$ and $\ell(x, d\mu)$ are respectively a measurable function and a kernel satisfying the conditions:

$$0 \le b(x) \le 1,\ 0 \le \int_0^\infty u \vee u^2 \ell(x, du) \le 1. \tag{3}$$

Throughout this paper we pose

$$\Phi(x, f(x)) = \frac{1}{2} b(x) f^2(x) + \int_0^\infty \mathcal{E}\,(uf)\,\ell(x, du) \tag{4}$$

where $\mathcal{E}(z) = e^{-z} + z - 1$. We call Φ a *branching mechanisms.* We use the notation $\Phi(x, f) := \Phi(x, f(x))$. In the same spirit as [12], we assume that for every $\varphi(x) \in \mathcal{D}(\mathrm{A})$, $\Phi(x, \varphi(x)) \in \mathcal{L}$. Moreover, we want that Φ be a *regular branching mechanisms*, that is, $t \mapsto \Phi(w_t, \varphi_t(w_t))$ is càdlàg when $t \mapsto w_t$ and $t \mapsto \varphi_t(w_t)$ are càdlàg trajectories.

Concerning the *branching rate*, we require:

1.3C $k(ds)$ is a continuous non negative additive functional of ξ satisfying the condition

$$h_t^r(x) := \pi_{r,x} k(r, t) \to 0 \text{ uniformly in } x \text{ as } t - r \to 0\ . \tag{5}$$

(Note that, since we consider only our processes during the time interval $[0, T]$, this is equivalent to the "admissibility condition" in [10] according to [10, Lemma 3.3.1]. (Such additive functionals are called *admissible additive functionals*). We assume that $h_t^r(.) \in \mathcal{L}$ for every r, t.

All measure valued processes considered in this paper will be canonical càdlàg processes and the triple $X = (X_t, \Im, P_{r,\mu})$ will always denote such processes:

- for $r \geq t$, X_t denotes the mapping from the space $D_{[r,\infty)}(\mathcal{M}_{\mathrm{f}})$ to the space $\mathcal{M}_{\mathrm{f}}$, which is defined by
$$X_t(\omega) = \omega_t, \text{ for } \omega \in D_{[r,\infty)}(\mathcal{M}_{\mathrm{f}})$$
- $P_{r,\mu}$ denotes a distribution on $D_{[r,\infty)}(\mathcal{M}_{\mathrm{f}})$; we *always* assume that $P_{r,\mu}(X_r = \mu) = 1$.
- $\Im$ denotes the collection of filtrations $\{\Im_t^r\}_{t\in[r,\infty)}$ defined by
$$\Im_t^r = \bigcap_{\varepsilon>0} \sigma\left(X_s : r \leq s \leq t+\varepsilon\right)^{P_{r,\mu}}$$
where the superscript ${}^{P_{r,\mu}}$ denotes the completion with respect to $P_{r,\mu}$.

1.4. Statement of the martingale problem.

Definition 1 [partial-martingale problem for ξ]. *A process $X = (X_t, \Im, P_{r,\mu})$ will be said to be a solution to the* partial martingale problem for ξ *if for every $\varphi \in \mathcal{D}(\mathrm{A})$*

$$t \mapsto \langle X_t, \varphi\rangle - \langle X_r, \varphi\rangle - \int_r^t \langle X_s, \mathrm{A}\varphi\rangle\, ds \tag{6}$$

is a $P_{r,\mu}$-martingale for $t \in [\,r, T]$.

The full martingale problem requires for its statement the notion of a lifting of an additive functional:

Definition 2 [Lifting]. *Let $X = (X_t, \Im, P_{r,\mu})$ be a (canonical càdlàg $\mathcal{M}_{\mathrm{f}}$-valued) process and let $a(ds)$ be an additive functional of ξ. A natural right continuous additive functional $A(ds)$ of X will be called a* lifting *of $a(ds)$ if for every $t \geq r$, the process*

$$s \mapsto A(r, s] + \langle X_s, \pi_{s,\cdot}a(s,t]\rangle$$

is a $P_{r,\mu}$-martingale for $s \in [r, t]$.

The following Proposition (which will be proved in a further section) guaranties the existence and uniqueness of liftings for every solution $X = (X_t, \Im, P_{r,\mu})$ to the partial martingale problem.

Proposition 3 [liftings existence and uniqueness]. *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem for ξ. Then for every additive functional $a(ds)$ of ξ satisfying (5), there exists a unique lifting $A(ds)$ of X. Moreover, it is a continuous additive functional.*

Notation 1. Let f be a progressively measurable bounded function and let Φ be a branching mechanism. Then the additive functional $\Phi(\xi_s, f(s, \xi_s))k(ds)$ satisfies (5), and we will denote by $K^{\Phi(f)dk}(ds)$ the lifting of $\Phi(\xi_s, f(s, \xi_s))k(ds)$.

The next result characterizes $(\xi, \Phi, k)-$superprocesses in terms of a martingale problems. It is the main result of this paper.

Theorem 4 [martingale problem]. *Let $r \in R_+$ and $\mu \in \mathcal{M}_{\mathrm{f}}$; let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem for ξ, and let $P_{r,\mu}^{(\xi,\Phi,k)}$ be the distribution of the $(\xi, \Phi, k)-$superprocess. The processes*

$$\begin{aligned} &t \mapsto \exp\left(-\langle X_t, \varphi\rangle\right) + \textstyle\int_r^t \exp\left(-\langle X_s, \varphi\rangle\right)\langle X_s, \mathrm{A}\varphi\rangle\, ds \\ &- \textstyle\int_r^t \exp\left(-\langle X_s, \varphi\rangle\right) K^{\Phi(\varphi)dk}(ds) \end{aligned} \tag{7}$$

are $P_{r,\mu}$ -martingales for every $\varphi \in \mathcal{D}(\mathrm{A})$, if and only if $P_{r,\mu} = P_{r,\mu}^{(\xi,\Phi,k)}$.

Definition 5. *A solution $X = (X_t, \Im, P_{r,\mu})$ to the partial martingale for ξ which is such that (7) is a $P_{r,\mu}$-martingale will be called a solution to the* **full martingale problem for** (ξ, Φ, k).

Remark 1. In this work, we will consider only the partial martingale problem for ξ, and only full martingale problem for (ξ, Φ, k). Thus we refer to them simply as the **partial martingale problem** and the **full martingale problem**.

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2. The full martingale problem: existence of a solution

In this section we prove the existence part of Theorem 4, that is, we show that the distribution $P_{r,\mu}^{(\xi,\Phi,k)}$ of the $(\xi, \Phi, k)-$superprocess is a solution to the full martingale problem. The proof is based on the following known results.

2.1. Some known results.

Theorem 6. *Let X be the superprocess with parameters (ξ, Φ, k). Then X is a Hunt process, the lifting of every natural additive functional with finite characteristic exists*[1] *and the lifting of $\ell(\xi_s, du)k(ds)$ is the modified Lévy measure $L(ds, d\mu)$ of X. In particular for every bounded measurable real valued function f we have*

$$P_{r,\mu} \int_r^t \int_{\mathcal{M}_{\mathrm{f}}} f\left(\langle \mu, \varphi\rangle\right) L(ds, d\mu) = \pi_{r,\mu} \int_r^t \int_0^\infty f\left(u\varphi(\xi_s)\right) \ell(\xi_s, du) k(ds). \tag{8}$$

The following moment formulae are satisfied:

[1] The notion of *lifting of additive functionals* is due to [11]. For the definition see section 1.1.

(i) $P_{r,\mu} \langle X_t, f \rangle = \pi_{r,\mu} f(\xi_t)$
(ii) $P_{r,\mu} \langle X_{t_1}, f_1 \rangle \langle X_{t_2}, f_2 \rangle \;=\; \pi_{r,\mu} f_1(\xi_{t_1}) \pi_{r,\mu} f_2(\xi_{t_2}) + \pi_{r,\mu} \int_r^t \pi_{s,\xi_s} f_1(\xi_{t_1}) \pi_{s,\xi_s} f_2(\xi_{t_2}) k^{(2)}(ds)$
where $k^{(2)}(ds) = \left(b(\xi_s) + \int_0^\infty u^2 \ell(\xi_s, du) \right) k(ds)$.

Proof. The E-valued process ξ is a Hunt process. Therefore the superprocess X is also a Hunt process (see [19, Th. 6.32]). The existence and uniqueness of liftings is due to [11] (see [10, p. 83]). The fact that the modified Lévy measure $L(ds, d\mu)$ is the lifting of $\ell(ds, d\mu) := \ell(\xi_s, du) k(ds)$ is also due to [11] (See [10, Th. 6.1.1 and Sect. 6.8.1]). The formula (8) follows from the definition of the lifting of a measure valued additive functional, see [10, equation 6.2.13a]. The moment formulae were established in [9, p.1163]. ■

2.2. Proof of the existence of a solution to the martingale problem. Let $P_{r,\mu}^{(\xi,\Phi,k)}$ be the distribution of the (ξ, Φ, k)-superprocess. Clearly, (6) is a $P_{r,\mu}^{(\xi,\Phi,k)}$-martingale for every $\varphi \in \mathcal{D}(\mathrm{A})$. Existence and uniqueness of liftings is given from Theorem 6. Let $C_t(\varphi)$ be the quadratic variation of the continuous martingale part of the semimartingale $\langle X_t, \varphi \rangle$. Then Itô's formula implies that

$$\begin{aligned} &t \mapsto \left(\langle X_t, \varphi \rangle \right)^2 - \left(\langle X_r, \varphi \rangle \right)^2 - 2 \int_r^t \langle X_s, \varphi \rangle \langle X_s, \mathrm{A}\varphi \rangle \, ds - C_t(\varphi) \\ &+ \sum_{r < s \le t} \Big(\left(\langle X_{s-} + \Delta X_s, \varphi \rangle \right)^2 - \left(\langle X_{s-}, \varphi \rangle \right)^2 - 2 \langle X_{s-}, \varphi \rangle \langle \Delta X_s, \varphi \rangle \Big) \end{aligned}$$

is a $P_{r,\mu}$ -martingale for every $\varphi \in \mathcal{D}(\mathrm{A})$. Simplifying we obtain that

$$\begin{aligned} &t \mapsto \left(\langle X_t, \varphi \rangle \right)^2 - \left(\langle X_r, \varphi \rangle \right)^2 - 2 \int_r^t \langle X_s, \varphi \rangle \langle X_s, \mathrm{A}\varphi \rangle \, ds \\ &- C_t(\varphi) + \sum_{r < s \le t} \left(\langle \Delta X_s, \varphi \rangle \right)^2 \end{aligned}$$

is a $P_{r,\mu}$ -martingale for every $\varphi \in \mathcal{D}(\mathrm{A})$. By definition of the modified Lévy measure, this is the same thing as saying that

$$\begin{aligned} &t \mapsto \left(\langle X_t, \varphi \rangle \right)^2 - \left(\langle X_r, \varphi \rangle \right)^2 - 2 \int_r^t \langle X_s, \varphi \rangle \langle X_s, \mathrm{A}\varphi \rangle \, ds \\ &- C_t(\varphi) + \int_r^t \int_{\mathcal{M}_\mathrm{f}} \langle \mu, \varphi \rangle^2 L(ds, d\mu) \end{aligned} \tag{9}$$

is a $P_{r,\mu}$ -martingale for every $\varphi \in \mathcal{D}(\mathrm{A})$, where $\int_{\mathcal{M}_\mathrm{f}} \langle \mu, \varphi \rangle^2 L(ds, d\mu)$ is the lifting of $\int_0^\infty (u\varphi)^2 (\xi_s) \ell(\xi_s, du) k(ds)$. Note that (by definition of lifting)

$$P_{r,\mu} \int_r^t \int_{\mathcal{M}_\mathrm{f}} \langle \mu, \varphi \rangle^2 L(ds, d\mu) = \pi_{r,\mu} \int_r^t \int_0^\infty (u\varphi)^2 (\xi_s) \ell(\xi_s, du) k(ds). \tag{10}$$

Since the $P_{r,\mu}$-expectation of martingale (9) is zero, we can use (10) and the moment formulae of Theorem 6 to calculate

$$P_{r,\mu} \left(C_t(\varphi) \right) = \pi_{r,\mu} \int_0^t b(\xi_s) \varphi^2(\xi_s) k(ds).$$

Thus

$$P_{r,\mu} \left(\frac{1}{2} C_t(\varphi) - \int_0^t \hat{Q}(\varphi)(ds) \right) = 0$$

where $\hat{Q}(\varphi)(ds)$ is the lifting of the additive functional $\frac{1}{2}b(\xi_s)\varphi^2(\xi_s)k(ds)$. Therefore, since X_t is a Markov process, this implies that

$$t \mapsto \frac{1}{2}C_t(\varphi) - \int_0^t \hat{Q}(\varphi)(ds)$$

is a martingale. But because $t \mapsto \frac{1}{2}C_t(\varphi) - \int_0^t \hat{Q}(\varphi)(ds)$ is also a right continuous predictable process of integrable variation, we obtain that $\frac{1}{2}C_t(\varphi) \equiv \int_0^t \hat{Q}(\varphi)(ds)$. We can apply Itô's formula which gives that

$$\begin{array}{l} t \mapsto \exp\left(-\langle X_t, \varphi\rangle\right) + \int_r^t \exp\left(-\langle X_s, \varphi\rangle\right)\langle X_s, \mathrm{A}\varphi\rangle\, ds \\ - \int_r^t \exp\left(-\langle X_s, \varphi\rangle\right)\hat{Q}(\varphi)(ds) - \int_r^t \exp\left(-\langle X_s, \varphi\rangle\right)\int_{\mathcal{M}_f} \mathcal{E}\left(-\langle \mu, \varphi\rangle\right) L(ds, d\mu) \end{array}$$

is a $P_{r,\mu}^{(\xi,\Phi,k)}$-martingale. But since $\hat{Q}(\varphi)(ds)$ is the lifting of $\frac{1}{2}b(\xi_s)\varphi^2(\xi_s)k(ds)$ and $\int_{\mathcal{M}_f} \exp\left(-\langle \mu, \varphi\rangle\right) L(ds, d\mu)$ is the lifting of $\int_0^\infty \mathcal{E}\left(u\varphi(\xi_s)\right)\ell(\xi_s, du)k(ds)$ this can be rewritten to give that

$$\begin{array}{l} t \mapsto \exp\left(-\langle X_t, \varphi\rangle\right) + \int_r^t \exp\left(-\langle X_s, \varphi\rangle\right)\langle X_s, \mathrm{A}\varphi\rangle\, ds \\ - \int_r^t \exp\left(-\langle X_s, \varphi\rangle\right) K^{\Phi(\varphi)dk}(ds) \end{array}$$

is a $P_{r,\mu}^{(\xi,\Phi,k)}$-martingale. ■

3. Approximation of superprocesses

As explained in §1.1, in order to prove that the full martingale problem has only one solution, we need to *approximate* superprocesses by other superprocesses with branching rate of the form $k^n(ds) = \lambda^n(s, \xi_s)ds$. This is done in Theorem 14 below which may have some independent interest. But before, some technical results are needed.

3.1. Some technical lemmas.

Lemma 7. *Let $(\Omega, \Im, P)$ be a filtered probability space. Suppose that $t \mapsto x_t^n$ is a sequence of right continuous processes such that*

$$\sup_{\tau \in [r,t]} P(|x_\tau^n|) \to 0 \tag{11}$$

where $\sup_{\tau \in [r,t]}$ indicates here the supremum over all stopping times τ such that $r \le \tau \le t$. Then

$$\sup_{s \in [r,t]} |x_s^n| \to 0 \text{ in } P\text{-probability}$$

Proof. Let $\eta > 0$. Let $\tau_\eta^n := \inf\{s \in [r,t] : |x_s^n| > \eta\}$, where $\inf \phi := t$. Then we have

$$\begin{array}{rcl} P\{\sup_{s\in[r,t]} |x_s^n| > \eta\} & \le & P\{\left|x_{\tau_\eta^n}^n\right| \ge \eta\} \\ & \le & \frac{1}{\eta}P(\left|x_{\tau_\eta^n}^n\right|) \end{array}$$

and this converges to zero according to (11). ■

Lemma 8. *Let $(\Omega, \Re, P)$ be a probability space and $a_n(ds)$ be a sequence of random measures on $\mathbf{R}_+$. If $a(ds)$ is such that $P\,|a_n([0,t]) - a([0,t])| \to 0$ for every $t \geq 0$, then there exists a subsequence a_{n_k} such that P-a.s. $a_{n_k} \Longrightarrow a$.*

Proof. With the use of Cantor's diagonalization method one finds a subsequence a_{n_k} such that $P\{|a_{n_k}[0,q] - a[0,q]| \to 0 \text{ for every rational } q \geq 0\} = 1$. But then, because the mappings $t \longmapsto a_n[0,t]$ are increasing, this implies that P-a.s. $a_{n_k} \Longrightarrow a$. ■

Lemma 9. *Let $k(ds)$ be any additive functional of an arbitrary right process $\xi = (\xi_t, \Im, \pi_{r,x})$. Let S_t^r be the time inhomogeneous semigroup generated by ξ. For every $0 \leq r \leq s \leq t$ we have that $S_s^r(h_t^s)(x) \leq h_t^r(x)$ and*

$$|S_s^r(h_t^s)(x) - h_t^r(x)| = h_s^r(x)$$

where $h_t^s(x) = \pi_{s,x} k(s,t)$.

Proof.

$$\begin{aligned} S_s^r(h_t^s)(x) &= \pi_{r,x}\left(\pi_{s,\xi_s} k(s,t]\right) \\ &= \pi_{r,x}\left(k(s,t]\right) \\ &= \pi_{r,x}\left(k(r,t]\right) - \pi_{r,x}\left(k(r,s]\right) \\ &= h_t^r(x) - h_s^r(x). \end{aligned}$$

■

3.2. A-smooth approximation of superprocesses. In §3.2 we introduce the concept of A-*smooth approximation of superprocesses*. The main result of §3.2 is Theorem 14 below, which states that, under the assumptions 1.3A-1.3C, an A-smooth approximation exists.

Definition 10. *A sequence $k^n(ds)$ of additive functionals of ξ is said to be* **uniformly admissible** *if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $s, t \in [0,T]$, $|s-t| < \delta$ implies that*

$$\sup_n \|{}^n h_t^s\|_\infty < \varepsilon$$

where ${}^n h_t^s(x) = \pi_{s,x} k^n(s,t)$.

Definition 11. *We say that a mapping $\psi : [0,T] \times E \to R_+$ is smooth for the strong generator $(\mathrm{A}, \mathcal{D}(\mathrm{A}))$ of ξ, or simply that ψ is* **smooth for** A*, if*

1) *$\psi(s,.)$ belongs to $\mathcal{D}(\mathrm{A})$ for every s*
2) *$\frac{\partial}{\partial s}\psi(s,x)$ exists for every s and $\left\|\frac{\psi(s+h,.)-\psi(s,.)}{h} - \frac{\partial}{\partial s}\psi(s,.)\right\|_\infty \to 0$*
3) *ψ, $\frac{\partial}{\partial s}\psi$ and $\mathrm{A}\psi$ are bounded and strongly continuous*

Definition 12. *We will say that a sequence of superprocesses X^n with parameters (ξ, Φ, k^n) is an* **A-smooth approximating sequence** *for the superprocess X with parameters (X, Φ, k), if:*

- *$k^n(ds)$ has the form $\lambda^n(s, \xi_s)ds$*

- *the log-Laplace functional* v^n *of* X^n *converges to the log-Laplace functional* v *of* X.
- *for every* $f \in \mathcal{D}(\mathrm{A})$, *the function* $\psi^n(s,x) := v^n_{s,T}(f)(x)$ *is smooth for* A,
- *for every* $f \in \mathcal{D}(\mathrm{A})$, $\mathrm{A}v^n_{s,T}(f)(x) + \frac{\partial}{\partial s}v^n_{s,T}(f)(x) \equiv \Phi(x, v^n_{s,T}(f)(x))\lambda^n(s,x)$.

The proof of existence of an A-smooth approximation, relies on the following Proposition where we show that additive functionals k can be approximated by additive functionals k^n of the form $k^n(ds) = \lambda^n(s,\xi_s)ds$.

Proposition 13. *Let* $k(ds)$ *be a (continuous) admissible additive functional of a right process* ξ. *There exists a sequence of additive functionals* $k^n(ds)$ *of the form*

$$k^n(ds) = \lambda^n(s,\xi_s)ds$$

such that

(i) $\sup_{0\le s<t\le T}\sup_{x\in E}|{}^n h^s_t(x) - h^s_t(x)|$ *converges to zero as* n *tends to infinity, where* ${}^n h^s_t(x) = \pi_{s,x}k^n(s,t)$ *and* $h^s_t(x) = \pi_{s,x}k(s,t)$;

(ii) *the sequence* $k^n(ds)$ *is uniformly admissible;*

(iii) $k^n(r,\tau]$ *converges to* $k(r,\tau]$ *in* $L^1(\pi_{r,x})$ *for every* r*-stopping time* τ *and every* r,x;

(iv) *for every* r,x *there exists a subsequence* $\{k^{n_k}(ds)\}_{k=1}^{\infty}$ *converging weakly to* $k(ds)$.

Proof. Let $t^n_i := \frac{i}{n}T$; Choose $\frac{1}{n}T > \delta_n > 0$ such that for every $\alpha \le \beta$ such that $|\alpha - \beta| \le \delta_n$ we have

$$\left\|h^\alpha_\beta\right\|_\infty \le \frac{1}{n^2}.$$

Let us denote by pC^∞_c the set of all infinitely differentiable non-negative functions $f : R_+ \to R_+$ with a compact support. We denote by $supp\{f\}$ the support of a function $f \in pC^\infty_c(R_+)$. Choose a function f^n_i in $pC^\infty_c(R_+)$ such that

1) $supp\{f^n_i\} \subset [t^n_i, t^n_i + \delta_n]$

2) $\int f^n_i(s)ds = 1$

3) (for simplicity) $f^n_i(s)$ is a translation of $f^n_j(s)$.

Let

$$k^n(ds) := \sum_{i=0}^{n-1} h^s_{t^n_{i+1}}(\xi_s) f^n_i(s)ds$$

and

$${}^n h^s_t(x) := \pi_{s,x}k(s,t].$$

Note that

$${}^n h^{t^n_j}_T(x) \quad = \quad \textstyle\sum_{i=j}^{n-1}\int_{t^n_i}^{t^n_i+\delta_n} f^n_i(s)ds S^{t^n_j}_s(h^s_{t^n_{i+1}})(x) \tag{12}$$

But for $s \in [t_i^n, t_i^n + \delta_n]$

$$\begin{aligned} S_s^{t_j^n}(h_{t_{i+1}^n}^s)(x) &= \pi_{t_j^n,x}\left(\pi_{s,\xi_s} k(s, t_{i+1}^n]\right) \\ &= \pi_{t_j^n,x}\left(k(s, t_{i+1}^n]\right) \\ &= \pi_{t_j^n,x}\left(k(t_i^n, t_{i+1}^n]\right) - \pi_{t_j^n,x}\left(k(t_i^n, s]\right) \\ &= \pi_{t_j^n,x}\left(\pi_{t_i^n,\xi_{t_i^n}} k(t_i^n, t_{i+1}^n]\right) - \pi_{t_j^n,x}\left(\pi_{t_i^n,\xi_{t_i^n}} k(t_i^n, s]\right) \\ &= \pi_{t_j^n,x}\left(h_{t_{i+1}^n}^{t_i^n}(\xi_{t_i^n})\right) - \pi_{t_j^n,x}\left(h_s^{t_i^n}(\xi_{t_i^n})\right); \end{aligned}$$

Thus

$$\left\| S_s^{t_j^n}(h_{t_{i+!}^n}^s)(x) - \pi_{t_j^n,x}\left(h_{t_{i+1}^n}^{t_i^n}(\xi_{t_i^n})\right)\right\|_\infty \le \max_{i=0,..,n-1}\left\| h_{t_i^n+\delta_n}^{t_i^n}\right\|_\infty \le \frac{1}{n^2}.$$

Returning to equation (12) we get that

$$\max_{j=0,..,n}\left\| {}^n h_T^{t_j^n}(x) - h_T^{t_j^n}(x)\right\|_\infty \le \frac{1}{n}.$$

Now if $s \in (t_{j-1}^n, t_j^n)$ we have hat

$$\begin{aligned} \|{}^n h_T^s(x) - h_T^s(x)\|_\infty &= \left\|\pi_{s,x}\left(k^n(s, t_j^n] - k(s, t_j^n]\right)\right\|_\infty \\ &= +\left\|\pi_{s,x}\left({}^n h_T^{t_j^n}(\xi_{t_j^n}) - h_T^{t_j^n}(\xi_{t_j^n})\right)\right\|_\infty \\ &\le 2 \sup_{s\in(t_{j-1}^n, t_j^n)}\left\| h_{t_j^n}^s\right\|_\infty + \tfrac{1}{n}; \end{aligned}$$

and the last expression tends to zero as n tends to infinity. Moreover, since

$$\begin{aligned} {}^n h_t^s(x) - h_t^s(x) &= \pi_{s,x}\left(k^n(s,T] - k(s,T]\right) \\ &\quad -\pi_{s,x}\left(\pi_{t,\xi_t} k(t,T] - \pi_{t,\xi_t} k^n(t,T]\right) \end{aligned}$$

we easily derive that

$$\sup_{0\le s,t\le T, x\in E} |{}^n h_t^s(x) - h_t^s(x)| \to 0$$

as n tends to infinity. This establishes that $k^n(ds)$ satisfies property (i). Property (ii) is an immediate consequence of (i).

It remains only to establish property (iii) and (iv). But property (i) implies that for every $r \ge 0$ and every $x \in E$, we have that

$$\sup_{\tau\in[r,T]} \pi_{r,x}(|{}^n h_T^\tau(\xi_\tau) - h_T^\tau(\xi_\tau)|) \to 0$$

where the supremum is taken over all r-stopping times τ such that $r \le \tau \le T$. Consequently, according to Lemma 7, we obtain that $\sup_{s\in[r,T]} |{}^n h_T^s(\xi_s) - h_T^s(\xi_s)| \to 0$ in $\pi_{r,x}$-probability. One verifies easily that all the hypothesis of Theorem 25 are verified, and this yields property (iii). Property (iv) is immediate from Lemma 8, and the proof is complete. ∎

Remark 2. *In Proposition 13, the sequence of additive functional $k^n(ds)$ can be chosen to have the form*

$$k^n(ds) = \sum_{i=0}^{n-1} h_{t_{i+1}^n}^{t_i^n}(\xi_s) f_i^n(s) ds,$$

where $f_i^n \in pC_c^\infty(R_+)$ for $n = 1, 2, ...$; $i = 0, ..., n-1$;

Proof. Choose δ_n such that for every $r \geq 0$ and every $\alpha \leq \beta \leq \alpha + \delta_n$ we have

$$\max_{i=1,...,n} \left\| S_\alpha^r(h_{t_{i+1}^n}^{t_i^n}) - S_\beta^r(h_{t_{i+1}^n}^{t_i^n}) \right\|_\infty + \left\| h_\beta^\alpha \right\|_\infty \leq \frac{1}{n^2}.$$

Proceed then exactly like in the proof of Proposition 13. Note that if $r \in \{t_0^n, ..., t_n^n\}$ then

$${}^n h_T^r(x) = \sum_{t_i^n \geq r} \int_{t_i^n}^{t_i^n + \delta_n} f_i^n(s) ds S_s^r(h_{t_{i+1}^n}^{t_i^n})(x)$$

But since for $s \in [t_i^n, t_i^n + \delta_n]$ we have

$$\left\| S_s^r(h_{t_{i+1}^n}^{t_i^n}) - S_{t_i^n}^r(h_{t_{i+1}^n}^{t_i^n}) \right\|_\infty \leq \frac{1}{n^2}$$

and since

$$\sum_{t_i^n \geq r} \int_{t_i^n}^{t_i^n + \delta_n} f_i^n(s) ds S_{t_i^n}^r(h_{t_{i+1}^n}^{t_i^n})(x) = h_T^r(x),$$

we get

$$\| {}^n h_T^r(x) - h_T^r(x) \| \leq \frac{1}{n}.$$

The rest is similar to the proof of Proposition 13. ■

Theorem 14. *There exists a uniformly admissible sequence of additive functionals $k^n(ds)$ with $k^n(ds) = \lambda^n(s, \xi_s) ds$ such that (ξ, Φ, k^n)-superprocesses form an* A*-smooth approximating sequence for the (ξ, Φ, k)-superprocess. For every $(r, x) \in R_+ \times E$ and every r-stopping time τ, $k^n(r, \tau]$ converges in $L^1(\pi_{r,x})$ to $k(r, \tau]$.*

Proof. Let $k^n(ds)$ be given as in Remark 2 and let $v_{r,t}^n(f)(x)$ be the log-Laplace functional of the (ξ, Φ, k^n)-superprocess, $n = 1, 2, ...$ According to Theorem 26, $v_{r,t}^n(f)(x) \to v_{r,t}(f)(x)$ where $v_{r,t}(f)(x)$ is the log-Laplace functional of the (ξ, Φ, k)-superprocess. According to Theorem 28, $v_{r,t}^n(f)(x)$ is smooth for A and $\mathrm{A} v_{s,T}^n(f)(x) + \frac{\partial}{\partial s} v_{s,T}^n(f)(x) = \Phi(x, v_{s,T}^n(f)(x)) \lambda^n(s, x)$ for every $0 \leq s \leq T$, $x \in E$ and $f \in \mathcal{D}(\mathrm{A})$. ■

4. The partial martingale problem

In this section we investigate some of the properties shared by all solutions $X = (X_t, \Im, P_{r,\mu})$ to the partial martingale problem. One of these properties is that for such processes, liftings exist, and therefore, the full martingale problem *can be stated.*

We also prove that the convergence of processes $s \mapsto F^n(s, \xi_s)$ to a process $s \mapsto F(s, \xi_s)$ can be *"lifted"* to obtain the uniform convergence of the processes $s \mapsto \langle X_s, F^n(s, .) \rangle$ to the process $s \mapsto \langle X_s, F(s, .) \rangle$.

4.1. Connection between X and its particle motion ξ. The following result is due to Fitzsimmons [15, Corollary 2.8]. It establishes -via the partial martingale problem- a link between solutions X to the partial martingale problem and their projection ξ.

Lemma 15. *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem and let S_t be the semigroup of ξ. If τ is a bounded $r-$stopping time then for all $f \in b\mathcal{B}$*

$$P_{r,\mu}^{\Im_\tau} \langle X_{\tau+t}, f\rangle = \langle X_\tau, S_t f\rangle, \text{ for every } t \geq 0$$

where $P_{r,\mu}^{\Im_\tau}$ denotes the conditional expectation with respect to $\Im_\tau$.

The following technical lemma will be used several times in this paper:

Lemma 16. *Let $X = (X_t, \Im, P_{r,\mu})$ be a solution to the partial martingale problem for* A. *Then for every $f \in b\mathcal{B}$, every $T > 0$ the process $t \mapsto \langle X_t, S_{T-t} f\rangle$ is a càdlàg martingale. In particular, for every bounded $r-$stopping time τ we have that*

$$P_{r,\mu} \langle X_\tau, S_{T-\tau} f\rangle = \langle \mu, S_{T-r} f\rangle$$

Proof. Form Lemma 15, we have that

$$P_{r,\mu}^{\Im_t} \langle X_{t+s}, S_{T-t-s}(f)\rangle = \langle X_t, S_{T-t}(f)\rangle$$

and hence the process $t \mapsto \langle X_t, S_{T-t}(f)\rangle$ is a martingale. Since it is dominated by $t \mapsto \|f\|_\infty \langle X_t, 1\rangle$, it belongs to class (D), according to [10, Lemma A.1.1]. If $f \in \mathcal{D}(\mathrm{A})$, then $S_t f \in \mathcal{D}(\mathrm{A})$ for every $t \geq 0$. Hence, for every t',the process $t \mapsto \langle X_t, S_{T-t'}(f)\rangle$ is a càdlàg process. Hence if Λ_n denotes a sequence of partitions $\{t_i^n\}_{i=0}^n$ of the interval $[r, T]$ with $mesh\{\Lambda_n\} \to 0$, then the process x_t^n defined by

$$t \mapsto x_t^n := \sum_{i=0}^{n-1} 1_{[t_i^n, t_{i+1}^n)}(t) \left\langle X_t, S_{T-t_i^n} f\right\rangle$$

is a càdlàg process. Because $f \in \mathcal{D}(\mathrm{A})$, we have that $1_{[t_i^n, t_{i+1}^n)}(t) S_{T-t_i^n} f(x)$ converges uniformly (in x and $t \in [r, T]$) to $S_{T-t}(f)(x)$. Therefore $t \mapsto x_t^n$ converges uniformly (in t) to $t \mapsto \langle X_t, S_{T-t} f\rangle$. Consequently, $t \mapsto \langle X_t, S_{T-t} f\rangle$ is a càdlàg martingale. From the optional sampling theorem we get that for every bounded r-stopping time τ

$$P_{r,\mu} \langle X_\tau, S_{T-\tau} f\rangle = \langle \mu, S_{T-r} f\rangle. \tag{13}$$

The extension of equality (13) to arbitrary $f \in b\mathcal{B}$ follows from the fact that $\mathcal{D}(\mathrm{A})$ is dense, for the bounded pointwise convergence, in $b\mathcal{B}$. From Lemma [10, A.1.1.D], we conclude from this equality that $t \mapsto \langle X_t, S_{T-t} f\rangle$ is a right continuous -and therefore càdlàg- martingale. ■

Corollary 17. *Let $\beta \in (r, T]$, $\alpha \in [r, T]$ and let $f(.) \in b\mathcal{B}$. Then the process*

$$t \mapsto x_t := 1_{[\alpha,\beta)}(t) \left\langle X_t, S_\beta^t f\right\rangle$$

is càdlàg, and moreover, for every $\delta > 0$ and every stopping time τ

$$P_{r,\mu} x_{\tau+\delta} = P_{r,\mu} 1_{[\alpha,\beta)}(\tau + \delta) \left\langle X_\tau, S_\beta^\tau f\right\rangle$$

Proof. The process $t \mapsto \left\langle X_{t\wedge\beta}, S_\beta^{t\wedge\beta} f \right\rangle$ is a martingale; so is therefore $t \mapsto 1_{[\alpha,\beta)}(t) \left\langle X_t, S_\beta^t f \right\rangle$.

Let τ be a stopping time and $\delta > 0$. Note that without lost of generality, we can assume that $\tau \leq \beta$: this is due to the facts that for $\tau > \beta$, we have $x_{\tau\wedge\beta+\delta} = x_{\tau+\delta} = 0$.

From the optional stopping time theorem, we get

$$P_{r,\mu}^{\Im_\tau^r} \left\langle X_{\tau+\delta}, S_\beta^{\tau+\delta} f \right\rangle = \left\langle X_\tau, S_\beta^\tau f \right\rangle$$

where $P_{r,\mu}^{\Im_\tau^r}(.)$ denotes the conditional expectation with respect to $\Im_\tau^r$.

Because $1_{[\alpha,\beta)}(\tau+\delta) \in \Im_\tau^r$ this completes the proof. ∎

Corollary 18. *Let $t_0 := r < t_1 < ... < t_n := T$ be a partition of $[r,T]$. Let $f^i(.) \in b\mathcal{B}$, that is a bounded $\mathcal{B}$-measurable function, for $i = 1, ..., n$. Then the process*

$$t \mapsto x_t := \sum_{i=0}^{n-1} 1_{[t_i,t_{i+1})}(t) \left\langle X_t, S_{t_{i+1}}^t f^{i+1} \right\rangle$$

is càdlàg, and for every stopping time τ and every $\delta > 0$ we have that

$$P_{r,\mu}(x_{\tau+\delta}) = P_{r,\mu} \sum_{i=0}^{n-1} 1_{[t_i,t_{i+1})}(\tau+\delta) \left\langle X_\tau, S_{t_{i+1}}^\tau f^{i+1} \right\rangle$$

Proof. This is immediate from the above Corollary. ∎

4.2. Liftings. Consider now the function $h_T^r(x) := \pi_{r,x} a(r,T]$ which is called the **characteristic** of the additive functional $a(ds)$. Assume that h is bounded. Note that by Markov property, for every $0 \leq r \leq s \leq T$ we have

$$S_s^r\left(h_T^s\right)(x) = \pi_{r,x}\left(\pi_{s,\xi_s} a(s,T]\right) = \pi_{r,x} a(s,T] \leq h_T^r(x).$$

We use this in the following proof of the existence and uniqueness of liftings for solutions to the partial martingale problem.

Proof of Proposition 3: According to Lemma 15

$$P_{r,\mu}^{\Im_t} \langle X_{t+s}, f \rangle = \langle X_t, S_s f \rangle\,,\ P_{r,\mu}\text{-almost surely for every } f \in b\mathcal{B}.$$

Consequently,

$$P_{r,\mu}^{\Im_t} \left\langle X_{t+s}, h_T^{t+s} \right\rangle = \left\langle X_t, S_s\left(h_T^{t+s}\right) \right\rangle \leq \left\langle X_t, h_T^t \right\rangle,$$

and therefore process $t \mapsto x_t := \langle X_t, h_T^t \rangle$ is a supermartingale.

Let $\Lambda_n := r = t_0^n < ... < t_n^n = T$ be a sequence of nested partitions of the interval $[r,T]$ with $mesh\{\Lambda_n\} \to 0$. According to Corollary 18, the processes

$$t \mapsto x_t^n := \sum_{i=0}^{n-1} 1_{[t_i^n,t_{i+1}^n)}(t) \left\langle X_t, S_{t_{i+1}^n}^t h_T^{t_{i+1}^n} \right\rangle$$

are càdlàg. Since $k(ds)$ is admissible, we have, according to Lemma 9

$$\max_{i=0,\dots,n-1} \sup_{t\in[t_i^n, t_{i+1}^n)} \left\| S_{t_{i+1}^n}^t h_T^{t_{i+1}^n} - h_T^t \right\|_\infty \to 0.$$

Moreover, due to the fact that $t \mapsto \langle X_t, 1\rangle$ is càdlàg,

$$\sup_{s\in[r,T]} \langle X_s, 1\rangle < \infty.$$

We can thus conclude that

$$\lim_{n\to\infty} \sup_{t\in[r,T]} |x_t^n - x_t| = 0.$$

The uniform limit of a sequence of càdlàg functions being also càdlàg, we conclude that $t \mapsto x_t$ is càdlàg.

Thus, by Doob-Meyer decomposition theorem (cf. [10, Th. A.1.1]),$t \mapsto x_t$ has a unique compensator $A(ds)$ (which is, by definition of lifting, the unique lifting of $a(ds)$) and

$$\begin{aligned} A(r,t] &= \lim_{\Lambda} \sum_{i=1}^{n-1} P_{r,\mu}^{\Im_{t_i}-} \left\{ \langle X_{t_i}, h_T^{t_i}\rangle - \left\langle X_{t_{i+1}}, h_T^{t_{i+1}} \right\rangle \right\} \\ &= \lim_{\Lambda} \sum_{i=1}^{n-1} P_{r,\mu}^{\Im_{t_i}-} \left\{ \langle X_{t_i}, h_T^{t_i}\rangle - \left\langle X_{t_i}, S_{t_{i+1}}^{t_i} h_T^{t_{i+1}} \right\rangle \right\} \\ &= \lim_{\Lambda} \sum_{i=1}^{n-1} P_{r,\mu}^{\Im_{t_i}-} \left\{ \left\langle X_{t_i}, h_{t_{i+1}}^{t_i} \right\rangle \right\} \end{aligned} \tag{14}$$

weakly in $L^1(P_\mu)$ as Λ runs over a standard sequence of partitions $\Lambda = \{r = t_0 < t_1 < \dots < t_n = t\}$ of the interval $[r,t]$. Moreover, the convergence in (14) is strong when A is continuous.

We now show that the lifting A of an admissible additive functional a is continuous. According to [10, Th. A.1.1], A is continuous if and only if for every sequence of $r-$stopping times $\tau_n \nearrow \tau$, with $\tau_n < \tau$, we have $E_{r,\mu} x_{\tau_n} \searrow E_{r,\mu} x_\tau$.

Let the $r-$stopping times τ_n increase to τ. Choose ε and pick δ such that $|\alpha - \beta| \le \delta$ implies $\left\| h_\beta^\alpha \right\|_\infty \le \varepsilon$.

We have

$$P_{r,\mu} x_{\tau_n} \ge P_{r,\mu} x_\tau \ge P_{r,\mu} x_{\tau\vee(\tau_n+\delta)} = P_{r,\mu} 1_{\{\tau_n+\delta<\tau\}} x_\tau + P_{r,\mu} 1_{\{\tau_n+\delta\ge\tau\}} x_{\tau_n+\delta}$$

But because x belongs to class (D), we have, for n big enough, that the right hand side of the above differs from $E_{r,\mu} x_{\tau_n+\delta}$ by a quantity which is less than or equal to ε. Therefore, for big n,

$$P_{r,\mu} x_{\tau_n+\delta} \le P_{r,\mu} x_\tau + \varepsilon. \tag{15}$$

On the other hand we get from Lemma 9

$$\sum_{i=0}^{N-1} P_{r,\mu} 1_{\{\frac{i}{N}T \le \tau+\delta < \frac{i+1}{N}T\}} \left\langle X_{\tau+\delta}, S_{\frac{i+1}{N}T}^{\tau+\delta} h_T^{\frac{i+1}{N}T} \right\rangle \quad \to \quad P_{r,\mu} \left\langle X_{\tau+\delta}, h_T^{\tau+\delta} \right\rangle \tag{16}$$

but by Corollary 18, the left hand side of (16) coincides with

$$\sum_{i=0}^{N-1} P_{r,\mu} 1_{\{\frac{i}{N}T \le \tau+\delta < \frac{i+1}{N}T\}} \left\langle X_\tau, S_{\frac{i+1}{N}T}^{\tau} h_T^{\frac{i+1}{N}T} \right\rangle.$$

An other use of Lemma 9 gives

$$\sum_{i=0}^{N-1} P_{r,\mu} 1_{\{\frac{i}{N}T \le \tau+\delta < \frac{i+1}{N}T\}} \left\langle X_\tau, S^{\tau}_{\frac{i+1}{N}T} h_T^{\frac{i+1}{N}T} \right\rangle \quad \to \quad P_{r,\mu} \left\langle X_\tau, S^{\tau}_{\tau+\delta} h_T^{\tau+\delta} \right\rangle,$$

and therefore

$$P_{r,\mu} \left\langle X_{\tau+\delta}, h_T^{\tau+\delta} \right\rangle = P_{r,\mu} \left\langle X_\tau, S^{\tau}_{\tau+\delta} h_T^{\tau+\delta} \right\rangle.$$

Thus, using (15), we have, for n big enough,

$$\begin{aligned} 0 \le P_{r,\mu} x_{\tau_n} - P_{r,\mu} x_\tau &\le \varepsilon + P_{r,\mu} \left(x_{\tau_n} - x_{\tau_n+\delta} \right) \\ &= \varepsilon + P_{r,\mu} \left(\langle X_{\tau_n}, h_T^{\tau_n} \rangle - \left\langle X_{\tau_n}, S^{\tau_n}_{\tau_n+\delta} h_T^{\tau_n+\delta} \right\rangle \right) \\ &= \varepsilon + P_{r,\mu} \left(\langle X_{\tau_n}, h^{\tau_n}_{\tau_n+\delta} \rangle \right) \\ &\le \varepsilon (1 + |\mu|). \end{aligned}$$

This shows that $P_{r,\mu} x_{\tau_n} \searrow P_{r,\mu} x_\tau$ and therefore, the compensator A of x is continuous. ■

4.3. Convergence for ξ versus convergence for X. Let $f^n(r,x)$ be a collection of nearly Borel functions, and (for fixed $T > 0$) consider the process $s \mapsto F^n_{s,T}(\xi_s)$, $s \in [0,T]$, where $F^n_{r,T}(x) := \pi_{r,x} f^n(T, \xi_T)$. To these processes, correspond the "lifted" processes given by $s \mapsto \langle X_s, F^n_{s,T} \rangle$. In this subsection, we establish a criterion under which the pointwise convergence of $F^n_{r,T}(x)$ to $F_{r,T}(x)$ implies the that the process $s \mapsto \langle X_s, F^n_{s,T} \rangle$ converges *uniformly in* s to the process $s \mapsto \langle X_s, F_{s,T} \rangle$.

We also establish that a criteria under which the convergence of the additive functionals $k^n(ds)$ to the additive functional $k(ds)$ implies the same convergence for their liftings $K^n(ds)$ and $K(ds)$.

In fact, we are particularly interested in the processes $s \mapsto \langle X_s, v^n_{s,T}(.) \rangle$ where v^n is the log-Laplace functional of an A-smooth approximating sequence for the superprocess with parameters (ξ, Φ, k). We want to show that $s \mapsto \langle X_s, v^n_{s,T}(.) \rangle$ converges in probability uniformly in s to $s \mapsto \langle X_s, v_{s,T}(.) \rangle$.

We also study the processes $s \mapsto \langle X_s, {}^n h^s_T(.) \rangle$ where ${}^n h^s_T(.)$ is the characteristic of an additive functional $k^n(ds)$. We derive from the uniform convergence of $s \mapsto \langle X_s, {}^n h^s_T(.) \rangle$ to $s \mapsto \langle X_s, h^s_T(.) \rangle$ that (under some assumptions) the liftings K^n of k^n converge (weakly a.s.) to the lifting K of k. This is crucial for the proof of uniqueness to the martingale problem.

Uniform convergence of sequences of "lifted processes".

Notation 2. Let z_s be a function of $s \in [r,T]$. In the following, the expression z^* will denote

$$z^* := \sup_{t \in [r,T]} z_t.$$

Lemma 19. *Let $f^n(t,x)$ be a sequence of uniformly bounded measurable functions satisfying the condition*

$$\sup_{0 \le t \le T, x, n} |S_{t+\delta} f^n(t+\delta)(x) - f^n(t,x)| \to 0 \text{ as } \delta \to 0. \tag{17}$$

Suppose that

$$f(t,x) = \lim_{n\to\infty} f^n(t,x). \tag{18}$$

Then the process

$$|x^n - x|^* := \sup_{t\in[r,T]} |\langle X_t, f^n(t,.)\rangle - \langle X_t, f(t,.)\rangle|$$

converges to zero in $P_{r,\mu}$-probability.

Proof. Let $t_i^m := r + \frac{i}{m}(T-r)$, for $i = 0, ..., m$. Define

$$\begin{array}{lcl} x_t^n & := & \langle X_t, f^n(t,.)\rangle \\ x_t^\infty & ;= & \langle X_t, f(t,.)\rangle \\ x_t^{n,m} & := & \sum_{i=0}^{m-1} 1_{[t_i^m, t_{i+1}^m)}(t) \left\langle X_t, S^t_{t_{i+1}^m} f^n(t_{i+1}^m,.)\right\rangle \\ x_t^{\infty,m} & := & \sum_{i=0}^{m-1} 1_{[t_i^m, t_{i+1}^m)}(t) \left\langle X_t, S^t_{t_{i+1}^m} f(t_{i+1}^m,.)\right\rangle. \end{array}$$

Recall that for every ω, $\sup_{t\in[r,T]} \langle X_t(\omega), 1\rangle < \infty$. Thus, (17) implies that for every $\varepsilon > 0$ and for every m big enough, we have

$$\sup_{n\in[1,\ldots,\infty]} |x^{n,m}(\omega) - x^n(\omega)|^* < \varepsilon.$$

Therefore, it suffices to prove that for every $m > 0$, $|x^{n,m} - x^{\infty,m}|^*$ converges to zero in $P_{r,\mu}$-probability.

This will clearly be verified if for every $c > 0$

$$\sup_{t\in[r,c]} \left|\left\langle X_t, S_c^t f^n(c,.)\right\rangle - \left\langle X_t, S_c^t f(c,.)\right\rangle\right|$$

converges to zero in $P_{r,\mu}$-probability. This is the case if

$$\sup_{t\in[r,c]} \left\langle X_t, S_c^t \left|f^n(c,.) - f(c,.)\right|\right\rangle$$

converges to zero in $P_{r,\mu}$-probability. To prove this, it suffices, according to Lemma 7, to check that

$$\lim_{n\to\infty} \sup_{r\le\tau\le c} P_{r,\mu} \left\langle X_\tau, S_c^\tau \left|f^n(c,.) - f(c,.)\right|\right\rangle = 0.$$

But for every $g \in \mathcal{B}$, the process $\langle X_t, S_c^t g\rangle$ is a càdlàg martingale. Hence, from the optional sampling theorem we get that for every stopping time τ

$$\begin{array}{rcl} P_{r,\mu} \left\langle X_\tau, S_c^\tau \left|f^n(c,.) - f(c,.)\right|\right\rangle & = & P_{r,\mu} \left\langle X_r, S_c^r \left|f^n(c,.) - f(c,.)\right|\right\rangle \\ & = & \left\langle \mu, S_c^r \left|f^n(c,.) - f(c,.)\right|\right\rangle. \end{array}$$

Because f^n converges to f, and because $\{f^n\}$ is uniformly bounded, the right hand side of the last equality tends to zero. ■

Corollary 20. *Let $k^n(ds), k(ds)$ be a collection of uniformly admissible additive functionals of a right process ξ and Φ a regular branching mechanism. Let v^n be the log-Laplace functional of the (ξ, Φ, k^n)-superprocess, and v the log-Laplace functional of the (ξ, Φ, k)-superprocess. Let $g \in \mathcal{L}$ and suppose that $v^n_{r,T}(g)(x)$ converges to $v_{r,T}(g)(x)$ for every r, x. Then $s \mapsto \langle X_s, v^n_{s,T}(g)\rangle$ converges uniformly to $s \mapsto \langle X_s, v_{s,T}(g)\rangle$ in $P_{r,\mu}$-probability.*

Proof. Let $g \in \mathcal{L}$. Note that the functions $v^n_{r,T}(g)(x)$ and $\Phi(x, v^n_{r,T}(g)(x))$ are uniformly bounded. This implies that the family of additive functionals $\{k^{*n}\}$, defined by $k^{*n}(ds) := \Phi(\xi_s, v^n_{s,T}(f)(\xi_s))k^n(ds)$, is uniformly admissible. It follows from Lemma 9 that (17) holds with $f^n(t,x) := \pi_{r,x}k^{*n}(r,T]$ and $f(t,x) := \pi_{r,x}k^*(r,T]$, where $k^*(ds) := \Phi(\xi_s, v_{s,T}(f)(\xi_s))k(ds)$. This yields (17) with $f^n(t,x) := v^n_{r,T}(f)(x)$ and $f(t,x) := v_{r,T}(f)(x)$. The assumption that $v^n_{r,T}(f)(x)$ converges to $v_{r,T}(f)(x)$ is identical to (18). An appeal to Lemma 19 completes the proof. ■

Convergence of additive functionals versus convergence of their liftings.

Proposition 21. *Let $k^n(ds), k(ds)$ be a collection of uniformly admissible additive functionals. Suppose that for every r, x we have that*

$$ {}^n h^r_T(x) := \pi_{r,x}k^n(r,T] \to \pi_{r,x}k(r,T] =: h^r_T(x). \tag{19}$$

Then for every r-stopping time τ, $K^n(r,\tau]$ converges to $K(r,\tau]$ in $L^1(P_{r,\mu})$, where $K^n(ds)$ (resp. $K(ds)$) is the lifting of $k^n(ds)$ (resp. $k(ds)$).

Proof. Because the additive functionals are uniformly admissible, we derive form Lemma 9 that condition (17) is verified with $f^n(t,x) ={}^n h^t_T(x)$ and $f(t,x) = h^t_T(x)$. Condition (19) is identical to condition (18) and therefore, according to Lemma 19,

$$\sup_{t\in[r,T]} \left|\langle X_t, {}^n h^t_T\rangle - \langle X_t, h^t_T\rangle\right| \to 0$$

in $P_{r,\mu}$ -probability.

Clearly, for every stopping time τ and every bounded random variable M

$$P_{r,\mu}\left(M\langle X_\tau, {}^n h^\tau_T\rangle\right) \to P_{r,\mu}\left(M\langle X_\tau, h^\tau_T\rangle\right).$$

We have already established, in §4.2, that processes

$$\begin{array}{lcl} t\mapsto x_t & := & \langle X_t, h^t_T\rangle \\ t\mapsto x^n_t & := & \langle X_t, {}^n h^t_T\rangle \end{array}$$

are right continuous supermartingales of class (D) whose compensators are the liftings $K^n(ds)$ of the additive functionals $k^n(ds)$. In fact, since the additive functionals $k^n(ds)$ are uniformly admissible, there characteristics ${}^n h^r_T(x)$ are uniformly bounded, so that the processes $t \mapsto x^n_t$ belong uniformly to class (D).

It suffices only to appeal to Theorem 25 to obtain the desired result. ■

Corollary 22. *Let $k^n(ds), k(ds)$ be a collection of uniformly admissible additive functionals of ξ, let Φ be a regular branching mechanism, let v^n be the log-Laplace functional of the (ξ, Φ, k^n)-superprocess, and v the log-Laplace functional of the (ξ, Φ, k)-superprocess. Let $f \in \mathcal{L}$ and assume that $v^n_{r,T}(f)(x)$ converges to $v_{r,T}(f)(x)$ for every r, x. Let $K^{\Phi(v^n)dk^n}(ds)$ be the lifting of $\Phi(s, \xi_s, v^n_{s,T}(\xi_s))k^n(ds)$. Then, for every r-stopping time τ, $K^{\Phi(v^n)dk^n}(r,\tau]$ converges to $K^{\Phi(v)dk}(r,\tau]$ in $L^1(P_{r,\mu})$.*

Proof. Clearly, $v^n_{r,T}(x) - \pi_{r,x} f(\xi_t)$ converges to $v_{r,T}(x) - \pi_{r,x} f(\xi_t)$ for every r, x. That is,

$$^n\tilde{h}^r_T(x) := \pi_{r,x} \int_r^T \Phi(s, \xi_s, v^n_{s,\xi_s}) k^n(ds) \to \pi_{r,x} \int_r^T \Phi(s, \xi_s, v_{s,\xi_s}) k(ds) =: \tilde{h}^r_T(x)$$

for every $(r,x) \in [0,T] \times E$. Moreover, the additive functionals $\Phi(s, \xi_s, v^n_{s,\xi_s}) k^n(ds)$ are uniformly admissible. An appeal to Proposition 21 completes the proof. ■

5. The full martingale problem: uniqueness of the solution

We now prove the uniqueness of the solution to the full martingale problem. Assume for now on that $(X_t, \Im, P_{r,\mu})$ is a solution to the full martingale problem. Our first goal, in this section, is to establish that $(X_t, \Im, P_{r,\mu})$ is a solution to an "extended" form of the full martingale problem.

5.1. Extension of the martingale problem to time dependent functions. The "extended" form of the full martingale problem for (ξ, Φ, k) is given in the following Lemma:

Lemma 23. *Let X_t be a solution to the full martingale problem for (ξ, Φ, k) and let ψ be smooth for* A. *Then*

$$\begin{array}{l} t \mapsto \exp\left(-\langle X_t, \psi_t\rangle\right) + \int_r^t \exp\left(-\langle X_s, \psi_s\rangle\right) \left\langle X_s, \mathrm{A}\psi_s + \frac{\partial}{\partial s}\psi_s \right\rangle ds \\ -\int_r^t \exp\left(-\langle X_s, \psi_s\rangle\right) K^{\Phi(\psi)dk}(ds) \end{array} \tag{20}$$

is a $P_{r,\mu}-$martingale, where $K^{\Phi(\psi)dk}(ds)$ is the lifting of $\Phi(\xi_s, \psi_s)k(ds)$.

Proof. The proof is a generalization of Lemma 8 in [12] (see also [13, Lemma 4.3.4]). First, for a measurable function $f(s,x)$, let us define (when the expressions makes sense)

$$\begin{array}{lcl} u_f(s, X_t) & := & \exp\left(-\langle X_t, f(s,.)\rangle\right) \\ v_f(s, X_t) & := & \exp\left(-\langle X_t, f(s,.)\rangle\right) \left\langle X_t, \frac{\partial}{\partial s} f(s,.)\right\rangle \\ w_f(s, X_t) & := & \exp\left(-\langle X_t, f(s,.)\rangle\right) \langle X_t, \mathrm{A} f(s,.)\rangle . \end{array}$$

Let ψ be smooth for A. Then we have

$$u_\psi(t_2, X_{t_2}) - u_\psi(t_1, X_{t_2}) \quad = \quad -\int_{t_1}^{t_2} v_\psi(s, X_{t_2}) ds$$

and

$$\begin{array}{rcl} E^{\Im_{t_t}}\left[u(t_1, X_{t_2}) - u(t_1, X_{t_1})\right] & = & -E^{\Im_{t_t}}\left[\int_{t_1}^{t_2} w(t_1, X_s) ds\right] \\ & & -E^{\Im_{t_t}}\left[\int_{t_1}^{t_2} u(t_1, X_s) K^{\Phi(\psi_{t_1})dk}(ds)\right] . \end{array} \tag{21}$$

Therefore, if Λ^n is a partition of $[t_1, t_2]$ with $mesh\{\Lambda^n\} \to 0$ and ψ^n and X^n are defined by

$$\begin{array}{lcl} \psi^n(s,x) & := & \sum_{i=1}^n \psi(t_i^n, x) 1_{[t_i^n, t_{i+1}^n)}(s) \\ X^n(s) & := & \sum_{i=1}^n X_{t_{i+1}^n} 1_{[t_i^n, t_{i+1}^n)}(s) \end{array}$$

then, clearly,

$$\sum_{i=1}^{n-1} 1_{[t_i^n,t_{i+1}^n)}(s)K^{\Phi(\xi_s,\psi_{t_i^n}(\xi_s))dk}(ds) \quad = \quad K^{\Phi(\xi_s,\psi_s^n(\xi_s))dk}(ds)$$

(where $K^{\eta(s,\xi_s)dk}(ds)$ denotes here the lifting of $\eta(s,\xi_s)k(ds))$ and we get (by summing the expressions in (21)) that

$$\begin{aligned} E^{\Im_{t_1}}\left[u(t_2,X_{t_2})-u(t_1,X_{t_1})\right] \quad = \quad & -E^{\Im_{t_1}}\left[\int_{t_1}^{t_2} v_\psi(s,X_s^n)ds\right] \\ & -E^{\Im_{t_1}}\left[\int_{t_1}^{t_2} w_{\psi^n}(s,X_s)ds\right] \\ & +E^{\Im_{t_1}}\left[\int_{t_1}^{t_2} u_{\psi^n}(s,X_s)K^{\Phi(\psi^n)dk}(ds)\right]. \end{aligned} \tag{22}$$

We want to show that

$$\begin{aligned} \lim_{n\to\infty} E^{\Im_{t_1}}\left[\int_{t_1}^{t_2} v_\psi(s,X_s^n)ds\right] \quad &= \quad E^{\Im_{t_1}}\left[\int_{t_1}^{t_2} v_\psi(s,X_s)ds\right] \\ \lim_{n\to\infty} E^{\Im_{t_1}}\left[\int_{t_1}^{t_2} w_{\psi^n}(s,X_s)ds\right] \quad &= \quad E^{\Im_{t_1}}\left[\int_{t_1}^{t_2} w_\psi(s,X_s)ds\right] \\ \lim_{n\to\infty} E^{\Im_{t_1}}\left[\int_{t_1}^{t_2} u_{\psi^n}(s,X_s)K^{\Phi(\psi^n)dk}(ds)\right] \quad &= \quad E^{\Im_{t_1}}\left[\int_{t_1}^{t_2} u_\psi(s,X_s)K^{\Phi(\psi)dk}(ds)\right] \end{aligned} \tag{23}$$

which would complete the proof of the Lemma.

1°)Let us first show that the first two limits of (23) are verified. From Lebesgue's theorem, it suffices to prove that for a fixed $s \geq t_1$, we have that $P_{r,\mu}$-almost surely

$$\begin{aligned} &\langle X_s^n,\psi_s\rangle \to \langle X_s,\psi_s\rangle\,, \\ &\langle X_s,\psi_s^n\rangle \to \langle X_s,\psi_s\rangle\,, \\ &\langle X_s,\mathrm{A}\psi_s^n\rangle \to \langle X_s,\mathrm{A}\psi_s\rangle\,, \\ &\left\langle X_s^n,\tfrac{\partial}{\partial s}\psi_s\right\rangle \to \left\langle X_s,\tfrac{\partial}{\partial s}\psi_s\right\rangle. \end{aligned}$$

Only the last convergence is not straightforward. Let Q_+ denote the set of non-negative rational numbers. Since $\frac{\psi_{s+h}-\psi_s}{h} \in \mathcal{D}(\mathrm{A})$, we have the processes

$$t \mapsto \left\langle X_t, \frac{\psi_{s+h}-\psi_s}{h}\right\rangle, \text{ where } s,h \in Q_+$$

are $P_{r,\mu}$-indistinguishable from right continuous processes. But since $\langle X_t,1\rangle$ is a càdlàg process,

$$\sup_{r\leq t\leq T}\langle X_t,1\rangle < \infty;$$

using the facts that $\frac{\psi_{s+h}-\psi_s}{h} \to \frac{\partial}{\partial s}\psi_s$ uniformly in x, we obtain that $P_{r,\mu}$-almost surely, the mappings $\{t \mapsto \left\langle X_t,\frac{\partial}{\partial s}\psi_s\right\rangle\}_{s\in Q_+}$ are uniform limits of right continuous mappings; they are therefore also a right continuous. Because the function $\frac{\partial}{\partial s}\psi_s$ is strongly continuous and bounded, it is easy to derive that $(t,s) \mapsto \left\langle X_t,\frac{\partial}{\partial s}\psi_s\right\rangle$ is jointly right continuous, $P_{r,\mu}$-almost surely. Hence, $\left\langle X_s^n,\frac{\partial}{\partial s}\psi_s\right\rangle \to \left\langle X_s,\frac{\partial}{\partial s}\psi_s\right\rangle$, $P_{r,\mu}$-almost surely, as wanted.

2°) We now show that the third limit of (23) holds. Note that, for every ω, $s \mapsto \langle X_s,\psi_s^n\rangle(\omega)$ converges uniformly (in $s\in[r,T]$) to $s\mapsto\langle X_s,\psi_s\rangle(\omega)$. According to Proposition 21, $K^{\Phi(\psi^n)dk}(r,\tau]$ converges to $K^{\Phi(\psi)dk}(r,\tau]$ in $L^1(P_{r,\mu})$. With Lemma 8, it is also possible to suppose (perhaps by taking a subsequence) that $K^{\Phi(\psi^n)dk}(ds)$ converges weakly to $K^{\Phi(\psi)dk}(ds)$. This yields (20) as wanted. ■

5.2. Proof of uniqueness for the full martingale problem. We are now ready to show that the solution to the full martingale problem for (ξ, Φ, k) is unique, as stated in Theorem 4.

Proof of the uniqueness part in Theorem 4:

Step 1) According to Lemma 14, we can choose a uniformly admissible sequence of additive functionals $k^n(ds) = \lambda^n(s, \xi_s)ds$ such that if $v^n_{r,T}(f)(x)$ (resp. $v_{r,T}(f)(x)$) is the log-Laplace of the (ξ, Φ, k^n)-superprocess (resp. (ξ, Φ, k^n)-superprocess) then

(i) $v^n_{r,T}(f)(x)$ converges to $v_{r,T}(f)(x)$ for every r, x, f

(ii) $v^n_{r,T}(x)$ is smooth for A, for every $f \in \mathcal{D}(\mathrm{A})$

(iii) $\mathrm{A}v^n_{s,T}(.) + \frac{\partial}{\partial s}v^n_{s,T}(.) = \Phi(., v^n_{s,T})\lambda^n(s, .)$, for every $f \in \mathcal{D}(\mathrm{A})$

(iv) for every r, x and every r-stopping time τ, $k^n(r, \tau]$ converges in $L^1(\pi_{r,x})$ to $k(r, \tau]$

Let us define

$$\begin{aligned} K^n_1(ds) &:= \left\langle X_s, \Phi(., v^n_{s,T})\lambda^n(s, .)\right\rangle ds \\ K^n_2(ds) &:= K^{\Phi(v^n_{.,T})dk}(ds). \end{aligned}$$

Note that $K^n_1(ds)$ is the lifting of the additive functional $\Phi(\xi_s, v^n_{s,T})\lambda^n(s, \xi_s)ds$. According to Proposition 21 and Corollary 22 we have that

(A) For every r-stopping time τ, both random variables $K^n_1(r, \tau]$ and $K^n_2(r, \tau]$ converge to $K^{\Phi(v_{.,T})dk}(r, \tau]$ in $L^1(P_{r,\mu,})$.

Invoking Lemma 8 we are also allowed to assume (by mean of taking a subsequence) that a.s.

(B) $K^n_1(ds)$ and $K^n_2(ds)$ converges weakly to $K^{\Phi(v_{.,T})dk}(ds)$.

Moreover, from Corollary 20, it is also possible to suppose (by mean of taking a subsequence) that

(C) $s \mapsto \left\langle X_s, v^n_{s,T}\right\rangle$ converges uniformly (in $s \in [r, T]$) to $s \mapsto \langle X_s, v_{s,T}\rangle$.

Step2) According to Lemma (23),

$$\begin{aligned} &t \mapsto \exp\left(-\left\langle X_t, v^n_{t,T}\right\rangle\right) + \textstyle\int_r^t \exp\left(-\left\langle X_s, v^n_{s,T}\right\rangle\right)\left\langle X_s, \mathrm{A}v^n_{s,T} + \frac{\partial}{\partial s}v^n_{s,T}\right\rangle ds \\ &- \textstyle\int_r^t \exp\left(-\left\langle X_s, v^n_{s,T}\right\rangle\right) K^{\Phi(v^n_{.,T})dk}(ds) \end{aligned}$$

is a martingale. Putting $x^n_t = \left\langle X_t, v^n_{t,T}\right\rangle$, the equality

$$\mathrm{A}v^n_{s,T} + \frac{\partial}{\partial s}v^n_{s,T} = \Phi(., v^n_{s,T})\lambda^n(s, .)$$

gives

$$e^{-x^n_t} \quad = \quad M^n_t(\varphi) + \textstyle\int_r^t e^{-x^n_s} K^n_1(ds) - \int_r^t e^{-x^n_s} K^n_2(ds) \tag{24}$$

Clearly $e^{-x^n_t} \to e^{-x_t}$ pointwise and in $L^1(P_{r,\mu})$ where $x_t := \langle X_t, v_{t,T}\rangle$ and v is the log-Laplace functional of the superprocess (ξ, Φ, k).

From (A), (B) and (C) we get that

$$\int_0^t e^{-x^n_s} K^n_1(ds) - \int_0^t e^{-x^n_s} K^n_2(ds) \to 0 \tag{25}$$

where the convergence holds in $L^1(P_{r,\mu})$.

That forces $M_t^n(\varphi)$ to converge in $L^1(P_{r,\mu})$ to a limit $M_t(\varphi)$ which has to be a martingale, and we get

$$P_{r,\mu}\left(e^{-x_T}\right) = P_{r,\mu}\left(e^{-x_r}\right),$$

Which is precisely

$$P_{r,\mu}\left(\exp\left(-\langle X_T, \varphi\rangle\right)\right) = \exp\left(-\langle \mu, v_{r,T}\rangle\right).$$

Since T is arbitrary, X is the superprocess with parameters (ξ, Φ, k). ■

6. Application to superprocesses with interactions

We now introduce a Dawson-Girsanov transformation (cf. [3] and [4, Th. 7.2.2]) for $(\xi, (.)^2, k)$-superprocesses. The purpose of this section is to generalize [4, Th. 7.2.2]. In fact, the difficulty[2] consists here in finding an appropriate way to *state* the martingale problem: the proof that it is well posed (Theorem 24) is identical to [4, Th. 7.2.2].

In addition to the hypotheses and notation of §1.3, we now restrict ourself to the *binary branching mechanism*, that is, we additionally assume that Φ has the form $\Phi(x, \lambda) = \lambda^2$. Throughout the rest of this paper we fix $r \geq 0$ and $\mu \in \mathcal{M}_{\mathrm{f}}$.

6.1. The $(\xi, (.)^2, k)$-superprocess. According to [19, Th.1.3 and Rem 1.1], there exists a continuous version of the $(\xi, (.)^2, k)$-superprocess. Let $X = (X_t, \Im, P_{r,\mu}^{(\xi,(.)^2,k)})$ denote the *canonical* $(\xi, (.)^2, k)$-superprocess realized on $C_{[r,\infty)}(\mathcal{M}_{\mathrm{f}})$, the subspace of $D_{[r,\infty)}(\mathcal{M}_{\mathrm{f}})$ consisting of continuous trajectories. It clearly follows from Theorem 4, that $P_{r,\mu}^{(\xi,(.)^2,k)}$ is the unique distribution on $C_{[r,\infty)}(\mathcal{M}_{\mathrm{f}})$ such that $P_{r,\mu}^{(\xi,(.)^2,k)}$ solves the full $(\xi, (.)^2, k)$-martingale problem. It follows from Itô's formula and uniqueness of the full $(\xi, (.)^2, k)$-martingale problem that $P_{r,\mu}^{(\xi,(.)^2,k)}$ is the only distribution on $C_{[r,\infty)}(\mathcal{M}_{\mathrm{f}})$ such that, for every $\varphi \in \mathcal{D}(\mathrm{A})$,

$$t \mapsto M_t(\varphi) := \langle X_t, \varphi\rangle - \langle X_r, \varphi\rangle - \int_r^t \langle X_s, \mathrm{A}\varphi\rangle\, ds.$$

is a square integrable $P_{r,\mu}^{(\xi,(.)^2,k)}$-martingale with quadratic variation $\hat{Q}(\varphi^2)(ds)$, where $\hat{Q}(\varphi^2)(ds)$ is the lifting of $\varphi^2(\xi_s)k(ds)$.

One easily checks that

$$\langle M(f), M(g)\rangle (ds) = \hat{Q}(fg)(ds). \tag{26}$$

We denote by M the (orthogonal) martingale measure (with intensity $\nu\left((r,t] \times A\right) = \int_r^t \hat{Q}(1_A)(ds)$) extending the martingales $M_t(\varphi)$. We denote by $Q(ds, dx, dy)$ the covariance functional of M. It is clear from (26) that $Q(ds, f, g) = \hat{Q}(fg)(ds)$ for every $f, g \in pb\mathcal{B}$. We set

$$\mathcal{P}_M := \{f : f(\omega, s, x) \in \mathcal{P} \times B,\ P_{r,\mu}^{(\xi,(.)^2,k)} \int_{R_+ \times E} f^2(\omega, s, x)\nu(\omega, ds, dx) < \infty\},$$

[2] See the discussion in §1.

where $\mathcal{P}$ is the predictable σ-algebra. For any $g : \mathcal{M}_{\mathrm{f}} \to \hat{C}(E)$ which is bounded and measurable we pose

$$Z_g(t) = \exp\left\{ \int_r^t \int_E g(X_s, y) M(ds, dy) - \frac{1}{2} \int_r^t \int_E \int_E g(X_s, x) g(X_s, y) Q(ds, dx, dy) \right\}.$$

6.2. Statement of the $(\xi, (.)^2, k, \Re)$-martingale problem. In addition to the notation of §6.1, let $\Re$ denote a bounded an measurable mapping from $\mathcal{M}_{\mathrm{f}}$ to $\hat{C}(E)$. Clearly, $Q(ds, dx, dy)$ is defined up to $P_{r,\mu}^{(\xi,(.)^2,k)}$-indistinguishability, but if $P_{r,\mu}$ is another distribution on $C_{[r,\infty)}(\mathcal{M}_{\mathrm{f}})$, it may not be the case that $Q(ds, dx, dy)$ is defined up to $P_{r,\mu}$-indistinguishability. Choose and fix any of the $P_{r,\mu}^{(\xi,(.)^2,k)}$-indistinguishable version of $Q(ds, dx, dy)$. The statement of the $(\xi, (.)^2, k, \Re)$-martingale problem is:

There exists one and only one distribution $P_{r,\mu}^{(\xi,(.)^2,k,\Re)}$ on $C_{[r,\infty)}(\mathcal{M}_{\mathrm{f}})$ such that for every $\varphi \in \mathcal{D}(\mathrm{A})$,

$$t \mapsto M_t^{\Re}(\varphi) := \langle X_t, \varphi \rangle - \langle X_r, \varphi \rangle - \int_r^t \langle X_s, \mathrm{A}\varphi \rangle \, ds - \int_r^t \int \int \Re(X_s)(y) \varphi(x) Q(ds, dx, dy)$$

is a continuous local martingale with increasing process

$$\int_r^t \int \int \varphi(y) \varphi(x) Q(ds, dx, dy),$$

and such that $t \mapsto Z_{-\Re}(t)$ is a martingale.

Remark 3. Theorem 24 below asserts that $P_{r,\mu}^{(\xi,(.)^2,k,\Re)}$ and $P_{r,\mu}^{(\xi,(.)^2,k)}$ are equivalent measures. Therefore the solution of the $(\xi, (.)^2, k, \Re)$-martingale problem does not depend on the choice of a particular version of $Q(ds, dx, dy)$.

6.3. $(\xi, (.)^2, k)$-superprocesses with interactions. In addition to the hypotheses and notation introduced so far, we now require that[3]:

6.3A For every $\theta > 0$, and every $t \geq r$

$$P_{r,\mu}^{(\xi,(.)^2,k)} \left(e^{\theta Q((r,t],E,E)} \right) = P_{r,\mu}^{(\xi,(.)^2,k)} \left(e^{\theta \hat{Q}(1)(r,t]} \right) < \infty.$$

Let $g : \mathcal{M}_{\mathrm{f}} \to \hat{C}(E)$ be bounded and measurable. Since X_s is continuous and g is bounded and measurable, $g(X_s, .) \in \mathcal{P}_M$ and by [4, Th. 7.1.6] the stochastic integral $\int_r^t \int g(X_s, y) M(ds, dy)$ is a continuous martingale with increasing process $\int_r^t \int_E \int_E g(X_s, x) g(X_s y) Q(ds, dx, dy)$. Therefore, by [4, Th. 7.1.7], $Z_g(t)$ is a continuous $P_{r,\mu}^{(\xi,(.)^2,k)}$-local martingale. It follows from [4, Th. 7.1.7] that, that under assumption 6.3A, $t \mapsto Z_g(t)$ is a $P_{r,\mu}^{(\xi,(.)^2,k)}$-martingale. The later is required in [4, Th. 7.2.2] and this is why assumption 6.3A was introduced.

[3] This condition can be compared to a condition in [9] which asserts that for every $t > 0$ and every $\theta > 0$, $\sup_{r<t} \sup_x \pi_{r,x} e^{\theta k(r,t)} < \infty$.

Theorem 24 [A Dawson-Girsanov transformation]. *The* $(\xi, (.)^2, k, \Re)$*-martingale problem is well posed, and its solution* $P_{r,\mu}^{(\xi,(.)^2,k,\Re)}$ *is a measure which is equivalent to* $P_{r,\mu}^{(\xi,(.)^2,k)}$.

Proof. The proof is *identical* to the proof in [4, Th. 7.2.2]. In Dawson's argument, one should only replace 1° $Q(X_s, dx, dy)ds$ by $Q(ds, dx, dy)$, 2° r by $\Re$ and 3° $R(X_s, dx)ds$ by $\int \Re(X_s, y)Q(ds, dx, dy)$. ■

7. Appendix

Theorem 25. *Let* x^n, $n = 1, ..., \infty$ *be right continuous supermartingales and* A^n, $n = 1, ..., \infty$ *their compensators. Assume the* x^n *belong uniformly to the class* (D)*. Assume also that for every stopping time* τ, x_τ^n *converges weakly in* L^1 *to* x_τ^∞ *and that* $\sup_{0 \le s \le T} |x_s^n - x_s|$ *converges to zero in probability. Then for every stopping time* τ, A_τ^n *converges to* A_τ^∞ *in* L^1.

Proof. See [8, VII.19 and 20]. ■

Theorem 26 [joint continuity in fdd]. Consider branching functionals $k^1, ..., k^\infty = k$ being uniformly of bounded characteristic. Suppose that for every starting point $(r, x) \in [0, T] \times E$ and every r–stopping time $\sigma \le T$ we know that $k^n(r, \sigma]$ converges to $k(r, \sigma]$ in $L^1(\pi_{r,x})$ as $n \to \infty$. *Then the related log-Laplace functionals converge:*

$$v_{r,t}^n(f)(x) \underset{n}{\rightarrow} v_{r,t}(f)(x), \qquad 0 \le r \le t \le T, \quad x \in E, \quad f \in b\mathcal{E}_+ .$$

Proof. See [6, Th. 23]. ■

Definition 27. *We say that a mapping* $\Gamma(s, x, \lambda)$ *is* **locally in λ strongly continuous** *if for every* $s \ge 0$ *and every* $\Lambda \ge 0$

$$\lim_{t \to s} \sup_{x \in E, 0 \le \lambda \le \Lambda} |\Gamma(s, x, \lambda) - \Gamma(t, x, \lambda)| = 0$$

Theorem 28. *Let* $(\xi, \mathcal{F}, \pi_{r,x})$ *be a time homogeneous right process with value in a metrizable Luzin space* $(E, \mathcal{E})$*. Let* S_t *denote the semigroup of* ξ *and let* $\mathcal{L} \subseteq b\mathcal{E}$ *denote the set of functions* $f \in b\mathcal{E}$ *such that* $S_t(f)(x)$ *is strongly continuous. Let* $(\mathrm{A}, \mathcal{D}(\mathrm{A}))$ *be the (strong) generator of* S*. Let* $\Phi(s, x, \lambda)$ *be a non negative mapping such that* $\Phi(s, x, \varphi(x)) \in \mathcal{L}$ *for every* $\varphi \in \mathcal{D}(\mathrm{A})$ *and such that for each* $\Lambda, T \in R_+$,

$$\|\Phi_s'\|_\infty \vee \|\Phi_s''\|_\infty \vee \|\Phi_\lambda'\|_\infty \vee \|\Phi_\lambda''\|_\infty =: M(\Lambda, T) =: M < \infty$$

where the supremum is taken over the triples (s, x, λ) *such that* $0 \le s \le T, x \in E, 0 \le \lambda \le \Lambda$*. Assume that* Φ *and its derivatives are locally in* λ *strongly continuous. Then for each* $\varphi \in \mathcal{D}(A)$*, there exists a unique solution* v *to the equation*

$$v_{t,T}(\varphi)(x) = S_{T-t}\varphi(x) - \int_t^T S_{r-t}\left[\Phi(r, v_{r,T}(\varphi))\right](x)dr.$$

v satisfies the properties

1) $v_{t,T}(\varphi)(x)$ *belongs to* $\mathcal{D}(\mathrm{A})$ *for every* t;
2) $\frac{\partial}{\partial t} v_{t,T}(\varphi)(x)$ *exists and* $\left\| \frac{v_{t+h,T}(\varphi)(.) - v_{t,T}(\varphi)(.)}{h} - \frac{\partial}{\partial t} v_{t,T}(\varphi)(.) \right\|_{\infty} \to 0$;
3) $v_{t,T}$, $\frac{\partial}{\partial t} v_{t,T}$ *and* $Av_{t,T}$ *are bounded and strongly continuous.*

Moreover

$$\frac{\partial}{\partial t} v_{t,T}(\varphi)(x) + Av_{t,T}(\varphi)(x) = \Phi(t, x, v_{t,T}(\varphi)(x)).$$

Proof. See [18, Th 2]. ■

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