# Continuous dependence of a class of superprocesses on branching parameters, and applications

Donald A. Dawson* Klaus Fleischmann Guillaume Leduc†

(WIAS preprint No. 317 of February 10, 1997)

Fields Institute for Mathematical Research
Toronto
Canada M5T 3J1
e-mail: ddawson@fields.utoronto.ca

Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstr. 39
D–10117 Berlin, Germany
e-mail: fleischmann@wias-berlin.de

Départment de mathématiques
Université du Québec à Montréal
case postale 8888, succursale Centre-Ville
Montréal, Canada H3C 3P8
e-mail: leduc@math.uqam.ca

*Mathematics Subject Classification* Primary 60J80; Secondary 60J40, 60G57

*Keywords* superprocess, branching functional of bounded characteristic, measure-valued branching, regular local branching mechanism, càdlàg right process, Hunt process, Feller process, weak convergence on path spaces, continuity theorem, log-Laplace functional, Skorohod path space, catalytic branching rate

*Running head* Continuous dependence, and applications

cont.tex typeset by LaTeX

*Supported by an NSERC grant.
†Research supported by NSERC.

### Abstract

A general class of finite variance critical $(\xi, \Phi, k)$–superprocesses $X$ in a Luzin space $E$ with càdlàg right motion process $\xi$, regular local branching mechanism $\Phi$, and branching functional $k$ of bounded characteristic are shown to continuously depend on $(\Phi, k)$. As an application we show that the processes with a classical branching functional $k(\mathrm{d}s) = \varrho_s(\xi_s)\mathrm{d}s$ (that is a branching functional $k$ generated by a classical branching rate $\varrho_s(y)$) are dense in the above class of $(\xi, \Phi, k)$–superprocesses $X$. Moreover, we show that, if the phase space $E$ is a compact metric space and $\xi$ is a Feller process, then always a Hunt version of the $(\xi, \Phi, k)$–superprocess $X$ exists. Moreover, under this assumption, we even get continuity in $(\Phi, k)$ in terms of weak convergence of laws on Skorohod path spaces.

# Contents

# 1 Introduction

## 1.1 Motivation, purpose, and main results

While the characterization of the class of $(\xi,\Phi,k)$–*superprocesses* $X$ is obviously a fundamental part of the theory of measure-valued branching processes, it cannot alone fully describe the reach structure of this class. In particular, it would be natural to define a meaningful metric in terms of only the parameters $(\xi,\Phi,k)$. Topological properties of this metric, such as for instance the description of dense or compact subsets, or such as the completeness property, would give further insight into the nature of superprocesses. As a long run goal, it seems to be desirable to express properties of $(\xi,\Phi,k)$–superprocesses in terms of the parameters $(\xi,\Phi,k)$ only, and this paper should be seen as a step in this direction.

Indeed, we focus here on the question of *jointly continuous dependence* on the branching mechanism $\Phi$ and the branching functional $k$. Once one has such a continuous dependence, it can for instance be used to derive certain properties of a class of superprocesses by starting from more elementary processes, rather than by a direct analysis. We will in fact include below such applications.

The problem of continuous dependence of superprocesses on their branching rate is not entirely new. For instance in Dawson and Fleischmann [DF91, Lemma 2.3.5 and its application in §§ 2.4 – 2.5], it was used to construct a class of one-dimensional superprocesses with *catalytic* branching rate $\varrho_s(\mathrm{d}y)$ by starting from superprocesses with classical branching rate $\varrho_s(y)\,\mathrm{d}y$. Or in [DF97, Propositions 6 and 12] continuity in $k$ was exploited to construct super-Brownian motions in $\mathsf{R}^d$ with (only) *locally* admissible branching functional $k$ by approximating them by (globally) admissible ones. In this way, a class of super-Brownian motions constructed by Dynkin [Dyn94] could be extended. Finally, in Fleischmann and Mueller [FM97], a truncation procedure of branching rate was applied to construct a one-dimensional super-Brownian motion with the locally *infinite* catalytic mass $|y|^{-2}\mathrm{d}y$. (In contrast to the present paper, this superprocess does not have a finite variance even though the branching mechanism is "binary critical".)

The question of continuous dependence of superprocesses on their branching mechanism $\Phi$ and branching functional $k$ is studied here for its own and partly in considerably more generality. But then we use this continuity to prove that for each $(\xi,\Phi,k)$–superprocess considered in this paper, a *Hunt version exists,* *provided that* the phase space is a *compact* metric space and the motion process $\xi$ is *Feller* (Theorem 44 at p.33). In this case we even get continuity in $(\Phi, k)$ in terms of weak convergence of the laws on Skorohod space of càdlàg paths (Theorem 46 at p.34).

The construction of superprocesses with *regularity properties of the paths* has a long history. Concerning recent general results, in the first place we refer to Fitzsimmons [Fit88], who proved the existence of a right or even Hunt version of a superprocess if the motion process is right or Hunt, respectively, provided that the branching mechanism is time-homogeneous and the branching functional is given by $k(\mathrm{d}s) = \mathrm{d}s$. Dynkin [Dyn93] and Kuznetsov [Kuz94] generalized Fitzsimmons' right version result. Finally, Leduc [Led97a] generalized Fitzsimmons' Hunt result to a general class of $(\xi, \Phi, k)$–superprocesses with finite variance and admissible (in the sense of Dynkin) functional $k$. One of our motivations was to obtain such result for *non-admissible* $k$ of bounded characteristic.

We finally mention that the results of the present paper play a crucial role in Leduc [Led97b] where a martingale problem is established for a class of $(\xi, \Phi, k)$–superprocesses under mild conditions.

## 1.2 Setup

Before going further, recall that main steps of the method of construction of superprocesses via the analysis of the related evolution equation (such as for instance in [Daw77, DF91, Dyn91, Dyn94, Led97a, FM97, DF97]) more or less resemble the following procedure. First, find for $n$ fixed a measure-valued process $X^n$ which log-Laplace functional $v^n = v^n(f) = v^n_{\cdot,t}(f)$ solves an *evolution equation*

$$v^n = \Psi^n(v^n). \tag{1}$$

Second, show that, for a certain norm $\|\cdot\|$ (typically a supremum norm $\|\cdot\|_\infty$, or a closely related one),

$$\|v^m - v^n\| \leq \tfrac{1}{2}\|v^m - v^n\| + q_{m,n}$$

where $q_{m,n}$ is a non-negative quantity converging to zero as $m,n \to \infty$. By completeness, this shows, that $v^n$ converges. It is usually possible to conclude

- that the limit $v$ again satisfies an evolution equation

$$v = \Psi(v), \tag{2}$$

- that $v$ is the unique solution to that equation,
- that each $v_{r,t}(x)$ is the log-Laplace functional of a random measure,
- and that $v$ determines a semigroup.

This semigroup then uniquely characterizes a superprocess $X$ (log-Laplace functional characterization).

Suppose now that (1) is the $(\xi, \Phi^n, k^n)$–*evolution equation* of the so-called $(\xi, \Phi^n, k^n)$–*superprocess* $X^n$. Here $\Psi^n$ is a functional of $\xi, \Phi^n, k^n$, where

- the particles' motion process $\xi = (\xi_t\,, \Im, \pi_{r,x})$ is càdlàg right Markov,
- $\Phi^n$ is a critical local branching mechanism with finite variance (see Assumption 17(f) at p.16), and
- the branching functional $k^n$ is a continuous additive functional of $\xi$ of bounded characteristic.

Our *key result* can briefly be described as follows. Suppose that $k^n$ converges to a continuous additive functional $k$ of $\xi$ in an appropriate sense, and the $\Phi^n$ converge uniformly to a regular branching mechanism $\Phi$, then the log-Laplace functionals $v^n$ converge to some $v$ solving the $(\xi, \Phi, k)$–evolution equation (2). As in Leduc [Led97a], this equation is then used to construct a $(\xi, \Phi, k)$–superprocess $X$ with $v$ as its log-Laplace functional. Since the convergence $v^n \rightarrow_n v$ of log-Laplace functionals implies the convergence $X^n \Rightarrow_n X$ in the sense of (weak) convergence of all finite-dimensional distributions (*fdd*), we get that the $(\xi, \Phi, k)$–superprocess *continuously depends* on $(\Phi, k)$ (Theorem 23 at p.18).

This fdd continuity theorem can be extended to weak convergence on some Skorohod path spaces, and several *applications* are supplied. In particular, if the phase space is a compact metric space and $\xi$ is Feller, we show that a *Hunt* version of $X$ exists and "classical" $\big(\xi, \Phi, \varrho_s(\xi_s)\mathrm{d}s\big)$–superprocesses are weakly dense in the set of all $(\xi, \Phi, k)$–superprocesses.

### 1.3 Outline

To prove the continuity theorem, in principal we follow the general idea which we described in the previous subsection. The norm which we use, is essentially the *norm* $\|\cdot\|_C$ defined to be the supremum over the set $C$ of all those points $(r, x)$ such that

$$\begin{array}{lrcl} (\alpha) & \pi_{r,x} \displaystyle\bigvee_{n=1}^{\infty} k^n(r,t] & < & \infty, \\ (\beta) & \pi_{r,x} \Big\{ k^n \underset{n}{\Longrightarrow} k \Big\} & = & 1 \end{array}$$

(recall $\pi_{r,x}$ refers to the law of the motion process $\xi$ with initial data $r, x$). Starting from a point $(r, x) \in C$, it is crucial to know that $\pi_{r,x}$–a.s. all points $(s, \xi_s),\ s > r$, also belong to $C$. This is essentially what we will cover in Section 2.

After introducing in the beginning of Section 3 more carefully the model we deal with in this paper, we formulate our *key result,* the fdd continuity Theorem 23 at p.18. Then we discuss the assumptions on the branching functional in that theorem, and review the log-Laplace functional characterization of $(\xi, \Phi, k)$–superprocesses. But the central part of our argument is Proposition 39 at p.26. It states that in the case $\Phi^n \equiv \Phi$, for small test functions $f$ (the parameter entering into the linear term of the evolution equation (2) coming from the log-Laplace functional), and for starting points $(r,x)$ in $C$, the log-Laplace functionals $v^n$ converge to some $v$.

The derived fdd continuity theorem has strong implications. First of all, as an *application* we establish in Theorem 26 at p.19 that each $(\xi,\Phi,k)$–superprocess can be *approximated* by ones with *"classical"* branching functional $k$. Classical here means that the branching functional $k$ can be represented as $k(\mathrm{d}s) = \varrho_s(\xi_s)\,\mathrm{d}s$ with $\varrho$ a bounded (classical) function. In this case, a particle at time $s$ at site $y$ splits with *branching rate* $\varrho_s(y)$. In other words, the approximating processes are *"classical"* superprocesses.

We mention that with *fdd convergence* of $X^n$ to $X$ we actually mean [1)]

$$E\exp\Big[\sum_{i=1}^m \big\langle X^n_{t_i}, -f_i\big\rangle\Big] \underset{n\to\infty}{\longrightarrow} E\exp\Big[\sum_{i=1}^m \big\langle X_{t_i}, -f_i\big\rangle\Big]$$

for *any* choice of bounded measurable non-negative functions $f_1,\dots,f_m$ on $E$. In other words, we have fdd convergence in *every* topology on $E$ compatible with the measurability structure $(E,\mathcal{E})$ of our Luzin space $E$.

Of course, one cannot expect that results in this generality should hold concerning weak convergence of laws on path spaces. In Section 4, in order to avoid expensive technicalities, we even restrict our attention to a much more restrictive situation: We consider the special case of a *Feller* motion process $\xi$ in a *compact* metric space $(E,d)$. Then the continuity and approximation theorems can be used to construct a *Hunt* version of the $(\xi,\Phi,k)$–superprocesses (Theorem 44 at p.33). These Hunt $(\xi,\Phi,k)$–superprocesses depend continuously on $(\Phi,k)$ in terms of weak convergence of the laws *on the Skorohod path spaces,* rather than only fdd (Theorem 46 at p.34).

In an appendix, we collect some results which are purely technical.

As a standard reference for weak convergence we refer to Ethier and Kurtz [EK86] and for $(\xi,\Phi,k)$–superprocesses to Dynkin [Dyn94].

## 1.4 Basic assumptions: motion process $\xi$ and branching functional $k$

In this paper, *'non-negative'* always means $\mathsf{R}_+$–valued, $\mathsf{R}_+ := [0,\infty)$. But at some places we need also to consider variables with values in the usual one-point compactification $\overline{\mathsf{R}}_+ := [0,\infty]$ of $\mathsf{R}_+$. In this case, we will explicitly refer to this.

**Assumption 1 (motion process)** Throughout this paper, the following assumptions are in force:

---

1) $\langle\mu,f\rangle$ abbreviates the integral $\int f(x)\,\mu(\mathrm{d}x)$.

**(a) (phase space)** The phase space $E$ is a *Luzin* space [2]. With this we mean a topological space $E$ which is homeomorphic to a Borel subset of a compact metrizable space. Let $\mathcal{E}$ denote the Borel $\sigma$–algebra of $E$, and $\mathcal{E}_+ = \mathcal{E}_+(E)$ the set of all $\overline{\mathsf{R}}_+$–valued measurable functions $f$ on $E$. Moreover, write $b\mathcal{E}_+ = b\mathcal{E}_+(E)$ for the subset of all bounded $f \in \mathcal{E}_+$, equipped with the topology of *bounded pointwise* convergence.

**(b) (measure space)** Let $\mathcal{M}_\mathrm{f} = \mathcal{M}_\mathrm{f}(E) = \mathcal{M}_\mathrm{f}(\mathcal{E})$ denote the set of all finite measures on $\mathcal{E}$. Endowed with the topology of *weak* convergence, $\mathcal{M}_\mathrm{f}$ is a Luzin space.

**(c) (time interval)** We consider first of all stochastic processes on a fixed finite interval $I := [0,T]$, $T > 0$, or on subintervals of $I$; later, in Section 4, we extend to $\mathsf{R}_+$.

**(d) (underlying particle's motion process $\xi$)** Once and for all, fix an $E$–valued process $\xi$ on $I$ satisfying the following conditions:

**(d1) (Markov process)** $\xi$ is a (time-inhomogeneous) *Markov process* $(\xi_t\,, \Im, \pi_{r,x})$ in Dynkin's [Dyn94, § 2.2.1] setting.

**(d2) (right process)** This Markov process $\xi$ is assumed to be a *right process:* [3]

**(i)** $t \mapsto \xi_t(\omega)$ is right continuous (in the Luzin $E$), for *each* $\omega$.

**(ii)** For $0 \le r \le t \le T$, $\mu \in \mathcal{M}_\mathrm{f}$, and $f \in \mathcal{E}_+$ fixed, the function $s \mapsto \pi_{s,\xi_s} f(\xi_t)$, $s \in [r,t)$, is right continuous $\pi_{r,\mu}$–almost everywhere.

**(d3) (càdlàg)** The process $\xi$ is required to be *càdlàg* (additionally to (i)), that is, for *each* $\omega$, the limits $\lim_{s\uparrow t} \xi_s =: \xi_{t-}$ exist in $E$ for all $t \in (0,T]$.

**(d4) (Hunt)** *Sometimes* we additionally assume that the càdlàg right Markov process $\xi$ is *Hunt.* In this case we work with $\mathsf{R}_+$ as the time axis.

**(e) (branching functional)** As a rule, the letter $k$ refers to a (non-negative) *continuous additive functional* of $\xi$ ([Dyn94, § 2.4.1]) of *bounded characteristic:*

$$\sup_{(r,x)\in I\times E} \pi_{r,x}\, k(r,T] < \infty. \tag{3}$$

We call such $k$ a *branching functional.* Intuitively, $k(\mathrm{d}s)$ is the rate of branching of a particle with position $\xi_s$ at time $s$. ◇

---

[2] Note that e.g. every complete separable metric space is Luzin (see, for instance, Sharpe [Sha88, p.370]).

[3] Note that our terminology differs slightly from Dynkin [Dyn94] we often quote: Dynkin includes the càdlàg property (d3) in his notion of a right process, but we speak in this situation more carefully of a càdlàg right process.

**Remark 2 (admissible functionals)** Note that condition (3) is weaker than Dynkin's [Dyn94, § 3.3.3] *admissibility* requirement

$$\sup_{x \in E} \pi_{r,x}\, k(r,t) \underset{r,t \to s}{\longrightarrow} 0, \qquad s \in I. \tag{4}$$

(In fact, read the proof of Lemma 3 in [DF97] with $\phi_p$ replaced by 1.) $\diamond$

**Remark 3 (natural functionals $k$)** Several partial results in the present paper remain valid if the (limiting) additive functional $k$ is only *natural* (instead of continuous). But we stress the fact that in our key Theorem 23 (p.18), the assumption on the continuity of $k$ *cannot* be dropped. Moreover, in [FL97] we will show that under mild conditions as in the present paper *all* branching functionals $k$ are continuous. $\diamond$

# 2 Path and preservation properties

In this section we investigate the following *question*. Suppose that for a *"starting point"* $(r,x)$ a certain *property* $\wp$ of particles' motion process $\xi$ holds $\pi_{r,x}$–a.s. When can we say that, $\pi_{r,x}$–a.s., the process $s \mapsto (s, \xi_s)$ passes only through those points $(s,y)$ such that the property $\wp$ is valid $\pi_{s,y}$–a.s.?

An *example* of that sort of questions is the following case (which will essentially interest us later in this section). Suppose that $k^1, k^2, \ldots$ are (continuous) additive functionals of the (càdlàg right Markov) process $\xi = (\xi_t\,, \Im, \pi_{r,x})$. Fix a starting point $(r,x) \in I \times E$. Assume that $\pi_{r,x}$–almost surely the (finite) measures $k^n$ (as measures on $[r,T]$) converge weakly to $k$ as $n \to \infty$. Is it then the case that $\pi_{r,x}$–almost surely, for every $s \in [r,T]$, with $\pi_{s,\xi_s}$–probability one, $k^n$ converges weakly to $k$ (as measures on $[s,T]$)?

With Proposition 12 at p.13, we will give a positive answer to this type of question. At this place it might be helpful to give a *heuristic reasoning* which indicates the strategy we will use. Suppose that the following expectation vanishes:

$$\pi_{r,x}\left(\sup_{s \in [r,T]} \limsup_{n} \left| k^n(s,T) - k(s,T) \right| \right) = 0.$$

Then, for any point $s \in [r,T]$, the Markov property gives that

$$\pi_{r,x}\left(\pi_{s,\xi_s}\left(\sup_{t \in [s,T]} \limsup_{n} \left| k^n(t,T) - k(t,T) \right| \right)\right) = 0.$$

Obviously, this remains true, for countably many $s$. Hence, if the process

$$s \mapsto \pi_{s,\xi_s}\left(\sup_{t \in [s,T]} \limsup_{n} \left| k^n(t,T) - k(t,T) \right| \right)$$

could be verified to be right continuous, we get that

$$\pi_{r,x}\Big(\sup_{s\in[r,T]}\pi_{s,\xi_s}\Big(\sup_{t\in[s,T]}\limsup_{n}\Big|k^n(t,T]-k(t,T]\Big|\Big)\Big) \;=\; 0,$$

as wanted.

This reasoning motivates in particular the following subsection.

## 2.1 Path properties of some class of processes

For convenience, we impose the following assumption (which will be in force throughout this subsection).

**Assumption 4** Fix a starting point $(r,x)\in I\times E$. For $s\in[r,T]$, let $Y_s$ and $Z_s$ be $\overline{\mathsf{R}}_+$–valued $\Im[s,T]$–measurable[4)] variables. Define $y_s := \pi_{s,\xi_s}Y_s$ and $z_s := \pi_{s,\xi_s}Z_s$ (which could be infinite at this stage). Suppose $\pi_{r,x}\,Y_r<\infty$. $\diamond$

Note that we do not assume $\pi_{s,y}Y_s<\infty$ for every $(s,y)\in[r,T]\times E$. Recall that 'non-negative' always means $\mathsf{R}_+$–valued. The main result of this subsection is:

**Proposition 5 (non-negative càdlàg process of class (D))** *Impose Assumption 4. Let in addition $s\mapsto Y_s$ be right continuous and non-increasing (for each $\omega$, as $\overline{\mathsf{R}}_+$–valued functions). Then the following statements hold:*

**(i)** *The process $y=\{y_s : r\le s\le T\}$ is $\pi_{r,x}$–indistinguishable from a non-negative càdlàg process of class $(D)$.*

**(ii)** *If additionally $Z\le Y$, and $s\mapsto Z_s$ is càdlàg (as $\overline{\mathsf{R}}_+$–valued functions), then $z=\{z_s : r\le s\le T\}$ is also $\pi_{r,x}$–indistinguishable from a non-negative càdlàg process of class $(D)$.*

Before providing the proof, we need some lemmas and definitions. Consider $Y$ as in the theorem. For every $c\in[0,\infty]$, define

$$y^c_s \;:=\; \pi_{s,\xi_s}Y^c_s, \qquad Y^c_s \;:=\; c\wedge Y_s\,.$$

Note that $Y^\infty_s=Y_s$ and $y^\infty_s=y_s$ .

**Lemma 6** *Let $c\in[0,\infty]$. Suppose that with respect to $\pi_{r,x}$ the process $y^c$ is indistinguishable from a non-negative process and belongs to class $(D)$. Then it is $\pi_{r,x}$–almost surely right continuous.*

**Proof** *Step* 1° We first establish that $y^c$ is optional. Given $\xi$ (with respect to $\pi_{r,x}$), for $n\ge1$ introduce the step function

$$y^{n,c}_s \;:=\; \sum_{n=0}^{n-1}\mathbf{1}_{[s^n_i,\,s^n_{i+1})}(s)\,\pi_{s,\xi_s}Y^c_{s^n_{i+1}}, \qquad r\le s\le T, \tag{5}$$

4) Note that $\Im^s_T=\Im[s,T]$ is *not* a filtration since $\Im^a_T\subset\Im^b_T$, $a\le b$, does *not* hold.

where $s_i^n := r+\frac{i}{n}(T-r)$, for $i = 0, ..., n$. Obviously, the $\pi_{r,x}$–almost surely non-negative process $y^{n,c}$ is $\pi_{r,x}$–a.s. right continuous [5] and thus optional. Clearly, pointwise $y_s^c = \lim_n y_s^{n,c}$ holds. Therefore $y^c$ is also optional.

*Step 2°* Let $\sigma_n \le T$ be $r$–stopping times non-increasing to (the $r$–stopping time) $\sigma$ as $n \to \infty$. Then by the definition of $y^c$, the strong Markov property, right continuity of $Y^c$, and the monotone convergence theorem, we have

$$\lim_n \pi_{r,x}\, y_{\sigma_n}^c \;=\; \lim_n \pi_{r,x}\, \pi_{\sigma_n,\xi_{\sigma_n}} Y_{\sigma_n}^c \;=\; \lim_n \pi_{r,x}\, Y_{\sigma_n}^c \;=\; \pi_{r,x}\, Y_\sigma^c \;=\; \pi_{r,x}\, y_\sigma^c\,.$$

Hence, according to [Dyn94, A.1.1.D, p.116], the $\pi_{r,x}$–a.s. non-negative process $y^c$ is $\pi_{r,x}$–a.s. right continuous. ■

**Corollary 7** *For every $c \in [0,\infty)$, the non-negative process $y^c$ is $\pi_{r,x}$–a.s. right continuous and belongs to class $(D)$.*

**Proof** This is immediate from the above lemma and the fact that these processes are bounded (by $c$). ■

**Lemma 8** *The $\overline{\mathsf{R}}_+$–valued process $y$ is $\pi_{r,x}$–indistinguishable from a non-negative process (that is, $\mathsf{R}_+$–valued process).*

**Proof** According to Corollary 7, for $c$ finite, the non-negative process $y^c$ is $\pi_{r,x}$–a.s. right continuous. Therefore, $\sup_{r\le s\le T} y_s^c$ is measurable, and monotonously converges to $\sup_{r\le s\le T} y_s$ as $c \uparrow \infty$. Hence, for $\eta > 0$,

$$\pi_{r,x}\big\{\sup\nolimits_{r\le s\le T}\, y_s > \eta\big\} \quad=\quad \lim\nolimits_{c\to\infty}\, \pi_{r,x}\big\{\sup\nolimits_{r\le s\le T}\, y_s^c > \eta\big\}.$$

We can thus invoke Proposition 58 of p.42 in the appendix, and continue with

$$\begin{aligned}
\pi_{r,x}\big\{\sup\nolimits_{r\le s\le T}\, y_s > \eta\big\} &\le \eta^{-1}\lim\nolimits_{c\to\infty}\, \sup\nolimits_{r\le\sigma\le T}\, \pi_{r,x}\, y_\sigma^c \\
&= \eta^{-1}\lim\nolimits_{c\to\infty}\, \sup\nolimits_{r\le\sigma\le T}\, \pi_{r,x}\, Y_\sigma^c \\
&\le \eta^{-1}\lim\nolimits_{c\to\infty}\, \pi_{r,x}\, Y_r^c \\
&\le \eta^{-1}\pi_{r,x}\, Y_r \;<\; \infty.
\end{aligned}$$

Letting $\eta \to \infty$ gives the claim. ■

**Lemma 9** *With $\pi_{r,x}$–probability 1, $y$ is non-negative, and it belongs to class $(D)$.*

[5] If $Y_s$ has the form $Y_s := f(s,\xi_s)$ for a measurable bounded $f$ then the $\pi_{r,x}$–a.s. right continuity of $y^{n,c}$ is immediate form the definition of a right process (see [Dyn94, p.27]). The more general case reduces to the just mentioned one by taking the conditional expectation.

**Proof** First of all, for $r$–stopping times $\sigma \leq T$,

$$\sup_{r \leq \sigma \leq T} \pi_{r,x}\, y_\sigma \;=\; \sup_{r \leq \sigma \leq T} \pi_{r,x}\, Y_\sigma \;\leq\; \pi_{r,x}\, Y_r \;<\; \infty$$

by the Markov property and monotonicity of $Y$.

Consider a collection of measurable sets $\Gamma_n$ with the property $\pi_{r,x}(\Gamma_n) \searrow 0$ as $n \to \infty$. Let us indicate by $\pi_{r,x}^{\Im_\sigma^r}$ the conditional expectation with respect to $\Im_\sigma^r := \Im[r,\sigma]$. We have that

$$\pi_{r,x} \mathbf{1}_{\Gamma_n} y_\sigma \;=\; \pi_{r,x} \pi_{r,x}^{\Im_\sigma^r} \mathbf{1}_{\Gamma_n} y_\sigma \;=\; \pi_{r,x}\big(\pi_{r,x}^{\Im_\sigma^r} \mathbf{1}_{\Gamma_n}\big) y_\sigma$$

since $y_\sigma$ is measurable with respect to $\Im_\sigma^r$. By the strong Markov property, we can continue with

$$\pi_{r,x}\big(\pi_{r,x}^{\Im_\sigma^r} \mathbf{1}_{\Gamma_n}\big) y_\sigma = \pi_{r,x}\big(\pi_{r,x}^{\Im_\sigma^r} \mathbf{1}_{\Gamma_n}\big) \pi_{\sigma,\xi_\sigma} Y_\sigma = \pi_{r,x}\big(\pi_{r,x}^{\Im_\sigma^r} \mathbf{1}_{\Gamma_n}\big) Y_\sigma \leq \pi_{r,x}\big(\pi_{r,x}^{\Im_\sigma^r} \mathbf{1}_{\Gamma_n}\big) Y_r\,.$$

Suppose it is the case that

$$\limsup_{n} \sup_{r \leq \sigma \leq T} \pi_{r,x}\big(\pi_{r,x}^{\Im_\sigma^r} \mathbf{1}_{\Gamma_n}\big) Y_r \;>\; 0. \tag{6}$$

Then we can find a sequence of $r$–stopping times $\sigma_n \leq T$ such that

$$\pi_{r,x}\Big(\pi_{r,x}^{\Im_{\sigma_n}^r} \mathbf{1}_{\Gamma_n}\Big) Y_r \;>\; \delta, \qquad n > 0, \tag{7}$$

for some $\delta > 0$. But

$$\pi_{r,x}\, \pi_{r,x}^{\Im_{\sigma_n}^r} \mathbf{1}_{\Gamma_n} \;=\; \pi_{r,x}(\Gamma_n) \;\underset{n}{\searrow}\; 0.$$

Hence $\pi_{r,x}^{\Im_{\sigma_n}^r} \mathbf{1}_{\Gamma_n}$ (strongly) converges to 0 in $L^1(\pi_{r,x})$. Since $0 \leq \pi_{r,x}^{\Im_{\sigma_n}^r} \mathbf{1}_{\Gamma_n} \leq 1$, we have that

$$0 \leq \Big(\pi_{r,x}^{\Im_{\sigma_n}^r} \mathbf{1}_{\Gamma_n}\Big) Y_r \leq Y_r \in L^1(\pi_{r,x}).$$

Consequently, appealing to the dominated convergence theorem (in the version of [EK86, Theorem A.1.2]), we obtain that

$$\pi_{r,x}\Big(\pi_{r,x}^{\Im_{\sigma_n}^r} \mathbf{1}_{\Gamma_n}\Big) Y_r \;\underset{n}{\longrightarrow}\; 0.$$

This contradicts (7). Therefore (6) is impossible, hence

$$\limsup_{n} \sup_{r \leq \sigma \leq T} \pi_{r,x} \mathbf{1}_{\Gamma_n} y_\sigma \;=\; 0.$$

That is, $y$ belongs to class $(D)$. ■

**Proof of Proposition 5** **(ii)** Besides $Y$, consider $Z$ as in the theorem. Immediately from the Lemmas 8, 9, and 6 it follows that $y$ is $\pi_{r,x}$–a.s. a non-negative right continuous process of class $(D)$. Since $0 \le Z_s \le Y_s$ we get that $0 \le z_s \le y_s$ , and therefore $z$ belongs to class $(D)$. We have to show that $z$ is $\pi_{r,x}$–a.s. càdlàg.

Consider

$$z_s^n := \sum_{n=0}^{n-1} \mathbf{1}_{[s_i^n,\, s_{i+1}^n)}(s)\, \pi_{s,\xi_s} Z_{s_{i+1}^n}\,, \qquad r \le s \le T,$$

where again $s_i^n := r + \frac{i}{n}(T-r)$, for $i = 0, ..., n$. The process $z^n$ is càdlàg $\pi_{r,x}$–a.s., and thus optional. We have that

$$\mathbf{1}_{[s_i^n,\, s_{i+1}^n)}(s) Z_{s_{i+1}^n} \le \mathbf{1}_{[s_i^n,\, s_{i+1}^n)}(s) Y_{s_{i+1}^n} \le \mathbf{1}_{[s_i^n,\, s_{i+1}^n)}(s) Y_s\,.$$

Since $y_s = \pi_{s,\xi_s} Y_s < \infty$, $\pi_{r,x}$–a.s., the above inequalities allow to invoke the dominated convergence theorem and we obtain

$$\sum_{n=0}^{n-1} \mathbf{1}_{[s_i^n,\, s_{i+1}^n)}(s)\, \pi_{s,\xi_s} Z_{s_{i+1}^n} \underset{n}{\longrightarrow} \pi_{s,\xi_s} Z_s\,.$$

That is $z_s^n \to_n z_s$ . Therefore the process $z$ is optional.

Let $\sigma_1, \sigma_2, ... \le T$ be a non-increasing sequence of $r$–stopping times converging to $\sigma$. Recall that by assumption $Z$ is $\overline{\mathsf{R}}_+$–valued càdlàg, and that

$$0 \le \sup_{r \le s \le T} Z_s \le Y_r \in L^1(\pi_{r,x}).$$

Hence, $Z$ is $\pi_{r,x}$–a.s. non-negative, and by definition,

$$\pi_{r,x}\, z_{\sigma_n} = \pi_{r,x}\, \pi_{\sigma_n,\xi_{\sigma_n}} Z_{\sigma_n} = \pi_{r,x}\, Z_{\sigma_n}\,.$$

Invoking the dominated convergence theorem, we get

$$\lim_n \pi_{r,x}\, z_{\sigma_n} = \lim_n \pi_{r,x}\, Z_{\sigma_n} = \pi_{r,x}\, Z_\sigma = \pi_{r,x}\, z_\sigma\,.$$

Hence, $z$ is $\pi_{r,x}$–a.s. right continuous (recall [Dyn94, A.1.1.D, p.116]). An analogous reasoning, invoking Lemma 57 from the appendix, shows that $z$ has also left limits $\pi_{r,x}$–a.s. Consequently, $z$ is $\pi_{r,x}$–a.s. non-negative càdlàg, proving **(ii)**.

**(i)** $Y$ itself satisfies the assumptions on $Z$ in **(ii)**, since it is in particular càdlàg. Hence, by the already proved statement **(ii)**, together with $z$ also $y$ is $\pi_{r,x}$–a.s. non-negative càdlàg, finishing the proof. ∎

## 2.2 The case of indistinguishability from zero

Recall that in this section we investigate conditions under which the following holds. If a certain property $\wp$ is true $\pi_{r,x}$–a.s., then $\pi_{r,x}$–a.s., the property $\wp$ is true $\pi_{s,\xi_s}$–a.s. for all $s \in [r,T]$. In this subsection now, $\wp$ is the property of being *indistinguishable from zero.*

**Lemma 10 (preservation of indistinguishability of zero)** *Fix a starting point* $(r,x) \in I \times E$. *Let* $Y_s\,,\ s \in [r,T]$, *again be* $\overline{\mathsf{R}}_+$*–valued* $\Im[s,T]$*–measurable variables. Suppose that* $Y = \{Y_s\}_{s\in[r,T]}$ *is non-increasing and right continuous, and that* $\pi_{r,x} Y_r < \infty$. *If* $\{Y_s\}_{s\in[r,T]}$ *is* $\pi_{r,x}$*–indistinguishable from zero, then*

$$\pi_{r,x}\Big\{\{Y_t\}_{t\in[s,T]} \text{ is } \pi_{s,\xi_s}\text{–indistinguishable from zero, } \forall s \in [r,T]\Big\} = 1, \quad (8)$$

*or equivalently*

$$\pi_{r,x}\bigg(\sup_{s\in[r,T]} \pi_{s,\xi_s}\Big(\sup_{t\in[s,T]} Y_t\Big)\bigg) \;=\; 0. \quad (9)$$

**Proof** According to Proposition 5 (i), the process $s \mapsto y_s = \pi_{s,\xi_s} Y_s$ is $\pi_{r,x}$–a.s. a non-negative càdlàg process of class $(D)$. By the strong Markov property, for every $r$–stopping time $\sigma \le T$,

$$\pi_{r,x}\, y_\sigma \;=\; \pi_{r,x}\, Y_\sigma \;=\; 0.$$

Hence, if $s \mapsto \mathbf{0}_s$ denotes the process which is constant and equal to 0, we have that for every $r$–stopping time $\sigma \le T$

$$\pi_{r,x}\, y_\sigma \;=\; \pi_{r,x}\, \mathbf{0}_\sigma\,. \quad (10)$$

Consider now $y$ as a process on the time axis $\mathsf{R}_+$ stopped at time $T$. Since $Y_T = 0$, $\pi_{r,x}$–a.s., then for every stopping time $\tau$ we have $Y_{T\wedge\tau} = Y_\tau$. Then (10) and [Dyn94, A.1.1.E, p.116] imply that $y$ is $\pi_{r,x}$–almost surely indistinguishable from zero. Since $Y$ is non-increasing, we have that $\sup_{s\le t\le T} Y_t := Y_s$. Hence $\pi_{s,\xi_s}\left(\sup_{s\le t\le T} Y_t\right)$ is $\pi_{r,x}$–a.s. indistinguishable from zero. This is exactly what we wanted to prove. ■

## 2.3 Preservation of initial properties for additive functionals

**Assumption 11 (initial properties of additive functionals)** Denote by $k^1, ..., k^\infty$ (non-negative) continuous additive functionals of our càdlàg right process $\xi = (\xi_t\,, \Im, \pi_{r,x})$. In the sequel we also write $k$ instead of $k^\infty$. We assume that, for the starting point $(r,x) \in I \times E$ we have

($\alpha$) $\pi_{r,x} \bigvee_{n=1}^{\infty} k^n(r,T] < \infty$,[6] and

($\beta$) with $\pi_{r,x}$–probability one, $k^n(s,T] \to_n k(s,T]$ for every $s \in [r,T]$. ◇

Note that the requirement "for every $s \in [r,T]$" in part ($\beta$) of Assumption 11 can be replaced by "for every *rational* $s \in (r,T]$ and $s = r$", hence it is a

[6] Note that we included $k^\infty$ in the definition of $k^n$, so that $k^\infty$ is also involved in such a supremum expression.

*measurable* assertion. In fact, $k^n(s,T]$ and $k(s,T]$ are monotone and continuous in $s$. Note also that $(\beta)$ *implies* that

$$\pi_{r,x}\text{–almost surely,}\quad k^n(s,t]\to_n k(s,t]\quad\text{whenever}\quad r\le s\le t\le T \tag{11}$$

(indeed, consider differences).

The main result of this section is:

**Proposition 12 (preservation of initial properties)** *Under Assumption 11, with $\pi_{r,x}$–probability one the process $s\mapsto(s,\xi_s)$, $s\in[r,T]$, will pass only through those points $(s,y)$ such that*

$(\alpha)$ $\pi_{s,y}\bigvee_{n=1}^{\infty}k^n(s,T]<\infty$, *and*

$(\beta)$ *with $\pi_{s,y}$–probability one, $k^n(t,T]\to_n k(t,T]$ for every $t\in[s,T]$.*

Before providing the proof of Proposition 12, we need to establish some preliminary results. For this purpose, for $s\in[r,T]$ introduce the following notation:

$$Y_s^1 := \bigvee_{n=1}^{\infty}k^n(s,T],\qquad Y_s^2 := \sup_{t\in(s,T]}\limsup_n\Big|k^n(t,T]-k(t,T]\Big|, \tag{12}$$

$$Y_s^3 := \limsup_n\Big|k^n(s,T]-k(s,T]\Big|, \tag{13}$$

and set $y_s^i := \pi_{s,\xi_s}Y_s^i$ for $i=1,2,3$.

**Lemma 13** *The variables $Y_s^i$, $i=1,2,3$, $s\in[r,T]$, are measurable.*

**Proof** We need only to consider $Y_s^2$. It suffices to show that for $a\ge 0$ fixed, we have $Y_s^2>a$ if and only if

$$\sup_{\text{rational } q\in(s,T]}\limsup_n\Big|k^n(q,T]-k(q,T]\Big| > a. \tag{14}$$

But $Y_s^2>a$ implies the existence of some $t\in(s,T]$ such that

$$\limsup_n\Big|k^n(t,T]-k(t,T]\Big| > a. \tag{15}$$

Given $\varepsilon>0$ small enough, the latter inequality yields

$$k^n(t,T]\ \ge\ k(t,T]+a+2\,\varepsilon\quad\text{for infinitely many } n,$$

or

$$k^n(t,T]\ \le\ k(t,T]-a-2\,\varepsilon\quad\text{for infinitely many } n.$$

In the first case, by the continuity of $k$, we find a rational $q \in (s,t)$ such that $k(q,t] \le \varepsilon$. Hence,

$$k^n(q,T] \;\ge\; k(q,T] + a + \varepsilon \quad \text{infinitely often.}$$

Together with an analogous argument in the second case, we arrive at the inequality (15) with $t$ replaced by $q$, giving (14). Note that (14) implies $Y_s^2 > a$, finishing the proof. ∎

**Lemma 14** *Under Assumption* 11*, the non-increasing* $\overline{\mathsf{R}}_+$*–valued processes* $Y^1$ *and* $Y^2$ *are right continuous.*

**Proof** Actually, we have only to prove that $Y^1$ is right continuous. Suppose on the contrary that for some $s$ (and a fixed $\omega$)

$$\bigvee_{n=1}^{\infty} k^n(s,T] \;:=\; \alpha \;>\; \beta \;:=\; \lim_{t\searrow s} \bigvee_{n=1}^{\infty} k^n(t,T].$$

Then, for every $n$,

$$\beta \;\ge\; \lim_{t\searrow s} k^n(t,T] \;=\; k^n(s,T],$$

since $k^n$ is a measure. Thus $\beta \ge \bigvee_{n=1}^{\infty} k^n(s,T] = \alpha$ which is a contradiction. Therefore $Y^1$ is right continuous. ∎

**Remark 15** Note that under Assumption 11, by Lemma 14 and according to Proposition 5 (i), the processes $y^1$ and $y^2$ are $\pi_{r,x}$–a.s. non-negative càdlàg and of class $(D)$. ◇

**Lemma 16** *Under Assumption* 11*, for* $\ell = 1,\dots,\infty$ *and* $s \in [r,T]$*, let* $\psi_s^\ell$ *be* $\Im[s,T]$*–measurable non-negative variables. Suppose that with respect to* $\pi_{r,x}$ *the* $\psi^1, \psi^2, \dots, \psi^\infty$ *are measurable processes uniformly bounded by a (non-random) constant. For* $r \le s \le T$*, and* $M \in \{1,\dots,\infty\}$ *put*

$$Z_s(M) \;:=\; \bigvee_{n=1}^{M} \left| \int_{(s,T]} \psi_t^n \, k^n(\mathrm{d}t) - \int_{(s,T]} \psi_t^\infty \, k^\infty(\mathrm{d}t) \right|$$

*and*

$$z_s(M) \;:=\; \pi_{s,\xi_s} Z_s(M).$$

*Then the process* $z(\infty)$ *is* $\pi_{r,x}$*–indistinguishable from a non-negative càdlàg process of class* $(D)$*.*

**Proof** Set $B := \sup_{n,s} |\psi_s^n|$, and let $M$ be finite. Note that

$$Z_s(M) \;\le\; 2B \bigvee_{n=1}^{\infty} k^n(s,T] \;=\; 2B\, Y_s^1 \in L^1(\pi_{r,x}) \tag{16}$$

and that $Z(M)$ is non-negative càdlàg. Hence, by Lemma 14 and Proposition 5 (ii), the process $z(M)$ is $\pi_{r,x}$–a.s. a non-negative càdlàg process of class $(D)$. By monotone convergence, $z_s(\infty) = \lim_M z_s(M)$, and therefore $z(\infty)$ is optional. For all $M$, from (16) we get $z_s(M) \leq 2B\, y_s^1$, and recalling Remark 15, we conclude that $z(\infty)$ is $\pi_{r,x}$–a.s. non-negative and belongs to class $(D)$. All that remains to be proved is that $z(\infty)$ is càdlàg, $\pi_{r,x}$–a.s. Clearly, because of (16) and the monotonicity of $Y^1$, we have that $Z(\infty)$ is $\pi_{r,x}$–a.s. non-negative.

From the elementary identity

$$\bigvee_n |a_n - a| = \Big(\bigvee_n a_n - a\Big) \vee \Big(a - \bigwedge_n a_n\Big)$$

we conclude with Lemma 14 and Corollary 60 from p.43 that $Z(\infty)$ is $\pi_{r,x}$–a.s. a right continuous non-negative process. Now, if $\sigma_n \leq T$ are $r$–stopping times non-increasing to $\sigma$, by the strong Markov property, $\pi_{r,x}\, z_{\sigma_n}(\infty) = \pi_{r,x}\, Z_{\sigma_n}(\infty)$. By right continuity, $Z_{\sigma_n}(\infty)$ converges to $Z_\sigma(\infty)$ as $n \to \infty$. Because of (16) we can invoke the dominated convergence theorem to derive that $\lim_n \pi_{r,x}\, z_{\sigma_n}(\infty) = \pi_{r,x}\, Z_\sigma(\infty)$. But again $\pi_{r,x}\, Z_\sigma(\infty) = \pi_{r,x}\, z_\sigma(\infty)$, and hence $\lim_n \pi_{r,x}\, z_{\sigma_n}(\infty) = \pi_{r,x}\, z_\sigma(\infty)$. This proves that $z$ is $\pi_{r,x}$–a.s. non-negative right continuous. A similar reasoning, invoking Lemma 57 at p.41 shows that $z$ also has left limits $\pi_{r,x}$–a.s. ■

**Proof of Proposition 12** *Step* 1° According to Remark 15, the processes $y^1$ and $y^2$ are $\pi_{r,x}$–a.s. non-negative càdlàg processes of class $(D)$. By Lemma 16, if we put for $N \geq 1$,

$$Y_s^3(N) := \bigvee_{n=N}^{\infty} \Big| k^n(s,T] - k(s,T] \Big|, \qquad y_s^3(N) := \pi_{s,\xi_s}\, Y_s^3(N),$$

then $y^3(N)$ is also $\pi_{r,x}$–a.s. a non-negative càdlàg process of class $(D)$. Since $Y_s^3(N) \leq Y_s^1 < \infty$, $\pi_{r,x}$–a.s., and $Y_s^3(N) \searrow Y_s^3$ (defined in (13)) as $N \to \infty$, we get by dominated convergence that $y_s^3(N) \searrow y_s^3$ as $N \to \infty$. This establishes that $y^3$ is a non-negative optional process of class $(D)$.

*Step* 2° Recall that $y^1$ is in particular $\pi_{r,x}$–indistinguishable from a non-negative process, by Remark 15. In other words: $\pi_{r,x}$–a.s. the process $s \mapsto (s, \xi_s)$ passes only through points $(s,y)$ such that $\pi_{s,y} \bigvee_{n=1}^{\infty} k^n(s,T] < \infty$.

*Step* 3° Recall that $Y^2$ defined in (12) is $\overline{\mathsf{R}}_+$–valued non-increasing and right continuous, and by Assumption 11, $\pi_{r,x}$–indistinguishable from 0. Hence, by Lemma 10, the statement (9) holds (with $Y^2$ instead of $Y$). In other words: With $\pi_{r,x}$–probability one, the process $(s, \xi_s)$ passes only through points $(s,y)$ such that $\pi_{s,y}$–almost surely, $k^n(t,T] \to_n k(t,T]$ for every $t \in (s,T]$. (Note that $t = s$ is not yet included in the statement.)

*Step* 4° From step 1° we know that $y^3$ is a non-negative optional process of class $(D)$. Moreover, by the strong Markov property, we have for every $r$–stopping time $\sigma \leq T$ that

$$\pi_{r,x} y^3_\sigma \;=\; \pi_{r,x} \limsup_n \left| k^n(\sigma, T] - k(\sigma, T] \right| \;=\; 0.$$

And therefore, according to [Dyn94, A.1.1.E, p.116], the process $y^3$ is $\pi_{r,x}$–a.s. indistinguishable from zero. In other words, with $\pi_{r,x}$–probability 1, the process $(s, \xi_s)$ passes only through points $(s, y)$ such that $\pi_{s,y}$–almost surely, $k^n(t, T] \to_n k(t, T]$, for $t \in [s, T]$. ■

# 3 Key result: fdd continuity in $(\Phi, k)$

After the preparations in the previous section, we turn to the continuous dependence of finite-dimensional distributions of $(\xi, \Phi, k)$–superprocesses on their regular branching mechanism $\Phi$ and branching functional $k$ (Theorem 23 at p.18). A key step in deriving this will be Proposition 39 at p.26 describing the convergence of log-Laplace functionals for those starting points $(r, x)$ such that $s \mapsto (s, \xi_s)$ will pass $\pi_{r,x}$–a.s. only through those points which preserve some moment and convergence property of the branching functionals in the sense of Proposition 12. As an *application* we prove that $(\xi, \Phi, k)$–superprocesses can fdd be approximated by "classical" superprocesses (Theorem 26 at p.19).

## 3.1 Basic assumptions: branching mechanism $\Phi$

**Assumption 17** Now we complement the basic Assumption 1 from p.5 (concerning the motion process $\xi$):

**(f) (branching mechanism)** $\Phi$ is always a (local) *branching mechanism* of the form

$$\Phi(r, x, \lambda) \;=\; b^r(x)\lambda^2 + \int_0^\infty e(u\lambda)\, n(r, x, \mathrm{d}u), \qquad (r, x, \lambda) \in I \times E \times \mathsf{R}_+\,,$$

where $e(z) := \mathrm{e}^{-z} + z - 1$, where $0 \leq b^r(x) \leq 1$ is measurable in $(r, x)$, and where $n$ is a kernel satisfying the condition

$$0 \leq \int_0^\infty u^2\, n(r, x, \mathrm{d}u) \leq 1, \qquad (r, x) \in I \times E.$$

Here 'kernel' means: $n : \mathsf{R}_+ \times E \to \mathcal{M}$ is measurable, where $\mathcal{M} = \mathcal{M}(0, \infty)$ is the set of all measures on the locally compact space $(0, \infty)$, finite on compact subsets, endowed with the topology of vague convergence (Polish space).

**(g) (regular $\Phi$)** Additionally, the branching mechanism $\Phi$ is often assumed to be *regular* in the following sense. If for each starting point $(r, x)$ in $I \times E$ the process $s \mapsto z_s$ is non-negative càdlàg with $\pi_{r,x}$–probability one, then so is $s \mapsto \Phi(s, \xi_s\,, z_s)$. ◇

The following result is taken from Leduc [Led97a, Theorem 1.2], who generalized Theorem 5.2.1 of [Dyn94] where the admissibility (4) on $k$ was imposed rather than only the boundedness (3) of characteristic.

**Lemma 18 ('unique' existence of the $(\xi,\Phi,k)$–superprocess)** *To each branching functional $k$ and branching mechanism $\Phi$, the $(\xi,\Phi,k)$–superprocess $X$ exists. More precisely, an $\mathcal{M}_{\mathrm{f}}$–valued (time-inhomogeneous) Markov process $(X_t\,,\mathcal{F},P_{r,\mu})$ exists (in the sense of Assumption 1 (d1)) with* log-Laplace transition functional

$$-\log P_{r,\mu}\exp\langle X_t\,,-f\rangle \;=\; \int v_{r,t}(f)(x)\,\mu(\mathrm{d}x) \tag{17}$$

$0\le r\le t\le T$, $x\in E$, $f\in b\mathcal{E}_+$, *where* $v=v(f)=v_{\cdot,t}(f)\ge 0$ *solves the* $(\xi,\Phi,k)$–evolution equation

$$v_{r,t}(f)(x) \;=\; \pi_{r,x}f(\xi_t)-\pi_{r,x}\int_{(r,t]}\Phi\bigl(s,\xi_s\,,v_{s,t}(\xi_s)\bigr)\,k(\mathrm{d}s), \tag{18}$$

*and is the* only *log-Laplace solution to that equation.*

*From now on we restrict our attention to such $(\xi,\Phi,k)$–superprocesses.* That is, speaking of a $(\xi,\Phi,k)$–superprocess we tacitly mean that $\xi$ is a càdlàg right process, $k$ a branching functional and $\Phi$ a branching mechanism, all according to our basic Assumptions 1 and 17. Moreover, since the log-Laplace transition functional (17) of the $(\xi,\Phi,k)$–superprocess $X$ is uniquely determined by $v$, for simplicity we call $v$ the *log-Laplace functional* related to $X$ (as we did already in Section 1).

**Remark 19 (projection, criticality, total mass process)** The motion process $\xi$ of the $(\xi,\Phi,k)$–superprocess $X$ (we consider in this paper) can be recovered by *projection (expectation formula):*

$$P_{r,\mu}\langle X_t\,,f\rangle \;=\; \pi_{r,\mu}f(\xi_t), \qquad 0\le r\le t\le T, \quad \mu\in\mathcal{M}_{\mathrm{f}}\,, \quad f\in b\mathcal{E}_+\,.$$

This in particular implies that $X$ is *critical,* that is, the total mass process $t\mapsto\langle X_t\,,1\rangle$ is a *martingale* (with respect to the natural filtration of $X$). ◇

**Remark 20 (finite variances)** These $(\xi,\Phi,k)$–superprocesses have *finite second moments:*

$$\sup_{r\le t} P_{r,\mu}\langle X_t\,,1\rangle^2 \;<\; \infty, \qquad t\in I, \quad \mu\in\mathcal{M}_f\,, \tag{19}$$

(i.e. with some uniformity in the starting time $r$). ◇

## 3.2 The fdd joint continuity theorem

The formulation of our main result will be based on the following definition.

**Definition 21 (uniformly of bounded characteristic)** If the branching functionals $k^1, ..., k^\infty = k$ satisfy

$$\bigvee_{n=1}^{\infty} \sup_{(r,x)\in I\times E} \pi_{r,x}\, k^n(r,T] < \infty, \tag{20}$$

they are called to be *uniformly of bounded characteristic.* $\diamond$

For convenience, we introduce the following assumption.

**Assumption 22** Consider branching mechanisms $\Phi^1, \Phi^2, ...$ converging uniformly to a *regular* branching mechanism $\Phi$. Moreover, consider branching functionals $k^1, ..., k^\infty = k$ being uniformly of bounded characteristic. Suppose that for every starting point $(r,x) \in I \times E$ and every $r$–stopping time $\sigma \leq T$ we know that $k^n(r,\sigma]$ converges to $k(r,\sigma]$ in $L^1(\pi_{r,x})$ as $n \to \infty$. $\diamond$

**Theorem 23 (joint continuity in fdd)** *Impose Assumption* 22. *Then the related log-Laplace functionals converge:*

$$v^n_{r,t}(f)(x) \underset{n}{\to} v_{r,t}(f)(x), \qquad 0 \leq r \leq t \leq T, \quad x \in E, \quad f \in b\mathcal{E}_+\,. \tag{21}$$

*Consequently, the related superprocesses converge fdd.*

The proof of this theorem requires some preparation, provided in the following subsections. We first completely concentrate on the case $\Phi^n \equiv \Phi$. For this, the final steps in § 3.9 then follow along the lines of construction of a general class of $(\xi, \Phi, k)$–superprocesses given in Leduc [Led97a, Proposition 4.20]. Then in § 3.10 we remove the $\Phi^n \equiv \Phi$ restriction by an approximation procedure.

Note that the requirement in Theorem 23 that the limiting $\Phi$ is *regular cannot be dropped*:

**Example 24 (fdd discontinuity for a non-regular $\Phi$)** Let $k^n$ be a (deterministic) absolutely continuous (with respect to Lebesgue measure) probability law on $I = [0,1]$ converging weakly as $n \to \infty$ to a singularly continuous law $k$ with support the Cantor set $\mathsf{C}$. Set $\Phi(s,x,\lambda) \equiv \lambda^2 \mathbf{1}_{I\setminus\mathsf{C}}(s)$, that is consider the "binary splitting" but only at time points $s$ outside the Cantor set $\mathsf{C}$. Note that $\Phi(s,\xi_s,\lambda)\, k^n(\mathrm{d}s) \equiv \lambda^2 k^n(\mathrm{d}s)$, for any motion process $\xi$. Hence, the $(\xi, \Phi, k^n)$–superprocess is precisely the $(\xi, \lambda^2, k^n)$–superprocess. Therefore, by Theorem 23, the $(\xi, \Phi, k^n)$–superprocesses converge fdd to the $(\xi, \lambda^2, k)$–superprocess as $n \to \infty$. Note that this limiting process is non-degenerate. In fact, it has non-zero variance: $\mathrm{Var}_{0,\delta_x} \langle X_1, 1\rangle \equiv 2k(I) = 2$. On the other hand, $\Phi(s,\xi_s,\lambda)\, k(\mathrm{d}s) \equiv 0$. Thus, the $(\xi, \Phi, k)$–superprocess is degenerate: It is the deterministic mass flow according to the semigroup of the motion process. Consequently, for this *non-regular* $\Phi$, fdd *continuity in $k$ is violated.* $\diamond$

For *fixed* branching functional $k$, the fdd continuity in the branching mechanism $\Phi$ can be sharpened by using a weaker convergence concept for $\Phi$, and by allowing *non-regular limiting* $\Phi$.

**Proposition 25 (fdd continuity in $\Phi$ only)** *Fix a branching functional $k$. If the branching mechanisms $\Phi^n$ converge boundedly pointwise to the branching mechanism $\Phi$ as $n \to \infty$, then the related log-Laplace functionals $v_n$ and $v$ converge as expressed in* (21).

The proof of this proposition is postponed to § 3.11.

## 3.3 Application: fdd approximation by classical processes

We can use our fdd continuity Theorem 23 to show that all the $(\xi, \Phi, k)$–superprocesses (of the present paper) with regular branching mechanism $\Phi$ can be approximated by superprocesses with a "classical" branching rate. Note that the approximating branching functionals $k^n$ are in particular *absolutely continuous* with respect to the Lebesgue measure.

**Theorem 26 (fdd approximation by classical processes)** *Let $\Phi$ be a regular branching mechanism, and $k$ be a branching functional. Then there exist bounded measurable functions $\varrho^n : I \times E \to R_+$, $n \geq 1$, such that the $(\xi, \Phi, k^n)$–superprocesses $X^n$ with "classical" branching functional*

$$k^n(\mathrm{d}s) := \varrho^n_s(\xi_s)\,\mathrm{d}s \tag{22}$$

*converge fdd to the $(\xi, \Phi, k)$–superprocess $X$ as $n \to \infty$.*

The proof of this theorem will be provided in § 3.12.

## 3.4 Convergence of branching functionals

Next we want to *reformulate the convergence of additive functionals* occurring in Assumption 22.

**Proposition 27 (convergence criterion for additive functionals)** *Let $k^1, \ldots, k^\infty = k$ be continuous additive functionals of $\xi$. Fix a time point $r \in I$, and a measure $\mu \in \mathcal{M}_\mathrm{f}$. The following two conditions are equivalent:*

**(i)** *$k^n(r, \sigma]$ converges to $k(r, \sigma]$ in $L^1(\pi_{r,\mu})$ as $n \to \infty$, for each $r$–stopping time $\sigma \leq T$.*

**(ii)** *For every subsequence $\{k^{n_m}\}$ of $\{k^n\}$ there exists a subsequence $\{k^{n_{m_i}}\}$ of $\{k^{n_m}\}$ such that*

$(\alpha)$ $\pi_{r,\mu} \bigvee_{i=1}^{\infty} k^{n_{m_i}}(r, T] < \infty$, *and*

$$(\beta) \qquad \sup_{s,t:\, r \le s \le t \le T} \left| k^{n_{m_i}}(s,t] - k(s,t] \right| \underset{i\to\infty}{\longrightarrow} 0, \quad \pi_{r,\mu}\text{-}a.e.$$

**Proof (i) $\Longrightarrow$ (ii)($\alpha$)** Let $\{k^{n_m}\}$ be a subsequence of $\{k^n\}$. Since $k^{n_m}(r,T]$ converges to $k(r,T]$ in $L^1(\pi_{r,\mu})$ as $m \to \infty$, we have by uniform integrability, that

$$\pi_{r,\mu}\left(\mathbf{1}\Big\{k^{n_m}(r,T] > k(r,T]+1\Big\}k^{n_m}(r,T]\right) \underset{m\to\infty}{\longrightarrow} 0.$$

By choosing a subsequence such that the above terms do not only converge to zero but form also a convergent series, we get (ii)($\alpha$).

**(i) $\Longrightarrow$ (ii)($\beta$)** Let $\{k^{n_m}\}$ be a subsequence of $\{k^n\}$. With the use of Cantor's diagonalization method one finds a subsequence $\{k^{n_{m_i}}\}$ such that

$$\left|k^{n_{m_i}}(r,q] - k(r,q]\right| \underset{i}{\to} 0 \text{ for every rational } q \in (r,T] \text{ and } q = T \tag{23}$$

$\pi_{r,\mu}$–a.e. But then, because the mappings $t \mapsto k^{n_{m_i}}(r,t]$ are non-decreasing, that implies that $\pi_{r,\mu}$–almost everywhere, $k^{n_{m_i}}(r,t] \to_i k(r,t]$ for all $t$ in $(r,T]$. In fact, fix $\omega$ such that (23) holds, and take $\varepsilon > 0$. Since $k$ is continuous by assumption, we may choose two rational numbers $q_1, q_2$ in $(r,T)$ such that $q_1 < t < q_2$ and $k(q_1, q_2] < \varepsilon$. Then

$$\begin{aligned} k(r,t]-\varepsilon \;&<\; k(r,q_1] \;=\; \lim_i k^{n_{m_i}}(r,q_1] \;\le\; \liminf_i k^{n_{m_i}}(r,t] \\ &\le\; \limsup_i k^{n_{m_i}}(r,t] \;\le\; \lim_i k^{n_{m_i}}(r,q_2] \;=\; k(r,q_2] \;<\; k(r,t]+\varepsilon. \end{aligned}$$

Therefore, since $\varepsilon$ can be made arbitrarily small, the $\liminf_i$ and $\limsup_i$ expressions must coincide with $k(r,t]$. Since all the functionals are monotone in $t$, and $k(r,t]$ is uniformly continuous in $t$, we get

$$\sup_{t\in(r,T]} \left|k^{n_{m_i}}(r,t] - k(r,t]\right| \underset{i\to\infty}{\longrightarrow} 0, \quad \pi_{r,\mu}\text{–a.e.}$$

The claim ($\beta$) then follows from a difference expression.

**(ii) $\Longrightarrow$ (i)** To show this implication, suppose that (i) is not verified. Then, for some $r$–stopping time $\sigma \le T$, it is possible to find an $\varepsilon > 0$ and a subsequence $\{k^{n_m}\}$ of $\{k^n\}$ such that for every $m$

$$\pi_{r,\mu}\left|k^{n_m}(r,\sigma] - k(r,\sigma]\right| > \varepsilon. \tag{24}$$

On the other hand, according to (ii), it is possible to choose a subsequence $\{k^{n_{m_i}}\}$ of $\{k^{n_m}\}$ such that (ii)($\alpha$) and (ii)($\beta$) are satisfied. Passing to differences, with Lebesgue's theorem this implies that $k^{n_{m_i}}(r,\sigma]$ converges to $k(r,\sigma]$ in $L^1(\pi_{r,\mu})$. This obviously contradicts (24), and the proof of the proposition is finished. ■

For applications of our main Theorem 23 the following *sufficient criterion* for the convergence of additive functionals might be helpful.

**Lemma 28 (sufficient criterion)** *Let $k^1, ..., k^\infty = k$ be branching functionals which are uniformly of bounded characteristic. Fix $r \in I = [0,T]$, and $\mu \in \mathcal{M}_{\mathrm{f}}$. Let $\pi_{r,\mu}$–almost everywhere $k^n$ weakly converge to $k$ as $n \to \infty$. Then the assertions* (i) *and* (ii) *in Proposition* 27 *hold.*

**Proof** We want to prove (ii). Let $\{k^{n_m}\}$ be any subsequence of $\{k^n\}$. Arguing as in the end of the step $(i) \Rightarrow (ii)(\beta)$ in the proof of Proposition 27, it remains only to show the existence of a subsequence $\{k^{n_{m_i}}\}$ of $\{k^{n_m}\}$ such that $(\alpha)$ holds.

If $\{a_m\}$ is a sequence of numbers converging to $a$ as $m \to \infty$, then either $\sup_m a_m \le a+1$, or there exists a smallest integer $M$ such that $\sup_m a_m = a_M > a+1$. Then

$$\sup_m a_m \;\le\; a+1+a_M \;=\; a+1+\sum_m a_m \mathbf{1}\{M=m\}.$$

Thus it suffices to show that there exists a subsequence $\{k^{n_{m_i}}\}$ such that

$$\pi_{r,\mu} \sum\nolimits_i k^{n_{m_i}}(r,T]\,\mathbf{1}\Big\{k^{n_{m_i}}(r,T] > k(r,T]+1\Big\} \;<\; \infty.$$

Interchange integration with $\pi_{r,\mu}$ and summation, apply the *Cauchy-Schwarz inequality,* and the fact that

$$\pi_{r,\mu}\big(k^{n_{m_i}}(r,T]\big)^2 \;\le\; 2\,\mu(E)\left|\,\bigvee_{n=1}^{\infty} \sup_{(s,y)\in I\times E} \pi_{s,y}\, k^n(s,T]\right|^2 \;<\; \infty$$

since the branching functionals $k^n$ are uniformly of bounded characteristic. Thus, it remains to show that

$$\sum\nolimits_i \left|\pi_{r,\mu}\Big(k^{n_{m_i}}(r,T] > k(r,T]+1\Big)\right|^{1/2} \;<\; \infty$$

for some subsequence $\{k^{n_{m_i}}\}$. But the measure expressions converge to 0 as $i \to \infty$ by the assumed $\pi_{r,\mu}$–a.e. convergence, implying the existence of the desired subsequence. ■

## 3.5 Review: the log-Laplace characterization of $(\xi, \Phi, k)$–superprocesses

For convenience, here we review the log-Laplace functional characterization of $(\xi, \Phi, k)$–superprocesses, and some related facts on log-Laplace functionals, the latter are versions of the Lemmas 4.23, 4.25 and 4.26 in Leduc [Led97a].

**Lemma 29 (log-Laplace characterization)** *Suppose that $f \mapsto v_{r,t}(f)(x)$, $f \in b\mathcal{E}_+$, is the log-Laplace functional of an $\mathcal{M}_{\mathrm{f}}$–valued random measure, for* *every choice of* $0 \le r \le t \le T$ *and* $x \in E$. *Moreover, let* $x \mapsto v_{r,t}(f)(x)$ *be measurable. Finally, let* $\{v_{r,t} : 0 \le r \le t \le T\}$ *form a semigroup on* $b\mathcal{E}_+$ :

$$v_{r,s}\big(v_{s,t}(f)\big)(x) = v_{r,t}(f)(x), \quad 0 \le r \le s \le t \le T,\ x \in E,\ f \in b\mathcal{E}_+ . \tag{25}$$

*Then there exists a unique (in the sense of finite-dimensional distributions)* $\mathcal{M}_\mathrm{f}$*–valued Markov process* $X$ *with log-Laplace functional* $v$ *(recall* (17)*).*

For $c > 0$, let us introduce the following set

$$b\mathcal{E}_+^c := \big\{f \in b\mathcal{E}_+ :\ f \le c\big\}. \tag{26}$$

**Lemma 30 (continuity in $f$)** *Let* $\Phi$ *be any branching mechanism. Fix* $t \in I$, *and* $\delta > 0$. *Let* $(r,x) \to v_{r,t}f(x)$ *be a non-negative solution of the* $(\xi,\Phi,k)$*–evolution equation* (18), *for each* $f \in b\mathcal{E}_+^{2\delta}$. *Moreover, let* $f \mapsto v_{\cdot,t}(f)$ *be increasing. Then, for each* $(r,x) \in [0,t] \times E$ *fixed, the functional* $f \mapsto v_{r,t}(f)(x)$ *is continuous on* $b\mathcal{E}_+^\delta$ *(in the topology of bounded pointwise convergence induced by* $b\mathcal{E}_+$*).*

**Lemma 31 (convergence of Laplace functionals)** *Assume that* $L_{P_n}$ *is the Laplace functional of some* $\mathcal{M}_\mathrm{f}$*–valued random variable, for each* $n \ge 1$. *Suppose there exists* $\delta > 0$ *such that* $L_{P_n}(f) \to L(f)$ *as* $n \to \infty$, *for every* $f \in b\mathcal{E}_+^\delta$ *and that* $L$ *is continuous on that set. Then there exists an extension of* $L$ *to all of* $b\mathcal{E}_+$ , *and a probability measure* $P_\infty$ *on* $\mathcal{M}_\mathrm{f}$ *such that* $L$ *is the Laplace functional of* $P_\infty$ , *and* $L_{P_n}(f) \to L(f)$ *as* $n \to \infty$, *for every* $f$ *in* $b\mathcal{E}_+$ .

**Lemma 32 (semigroup property of solutions)** *Suppose*

$$f \mapsto \langle \mu, v_{r,t}(f) \rangle, \qquad f \in b\mathcal{E}_+ ,$$

*is the log-Laplace functional of an* $\mathcal{M}_\mathrm{f}$*–valued random measure, for every choice of* $0 \le r \le t \le T$ *and* $\mu \in \mathcal{M}_\mathrm{f}$. *Moreover, let* $\Phi$ *be a branching mechanism,* $k$ *be a branching functional, and let* $(r,x) \to v_{r,t}f(x)$ *solve the* $(\xi,\Phi,k)$*–evolution equation* (18), *for each* $t \in I$ *and* $f \in b\mathcal{E}_+$ *fixed. Then the semigroup property* (25) *holds.*

## 3.6 Solutions to the evolution equation in the case of small $f$

By a slight abuse of notation, we adopt the following convention.

**Convention 33** For convenience, we will often write $\|g(r,x)\|_\infty$ instead of $\|g(\cdot,\cdot)\|_\infty = \sup_{r,x} |g(r,x)|$. That is, even though the time space variable $(r,x)$ in $I \times E$ appears under the norm sign, the supremum is always taken over them, also if additionally other parameters are eventually involved, as $N$ etc. $\diamond$

The following lemma is taken from Leduc [Led97a, Lemma 4.21].

**Lemma 34 (local Lipschitz continuity)** *Let $\Phi$ be a branching mechanism. Then, $\Phi(r,x,0) \equiv 0$. Moreover, for every $c > 0$ and $\lambda_1, \lambda_2 \in [0,c]$,*

$$\|\Phi(r,x,\lambda_1) - \Phi(r,x,\lambda_2)\|_\infty \leq 3\,c\,|\lambda_1 - \lambda_2|\,. \tag{27}$$

*Finally, if $0 \leq \lambda_1 \leq \lambda_2$ then $0 \leq \Phi(r,x,\lambda_1) \leq \Phi(r,x,\lambda_2)$, $(r,x) \in I \times E$.*

As a first step towards the proof of our main theorem, here we want to give an independent construction of a solution to the $(\xi,\Phi,k)$–evolution equation (18) in the case of small $f$.

**Proposition 35 (solution for small $f$)** *Fix $t \in I$, a regular branching mechanism $\Phi$, and a branching functional $k$. Let $\delta > 0$ satisfy*

$$3\,\delta \sup_{(r,x)\in[0,t]\times E} \pi_{r,x}\, k(r,t] \leq \tfrac{1}{2}\,. \tag{28}$$

*Then, for $f \in b\mathcal{E}_+^\delta$,*

**(i) (unique existence)** *a unique measurable function $v_{\cdot,t}(f) \geq 0$ exists which solves the $(\xi,\Phi,k)$–evolution equation (18), and*

**(ii) (càdlàg regularity)** *the process $s \mapsto v_{s,t}(f)(\xi_s)$, $s \in [r,t]$, is càdlàg $\pi_{r,x}$–a.s., for every starting point $(r,x) \in [0,t] \times E$.*

**Proof** Fix $t, \Phi, k, f$ as in the proposition. Let $\mathcal{B}^{t,\delta}$ be the set of all measurable mappings $u$ from $[0,t] \times E$ to $[0,\delta]$ such that $s \mapsto u_s(\xi_s)$ is càdlàg. Equipped with the metric generated by the supremum norm $\|\cdot\|_\infty$, this is a complete metric space. Define an operator $G$ on $\mathcal{B}^{t,\delta}$ by

$$G(u)(r,x) := \pi_{r,x} f(\xi_t) - \pi_{r,x} f(\xi_t) \wedge \pi_{r,x} \int_{(r,t]} \Phi\big(s,\xi_s,u_s(\xi_s)\big)\, k(\mathrm{d}s).$$

We want to show that $G$ maps into $\mathcal{B}^{t,\delta}$. Let $\sigma_n \leq T$ be $r$–stopping times monotonously converging to $\sigma$ as $n \to \infty$. Only by the Markov property,

$$\lim_{n\to\infty} \pi_{r,x}\pi_{\sigma_n,\xi_{\sigma_n}} f(\xi_t) = \pi_{r,x}\pi_{\sigma,\xi_\sigma} f(\xi_t).$$

Similarly, by a property of measures,

$$\begin{aligned}\lim_{n\to\infty} \pi_{r,x}\pi_{\sigma_n,\xi_{\sigma_n}} &\int_{(\sigma_n,t]} \Phi\big(s,\xi_s,u_s(\xi_s)\big)\, k(\mathrm{d}s)\\ &= \pi_{r,x}\pi_{\sigma,\xi_\sigma} \int_{(\sigma,t]} \Phi\big(s,\xi_s,u_s(\xi_s)\big)\, k(\mathrm{d}s).\end{aligned}$$

By [Dyn94, A.1.1.D, p.116] and Lemma 57 at p.41, this establishes that the processes

$$s \mapsto \pi_{s,\xi_s} f(\xi_t) \quad \text{and} \quad s \mapsto \pi_{s,\xi_s} \int_{(s,t]} \Phi\big(s',\xi_{s'},u_{s'}(\xi_{s'})\big)\, k(\mathrm{d}s')$$

are càdlàg $\pi_{r,x}$–a.s., for every starting point $(r,x) \in I \times E$. Thus

$$\lim_{n\to\infty} \pi_{r,x} G(u)(\sigma_n, \xi_{\sigma_n}) \quad = \quad \pi_{r,x} G(u)(\sigma, \xi_\sigma),$$

showing that $s \mapsto G(u)(s, \xi_s)$ is càdlàg. Hence, $G$ maps $\mathcal{B}^{t,\delta}$ into itself.

Let $z^1$ and $z^2$ be two mappings in $\mathcal{B}^{t,\delta}$. From (27), we get

$$\left|\Phi\left(s, \xi_s, z_s^1(x)\right) - \Phi\left(s, \xi_s, z_s^2(x)\right)\right| \le 3\,\delta \|z^1 - z^2\|_\infty.$$

Thus,

$$\begin{aligned}\left|G(z^1)(r,x) - G(z^2)(r,x)\right| &\le 3\,\delta\, \pi_{r,x} \textstyle\int_{(r,t]} \|z^1 - z^2\|_\infty \, k(\mathrm{d}s) \\ &\le 3\,\delta \|z^1 - z^2\|_\infty \sup_{r,x} \pi_{r,x}\, k(r,t] \\ &\le \tfrac{1}{2} \|z^1 - z^2\|_\infty ,\end{aligned}$$

where we used (28). Hence, $G$ is a *contraction* on $\mathcal{B}^{t,\delta}$. By the Banach fixed point theorem, there exists a (unique) element $u$ in $\mathcal{B}^{t,\delta}$ which solves

$$u_r(x) = G(u)(r,x) = \pi_{r,x} f(\xi_t) - \pi_{r,x} f(\xi_t) \wedge \pi_{r,x} \textstyle\int_{(r,t]} \Phi\big(s, \xi_s, u_s(\xi_s)\big)\, k(\mathrm{d}s)$$

on $I \times E$. Let us now show that, indeed, $u$ solves (18). To do this, let

$$\sigma^r := \inf\Big\{ s \in (r,t] : \pi_{s,\xi_s} \textstyle\int_{(s,t]} \Phi\big(s', \xi_{s'}, u_{s'}(\xi_{s'})\big)\, k(\mathrm{d}s') \le \pi_{s,\xi_s} f(\xi_t) \Big\}$$

Note that $u_s(s, \xi_s) = G(u)(s, \xi_s) = 0$ for $s \in (r, \sigma^r]$, hence $\Phi\big(s, \xi_s, u_s(\xi_s)\big)$ vanishes for those $s$. Thus, using the strong Markov property, we are allowed to write

$$u_r(x) = \pi_{r,x} f(\xi_t) - \pi_{r,x} f(\xi_t) \wedge \pi_{r,x} \pi_{\sigma^r, \xi_{\sigma^r}} \textstyle\int_{(\sigma^r, t]} \Phi\big(s, \xi_s, u_s(\xi_s)\big)\, k(\mathrm{d}s),$$

for all $r, x$. But, by definition of $\sigma^r$,

$$\pi_{r,x} \pi_{\sigma^r, \xi_{\sigma^r}} \textstyle\int_{(\sigma^r, t]} \Phi\big(s, \xi_s, u_s(\xi_s)\big)\, k(\mathrm{d}s) \le \pi_{r,x} \pi_{\sigma^r, \xi_{\sigma^r}} f(\xi_t) = \pi_{r,x} f(\xi_t).$$

Consequently,

$$\pi_{r,x} f(\xi_t) \wedge \pi_{r,x} \textstyle\int_{(r,t]} \Phi\big(s, \xi_s, u_s(\xi_s)\big)\, k(\mathrm{d}s) = \pi_{r,x} \textstyle\int_{(r,t]} \Phi\big(s, \xi_s, u_s(\xi_s)\big)\, k(\mathrm{d}s).$$

Therefore, $u$ solves (18), proving the existence part of the proposition.

In the definition of $\mathcal{B}^{t,\delta}$, drop now the càdlàg requirement and allow values in $[-\delta, \delta]$. Then the r.h.s. of the evolution equation (18) still defines a contraction, yielding also the uniqueness claim. This finishes the proof. ∎

## 3.7 Special notation

For convenience, we introduce the following special notation.

**Notation 36** Consider a regular branching mechanism $\Phi$, and branching functionals $k^1, ..., k^\infty = k$ of uniformly bounded characteristic. For $n \geq 1$, let $v^n$ denote the log-Laplace functional related to the $(\xi, \Phi, k^n)$–superprocess.

**(i) (nice starting points)** Denote by $C = C(k^1, ..., k^\infty)$ the set of all points $(r, x) \in [0, T] \times E$ such that

($\alpha$) $\pi_{r,x} \bigvee_{n=1}^{\infty} k^n(r, T] < \infty$, and

($\beta$) $\pi_{r,x}$–a.s., $k^n(s, t] \to_n k(s, t]$ whenever $r \leq s \leq t \leq T$.

**(ii) (special norm)** For any mapping $h : [0, T] \times E \to \mathsf{R}$, we set

$$\|h(r, x)\|_C := \sup_{(r,x)\in C} |h(r, x)|$$

(applying the Convention 33 introduced for $\|\cdot\|_\infty$ analogously to $\|\cdot\|_C$).

**(iii)** For $t \in I$ and $f \in b\mathcal{E}_+$ fixed, for $n \geq 1$ and $r \in I$ we pose

$$\begin{aligned} v_r^n &:= v_{r,t}^n(\xi_r); & v_r &:= v_{r,t}(\xi_r); \\ \Phi_r^n &:= \Phi(r, \xi_r, v_{r,t}^n(\xi_r)); & \Phi_r &:= \Phi(r, \xi_r, v_{r,t}(\xi_r)); \\ S_r^n &:= \sup_{\ell \geq n} \left| \int_{(r,t]} \Phi_s^\ell \, k^\ell(\mathrm{d}s) - \int_{(r,t]} \Phi_s \, k(\mathrm{d}s) \right|, \end{aligned}$$

with reading such quantities as $0$ if $r > t$.

**(iv)** $B$ will denote the following supremum expression:

$$\sup_{t\in I} \left\{ \|\pi_{r,x} \lim_n S_r^n\|_C \vee \sup_\ell \left\| \pi_{r,x} \left| \int_{(r,t]} \Phi_s^\ell \, k^\ell(\mathrm{d}s) - \int_{(r,t]} \Phi_s \, k(\mathrm{d}s) \right| \right\|_\infty \right\}.$$

◇

**Lemma 37** *We have* $B < \infty$.

**Proof** First pass from the minus sign to a plus sign in the definition of $S_r^n$. From the definition of $\Phi$ in Assumption 17 (f) we obtain

$$\|\Phi(r, x, \lambda)\|_\infty \leq \tfrac{3}{2} \lambda^2, \tag{29}$$

since $0 \leq e(z) \leq z^2/2$, $z \geq 0$. Recall that the log-Laplace functionals $v^n$ solve the $(\xi, \Phi, k^n)$–evolution equation (18) (with $k$ replaced by $k^n$). Hence,

$$0 \leq v_{r,t}^n(f)(x) \leq \|f\|_\infty . \tag{30}$$

Using this domination, altogether we get the estimate

$$\left| \int_{(r,t]} \Phi_s^\ell \, k^\ell(\mathrm{d}s) - \int_{(r,t]} \Phi_s \, k(\mathrm{d}s) \right| \leq \tfrac{3}{2} \|f\|_\infty^2 \left( k^\ell(r, T] + k(r, T] \right). \tag{31}$$

Taking the $\pi_{r,x}$–expectation, the finiteness of the second part in the definition of $B$ immediately follows from (20). On the other hand, for the first part, take the supremum on $\ell \geq n$ and the limit as $n \to \infty$ of the r.h.s. of (31) to get $3\,\|f\|_\infty^2\, k(r,T]$ with $\pi_{r,x}$–probability one, for each $(r,x) \in C$. Hence,

$$\|\pi_{r,x} \lim_n S_r^n\|_C \leq \text{const}\, \|\pi_{r,x}\, k(r,T]\|_\infty$$

which is finite, again by (20). ∎

Later we need also the following simple fact.

**Lemma 38 (convergence of functionals)** *Fix a starting point* $(r,x) \in C$ *(with* $C$ *defined in Notation* 36 (i)) *and* $t \in [r,T]$. *For* $s \in [r,t]$, *let* $\psi_s$ *denote* $\Im[s,t]$*–measurable non-negative variables, and let* $s \mapsto \psi_s$ *be* $\pi_{r,x}$*–indistinguishable from a càdlàg process, bounded by a (non-random) constant. Then,*

$$\int_{(r,t]} \psi_s\, k^\ell(\mathrm{d}s) \;\to_\ell\; \int_{(r,t]} \psi_s\, k(\mathrm{d}s)$$

*with* $\pi_{r,x}$*–probability one.*

**Proof** This e.g. immediately follows from [Bil68, Theorem 5.1]. ∎

## 3.8 Key step: convergence of log-Laplace functionals for nice starting points

The central part in deriving our key result is the following proposition concerning the convergence of log-Laplace functionals for small test functions $f$, and for starting points in $C$ (guaranteeing some convergence of the functionals $k^n$).

**Proposition 39 (convergence if start in $C$)** *Consider a regular branching mechanism* $\Phi$ *and branching functionals* $k^1, \ldots, k^\infty = k$ *which are uniformly of bounded characteristic. Let* $f \in b\mathcal{E}_+$ *be such that*

$$3\,\|f\|_\infty\, \|\pi_{r,x}\, k(r,T]\|_\infty \leq \tfrac{1}{2}\,. \tag{32}$$

*Then for the log-Laplace functionals* $v^n(f) = v^n$, $n \geq 1$, *of* (17) *related to* $k^1, k^2, \ldots$, *respectively, we have*

$$\lim_n v_{r,t}^n(x) = v_{r,t}(x), \qquad (r,x) \in C, \quad t \in [r,T],$$

*with* $v = v(f)$ *the (unique) "small solution" of the* $(\xi,\Phi,k)$*–evolution equation* (18) *constructed in Proposition* 35, p.23.

In order to explain the *concept of proof,* recall in particular the symbols $S_r^n$ and $B$ introduced in (iii) and (iv) of Notation 36. For $r,x,t$ as in the proposition, we clearly have

$$\left|v_{r,t}^n(x) - v_{r,t}(x)\right| \quad\leq\quad B \wedge \pi_{r,x} \left|\int_{(r,t]} \Phi_s^n\, k^n(\mathrm{d}s) - \int_{(r,t]} \Phi_s\, k(\mathrm{d}s)\right|$$

and thus

$$\left|v^n_{r,t}(x) - v_{r,t}(x)\right| \quad \leq \quad B \wedge \pi_{r,x} S^n_r. \tag{33}$$

Assume for the moment that we already showed the following lemma.

**Lemma 40** *Under the assumptions of Proposition* 39,

$$\lim_n \pi_{r,x} S^n_r = 0, \quad \text{for all} \quad (r,x) \in C \quad \text{and} \quad t \in [r,T].$$

Then Lemma 37 and (33) will establish the claim in Proposition 39. So it remains to verify Lemma 40 (for which the bound $B$ in (33) will be essential).

**Proof of Lemma 40** *Step* 0° For the moment, fix $t \in I$. For $n \geq 0$, $r \in [0,t]$, and $x \in E$, set

$$o^n_{r,x} \;:=\; \pi_{r,x} \, \sup\nolimits_{\ell \geq n} \left| \int_{(r,t]} \Phi_s \, k^\ell(\mathrm{d}s) - \int_{(r,t]} \Phi_s \, k(\mathrm{d}s) \right|. \tag{34}$$

Just as we derived (31),

$$\sup\nolimits_{\ell \geq n} \left| \int_{(r,t]} \Phi_s \, k^\ell(\mathrm{d}s) - \int_{(r,t]} \Phi_s \, k(\mathrm{d}s) \right| \;\leq\; 3 \, \|f\|^2_\infty \bigvee_{\ell=1}^{\infty} k^\ell(r,t] \in L^1(\pi_{r,x}).$$

Therefore, we can invoke Lebesgue's theorem, Proposition 35 (ii), the regularity of $\Phi$, and Lemma 38, to obtain that for every $(r,x) \in C$, $r \leq t$,

$$\lim\nolimits_n o^n_{r,x} = 0.$$

*Step* 1° We next establish that for $(r,x) \in C$, $r \leq t$, and $n \geq m$,

$$\pi_{r,x} S^n_r \quad \leq \quad 3 \, \|f\|_\infty \, \pi_{r,x} \Big( \sup\nolimits_{\ell \geq n} \int_{(r,t]} B \wedge (\pi_{s,\xi_s} S^m_s) \, k^\ell(\mathrm{d}s) \Big) + o^n_{r,x} \, . \tag{35}$$

In fact, we have

$$S^n_r \;\leq\; \sup\nolimits_{\ell \geq n} \left| \int_{(r,t]} (\Phi^\ell_s - \Phi_s) \, k^\ell(\mathrm{d}s) \right| + \sup\nolimits_{\ell \geq n} \left| \int_{(r,t]} \Phi_s \, k^\ell(\mathrm{d}s) - \int_{(r,t]} \Phi_s \, k(\mathrm{d}s) \right|,$$

and therefore (by notation (34)),

$$\pi_{r,x} S^n_r \quad \leq \quad \pi_{r,x} \Big( \sup\nolimits_{\ell \geq n} \left| \int_{(r,t]} (\Phi^\ell_s - \Phi_s) \, k^\ell(\mathrm{d}s) \right| \Big) + o^n_{r,x} \, .$$

Using the Lipschitz inequality (27) and domination (30) we can continue with

$$\pi_{r,x} S^n_r \quad \leq \quad 3 \, \|f\|_\infty \, \pi_{r,x} \Big( \sup\nolimits_{\ell \geq n} \int_{(r,t]} \left| v^\ell_s - v_s \right| k^\ell(\mathrm{d}s) \Big) + o^n_{r,x}$$

and thus, from (33)

$$\pi_{r,x} S^n_r \quad \leq \quad 3 \, \|f\|_\infty \, \pi_{r,x} \Big( \sup\nolimits_{\ell \geq n} \int_{(r,t]} B \wedge \big( \pi_{s,\xi_s} S^\ell_s \big) \, k^\ell(\mathrm{d}s) \Big) + o^n_{r,x} \, . \tag{36}$$

But for $\ell \geq n \geq m$, we have $S^\ell \leq S^n \leq S^m$, and (36) yields (35).

*Step* 2° We will now derive from (35) that for $(r,x) \in C$ and $t \in [r,T]$ fixed,

$$\pi_{r,x} \lim_n S_r^n \quad \leq \quad 3\,\|f\|_\infty \, \pi_{r,x}\Big( \int_{(r,t]} B \wedge \big(\pi_{s,\xi_s} \lim_n S_s^n\big)\, k(\mathrm{d}s)\Big). \tag{37}$$

Indeed, $s \mapsto B \wedge \pi_{s,\xi_s} S_s^m$ is càdlàg $\pi_{r,x}$–a.s., according to Lemma 16. Therefore, in view of Lemma 38,

$$\int_{(r,t]} B \wedge \big(\pi_{s,\xi_s} S_s^m\big)\, k^\ell(\mathrm{d}s) \ \to_\ell \ \int_{(r,t]} B \wedge \big(\pi_{s,\xi_s} S_s^m\big)\, k(\mathrm{d}s) \tag{38}$$

with $\pi_{r,x}$–probability one. Note that

$$0 \ \leq \ \sup_{\ell \geq n} \int_{(r,t]} B \wedge \big(\pi_{s,\xi_s} S_s^m\big)\, k^\ell(\mathrm{d}s) \ \leq \ B \bigvee_{\ell=1}^{\infty} k^\ell(r,T] \in L^1(\pi_{r,x}).$$

Hence, from monotone convergence, inequality (35), Lebesgue's theorem, and (38), we get

$$\pi_{r,x} \lim_n S_r^n \quad = \quad \lim_n \pi_{r,x} S_r^n \quad \leq \quad 3\,\|f\|_\infty \, \pi_{r,x}\Big( \int_{(r,t]} B \wedge \big(\pi_{s,\xi_s} S_s^m\big)\, k(\mathrm{d}s)\Big).$$

Passing to the monotone limit as $m \to \infty$, this yields (37).

*Step* 3° We will show that (37) implies

$$\|\pi_{r,x} \lim_n S_r^n\|_C \quad \leq \quad 3\,\|f\|_\infty \, \|\pi_{r,x}\, k(r,T]\|_\infty \, \|\pi_{r,x} \lim_n S_r^n\|_C \, . \tag{39}$$

In fact, according to Proposition 12, for every point $(r,x) \in C$,

$$\pi_{r,x}\Big\{ (s,\xi_s) \in C \text{ for every } s \in [r,T] \Big\} = 1.$$

Moreover, for any point $(r,x) \in C$, we have, by definition of $B$, that

$$B \wedge \pi_{r,x} \lim_n S_r^n \ = \ \pi_{r,x} \lim_n S_r^n.$$

Hence, for any point $(r,x) \in C$, inequality (37) implies that

$$\pi_{r,x} \lim_n S_r^n \quad \leq \quad 3\,\|f\|_\infty \, \|\pi_{r,x} \lim_n S_r^n\|_C \, \pi_{r,x}\, k(r,T].$$

Taking the supremum over $(r,x) \in C$ we obtain (39).

*Step* 4° Recall that according to Lemma 37, $\|\pi_{r,x} \lim_n S_r^n\|_C \leq B < \infty$. Using assumption (32), therefore (39) implies that $\|\pi_{r,x} \lim_n S_r^n\|_C = 0$, and in particular $\pi_{r,x} \lim_n S_r^n = 0$ for $r, x, t$ as considered in the lemma. By monotone convergence, this completes the proof of Lemma 40. ■

## 3.9 Final steps of proof of fdd continuity if $\Phi^n \equiv \Phi$

Here we complete the proof of Theorem 23 in the case $\Phi^n \equiv \Phi$. Consider branching functionals $k^1, ..., k^\infty = k$ which are uniformly of bounded characteristic. Let $f \in b\mathcal{E}_+$ satisfy the smallness property (32). Fix a starting point $(r, x) \in I \times E$. Consider a subsequence $\{k^{n_m}\}$ of $\{k^n\}$. By assumption, and by the convergence criterion Proposition 27 there exists a subsequence $\{k^{n_{m_i}}\}$ of $\{k^{n_m}\}$ such that $(\alpha)$ and $(\beta)$ in (ii) of this proposition hold. We conclude that $(r, x)$ belongs to the set $C$ introduced in Notation 36 (i), related to this sequence $\{k^{n_{m_i}}\}$. By Proposition 39, we then get that $v_{r,t}^{n_{m_i}}(f)(x)$ converges to $v_{r,t}(f)(x)$ as $i \to \infty$ for each $t \in [r, T]$, with $v_{\cdot,t}(f)$ the (unique) small solution to (18). Hence, the limit is independent of the choice of the subsequences, and we get the latter convergence statement along the whole sequence $\{k^n\}$.

But each $v_{r,t}^n(f)(x)$ is monotonic as a functional of $f$ satisfying assumption (32) (since it is a log-Laplace functional), and therefore this property is shared by $v_{r,t}(f)(x)$. According to Lemma 30, the mapping $f \mapsto v_{r,t}(f)(x)$ must then be continuous, for all sufficiently small $f$. As a consequence, Lemma 31 implies that $v_{r,t}^n(f)(x)$ converges to some $v_{r,t}(f)(x)$ as $i \to \infty$, for *any* $f$ in $b\mathcal{E}_+$, where $v_{r,t}(\cdot)(x)$ is the log-Laplace functional of some random measure. In order to finish the proof, it suffices to show according to Lemma 29 that the family $\{v_{r,t} : 0 \le r \le t \le T\}$ determines a semigroup on $b\mathcal{E}_+$, and that in fact $v_{\cdot,t}(f)$ solves the $(\xi, \Phi, k)$–evolution equation (18).

Recall that $v_{\cdot,t}(f)$ solves (18) for $f$ small in the sense of (32). On the other hand, for any $f \in b\mathcal{E}_+$, the mapping $\theta \mapsto v_{r,t}(\theta f)(x)$ is analytic on the half line $(0, \infty)$, since $v_{r,t}(\cdot)(x)$ is a log-Laplace functional. By replacing $f$ by $\theta f$, we get that both sides of the $(\xi, \Phi, k)$–evolution equation (18) are analytic mappings of $\theta$ (since $\Phi$ is analytic in its third variable, and by the imposed moment assumptions). Since both sides of (18) coincide for small values of $\theta$, by the uniqueness of analytic continuation they are hence equal for every $\theta$. Specializing to $\theta = 1$, this shows that $v_{\cdot,t}(f)$ solves (18) not only for small $f$ but in fact for every $f \in b\mathcal{E}_+$. Since $(r, x)$ is arbitrary, by Lemma 32 the proof is finished. ■

## 3.10 Extension to fdd joint continuity

To complete the proof of Theorem 23 altogether, we have to remove the $\Phi^n \equiv \Phi$ restriction. Consider $\Phi^1, ..., \Phi^\infty = \Phi$ and $k^1, ..., k^\infty = k$ as in Assumption 22. Fix $f \in b\mathcal{E}_+$. Write $v^{n,m} = v^{n,m}(f)$ for the log-Laplace functional related to $\Phi^n, k^m$, $n, m = 1, ..., \infty$. For $0 \le r \le t \le T$ and $x \in E$, consider

$$\left| v_{r,t}^{n,n}(x) - v_{r,t}^{\infty,n}(x) \right|. \tag{40}$$

We take the abbreviation $\Phi^i(v^{n,m})$ for $\Phi^i\big(r,\xi_r\,,v^{n,m}_{r,t}(\xi_r)\big)$, $i,n,m=1,...,\infty$. Using the evolution equation (18), we may estimate (40) to

$$\leq\ \pi_{r,x}\int_{(r,t]}\Big|\Phi^n(v^{n,n})-\Phi^\infty(v^{\infty,n})\Big|k^n(\mathrm{d}s).$$

Compare additionally with the analogous term involving $\Phi^n(v^{\infty,n})$. In the first case, by the Lipschitz property (27) and the domination (30), we get the bound

$$3\,\|f\|_\infty\,\|v^{n,n}_{\cdot,t}-v^{\infty,n}_{\cdot,t}\|_\infty\,\pi_{r,x}\,k^n(r,t].$$

The other part is bounded by $\|\Phi^n-\Phi^\infty\|_\infty\,\pi_{r,x}\,k^n(r,t]$. Since all the branching functionals are uniformly of bounded characteristic, and $\Phi^n\to\Phi$ in uniform convergence, putting both together, for $\|f\|_\infty$ small enough we get

$$\lim_{n\to\infty}\|v^{n,n}_{\cdot,t}-v^{\infty,n}_{\cdot,t}\|_\infty\ =\ 0.$$

But $v^{\infty,n}_{s,t}(x)$ converges pointwise to $v^{\infty,\infty}_{s,t}(x)$ as $n\to\infty$, hence $v^{n,n}_{s,t}(x)$ approaches $v^{\infty,\infty}_{s,t}(x)$ as $n\to\infty$, too, for all sufficiently small $f$. By Lemma 31, this extends to all $f\in b\mathcal{E}_+$, finishing the proof of Theorem 23. ■

**Remark 41 (indexed sequences of branching functionals)** In the beginning of § 3.9, we fixed a starting point $(r,x)$, constructed then $v_{r,t}(f)(x)$, for any $t$ and $f$, and verified the properties we needed. Note that all the arguments would work, if the sequence of branching functionals $k^1,k^2,...$ we started from depended on $(r,x)$, provided that only the "limiting" $k^\infty=k$ is independent of $(r,x)$. Hence, the fact that in Theorem 23 the sequence $\{k^n\}$ of branching functionals is assumed to be independent of the choice of the starting point $r,x$ is not substantial. One could conversely consider a *family* of sequences $\{k^n_{r,x}\}$ indexed by $(r,x)$, if only the "limiting" $k^\infty=k$ does not depend on $(r,x)$. ◇

## 3.11 Fdd continuity in only the branching mechanism

The purpose of this subsection is to provide the *Proof of Proposition* 25. First note that the log-Laplace functionals $v_n$ and $v$ exist uniquely by Lemma 18. Set

$$\overline{v}_{r,t}(f)(x)\ :=\ \limsup_n v^n_{r,t}(f)(x),\qquad \underline{v}_{r,t}(f)(x)\ :=\ \liminf_n v^n_{r,t}(f)(x).$$

By the evolution equation (18), we have

$$\overline{v}_{r,t}(f)(x)\ =\ \pi_{r,x}f(x)-\liminf_n\pi_{r,x}\int_r^t\Phi^n\big(s,\xi_s\,,v^n_{s,t}(f)(\xi_s)\big)\,k(\mathrm{d}s).$$

Since $\Phi$ is non-decreasing in its third variable, for each $M\geq1$, we may continue with

$$\leq\ \pi_{r,x}f(x)-\liminf_n\pi_{r,x}\int_r^t\Phi^n\big(s,\xi_s\,,\inf_{m\geq M}v^m_{s,t}(f)(\xi_s)\big)\,k(\mathrm{d}s)$$

which equals

$$\pi_{r,x} f(x) - \pi_{r,x} \int_r^t \Phi\Big(s, \xi_s \, , \inf\nolimits_{m \geq M} v_{s,t}^m(f)(\xi_s)\Big) \, k(\mathrm{d}s).$$

Letting $M \to \infty$, we conclude for

$$\overline{v}_{r,t}(f)(x) \;\leq\; \pi_{r,x} f(x) - \pi_{r,x} \int_r^t \Phi\big(s, \xi_s \, , \underline{v}_{\,s,t}(f)(\xi_s)\big) \, k(\mathrm{d}s). \tag{41}$$

Analogously,

$$\underline{v}_{\,r,t}(f)(x) \;\geq\; \pi_{r,x} f(x) - \pi_{r,x} \int_r^t \Phi\big(s, \xi_s \, , \overline{v}_{s,t}(f)(\xi_s)\big) \, k(\mathrm{d}s). \tag{42}$$

By the local Lipschitz Lemma 34, from this we get

$$\overline{v}_{r,t}(f)(x) - \underline{v}_{\,r,t}(f)(x) \;\leq\; 3 \, \|f\|_\infty \, \pi_{r,x} \int_r^t \big(\overline{v}_{s,t}(f)(\xi_s) - \underline{v}_{\,s,t}(f)(\xi_s)\big) \, k(\mathrm{d}s).$$

Hence,

$$\begin{aligned} &\big\|\overline{v}_{r,t}(f)(x) - \underline{v}_{\,r,t}(f)(x)\big\|_\infty \\ &\leq\; 3 \, \|f\|_\infty \, \big\|\overline{v}_{r,t}(f)(x) - \underline{v}_{\,r,t}(f)(x)\big\|_\infty \, \|\pi_{r,x} \, k(r,t]\|_\infty \end{aligned}$$

(recall Convention 33). Thus, for $f$ small enough, the limit of the l.h.s. in (21) exists. Repeating the argument working with $v$ instead of $\overline{v}$ and $\underline{v}$ we conclude that the inequalities (41) and (42) hold for $v$. That is, $v$ solves the log-Laplace equation (17). By uniqueness, we arrive at the desired limit $v(f)$ in (21), for these small $f$.

$v(f)$ is the limit of functionals which are monotone in $f$ and is therefore monotone in $f$. The rest of the proof is identical to the arguments to our main Theorem in the end of § 3.9. ■

## 3.12 Proof of the fdd approximation by classical processes

For the *proof of Theorem* 26, by Theorem 23 it obviously suffices to verify the following lemma.

**Lemma 42 (approximation by classical branching functionals)** *Let $k$ be a branching functional. Then there exist bounded measurable functions $\varrho^n : I \times E \to R_+$, $n \geq 1$, such that the classical branching functionals $k^n(\mathrm{d}s) = \varrho^n_s(\xi_s) \, \mathrm{d}s$ of* (22) *are uniformly of bounded characteristic and have the following property:*

*For every starting point $(r,x) \in I \times E$ and every $r$–stopping time $\sigma \leq T$ fixed, $k^n(r,\sigma]$ converges to $k(r,\sigma]$ in $L^1(\pi_{r,x})$ as $n \to \infty$.*

**Proof** Fix $k, r, x$ as in the lemma. Consider $\pi_{r,x}$. To the branching functional $k$ there corresponds the *supermartingale*

$$t \mapsto h_T^t(\xi_t) \;:=\; \pi_{t,\xi_t} k(t,T], \qquad t \in [r,T],$$

with *compensator* $t \mapsto k(r,t]$. Following Dellacherie and Meyer [DM83, Remark VII.22 b)], we also consider the approximating sequence of supermartingales

$$t \mapsto {}^n h_T^t(\xi_t) \;:=\; \pi_{t,\xi_t} n \int_0^{\frac{1}{n}\wedge(T-t)} h_T^{t+u}(\xi_{t+u})\,\mathrm{d}u \;=\; \pi_{t,\xi_t} n \int_0^{1/n} k(t+u,T]\,\mathrm{d}u$$

with compensator

$$t \mapsto k^n(r,t] \;:=\; n \int_r^t \Big( h_T^s(\xi_s) - \pi_{s,\xi_s} k(s+\tfrac{1}{n}, T]\Big)\mathrm{d}s \;=\; n \int_r^t h^s_{(s+\frac{1}{n})\wedge T}(\xi_s)\,\mathrm{d}s,$$

$n \geq 1$. Note that ${}^n h_T^t(\xi_t)$ increases to $h_T^t(\xi_t)$ as $n \to \infty$. It follows from Proposition 5 (i) (with $Y_s = k(s,T]$) that $s \mapsto h_T^s(\xi_s)$ is $\pi_{r,x}$–indistinguishable from a non-negative càdlàg process of class (D). Moreover, for every $r$–stopping time $\sigma \leq T$, by the strong Markov property,

$$\pi_{r,x}\Big(h_T^{\sigma}(\xi_\sigma) - h_T^{(\sigma+\delta)\wedge T}(\xi_{(\sigma+\delta)\wedge T})\Big) \;=\; \pi_{r,x}\Big(k(\sigma,T] - k(\sigma+\delta,T]\Big)$$

which converges to 0 as $\delta \downarrow 0$, uniformly in $\sigma$. In fact, $s \mapsto k(s,T]$ is uniformly continuous, and the integrand is bounded by $2k(r,T] \in L^1(\pi_{r,x})$. By Proposition 58 (p.42) their uniform convergence to zero implies that

$$\pi_{r,x}\Big\{ \sup_{t\in[r,T]} \Big| h_T^t(\xi_t) - h_T^{(t+\delta)\wedge T}(\xi_{(t+\delta)\wedge T})\Big| > \varepsilon \Big\} \underset{\delta\downarrow 0}{\longrightarrow} 0,$$

for all $\varepsilon > 0$. Hence, for any sequence of $r$–stopping times $\sigma_n \leq T$, and $\varepsilon > 0$,

$$\pi_{r,x}\Big\{ \Big| h_T^{\sigma_n}(\xi_{\sigma_n}) - h_T^{(\sigma_n+\delta)\wedge T}(\xi_{(\sigma_n+\delta)\wedge T})\Big| > \varepsilon \Big\} \underset{n\to\infty}{\longrightarrow} 0.$$

In other words, the process $t \mapsto h_T^t(\xi_t)$ satisfies Aldous's criterion, hence it is quasi-left continuous (see Jacod and Shiryaev [JS87, Remark VI.4.7, p.321]). We can then invoke Theorem VII.20 of [DM83] to conclude that $k^n(r,\sigma]$ converges to $k(r,\sigma]$ in $L^1(\pi_{r,x})$ as $n \to \infty$, for every $r$–stopping time $\sigma \leq T$. Finally, it is easy to see that the $k^n$ are uniformly of bounded characteristic (recall Definition 21), finishing the proof. ■

# 4 Special case: Feller $\xi$ on a compactum

Since $T$ is arbitrary, the $(\xi,\Phi,k)$–superprocesses on the interval $I = [0,T]$ considered so far, can easily be extended to the whole time half axis $\mathsf{R}_+$. This we will actually do from now on. Of course, conditions as (3) and (20) are then required to hold *for all* $T > 0$.

Recall that a càdlàg right Markov process $\xi = (\xi_t, \Im, \pi_{r,x})$ in a Luzin space is called a *Hunt* process if it is *quasi-left continuous.* That is, for $0 \leq r \leq T < \infty$

and $\mu \in \mathcal{M}_{\mathrm{f}}$ fixed, we have $\xi_{\sigma_n} \to_n \xi_\sigma$ , $\pi_{r,\mu}$–a.e. for every sequence of $r$–stopping times $\sigma_n \le T$ non-decreasing to (the $r$–stopping time) $\sigma$ as $n \to \infty$.

From now on we will pay attention to the following special case, although *some* of our results below – such as the existence of a Hunt version – can be extended to more general situation by making use of Ray-Knight methods as exploited in [Led97a]. But this would require considerably more technical proofs, and the Feller case on a compact space *perfectly* illustrates our method.

**Assumption 43 (Feller on a compactum)** Suppose that the phase space is a *compact* metric space $(E, d)$. Moreover, let $\xi$ be *time-homogeneous* and indeed be a *Feller* process. $\diamond$

Note however, that nevertheless a related $(\xi, \Phi, k)$–superprocess is in general time-*in*homogeneous.

Recall that we introduced in $\mathcal{M}_{\mathrm{f}} = \mathcal{M}_{\mathrm{f}}(\mathcal{E})$ the weak topology (Assumption 1(b)). It can be generated by the *Prohorov metric* in the sense of [EK86, Problem 9.5.6, p.408], we denote by $\mathrm{w}_d$. Recall that $(\mathcal{M}_{\mathrm{f}}, \mathrm{w}_d)$ is *separable* ([EK86, Theorem 3.1.7]).

Moreover, for each $r \ge 0$ we will introduce the *Skorohod spaces* $\mathcal{D}_r = \mathcal{D}\big[[r,\infty), \mathcal{M}_{\mathrm{f}}\big]$, of all $\mathcal{M}_{\mathrm{f}}$–valued càdlàg functions on $[r,\infty)$ equipped with the *Skorohod metric* $\mathrm{s}_d$, based on $d$ (actually on $\mathrm{w}_d$). Recall that $(\mathcal{D}_r, \mathrm{s}_d)$ is *separable* ([EK86, Theorem 3.5.6]), since $\mathcal{M}_{\mathrm{f}}$ is separable.

## 4.1 Results under the Feller assumption

So far we considered a $(\xi, \Phi, k)$–superprocess only as some Markov process in the sense of Assumption 1(d1). Now we will be concerned with *regularity properties* of its (measure-valued) paths. In fact, in this section, under Assumption 43, we extend the fdd convergence results of Section 3 to convergence in law on path space. Also, we show that for our $(\xi, \Phi, k)$–superprocesses a Hunt version exists.

**Theorem 44 (existence of a Hunt version)** *Impose Assumption* 43. *Let* $\Phi$ *be a branching mechanism and* $k$ *be a branching functional. Then there exists a Hunt version of the* $(\xi, \Phi, k)$*–superprocess.*

The proof of this theorem is postponed to § 4.4.1.

As an *application* of the previous Theorem 44, using an argument from [Dyn94, Chapter 6], we show that under the present Feller assumption the $(\xi, \Phi, k)$–superprocess is *continuous* exactly in the *"binary splitting"* case, regardless of the choice of the branching functional $k$ :

**Corollary 45 (characterization of continuous processes)** *Under the assumptions of Theorem* 44, *the (Hunt)* $(\xi, \Phi, k)$*–superprocess* $X$ *has almost surely continuous paths if and only if* $\Phi$ *has the form* $\Phi(s, x, \lambda) = b^s(x)\lambda^2$ *(recall Assumption* 17 (f)).

**Proof** $X$ is Hunt by the previous theorem. $X$ is almost surely continuous if and only if its modified Lévy measure vanishes, which occurs if and only if the projection of the latter ([Dyn94, § 6.8.1]) disappears. But this happens if and only if $n = 0$ in the definition of $\Phi$ (recall Assumption 17 (f)). ∎

Based on Theorem 44, our fdd continuity Theorem 23 can be sharpened in terms of convergence in law on Skorohod path spaces:

**Theorem 46 (continuity in law on path spaces)** *Under Assumptions* 43 *and* 22 (p.18), *for $r, \mu$ fixed, the laws $P^n_{r,\mu}$ on the Skorohod space $\mathcal{D}_r$ of the Hunt $(\xi, \Phi^n, k^n)$–superprocesses converge weakly towards the law $P_{r,\mu}$ of the Hunt $(\xi, \Phi, k)$–superprocess.*

The proof of this theorem will follow in § 4.4.2.

For fixed branching functional $k$, the continuity in the branching mechanism $\Phi$ can be sharpened by using a weaker convergence concept for $\Phi$, just as in the fdd case (Proposition 25):

**Proposition 47 (continuity on path spaces concerning $\Phi$ only)** *Fix a branching functional $k$. If the branching mechanisms $\Phi^n$ converge boundedly pointwise to a (not necessarily regular) branching mechanism $\Phi$ as $n \to \infty$, then, under Assumption* 43, *the related superprocesses converge in law on the Skorohod path spaces $\mathcal{D}_r$.*

The proof of this result is postponed to § 4.4.3.

We can combine Theorem 46 with Lemma 42 to conclude for the following approximation in law by classical superprocesses (detailed arguments will follow in § 4.4.4).

**Theorem 48 (approximation by classical processes)** *Impose Assumption* 43. *If $\Phi$ is a regular branching mechanism, then, on Skorohod spaces $\mathcal{D}_r$, any $(\xi, \Phi, k)$–superprocess $X$ can be approximated in law by classical Hunt superprocesses $X^n$ (based on the classical branching functionals* (22)*). If $\Phi$ is an arbitrary branching functional, then, for every $r \geq 0$ and $\mu \in \mathcal{M}_{\mathrm{f}}$, there exists a collection of regular branching mechanisms $\Phi^n$ and classical branching functionals $k^n$ such that the laws $P^n_{r,\mu}$ on $\mathcal{D}_r$ of the $(\xi, \Phi^n, k^n)$–superprocesses $X^n$ converge weakly to the law $P_{r,\mu}$ on $\mathcal{D}_r$ of the $(\xi, \Phi, k)$–superprocess $X$.*

## 4.2 A sufficient criterion for tightness on path space

A basic step in the proofs is the verification of the following criterion, which extends a result from [Led97a, Proposition 6.39]. Write $C_d(E)$ for the set of all non-negative $d$–uniformly continuous functions defined on $E$.

**Proposition 49 (tightness on path space)** *Let $\Phi^1, \Phi^2, \dots$ be a collection of branching mechanisms and let $k, k^1, k^2, \dots$ be branching functionals which are uniformly of bounded characteristic (on bounded intervals). Assume that for each starting point $(r,x) \in \mathsf{R}_+ \times E$, each $T \geq r$, and each $r$–stopping time $\sigma \leq T$ we know that $k^n(r,\sigma]$ converges to $k(r,\sigma]$ in $L^1(\pi_{r,x})$ as $n \to \infty$. Suppose that each $X^n = (X^n, \mathcal{F}, P^n_{r,\mu})$ is a càdlàg right $(\xi, \Phi^n, k^n)$–superprocess, $n \geq 1$. Then, for $r \geq 0$ and $\mu \in \mathcal{M}_\mathrm{f}$ fixed, the laws $P^n_{r,\mu}$ of the $X^n$, as measures on the Skorohod space $\mathcal{D}_r$, are tight. Moreover, for $T \geq r$ and $r$–stopping times $\mathcal{T}_n$ bounded by $T$, and $\delta_n \searrow 0$, we have*

$$\lim_{n\to\infty} P^n_{r,\mu}\Big| \exp\big\langle X^n_{\mathcal{T}_n}, -f\big\rangle - \exp\big\langle X^n_{\mathcal{T}_n+\delta_n}, -f\big\rangle \Big|^2 \;=\; 0, \tag{43}$$

*for each $f \in \mathcal{C}_d(E)$.*

To prepare for the proof, define $\mathbb{F}$ as the linear span of all functions $F_f$,

$$F_f(\mu) \;:=\; \exp\langle \mu, -f\rangle, \qquad \mu \in \mathcal{M}_\mathrm{f},$$

where $f$ varies in $\mathcal{C}_d(E)$.

**Lemma 50 (separation of points)** *Each $F \in \mathbb{F}$ is a bounded non-negative continuous function on $\mathcal{M}_\mathrm{f}$. Moreover, $\mathbb{F}$ separates the points of $\mathcal{M}_\mathrm{f}$.*

**Proof** Note that $\mathbb{F}$ separates points if the collection of all functions $-\log F_f$, $f \in \mathcal{C}_d(E)$, is separating. Therefore, it suffices to show that $\mathcal{C}_d(E)$ separates the points of $E$ ([EK86, Theorem 3.4.5 (a)]). But this is obvious (use $d$). ■

**Proof of Proposition 49** Fix $r \geq 0$ and $\mu$ in $\mathcal{M}_\mathrm{f}$, and consider the laws $P^n_{r,\mu}$ on $\mathcal{D}_r$ of the $X^n$, $n \geq 1$. We will use *Jakubowski's criterion* (see e.g. [Daw93, Theorem 3.6.4]) to verify the tightness of these laws.

To check the first condition in Jakubowski's criterion, we show that the processes $X^n$ "almost live" on a common compact subset of $\mathcal{M}_\mathrm{f}$. More precisely, we verify that for $T > r$ and $\varepsilon > 0$ fixed,

$$P^n_{r,\mu}\Big( \sup_{s\in[r,T]} \langle X^n_s, 1\rangle > \tfrac{1}{\varepsilon}\Big) \;\leq\; \varepsilon\,\langle \mu, 1\rangle, \qquad n \geq 1. \tag{44}$$

But using the Doob type inequality of Proposition 58, the l.h.s. can be estimated by

$$\leq\; \varepsilon\, \sup\nolimits_{\mathcal{T}} P^n_{r,\mu}\,\langle X^n_{\mathcal{T}}, 1\rangle \tag{45}$$

with the supremum running over all $r$–stopping times $\mathcal{T} \leq T$. But the right superprocesses $X^n$ are critical, hence the processes $t \to \langle X^n_t, 1\rangle$ are right continuous martingales (recall Remark 19). So our estimate (45) equals $\varepsilon P^n_{r,\mu}\,\langle X^n_r, 1\rangle = \varepsilon\,\langle \mu, 1\rangle$, proving (44).

Next, for the second condition in Jakubowski's criterion, using the separation Lemma 50 it is sufficient to check the tightness of the laws of the càdlàg processes $t \mapsto F_f(X_t^n)$, $n \geq 1$, on the Skorohod space $\mathcal{D}[[r,\infty),\mathsf{R}_+]$, for each fixed $f$ in $C_d(E)$. For this purpose, we use *Aldous's criterion* (see, for instance, [Daw93, Theorem 3.6.5]), from which we get that it suffices to show that, given $T \geq r$ and $r$–stopping times $\mathcal{T}_n$ bounded by $T$, and $\delta_n \searrow 0$, claim (43) holds. But expanding the binomial in (43), we get, in particular, a term $\exp\langle X^n_{\mathcal{T}_n+\delta_n}, -2f\rangle$. Its $P^n_{r,\mu}$–expectation can be written as

$$P^n_{r,\mu} \exp\Big\langle X^n_{\mathcal{T}_n}, -v^n_{\mathcal{T}_n,\mathcal{T}_n+\delta_n}(2f)\Big\rangle,$$

using the strong Markov property at time $\mathcal{T}_n$, and the log-Laplace transition functional representation (17). Here $v^n(2f)$ solves the evolution equation (18) with $f, \Phi, k$ replaced by $2f, \Phi^n, k^n$, respectively. We will compare this term with

$$P^n_{r,\mu} \exp\Big\langle X^n_{\mathcal{T}_n}, -2\, v^n_{\mathcal{T}_n,\mathcal{T}_n+\delta_n}(f)\Big\rangle.$$

Calculating the other term similarly, for the expectation expression in (43) we get

$$\begin{aligned} &P^n_{r,\mu}\Big|\exp\langle X^n_{\mathcal{T}_n}, -f\rangle - \exp\Big\langle X^n_{\mathcal{T}_n}, -v^n_{\mathcal{T}_n,\mathcal{T}_n+\delta_n}(f)\Big\rangle\Big|^2 \\ &+ P^n_{r,\mu}\Big(\exp\Big\langle X^n_{\mathcal{T}_n}, -v^n_{\mathcal{T}_n,\mathcal{T}_n+\delta_n}(2f)\Big\rangle - \exp\Big\langle X^n_{\mathcal{T}_n}, -2\, v^n_{\mathcal{T}_n,\mathcal{T}_n+\delta_n}(f)\Big\rangle\Big). \end{aligned}$$

To get an upper bound for this, we may drop the exponent 2, and continue with

$$\begin{aligned} \leq \quad & P^n_{r,\mu}\Big\langle X^n_{\mathcal{T}_n}, \Big|f - v^n_{\mathcal{T}_n,\mathcal{T}_n+\delta_n}(f)\Big|\Big\rangle \\ & + P^n_{r,\mu}\Big\langle X^n_{\mathcal{T}_n}, \Big|v^n_{\mathcal{T}_n,\mathcal{T}_n+\delta_n}(2f) - 2\, v^n_{\mathcal{T}_n,\mathcal{T}_n+\delta_n}(f)\Big|\Big\rangle. \end{aligned}$$

Using again [Dyn94, Theorem 6.2.1], to each $\mathcal{T}_n$ there exists an $r$-randomized stopping time $\tau_n \leq T$ for $\xi$ such that the latter equals

$$\left.\begin{aligned} = \quad & \pi_{r,\mu}\Big|f(\xi_{\tau_n}) - v^n_{\tau_n,\tau_n+\delta_n}(f)(\xi_{\tau_n})\Big| \\ & + \pi_{r,\mu}\Big|v^n_{\tau_n,\tau_n+\delta_n}(2f)(\xi_{\tau_n}) - 2\, v^n_{\tau_n,\tau_n+\delta_n}(f)(\xi_{\tau_n})\Big|. \end{aligned}\right\} \tag{46}$$

Applying the evolution equation (18), and the strong Markov property for $\xi$, for the first term in (46) we get the bound

$$\left.\begin{aligned} & \pi_{r,\mu}\Big|f(\xi_{\tau_n}) - \pi_{\tau_n,\xi_{\tau_n}} f(\xi_{\tau_n+\delta_n})\Big| \\ & + \pi_{r,\mu}\int_{\tau_n}^{\tau_n+\delta_n} \Phi^n\big(s, \xi_s, v^n_{s,\tau_n+\delta_n}(f)(\xi_s)\big)\, k^n(\mathrm{d}s). \end{aligned}\right\} \tag{47}$$

Since $\xi$ is a time-homogeneous strong Markov process, the first term is bounded by $\langle\mu,1\rangle \sup_x \big|f(x) - \pi_{0,x} f(\xi_{\delta_n})\big|$, and by the Feller property this will disappear as $n \to \infty$. If now $\{k^{n_m}\}$ is a subsequence of $\{k^n\}$, by the reformulation Proposition 27, there exists a subsequence $\{k^{n_{m_i}}\}$ of $\{k^{n_m}\}$ such that

$$(\alpha) \quad \pi_{r,\mu} \bigvee_{i=1}^{\infty} k^{n_{m_i}}(r,T] < \infty,$$

$$(\beta) \quad \sup_{s\in[r,T]} \left| k^{n_{m_i}}(s,T] - k(s,T] \right| \underset{i\to\infty}{\longrightarrow} 0, \quad \pi_{r,\mu}\text{–a.e.}$$

Combined with (29) and (30), we get that the second term in (47) will vanish as $n_{m_i} \to \infty$, hence as $n \to \infty$. So (47) will disappear in the limit.

The proof that the second term in (46) goes to zero is similar. Consequently, (46) will vanish in the limit, hence (43) is true, and Jakubowski's criterion is fulfilled. ■

**Corollary 51 (convergence on path space)** *Suppose in addition to the hypotheses of Proposition* 49 *that the* $X^n = (X^n_t, \mathcal{F}, P^n_{r,\mu})$ *converge fdd to a* $(\xi, \Phi, k)$*–superprocess* $X$ *with a regular branching mechanism* $\Phi$*. Then for each* $r, \mu$*, the laws* $P^n_{r,\mu}$ *on* $\mathcal{D}_r$ *converge weakly to* $P^\infty_{r,\mu}$.

**Proof** Since tightness plus fdd convergence implies weak convergence, we immediately get from Proposition 49 and the assumed fdd convergence that $P^n_{r,\mu}$ converges weakly to $P^\infty_{r,\mu}$ as $n \to \infty$. ■

## 4.3 Existence of a càdlàg right version $X$

Recall that $(E, d)$ is a compact metric space. For convenience, we introduce the following notion.

**Definition 52 (almost sure notions)** For the moment, consider an $\mathcal{M}_\mathrm{f}$–valued Markov process $X = \left(X_t, \mathcal{F}, P_{r,\mu}\right)$ with phase space $(E, d)$. We say that $X$ is an *a.s. càdlàg right process* if

**(i)** for $r \ge 0$ and $\mu \in \mathcal{M}_\mathrm{f}$,

$$P_{r,\mu}\Big\{ t \to X_t \text{ is càdlàg}, \quad t \in [r,\infty) \Big\} = 1,$$

**(ii)** for $0 \le r < t$, for $\mu \in \mathcal{M}_f$, and for measurable $F : \mathcal{M}_\mathrm{f} \to \mathsf{R}_+$, the function

$$s \mapsto \mathbf{1}_{s<t} P_{s,X_s} F(X_t), \qquad s \in [r,t),$$

is $P_{r,\mu}$–a.s. right continuous.

An a.s. càdlàg right process $X$ is said to be an *a.s. Hunt process* if it is quasi-left continuous. ◇

As shown in [Led97a, Lemma 5.28] the two introduced a.s. notions are not substantially different from the ones without 'a.s.':

**Lemma 53 (dropping 'a.s.')** *Let $X$ be an a.s. Hunt (respectively a.s. càdlàg right) process. Then there exists a Hunt (respectively càdlàg right) version of $X$.*

Now we are ready to state the following result.

**Lemma 54 (càdlàg right version)** *Impose Assumption 43. Let $\Phi$ be a branching mechanism and $k$ a branching functional. Then there exists a càdlàg right version of the $(\xi,\Phi,k)$–superprocess.*

**Proof** Recall that the $(\xi,\Phi,k)$–superprocess $X$ exists by Lemma 18. According to [Dyn93, Theorem 2.1], there is a right version $X=(X_t\,,\mathcal{F},P_{r,\mu})$ of this process. Let $(A,\mathcal{D}(A))$ be the strong generator of the Feller process $\xi$. Recall that $\mathcal{D}(A)\subseteq\mathcal{C}_d(E)$. Fix $r\geq 0$ and $\mu\in\mathcal{M}_{\mathrm{f}}$. Note that for $f\in\mathcal{D}(A)$ the processes $t\mapsto\langle X_t\,,f\rangle-\int_r^t\langle X_s\,,Af\rangle\,\mathrm{d}s,\ t\geq r$, are right continuous $P_{r,\mu}$–martingales, and therefore, with $P_{r,\mu}$–probability one, càdlàg martingales. Hence, the process $t\mapsto\langle X_t\,,f\rangle\,,\ t\geq r,$ is $P_{r,\mu}$–a.s. càdlàg. Let $\{f_m:m\geq 1\}\subseteq\mathcal{D}(A)$ be a convergence determining set (for the weak topology in $\mathcal{M}_{\mathrm{f}}$). Recall that $\{f_m:m\geq 1\}$ is separating. Let

$$\Omega_r\ :=\ \Bigl\{\omega:\ t\mapsto\langle X_t(\omega),f_n\rangle\,,\ \ t\geq r,\ \ \text{is càdlàg},\ n\geq 1\Bigr\}.$$

Note that $P_{r,\mu}(\Omega_r)=1$. Recall also that on every bounded interval $[r,T]$, the càdlàg trajectory $t\mapsto\langle X_t(\omega),1\rangle$ is bounded. Also, the sets $\bigl\{\mu:\langle\mu,1\rangle\leq N\bigr\}$ are compact in $\mathcal{M}_{\mathrm{f}}$. Consider $\omega\in\Omega_r\,,\ t>r$, and let $t_n\uparrow t,\ t_n<t$. It follows that the family $\{X_{t_n}(\omega)\}_{n\geq 1}\subseteq\mathcal{M}_{\mathrm{f}}$ is tight. Hence, it has an accumulation point $X_{t-}(\omega)$. But since $\omega\in\Omega_r$, this accumulation point is unique and independent of the choice of the sequence $\{t_n:n\geq 1\}$. Thus $\lim_{s\uparrow t}X_s(\omega)=X_{t-}(\omega)$. Since $t$ was arbitrary, it follows that $t\mapsto X_t(\omega)$ is càdlàg, for $\omega\in\Omega_r$. An appeal to Lemma 53 completes the proof. ■

## 4.4 Remaining proofs

### 4.4.1 Proof of existence of a Hunt version

The next result is taken from [Led97a, Lemma 6.38].

**Lemma 55** *Let $\{y_t:0\leq t\leq T\}$ and $\{z_t:0\leq t\leq T\}$ be $[0,1]$–valued stochastic processes over a filtered probability space $(\Omega,\Im,P)$. Suppose that $y$ is $P$-indistinguishable from a right continuous process. Let $\tau_n\leq T$ be stopping times converging to some stopping time $\tau$ as $n\to\infty$. Then there exists a sequence $\delta_n\searrow_n 0$ such that*

$$\lim_{n\to\infty}P\left|\,z_{\tau_n}y_\tau-z_{\tau_n}y_{\tau_n+\delta_n}\right|\ =\ 0.$$

Recall that a càdlàg right process $X = (X_t, \mathcal{F}, P_{r,\mu})$ is a *Hunt* process if and only if $P_{r,\mu}\{X_{\mathcal{T}-} = X_{\mathcal{T}}\} = 1$ for $r \geq 0$, $\mu \in \mathcal{M}_{\mathrm{f}}$, and every bounded *predictable* $r$–stopping time $\mathcal{T}$.

**Proof of Theorem 44** Take $\xi, \Phi, k$ as in the theorem. Recalling Lemma 54, let $X = (X_t, \mathcal{F}, P_{r,\mu})$ be a càdlàg right version of the $(\xi, \Phi, k)$–superprocess. Fix $r \geq 0$, $\mu \in \mathcal{M}_{\mathrm{f}}$, and $f \in \mathcal{C}_d(E)$. Consider a collection of $r$-stopping times $\mathcal{T}_n < \mathcal{T}$ non-decreasing to the bounded predictable stopping time $\mathcal{T}$. From Lemma 55 we conclude that there exists $\delta_n \downarrow 0$ such that

$$\left.\begin{aligned} &\lim_{n\to\infty} P_{r,\mu}\Big| \exp\langle X_{\mathcal{T}_n}, -f\rangle - \exp\langle X_{\mathcal{T}}, -f\rangle \Big| \\ &= \lim_{n\to\infty} P_{r,\mu}\Big| \exp\langle X_{\mathcal{T}_n}, -f\rangle - \exp\langle X_{\mathcal{T}_n+\delta_n}, -f\rangle \Big| \end{aligned}\right\} \tag{48}$$

Applying the tightness Proposition 49 with $X^n \equiv X$ we obtain

$$\lim_{n\to\infty} P_{r,\mu}\Big| \exp\langle X_{\mathcal{T}_n}, -f\rangle - \exp\langle X_{\mathcal{T}_n+\delta_n}, -f\rangle \Big|^2 = 0,$$

which implies that (48) vanishes. Using Fatou's lemma, we conclude

$$P_{r,\mu}\Big| \exp\langle X_{\mathcal{T}-}, -f\rangle - \exp\langle X_{\mathcal{T}}, -f\rangle \Big| = 0.$$

Hence $\langle X_{\mathcal{T}-}, f\rangle = \langle X_{\mathcal{T}}, f\rangle$ with $P_{r,\mu}$–probability 1. Arguing with a separating sequence of functions $f \in \mathcal{C}_d(E)$ yields $X_{\mathcal{T}-} = X_{\mathcal{T}}$ with $P_{r,\mu}$–probability 1, finishing the proof. ■

### 4.4.2 Proof of the joint continuity result

*Theorem* 46 directly follows from Theorem 44 (the process is Hunt), Theorem 23 (which guaranties fdd convergence) and Corollary 51 (from which we conclude the weak convergence). ■

### 4.4.3 Proof of the continuity in $\Phi$ only

*Proposition* 47 is derived from Theorem 44 (which guaranties the existence of a Hunt version), from Proposition 25 (which yields the fdd continuity in $\Phi$) and from Corollary 51 (from which we conclude the desired weak convergence). ■

### 4.4.4 Proof of approximation by classical superprocesses

We will need the following lemma:

**Lemma 56 ("approximation" by regular $\Phi$)** *Every branching mechanism $\Phi$ belongs to the bp–closure of the set of all regular branching mechanisms.*

**Proof** If the maps $(s,x) \mapsto b^s(x)$ and $(s,x) \mapsto n(s,x,\mathrm{d}u)$ in Assumption 17 (f) on a branching mechanism $\Phi$ are additionally continuous, then the corresponding branching mechanisms $\Phi$ are regular. Thus, the *bp*–closure of all regular branching mechanisms contains all ([0,1]–valued) measurable $(s,x) \mapsto b^s(x)$ and continuous $(s,x) \mapsto n(s,x,\mathrm{d}u)$ ([EK86, Proposition 3.4.2]). In particular, this is true for $n(s,x,\mathrm{d}u)$ of the form $f(s,x)\,n(\mathrm{d}u)$, where $f$ is continuous. Hence, the *bp*–closure contains all measurable functions $(s,x) \mapsto b^s(x)$ and $(s,x) \mapsto \mathbf{1}_A(s,x)\,n(\mathrm{d}u)$ with $A$ denoting a measurable subset of $\mathsf{R}_+ \times E$. Now let $n^1(\mathrm{d}u), n^2(\mathrm{d}u), \ldots$ be a dense subset of $\mathcal{M} = \mathcal{M}(0,\infty)$ (introduced in Assumption 17 (f)). Then every $n(s,x,\mathrm{d}u)$ is the pointwise limit of kernels of the form $n_N(s,x,\mathrm{d}u) := \sum_{\ell=1}^{\infty} \mathbf{1}_{A_N^\ell}(s,x)\,n^\ell(\mathrm{d}u)$ where

$$A_N^\ell := \Big\{(s,x) :\ d_{\mathrm{v}}(n^\ell,n) < \tfrac{1}{N} \ \text{ and } \ d_{\mathrm{v}}(n^i,n) \geq \tfrac{1}{N}\,,\ \ i=1,\ldots,\ell-1\Big\},$$

with $d_{\mathrm{v}}$ denoting a metric on $\mathcal{M}$ which generates the vague topology in $\mathcal{M}$. Using this fact completes the proof. ∎

**Proof of Theorem 48** *Step* 1° First we start from a $(\xi,\Phi,k)$–superprocess $X$ where $\Phi$ is regular. Note that, from Theorem 26 and Lemma 42, we can fdd approximate $X$ by classical $(\xi,\Phi,k)$–superprocesses $X^n$ in such a way that the $k^n$ satisfy the conditions imposed in Proposition 49. Note that the $X^n$ are Hunt. It suffices to invoke Corollary 51 to conclude that $P^n_{r,\mu} \Rightarrow P^\infty_{r,\mu}$.

*Step* 2° Suppose now that $\Phi$ is arbitrary. Fix $r \geq 0$, $\mu \in \mathcal{M}_{\mathrm{f}}$, and denote by $P^{(\xi,\Phi,k)}_{r,\mu}$ the law on $\mathcal{D}_r$ of the $(\xi,\Phi,k)$–superprocess with initial data $(r,\mu)$. Let $\mathcal{K}$ refer to the closure of the set of all laws $P^{(\xi,\Phi,k)}_{r,\mu}$ for which the branching functional $k$ is classical (recall (22)) and the branching mechanism $\Phi$ is regular. As shown in step 1°, the set $\mathcal{K}$ contains all $P^{(\xi,\Phi,k)}_{r,\mu}$ with arbitrary $k$ and regular $\Phi$. Consider the set $\mathbf{\Phi}_{\xi,k}$ of all $\Phi$ such that $P^{(\xi,\Phi,k)}_{r,\mu}$ belongs to $\mathcal{K}$. From Theorem 44 (Hunt) and Propositions 25 (fdd convergence) we can invoke Corollary 51 (weak convergence), and therefore conclude that the set $\mathbf{\Phi}_{\xi,k}$ is *bp*–closed. Therefore, since it contains all regular branching mechanisms, $\mathbf{\Phi}_{\xi,k}$ finally contains *all* branching mechanisms, by Lemma 56. In other words, all $P^{(\xi,\Phi,k)}_{r,\mu}$ belong to $\mathcal{K}$. Hence, for every $(k,\Phi)$ there exists a sequence $(k^n,\Phi^n)$ with classical $k^n$ and regular $\Phi^n$ such that

$$P^{(\xi,\Phi^n,k^n)}_{r,\mu} \Longrightarrow P^{(\xi,\Phi,k)}_{r,\mu} \quad \text{as} \quad n \to \infty.$$

This finishes the proof. ∎

# 5 Appendix

Here we collect some technical results. The following is a slight modification of [Dyn94, A.1.1.A, p.116].

**Lemma 57 (characterization of left continuity)** *Let* $y = \{y_t : 0 \leq t \leq T\}$ *denote a non-negative right continuous process of class* $(D)$ *over a filtered space* $(\Omega, \Im, P)$. *Then* $y$ *is* $P$*–a.s. càdlàg if and only if for every sequence of non-decreasing stopping times* $\sigma_n \leq T$ *we have that* $\lim_n P y_{\sigma_n}$ *exists.*

**Proof** $\Longrightarrow$ Suppose that $y$ is càdlàg. Let $y_{s-}$ denote the left limit $\lim_{t\uparrow s} y_t$. Hence if $\sigma_n \nearrow \sigma$ as $n \to \infty$ then $\lim_n y_{\sigma_n} = y_{\sigma-}$. But since $y$ belongs to class $(D)$,

$$P y_{\sigma-} \;=\; P \lim_n y_{\sigma_n} \;=\; \lim_n P y_{\sigma_n}.$$

Therefore $\lim_n P y_{\sigma_n}$ exists.

$\Longleftarrow$ Suppose now that $y$ is not $P$–a.s. càdlàg, but assume that for every sequence of non-decreasing stopping times $\sigma_n \leq T$, the limit $\lim_n P y_{\sigma_n}$ exists. Recall that by assumption, $y$ is right continuous and belongs to class $(D)$. Hence, there exists a set $\mathcal{N}$ of positive $P$-probability such that for every $\omega \in \mathcal{N}$

(i) the process $y_{.}(\omega)$ has a left oscillations, or

(ii) the process $y_{.}(\omega)$ has a left explosion.

We will show that each of these statements yield a contradiction.

(i) Suppose that the (right continuous) trajectory $y(\omega)$ has a left oscillation. Then there exist numbers $q, \delta$ in the set $Q_+$ of all non-negative rationales such that $y(\omega)$ oscillates around $q$ with oscillations of magnitude larger than $\delta$. In other words, the sequence $\{\sigma_n^{q,\delta}(\omega)\}_{n=0}^{\infty}$ defined by $\sigma_0^{q,\delta}(\omega) := 0$ and, for $m \geq 0$,

$$\begin{aligned}
\sigma_{2m+1}^{q,\delta}(\omega) \;&:=\; \inf\{t > \sigma_{2m}^{q,\delta}(\omega) :\; y_t(\omega) - q > \delta\},\\
\sigma_{2m+2}^{q,\delta}(\omega) \;&:=\; \inf\{t > \sigma_{2m+1}^{q,\delta}(\omega) :\; y_t(\omega) - q < -\delta\}
\end{aligned}$$

has the property that $\sigma_0^{q,\delta}(\omega) < \ldots < \sigma_n^{q,\delta}(\omega) < \sigma_{n+1}^{q,\delta}(\omega) < \ldots < T$. Setting again $\inf \emptyset := T$, then clearly, the random times $\sigma_n^{q,\delta}$ are stopping times. Let us define

$$A_{q,\delta} := \{\omega : \sigma_0^{q,\delta}(\omega) < \ldots < \sigma_n^{q,\delta}(\omega) < \sigma_{n+1}^{q,\delta}(\omega) < \ldots < T\}.$$

Moreover, let $y_t^*(\omega) := 1_{A_{q,\delta}^c}(\omega) y_t(\omega)$ where $A_{q,\delta}^c := \Omega - A_{q,\delta}$. Note that for $\omega \in A_{q,\delta}^c$, the sequence $\sigma_n^{q,\delta}(\omega)$ eventually reaches $T$. Thus $y^*_{\sigma_n^{q,\delta}}$ converges to $y_T^*(\omega)$. Because $y^*$ belongs to class $(D)$, this implies that

$$\lim_{n\to\infty} P\left(y^*_{\sigma_{n+1}^{q,\delta}} - y^*_{\sigma_n^{q,\delta}}\right) = 0. \tag{49}$$

On the other hand, we have

$$\lim_{n\to\infty} P\left(\mathbf{1}_{A_{q,\delta}}(y_{\sigma_{2n+1}^{q,\delta}} - y_{\sigma_{2n}^{q,\delta}})\right) \;\geq\; 2\delta P(A_{q,\delta}).$$

From (49) and the assumption that $\lim_{n\to\infty} P\Big(y_{\sigma^{q,\delta}_{2n+1}} - y_{\sigma^{q,\delta}_{2n}}\Big) = 0$, we conclude that $P(A_{q,\delta}) = 0$. Therefore, we obtain

$$P\Big(\bigcup_{q,\delta\,\in\, Q_+} A_{q,\delta}\Big) = 0.$$

That is, with probability one, there is no left oscillation, yielding a contradiction.

(ii) The proof is analogous. Write $\sigma_0 := 0$, and for $n \geq 0$, define $\sigma_{n+1} := \inf\{t > \sigma_n : y_t > n\}$. (Here again, $\inf\emptyset := T$.) We put

$$A := \{\sigma_n < T \text{ for every } n \geq 0\}.$$

In the same way as in (i) we have that the existing limit of $P(y_{\sigma_n})$ implies that $P(A) = 0$. Thus there is no explosions towards $+\infty$. ■

**Proposition 58 (a Doob type inequality)** *Let $\big\{y_t : t \in [0,T]\big\}$ denote a real-valued right continuous process of class* (D) *on a filtered probability space* $(\Omega, \mathcal{F}, P)$. *Then, for each* $\eta > 0$,

$$P\Big\{\sup\nolimits_{s\leq T} |y_s| > \eta\Big\} \quad\leq\quad \Big(\tfrac{2}{\eta}\sup\nolimits_\sigma |Py_\sigma| + P|y_T|\Big) \wedge \Big(\tfrac{1}{\eta}\sup\nolimits_\sigma P|y_\sigma|\Big)$$

*where $\sigma$ denotes any stopping time* (*bounded by* $T$).

**Proof** Let $\sigma_+^\eta := \inf\big\{s \in I : y_s > \eta\big\}$. Then by Markov's inequality,

$$P\big\{\sup\nolimits_s y_s > \eta\big\} \leq P\{y_{\sigma_+^\eta} \geq \eta\} \leq \tfrac{1}{\eta}\Big(Py_{\sigma_+^\eta} + P|y_T|\Big).$$

On the other hand, with $\sigma_-^\eta := \inf\big\{s \in I : y_s < -\eta\big\}$,

$$P\big\{\inf\nolimits_s y_s < -\eta\big\} \leq P\{y_{\sigma_-^\eta} \leq -\eta\} = P\{-y_{\sigma_-^\eta} \geq \eta\} \leq \tfrac{1}{\eta}\Big(-Py_{\sigma_-^\eta} + P|y_T|\Big).$$

Adding both cases, the first part of the claim follows. To get the other one, start with $\sigma^\eta := \inf\big\{s \in I : |y_s| > \eta\big\}$, and proceed directly in order to finish the proof. ■

**Lemma 59** *Let $a_n$, $b_n$ be real numbers. Then*

$$\Big|\bigwedge_{n=1}^{\infty} a_n - \bigwedge_{n=1}^{\infty} b_n\Big| \leq \bigvee_{n=1}^{\infty} |a_n - b_n|$$

*provided that at least one of the infimum expressions is finite.*

**Proof** The proof goes by induction. Without loss of generality, suppose that $\bigwedge_{n=1}^2 a_n \geq \bigwedge_{n=1}^2 b_n$. Again without loss of generality, suppose that $\bigwedge_{n=1}^2 a_n = a_1$. Then, two cases should be considered.

Case 1) $\bigwedge_{n=1}^2 b_n = b_1$ Then we have $\left|\bigwedge_{n=1}^2 a_n - \bigwedge_{n=1}^2 b_n\right| = |a_1 - b_1|$.

Case 2) $\bigwedge_{n=1}^2 b_n = b_2$ Since $\bigwedge_{n=1}^2 a_n = a_1$, we have $a_1 \leq a_2$. Thus

$$\left|\bigwedge_{n=1}^{2} a_n - \bigwedge_{n=1}^{2} b_n\right| = a_1 - b_2 \leq a_2 - b_2 \leq |a_2 - b_2|.$$

Consequently,

$$\left|\bigwedge_{n=1}^{2} a_n - \bigwedge_{n=1}^{2} b_n\right| \leq \bigvee_{n=1}^{2} |a_n - b_n|. \tag{50}$$

Let $N \geq 3$. To show that

$$\left|\bigwedge_{n=1}^{N} a_n - \bigwedge_{n=1}^{N} b_n\right| \leq \bigvee_{n=1}^{N} |a_n - b_n|, \tag{51}$$

just put

$$a_1^* := a_1, \quad b_1^* := b_1, \qquad a_2^* := \bigwedge_{n=2}^{N} a_n, \quad b_2^* := \bigwedge_{n=2}^{N} b_n.$$

By (50) we have

$$\begin{aligned} \left|\bigwedge_{n=1}^{N} a_n - \bigwedge_{n=1}^{N} b_n\right| &= \left|\bigwedge_{n=1}^{2} a_n^* - \bigwedge_{n=1}^{2} b_n^*\right| \leq \bigvee_{n=1}^{2} |a_n^* - b_n^*| \\ &= |a_1 - b_1| \vee \left|\bigwedge_{n=2}^{N} a_n - \bigwedge_{n=2}^{N} b_n\right|. \end{aligned}$$

Then by induction on $N$ the claim (51) follows. Letting $N$ tend to infinity gives the desired result. ■

**Corollary 60** *Suppose $k^n(\mathrm{d}s), k(\mathrm{d}s)$ are finite (deterministic) measures on $I = [0,T]$ such that $k^n(r,t]$ converges to $k(r,t]$ as $n \to \infty$, for every $r < t \leq T$. For each $n \geq 1$, let $s \mapsto \psi_s^n$ be uniformly bounded non-negative measurable functions on $I$. Then the function $t \mapsto F(t) := \bigwedge_{n=1}^{\infty} \int_{(t,T]} \psi_s^n \, k^n(\mathrm{d}s)$ is right continuous.*

**Proof** Consider $t < t + \delta \leq T$, and set $B := \sup_n \|\psi^n\|_\infty$. By Lemma 59 we have

$$\begin{aligned} |F(t) - F(t+\delta)| &\leq \bigvee_{n=1}^{\infty} \left| \int_{(t,T]} \psi_s^n \, k^n(\mathrm{d}s) - \int_{(t+\delta,T]} \psi_s^n \, k^n(\mathrm{d}s) \right| \\ &= \bigvee_{n=1}^{\infty} \int_{(t,t+\delta]} \psi_s^n \, k^n(\mathrm{d}s). \end{aligned}$$

Thus,

$$|F(t) - F(t+\delta)| \le B \bigvee_{n=1}^{\infty} k^n(t, t+\delta]. \tag{52}$$

Take any $\varepsilon > 0$ and choose $\delta$ so small that $k(t, t+\delta] \le \varepsilon$. Then there exists $N = N_{\varepsilon,\delta}$ such that for every $n \ge N$ we have $\left| k_n(t, t+\delta] - k(t, t+\delta] \right| \le \varepsilon$. Thus

$$\bigvee_{n=N}^{\infty} k^n(t, t+\delta] \le k(t, t+\delta] + \varepsilon \le 2\varepsilon.$$

But for $\delta_0 \in (0, \delta)$ small enough, we have $\bigvee_{n=1}^{N-1} k^n(t, t+\delta_0] \le 2\varepsilon$. Consequently, for $\delta_0 > 0$ sufficiently small,

$$\bigvee_{n=1}^{\infty} k^n(t, t+\delta_0] \le 2\varepsilon.$$

Returning to (52), the proof is complete. ∎

## References


[Bil68] P. Billingsley. *Convergence of probability measures.* Wiley, New York, 1968.

[Daw77] D.A. Dawson. The critical measure diffusion process. *Z. Wahrsch. verw. Gebiete*, 40:125–145, 1977.

[Daw93] D.A. Dawson. Measure-valued Markov processes. In P.L. Hennequin, editor, *École d'été de probabilités de Saint Flour XXI–1991*, volume 1541 of *Lecture Notes in Mathematics*, pages 1–260. Springer-Verlag, Berlin, 1993.

[DF91] D.A. Dawson and K. Fleischmann. Critical branching in a highly fluctuating random medium. *Probab. Theory Relat. Fields*, 90:241–274, 1991.

[DF97] D.A. Dawson and K. Fleischmann. A continuous super-Brownian motion in a super-Brownian medium. WIAS Berlin, Preprint Nr. 165, 1995, *Journ. Theoret. Probab.*, to appear 1997.

[DM83] C. Dellacherie and P.-A. Meyer. *Probabilités et potentiel: Chaptitres V à VIII Théorie des martingales.* Hermann, Paris, 1983.

[Dyn91] E.B. Dynkin. Branching particle systems and superprocesses. *Ann. Probab.*, 19:1157–1194, 1991.

[Dyn93] E.B. Dynkin. On regularity of superprocesses. *Probab. Theory Relat. Fields*, 95:263–281, 1993.

[Dyn94] E.B. Dynkin. *An Introduction to Branching Measure-Valued Processes.* volume 6 of *CRM Monograph Series.* Centre de Recherches Mathématiques, Université de Montréal, Amer. Math. Soc., Providence, 1994.

[EK86] S.N. Ethier and T.G. Kurtz. *Markov Processes: Characterization and Convergence.* Wiley, New York, 1986.

[Fit88] P.J. Fitzsimmons. Construction and regularity of measure-valued Markov branching processes. *Israel J. Math.*, 64(3):337–361, 1988.

[FL97] K. Fleischmann and G. Leduc. The problem of discontinuous branching functional for superprocesses. In preparation, 1997.

[FM97] K. Fleischmann and C. Mueller. A super-Brownian motion with a locally infinite catalytic mass. WIAS Berlin, Preprint Nr. 197, 1995, *Probab. Theory Relat. Fields*, to appear 1997.

[JS87] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin, 1987.

[Kuz94] S.E. Kuznetsov. Regularity properties of a supercritical superprocess. In *Collection: The Dynkin Festschrift*, volume 34 of *Progress Probab.*, pages 221–235. Birkhäuser, Boston, 1994.

[Led97a] G. Leduc. A complete characterization of a class of branching measure-valued processes. Technical Report No. 275, Lab. Research Stat. Probab., Carleton Univ. Ottawa, 1995, revised 1997.

[Led97b] G. Leduc. Martingale problem for superprocesses. In preparation, 1997.

[Sha88] M. Sharpe. *General Theory of Markov Processes.* Academic Press, Boston, 1988.