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Existence and weak-strong uniqueness for damage systems in viscoelasticity

Robert Lasarzik[1,2], Elisabetta Rocca[3,4], Riccarda Rossi[4,5]

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[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin, Germany
E-Mail: robert.lasarzik@wias-berlin.de

[2] Freie Universität Berlin
Department of Mathematics and Computer Science
Arnimallee 9
14195 Berlin, Germany

[3] Università di Pavia
Dipartimento di Matematica “F. Casorati”
Via Ferrata 5
I-27100 Pavia, Italy
E-Mail: elisabetta.rocca@unipv.it

[4] IMATI – C.N.R.
Via Ferrata 5
I-27100 Pavia, Italy

[5] Università di Brescia
Dipartimento di Ingegneria
Meccanica e Industriale
Via Branze 38
I-25133 Brescia, Italy
E-Mail: riccarda.rossi@unibs.it

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

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Abstract

In this paper we investigate the existence of solutions and their weak-strong uniqueness property for a PDE system modelling damage in viscoelastic materials. In fact, we address two solution concepts, *weak* and *strong* solutions. For the former, we obtain a global-in-time existence result, but the highly nonlinear character of the system prevents us from proving their uniqueness. For the latter, we prove local-in-time existence. Then, we show that the strong solution, as long as it exists, is unique in the class of weak solutions. This *weak-strong uniqueness* statement is proved by means of a suitable relative energy inequality.

1 Introduction

In this paper we address the following PDE system

$$\boldsymbol{u}_{tt} - \operatorname{div}\big(a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}) + b(\chi)\mathbb{V}\varepsilon(\boldsymbol{u}_t)\big) = \boldsymbol{f} \qquad \text{a.e. in } \Omega\times(0,T), \tag{1.1a}$$

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) - \Delta\chi + \frac{1}{2}a'(\chi)\varepsilon(\boldsymbol{u})\mathbb{C}\varepsilon(\boldsymbol{u}) + \partial W(\chi) \ni 0 \qquad \text{a.e. in } \Omega\times(0,T), \tag{1.1b}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{v}_0, \quad \chi(0) = \chi_0 \qquad \text{a.e. in } \Omega, \tag{1.1c}$$

$$\chi \geq 0, \quad \chi_t \leq 0 \qquad \text{a.e. in } \Omega\times(0,T), \tag{1.1d}$$

coupled with homogeneous Neumann boundary conditions for χ

$$\partial_{\boldsymbol{n}}\chi = 0 \qquad \text{a.e. on } \partial\Omega\times(0,T), \tag{1.1e}$$

and with Robin-type boundary conditions for $\boldsymbol{u}$

$$\gamma_0\boldsymbol{n}\cdot\left(a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}) + b(\chi)\mathbb{V}\varepsilon(\boldsymbol{u}_t)\right) + \gamma_1\boldsymbol{u}_t + \gamma_2\boldsymbol{u} = \boldsymbol{g} \qquad \text{a.e. on } \partial\Omega\times(0,T)\,, \tag{1.1f}$$

tuned by coeffcients

$$\gamma_0,\,\gamma_1,\,\gamma_2 \geq 0\,.$$

System (1.1) models damage processes in a viscoelastic material occupying a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 1,2,3$. We consider the evolution of the phenomenon in a time-interval $(0,T)$ and set $Q := \Omega\times(0,T)$ and $\Sigma := \partial\Omega\times(0,T)$. The state variables are the vector of small displacements $\boldsymbol{u}$, satisfying the momentum balance (1.1a), and the damage parameter χ, representing the local proportion of damage: $\chi = 1$ means that the material is completely safe, while $\chi = 0$ means it is completely damaged. We formulate the damage flow rule in the framework of the theory of M. FRÉMOND [16] and so we allow the phase parameter χ to assume also intermediate values inbetween 0 and 1 in the points of the domain Ω where only partial damage occurs.

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In (1.1a), $\varepsilon(\boldsymbol{u})_{ij} := (\boldsymbol{u}_{i,j} + \boldsymbol{u}_{j,i})/2$ denotes the linearized symmetric strain tensor, while $\mathbb{C}$ and $\mathbb{V}$ are the elastic and viscosity tensors, respectively. The χ-dependent coefficients $a, b \in \mathrm{C}^1(\mathbb{R})$ mark the damage dependence of the elasticity and viscosity modula, respectively; we will precisely specify our conditions on $\mathbb{C}$, $\mathbb{V}$, a, and b, in Section 2 ahead. The momentum balance is supplemented by the the Robin-type boundary condition (1.1f), where the parameters $\gamma_0, \gamma_1, \gamma_2$ in principle may be tuned in such a way as to yield a variety of boundary conditions for $\boldsymbol{u}$, among which

$$\begin{cases} \text{Neumann boundary conditions} & \text{for } \gamma_0 \neq 0,\ \gamma_1 = \gamma_2 = 0, \\ \text{time-dependent Dirichlet boundary conditions} & \text{for } \gamma_0 = 0, \min\{\gamma_1, \gamma_2\} > 0\,. \end{cases}$$

Later on, we will point out to which extent we can encompass the *general* conditions (1.1f) in our analysis.

The damage flow rule (1.1b) has a doubly nonlinear structure. Indeed, it features the subdifferential term $\partial I_{(-\infty,0]}(\chi_t)$, with $\partial I_{(-\infty,0]} : \mathbb{R} \rightrightarrows \mathbb{R}$ the (convex analysis) subdifferential of the indicator function $I_{(-\infty,0]}$, which serves to the purpose of enforcing unidirectionality of damage evolution via the constraint $\chi_t \leq 0$ a.e. in Q. In turn, the "double-well" type potential $W := \breve{W} + \hat{W}$ is assumed to be the sum of a convex (possibly non-smooth) part $\breve{W}$ and non-convex (but regular) part $\hat{W}$. Typical choices for W which we can include in our analysis are the logarithmic potential

$$W(r) := r\ln(r) + (1-r)\ln(1-r) - c_1 r^2 - c_2 r - c_3 \quad \forall\, r \in (0,1), \tag{1.2}$$

where c_1 and c_2 are positive constants, as well as the sum of the indicator function $\breve{W} = I_{[0,1]}$, forcing χ to range between 0 and 1, with a smooth non convex $\hat{W}$. Therefore, the subdifferential ∂W includes the (possibly) multivalued subdifferential $\partial\breve{W}$. We note that the upper wall of the well at 1 will already be respected by the unidirectional damage evolution $\chi_t \leq 0$ together with the condition on the initial value $\chi_0 \leq 1$ in Ω. The coupling with (1.1a) occurs through the term $\varepsilon(\boldsymbol{u})\mathbb{C}\varepsilon(\boldsymbol{u})$, which is a short-hand for the colon product $\varepsilon(\mathbf{u})\colon \mathbb{C}\varepsilon(\mathbf{u})$.

System (1.1) can be derived in the frame of the modelling approach by Frémond [16] (cf. also [2, 3, 4]) from of the following choices of the free-energy functional and of the pseudo-potential of dissipation:

$$\begin{aligned} \mathcal{E}(\boldsymbol{u}, \chi, \boldsymbol{u}_t) &:= \int_\Omega \left\{ \frac{1}{2}|\boldsymbol{u}_t|^2 + \frac{1}{2}a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u}) + \frac{1}{2}|\nabla\chi|^2 + W(\chi) \right\} \mathrm{d}x + \int_{\partial\Omega} \frac{\gamma_2}{2}|\boldsymbol{u}|^2 \,\mathrm{d}S\,, \\ \mathcal{D}(\chi, \boldsymbol{u}_t, \chi_t) &:= \int_\Omega \left\{ b(\chi)\mathbb{V}\varepsilon(\boldsymbol{u}_t){:}\varepsilon(\boldsymbol{u}_t) + |\chi_t|^2 + I_{(-\infty,0]}(\chi_t) \right\} \mathrm{d}x + \int_{\partial\Omega} \gamma_1 |\boldsymbol{u}_t|^2 \,\mathrm{d}S\,. \end{aligned}$$

Mathematical difficulties

The main mathematical hurdles encountered in the study of this system are related to the χ-dependence in the viscosity and elastic coffiecients a and b in (1.1a), and to the nonlinear features of equation (1.1b). In particular, the simultaneous presence of the non-smooth subdifferentials of $I_{(-\infty,0]}$, and W, and the quadratic term $\frac{1}{2}a'(\chi)\varepsilon(\boldsymbol{u})\mathbb{C}\varepsilon(\boldsymbol{u})$ occurring in (1.1b), impart a strongly nonlinear character to the system, so that the related analysis turns out to be nontrivial.

In the pioneering papers [3, 4], the momentum balance equation (with *scalar* displacements) had a degenerating character due to the loss of ellipticity in regions where $a(\chi) = b(\chi) = 0$. Consequently, only local-in-time existence results were proven. However, in most papers *complete damage* is avoided, and non-degenerating coefficients in front of either the elasticity or viscosity tensors are considered: we will also adopt this assumption hereafter.

Still, the highly nonlinear coupling between the momentum balance and the damage flow rule poses a major hurdle to global-in-time existence as already shown in [2], where the coupling with thermal effects was also encompassed. As a remedy to that, the flow rule for χ has been often regularized by means of a nonlinear p-Laplacian operator, with the exponent p greater than the space dimension (or a linear fractional Laplacian, [25]), in place of the usual Laplacian acting on χ. Indeed, this leads to higher spatial regularity for the damage variable and, as a consequence, paves the way for enhanced elliptic regularity estimates in the momentum balance, as well. This strategy has led to *global-in-time* existence for damage models in thermoviscoelastic materials [23, 30, 31], even encompassing phase separation [22].

Finally, let us also mention that in [31] we addressed the asymptotic analysis of the damage system with p-Laplacian regularization, where the case of the Laplacian operator was considered as a limit for $p \searrow 2$ in the p-Laplacian term. In that case, we showed that that the limit damage system needs to be formulated in a weaker fashion. We will dwell on this solvability concept later on.

The main aim of this paper is to cope with the analysis of system (1.1) *without* resorting to any higher-order regularization of the damage flow rule. In this context:

1 We will contend with *global-in-time solvability* for (1.1). As the literature available up to now suggest, global existence may be expected only for weak solutions to (1.1): we will carefully introduce our solvability concept and provide a set of conditions on the constitutive functions of the model, on the forces, and on the initial data, guaranteeing the existence of global-in-time solutions.

2 We will then turn to handling *strong* solutions, with the displacement $\boldsymbol{u}$ and the damage variable χ sufficiently regular in such a way as to satisfy system (1.1) pointwise. We will prove that such solutions exist at least locally in time.

3 We finally show that strong solutions are unique, as long as they exist, within the class of weak solutions.

The latter property goes under the name of *weak-strong* uniqueness. In this regard, let us mention that there is nowadays a consolidated literature on weak-strong uniqueness results in the context of fluid dynamics, such as SERRIN's uniqueness result [32] for LERAY's weak solutions [29] to the incompressible NAVIER–STOKES equation in three space dimensions, or the weak-strong uniqueness for suitable weak-solutions to the incompressible NAVIER–STOKES system [11] or to the full NAVIER–STOKES–FOURIER system [12]. The formulation of a relative energy inequality entailing weak-strong uniqueness of solutions for thermodynamical systems goes back to DAFERMOS [8]. In the context of fluid dynamics, this method has also been used to show the stability of a stationary solution [10], the convergence to a singular limit [13], or to derive *a posteriori* estimates for simplified models [15]. Even though the method is consolidated, there are fewer articles dealing with the case of nonconvex energies, and most of them are related to liquid crystals models (cf., e.g., [14, 9, 27, 26]). Finally we can quote the more recent papers [28] and [1], where the weak-strong uniqueness of solutions is obtained for the first time for a Frémond model of phase transitions accounting also for the temperature-evolution and for some Oldroyd-B type models for viscoelasticity at large strains, respectively.

Our results

Firstly, let us specify the notion of weak solution we will address. Our concept couples a standard variational formulation of the momentum balance, with the damage flow rule weakly formulated in terms of a one-sided variational inequality

$$\int_\Omega \big(\chi_t(t)\psi+\nabla\chi(t)\cdot\nabla\psi+\tfrac{1}{2}a'(\chi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)):\varepsilon(\boldsymbol{u}(t))\psi+W'(\chi(t))\psi\big)\,\mathrm{d}x \geq 0;$$

for all $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$ with $\psi \le 0$ a.e. in Ω, which is coupled with an energy-dissipation inequality

$$\begin{aligned} &\mathcal{E}(\boldsymbol{u}(t), \chi(t), \boldsymbol{u}_t(t)) + \int_0^t \mathcal{D}(\chi(s), \boldsymbol{u}_t(s), \chi_t(s))\,\mathrm{d}s \\ &\le \mathcal{E}(\boldsymbol{u}_0, \chi_0, \boldsymbol{v}_0) + \int_0^t \langle \boldsymbol{f}(s), \boldsymbol{u}_t(s)\rangle_{H^1(\Omega)}\,\mathrm{d}s + \int_0^t \langle \boldsymbol{g}(s), \boldsymbol{u}_t(s)\rangle_{H^{1/2}(\partial\Omega)}\,\mathrm{d}s\,. \end{aligned}$$

In Section 2 ahead we will provide more insight into this notion of solution, which was first introduced [20, 21] for PDE systems modelling damage in bodies at elastic equilibrium (hence, without inertial and viscous terms in the displacement equation), undergoing phase separation. Our first main result, Theorem 2.3 below, states the existence of *global-in-time weak solutions*. Its proof, carried out in Section 3, relies on a time discrete approximation scheme suitably tailored in order to obtain, as a byproduct, the non-negativity of the damage parameter χ.

The existence of local-in-time weak solutions, cf. Theorem 2.9, will be proved throughout Section 4. It relies on careful estimates, yielding higher spatial regularity for $\boldsymbol{u}$ and χ. The latter cannot be rigorously rendered on a time-discretization scheme, as they rely on a local-in-time Gronwall estimate that is not available on the time discrete level. In fact, we will resort to a different method based on *spatial* discretization (via a Faedo-Galerkin scheme) for a suitable approximation of system (1.1). For this approximate system we will prove local existence via a fixed point argument, and accordingly obtain local-in-time solutions to (1.1) by passing to the limit.

Our weak-strong uniqueness result, Theorem 2.12, will be obtained in the case of a regular potential W by means of the proof of a suitable relative energy inequality (cf. Proposition 5.1). The proof of such a result in case of a non-smooth potential W is still an open problem even for simpler semilinear equations.

2 Main results

In this section we lay the ground for our main results, stating the existence of global-in-time weak solutions and of local-in-time strong solutions to the damage PDE system, as well as the weak-strong uniqueness property for (1.1).

Preliminarily, we settle some general notation that will be used throughout the paper.

Notation 1. Given a Banach space X, we will denote by $\langle \cdot, \cdot \rangle_{\mathrm{X}}$ both the duality pairing between X^* and X and that between $(\mathrm{X}^d)^*$ and X^d; we will just write $\langle \cdot, \cdot \rangle$ for the inner Euclidean product in $\mathbb{R}^d$. Analogously, we will indicate by $\|\cdot\|_{\mathrm{X}}$ the norm in X and, most often, use the same symbol for the norm in X^d, while we will just write $|\cdot|$ for the Euclidean norm in $\mathbb{R}^d$.

Hereafter, we will use the symbols c, c', C, C', etc., whose meaning may vary even within the same line, to denote various positive constants depending only on known quantities. Furthermore, the symbols I_i, $i = 0, 1, ...$, will be used as place-holders for several integral terms (or sums of integral terms) appearing in the various estimates: we will not be consistent with the numbering, so that, for instance, the symbol I_1 will occur several times with different meanings.

2.1 Existence of weak solutions

We collect the first basic set of conditions on the tensors $\mathbb{C}$ and $\mathbb{V}$ and on the constitutive functions a, b and W.

Hypothesis A (Constitutive functions). *The elasticity and viscosity tensors $\mathbb{C},\, \mathbb{V} \in \mathbb{R}^{d\times d\times d\times d}$ are symmetric and positive definite in the sense that*

$$\begin{cases} \mathbb{E}_{ijkl} = \mathbb{E}_{klij} = \mathbb{E}_{jikl} = \mathbb{E}_{ijlk} \text{ for } i,j,l,k \in \{1,\dots,d\} \\ \exists\, \eta_{\mathbb{E}} > 0 \ \ \forall\, A \in \mathbb{R}^{d\times d} \ : \qquad \mathbb{E}A : A \geq \eta_{\mathbb{E}}|A|^2 \end{cases} \qquad \text{for } \mathbb{E} \in \{\mathbb{C},\mathbb{V}\}. \tag{2.1}$$

For the coefficient a we require that

$$a \in \mathrm{C}^1(\mathbb{R}), \tag{2.2a}$$

$$a \text{ is non-decreasing},\ \ a(r) \equiv 0 \text{ for } r \in (-\infty, 0] \tag{2.2b}$$

$$a \text{ is convex}, \tag{2.2c}$$

while we impose that

$$b \in \mathrm{C}^0(\mathbb{R}) \text{ and } \exists\, b_0 > 0\, \forall\, r \in \mathbb{R} \ : \quad b(r) \geq b_0. \tag{2.3}$$

Finally, we assume that

$$W \in \mathrm{C}^1(\mathbb{R}), \tag{2.4a}$$

$$\exists\, \ell \geq 0 \ : \ \text{the mapping } r \mapsto W(r) + \frac{\ell}{2}|r|^2 \text{ is convex,} \tag{2.4b}$$

$$W(0) \leq W(r) \text{ for all } r \leq 0. \tag{2.4c}$$

We now specify our conditions on the volume force and on the initial data for the existence of weak solutions.

Hypothesis B (Force and data). *We require that*

$$\begin{aligned} &\boldsymbol{f} \in L^2(0,T;H^1(\Omega;\mathbb{R}^d)^*), \qquad \boldsymbol{g} \in L^2(0,T;H^{-1/2}(\partial\Omega;\mathbb{R}^d))\,, \\ &\boldsymbol{u}_0 \in H^1(\Omega;\mathbb{R}^d), \quad \boldsymbol{v}_0 \in L^2(\Omega;\mathbb{R}^d), \quad \chi_0 \in H^1(\Omega) \text{ with } \chi_0 \in [0,1] \text{ a.e. in } \Omega. \end{aligned} \tag{2.5a}$$

Clearly, in the case of homogeneous Dirichlet boundary conditions for $\boldsymbol{u}$, (2.5a) should have to be suitably modified by requiring, for instance, $\boldsymbol{u}_0 \in H^1_0(\Omega;\mathbb{R}^d)$.

Remark 2.1. A few comments on Hyp. **A** are in order:

1 We have confined to spatially homogeneous tensors $\mathbb{C}$ and $\mathbb{V}$, but for the analysis of weak solutions we could indeed handle a suitable dependence on x, cf. Remark 2.4 ahead.
2 Clearly, it follows from (2.2b) that $a(r) \geq 0$ for all $r \in \mathbb{R}$; the possible degeneracy $a(\chi) = 0$ for the coefficient modulating the elasticity tensor is compensated by the fact that the coefficient $b(\chi)$ stays strictly positive by (2.3).
3 It follows from (2.4b) that W admits the *convex/concave* decomposition

$$W = \breve{W} + \hat{W} \qquad \text{with } \begin{cases} \breve{W}(r) = W(r) + \frac{\ell}{2}r^2, \\ \hat{W}(r) = -\frac{\ell}{2}r^2\,. \end{cases} \tag{2.6a}$$

Obviously, since $W \in \mathrm{C}^1(\mathbb{R})$, we have that $\breve{W},\, \hat{W} \in \mathrm{C}^1(\mathbb{R})$; we remark for later use that

$$\begin{cases} \breve{W}(0) \leq \breve{W}(r) & \text{for all } r \leq 0, \\ \hat{W}'(r) \leq 0 & \text{for all } r \in [0,1], \end{cases} \tag{2.6b}$$

where the first of (2.6b) obviously derives from (2.4c).

4 Our requirements on a are designed in such a way as to construct, via time discretization, weak solutions $(\boldsymbol{u}, \chi)$ to system (1.1) fulfilling $\chi \geq 0$ a.e. in Q, as well as the associated energy-dissipation inequality, cf. (2.12) ahead. In fact, while postponing all details to Section 3, we may mention that condition (2.2b) is exploited in the proof of the positivity of the discrete damage variable via a maximum principle argument, cf. Lemma 3.1. In turn, the convexity of a allows us to tailor the time discretization scheme for (1.1) in such a way as to guarantee the validity of a discrete energy-dissipation inequality, cf. Lemma 3.2 ahead.

5 We will also resort to the convex/concave splitting (2.6a) of W in the proof of Lemma 3.2, while properties (2.6b) of $\breve{W}$ and $\hat{W}$ will be used for the proof of the positivity of χ.

Our notion of weak solution features the following energy and dissipation functionals

$$\mathcal{E}(\boldsymbol{u}, \chi, \boldsymbol{u}_t) := \int_\Omega \left\{ \frac{1}{2}|\boldsymbol{u}_t|^2 + \frac{1}{2}a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u}) + \frac{1}{2}|\nabla\chi|^2 + W(\chi) \right\} \mathrm{d}x + \int_{\partial\Omega} \frac{\gamma_2}{2}|\boldsymbol{u}|^2 \,\mathrm{d}S\,, \tag{2.7}$$

$$\mathcal{D}(\chi, \boldsymbol{u}_t, \chi_t) := \int_\Omega \left\{ b(\chi)\mathbb{V}\varepsilon(\boldsymbol{u}_t):\varepsilon(\boldsymbol{u}_t) + |\chi_t|^2 + I_{(-\infty,0]}(\chi_t) \right\} \mathrm{d}x + \int_{\partial\Omega} \gamma_1 |\boldsymbol{u}_t|^2 \,\mathrm{d}S\,. \tag{2.8}$$

In fact, while $\mathcal{E}$ subsumes the contributions of the kinetic and elastic energies, of the volume force, and of the gradient regularization and potential energy for the damage variable, $\mathcal{D}$ encompasses the dissipation due to viscous damping and the quadratic dissipation for the damage gradient flow, with the indicator term enforcing unidirectionality. The weak solvability concept that we specify in Definition 2.2 below has been introduced, for (purely) elastic damage models possibly coupled with other diffusion processes, in [20, 21]. According to this notion, the (standard variational formulation of) the momentum balance is coupled with the damage flow rule, weakly formulated in terms of (2.11) & (2.12). This formulation reflects the fact that, if the subdifferential in (1.1b) is lifted to an operator $\partial I_{(-\infty,0]} : H^1(\Omega) \rightrightarrows H^1(\Omega)^*$, then (1.1b) rephrases as

$$\begin{cases} \langle -\{\chi_t - \Delta\chi + \frac{1}{2}a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u}) + W'(\chi)\}, \psi \rangle_{H^1(\Omega)} \leq \int_\Omega I_{(-\infty,0]}(\psi)\,\mathrm{d}x & \text{for all } \psi \in H^1(\Omega), \\ \langle -\{\chi_t - \Delta\chi + \frac{1}{2}a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u}) + W'(\chi)\}, \chi_t \rangle_{H^1(\Omega)} \geq \int_\Omega I_{(-\infty,0]}(\chi_t)\,\mathrm{d}x, & \end{cases}$$

both inequalities holding a.e. in $(0, T)$. Note that we used the 1-homogeneity of $I_{(-\infty,0]}$ in order to deduce the two above inequalities from (1.1b). Then, restricting the first inequality to negative test functions $\psi \in L^\infty(\Omega)$ (in order to have the term $\int_\Omega \frac{1}{2}a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u})\psi\,\mathrm{d}x$ well defined) yields (2.11). Adding the second inequality with the weak momentum balance tested by $\boldsymbol{u}_t$ and integrating in time leads to (2.12), which is termed an *upper* energy-dissipation inequality to emphasize that the overall energy $\mathcal{E}(\boldsymbol{u}(t), \chi(t), \boldsymbol{u}_t(t))$ at the current process time is estimated from above by the initial energy and the work of the external forces.

Definition 2.2 (Weak solution)**.** *We call a pair* $(\boldsymbol{u}, \chi)$ *a weak solution to the Cauchy problem for system* (1.1) *if*

$$\boldsymbol{u} \in H^1(0,T; H^1(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T; L^2(\Omega;\mathbb{R}^d)) \cap H^2(0,T; H^1(\Omega;\mathbb{R}^d)^*), \tag{2.9a}$$

$$\chi \in L^\infty(0,T; H^1(\Omega)) \cap L^\infty(Q) \cap H^1(0,T; L^2(\Omega)) \tag{2.9b}$$

satisfy initial conditions (1.1c)*, constraints* (1.1d)*, and*

- *the* weak momentum balance *for almost all* $t \in (0, T)$*, i.e.,*

$$\begin{aligned} &\langle \boldsymbol{u}_{tt}(t), \boldsymbol{\varphi} \rangle_{H^1(\Omega)} + \int_\Omega \big(b(\chi(t))\mathbb{V}\varepsilon(\boldsymbol{u}_t(t)) : \varepsilon(\boldsymbol{\varphi}) + a(\chi(t))\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{\varphi}) \big)\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} \big(\gamma_1 \boldsymbol{u}_t + \gamma_2 \boldsymbol{u}\big)\boldsymbol{\varphi} = \langle \boldsymbol{f}(t), \boldsymbol{\varphi} \rangle_{H^1(\Omega)} + \langle \boldsymbol{g}(t), \boldsymbol{\varphi} \rangle_{H^{1/2}(\partial\Omega)} \end{aligned} \tag{2.10}$$

  for all $\boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^d)$*;*

- *the* one-sided variational inequality *for the damage flow rule, i.e., for almost all* $t \in (0,T)$

$$\int_\Omega \big(\chi_t(t)\psi+\nabla\chi(t)\cdot\nabla\psi+\tfrac{1}{2}a'(\chi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)):\varepsilon(\boldsymbol{u}(t))\psi+W'(\chi(t))\psi\big)\,\mathrm{d}x \geq 0; \tag{2.11}$$

for all $\psi \in H^1(\Omega)\cap L^\infty(\Omega)$ *with* $\psi \leq 0$ *a.e. in* Ω*;*

- *the (overall)* upper energy-dissipation inequality*, i.e., for all* $t \in [0,T]$

$$\begin{aligned}&\mathcal{E}(\boldsymbol{u}(t),\chi(t),\boldsymbol{u}_t(t)) + \int_0^t \mathcal{D}(\chi(s),\boldsymbol{u}_t(s),\chi_t(s))\,\mathrm{d}s\\ &\leq \mathcal{E}(\boldsymbol{u}_0,\chi_0,\boldsymbol{v}_0) + \int_0^t \langle \boldsymbol{f}(s),\boldsymbol{u}_t(s)\rangle_{H^1(\Omega)}\,\mathrm{d}s + \int_0^t \langle \boldsymbol{g}(s),\boldsymbol{u}_t(s)\rangle_{H^{1/2}(\partial\Omega)}\,\mathrm{d}s\,.\end{aligned} \tag{2.12}$$

Theorem 2.3 (Global existence of weak solutions)**.** *Assume Hypotheses **A** & **B**. Then, the Cauchy problem for system* (1.1) *admits a weak solution* $(\boldsymbol{u},\chi)$ *in the sense of Definition 2.2.*

The *proof* will be carried out in Section 3.

Remark 2.4 (Extensions)**.** Theorem 2.3 may be extended to the non-homogeneous case, *i.e.*, with spatially dependent tensors $\mathbb{V},\mathbb{C} \in L^\infty(\Omega;\mathbb{R}^{d\times d\times d\times d})$.

Let us emphasize that, so far, we have not specified other conditions on the parameters γ_i, $i \in \{0,1,2\}$, besides $\gamma_1,\gamma_2 \geq 0$. Thus, as pointed out in the Introduction, the existence statement of Thm. 2.3 in particular encompasses the case of null Dirichlet boundary conditions on $\partial\Omega$, correspoding to $\gamma_0=\gamma_1=0$, $\gamma_2>0$, $\boldsymbol{g}\equiv\mathbf{0}$ in (1.1f). Clearly, in that case the weak momentum balance would feature test functions $\varphi \in H_0^1(\Omega;\mathbb{R}^d)$. We could also allow for a suitable time-dependent Dirichlet loading $\mathbf{w}$ enforcing the condition

$$\boldsymbol{u}=\mathbf{w} \qquad \text{on } \Sigma=\partial\Omega\times(0,T). \tag{2.13}$$

Indeed, (2.13) would correspond to the case $\gamma_0=0$, $\boldsymbol{g}\equiv\mathbf{0}$, $\min\{\gamma_1,\gamma_2\}>0$, with

$$\mathbf{w}(t)=\mathbf{c}\exp\left(-\frac{\gamma_2}{\gamma_1}t\right) \qquad \text{for some vector } \mathbf{c}\in\mathbb{R}^d\,.$$

To handle (2.13), it would be sufficient to formulate the momentum balance (2.10) for $\boldsymbol{u}=\widehat{\boldsymbol{u}}+\mathbf{w}$, with $\widehat{\boldsymbol{u}}(t) \in H_0^1(\Omega;\mathbb{R}^d)$ for all $t \in [0,T]$, and seek for a solution $\widehat{\boldsymbol{u}}$ complying with homogeneous Dirichlet boundary conditions.

A closer perusal of the proof of Thm. 2.3 also reveals that, since our estimates do not hinge on elliptic regularity arguments for the displacement variable, mixed boundary conditions could be also considered for $\boldsymbol{u}$: in particular, the body could be clamped on a portion Γ_{D} of the boundary, while an assigned traction could be applied on $\Gamma_{\mathrm{N}}=\partial\Omega\backslash\Gamma_{\mathrm{D}}$.

The extension to the case of a nonsmooth potential W is more delicate; it will be addressed in Section 3.3 ahead.

2.2 Existence of strong solutions

We start by specifying our notion of *strong* solvability for system (1.1) which, we recall, we address in the case of a possibly nonsmooth convex potential $\breve{W}$. In Definition 2.5 below, we ask for enhanced regularity and integrability properties for $\boldsymbol{u}$, which as a consequence ensure that the term $a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u})$ in the damage flow rule belongs to $L^2(\Omega)$. Then, both the momentum balance and the flow rule for χ make sense pointwise in space and time. Moreover, by comparison, $H^2(\Omega)$-regularity follows for χ.

Definition 2.5 (Strong solution). *We call a pair $(\boldsymbol{u},\chi)$ a strong solution if it enjoys the regularity properties*

$$\begin{aligned} &\boldsymbol{u}\in H^1(0,T;H^3(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;H^2(\Omega;\mathbb{R}^d))\cap H^2(0,T;H^1(\Omega;\mathbb{R}^d)),\\ &\chi\in L^\infty(0,T;H^2(\Omega))\cap H^1(0,T;H^1(\Omega))\,, \end{aligned} \tag{2.14}$$

and system (1.1)*, with the boundary condition*

$$\boldsymbol{n}\cdot\mathbb{C}\varepsilon(\boldsymbol{u})=0\qquad \text{a.e. on } \Sigma\,, \tag{2.15}$$

is satisfied pointwise a.e. in Q, which for the damage flow rule means that

$$\begin{aligned} &\exists\,\eta,\,\xi\in L^2(Q)\quad \text{with}\begin{cases}\eta\in\partial\partial I_{(-\infty,0]}(\chi_t),\\ \xi\in\partial\breve{W}(\chi)\end{cases}\qquad \text{a.e. in } Q\text{, such that}\\ &\chi_t-\Delta\chi+\frac{1}{2}a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u})+\xi+\eta+\hat{W}'(\chi)=0\quad \text{a.e. in } Q. \end{aligned} \tag{2.16}$$

Remark 2.6. It is straightforward to check that any strong solution satisfies inequality (2.12) (in which the coefficients γ_1 and γ_2 in the dissipation potential $\mathcal{D}$ and in the energy functional $\mathcal{E}$, respectively, are null due to the boundary condition (2.15)), as an energy-dissipation *balance*.

We will prove the existence of local-in-time strong solutions under an additional smoothness condition for the spatial domain Ω to allow for regularity estimates. Namely, we require that

$$\Omega\subset\mathbb{R}^d \text{ is a bounded domain of class } \mathrm{C}^1, \tag{H_Ω}$$

and under the following strengthened versions of Hypotheses **A** and **B** (although we no longer need to assume a convex, cf. (2.17b) below).

Hypothesis C (Constitutive functions). *In addition to* (2.1) *for the elasticity and viscosity tensors, we assume that*

$$\mathbb{C}=\mathbb{V}\,,\qquad a\in\mathrm{C}^2(\mathbb{R}), \tag{2.17a}$$

$$\exists\,\kappa_1>0\;\exists\,p\geq 1\;\;\forall\,r\in\mathbb{R}:\quad |a''(r)|\leq\kappa_1(|r|^p{+}1), \tag{2.17b}$$

and analogously we impose that, in addition to (2.3)*,*

$$b\in\mathrm{C}^2(\mathbb{R}),\ \text{and}\;\exists\,\kappa_2>0\;\exists\,q\geq 1\;\;\forall\,r\in\mathbb{R}:\quad |b''(r)|\leq\kappa_2(|r|^q{+}1). \tag{2.18}$$

As for W, we require that $W:\mathbb{R}\to(-\infty,+\infty]$, with $\mathrm{dom}\,W\neq\emptyset$, is ℓ-convex as in (2.4b)*.*

x

Remark 2.7. Let us motivate the above conditions and compare them with Hypothesis A:

1 The enhanced regularity required of the coefficients a and b will be instrumental in performing enhanced regularity estimates for the solutions. To carry them out, we will also resort to the polynomial growth conditions (2.17b) and (2.18), which obviously imply analogous growth conditions for a', b' and $a,\ b$, namely

$$\begin{cases} \exists\, \widehat{\kappa}_i > 0\ \forall\, r \in \mathbb{R}: & |\zeta'(r)| \leq \widehat{\kappa}_i(|r|^{\rho+1}{+}1), \\ \exists\, \hat{\hat{\kappa}}_i > 0\ \forall\, r \in \mathbb{R}: & |\zeta(r)| \leq \hat{\hat{\kappa}}_i(|r|^{\rho+2}{+}1), \end{cases} \qquad \text{for } i \in \{1,2\},\ \zeta \in \{a,b\},\ \rho \in \{p,q\}\,. \tag{2.19}$$

2 Let us emphasize that Hypothesis C allows for nonsmoothness of W (or, equivalently, of $\breve{W}$): in particular, in this context we can encompass the case in which $\breve{W} = I_{[0,\infty)}$, and positivity of χ is automatically enforced.

3 Condition (2.4b) guarantees the convex/concave decomposition $W = \breve{W}{+}\hat{W}$, with $\hat{W}(r) = -\frac{\ell}{2}r^2$, which we are going to use for the analysis of strong solutions, too.

Our conditions on the force and on the initial data will be enhanced as well. The compatibility condition (2.20b) below reflects that we confine our analysis of strong solutions to the homogeneous Neumann boundary conditions (2.15).

Hypothesis D (Force and data)**.** *We require that*

$$\boldsymbol{f} \in L^2(0,T;H^1(\Omega;\mathbb{R}^d)), \quad \boldsymbol{u}_0 \in H^3(\Omega;\mathbb{R}^d) \quad \boldsymbol{v}_0 \in H^2(\Omega;\mathbb{R}^d), \tag{2.20a}$$

$$\boldsymbol{n}\cdot\mathbb{C}\varepsilon(\boldsymbol{u}_0) = 0 \quad \textit{a.e. on } \partial\Omega. \tag{2.20b}$$

We take $\gamma_1 = \gamma_2 = 0$, $\boldsymbol{g} \equiv \boldsymbol{0}$, and $\gamma_0 \neq 0$ in (1.1f)*, and assume*

$$\chi_0 \in H^2(\Omega) \textit{ with } \begin{cases} \chi_0(x) \leq 1 \textit{ for all } x \in \overline{\Omega} \\ |\partial\breve{W}^\circ|(\chi_0) \in L^2(\Omega) \end{cases} \tag{2.20c}$$

where

$$|\partial\breve{W}^\circ|(\chi_0(x)) := \inf\{|\xi| \,:\, \xi \in \partial\breve{W}(\chi_0(x))\} \qquad \textit{for a.a. } x \in \Omega\,.$$

Remark 2.8 (On (2.20c))**.** First of all, it is immediate to check that the $\inf$ in the definition of $|\partial\breve{W}^\circ|(\chi_0(x))$ is indeed a $\min$. Furthermore, the von Neumann-Aumann selection theorem yields that there exists a *measurable* selection

$$\Omega \ni x \mapsto \xi^\circ(x) \in \mathrm{Argmin}\{|\xi| \,:\, \xi \in \partial\breve{W}(\chi_0(x))\}\,, \tag{2.21a}$$

so that (2.20c) is indeed equivalent to requiring that

$$\xi^\circ \in L^2(\Omega)\,. \tag{2.21b}$$

From this there follows that $W(\chi_0) \in L^1(\Omega)$: in fact, taking into account that $\hat{W}(\chi_0) \in L^\infty(\Omega)$ by the quadratic growth of $\hat{W}$, it is sufficient to show that $\breve{W}(\chi_0) \in L^1(\Omega)$. This is a consequence of the estimate

$$\int_\Omega [\breve{W}(\chi_0) - \breve{W}(r_o)]\,\mathrm{d}x \leq \int_\Omega \xi^\circ(x)(\chi_0(x){-}r_o)\,\mathrm{d}x$$

where r_o is *any* element in $\mathrm{dom}\,\breve{W}$.

Throughout Section 4 we will prove the following result.

Theorem 2.9 (Local existence of strong solutions)**.** *Assume Hypotheses **C** & **D**; let Ω fulfill condition* (H_Ω)*.*

Then, there exists $\widehat{T} \in (0,T]$ such that the Cauchy problem for system (1.1) *admits strong solution $(\boldsymbol{u},\chi)$ in the sense of Definition 2.5 on the interval $(0,\widehat{T})$.*

Remark 2.10 (Positivity for strong solutions)**.** As previously pointed out, our analysis of strong solutions encompasses the choice of a nonsmooth potential $\breve{W}$. In that case, if we additionally have $\mathrm{dom}(\breve{W}) \subset [0,\infty)$, then we immediately obtain that $\chi(x,t) \geq 0$ for all $(x,t) \in \overline{\Omega} \times [0,T]$.

An alternative way for obtaining nonnegativity of strong solutions is via the weak-strong uniqueness guaranteed by Thm. 2.12 ahead: in this way, we deduce $\chi \geq 0$ for the strong solution, since it is coincides with the weak one, which is known to be positive for instance under the assumptions of Thm. 2.3.

Outside these two cases, we do not claim positivity of χ and it is actually not needed in the analysis of strong solutions. Especially, in the case of nonmonotone a, we do not expect such a property due to the negative contribution on the left-hand side of the damage flow rule.

We conclude this section with a consistency result, useful for the proof of Thm. 2.9, showing that, for a sufficiently regular pair $(\boldsymbol{u},\chi)$, the pointwise flow rule may be proved by just checking a variational inequality, cf. (2.22) below, joint with the energy-dissipation inequality (2.12).

Proposition 2.11. *Let $(\boldsymbol{u},\chi)$ enjoy the regularity properties* (2.14) *and fulfill the weak momentum balance* (2.10) *and the energy-dissipation inequality* (2.12)*.*

Then, $(\boldsymbol{u},\chi)$ satisfies the pointwise flow rule (2.16)*, joint with a selection $\xi \in \partial\breve{W}(\chi)$ a.e. in Q, if and only if it complies with the variational inequality*

$$\int_\Omega \Big(\chi_t(t)\psi+\nabla\chi(t)\cdot\nabla\psi+\tfrac{1}{2}a'(\chi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)):\varepsilon(\boldsymbol{u}(t))\psi+(\xi+\hat{W}'(\chi(t)))\psi\Big)\,\mathrm{d}x \geq 0 \tag{2.22}$$

for all $\psi \in H^1(\Omega)\cap L^\infty(\Omega)$ with $\psi \leq 0$ a.e. in Ω.

Proof. Clearly, it suffices to show that from (2.22) we can derive (2.16). For this, we start by observing that, by the assumed regularity (2.14),

$$\eta := -\left(\chi_t-\Delta\chi+\frac{1}{2}a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u})+\xi+\hat{W}'(\chi)\right) \in L^2(Q)\,. \tag{2.23}$$

Choosing $\boldsymbol{\varphi} = \boldsymbol{u}$ in (2.10) and subtracting this from (2.12), we find

$$\int_\Omega \frac{1}{2}a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u})+\frac{1}{2}|\nabla\chi|^2+W(\chi)\,\mathrm{d}x\Big|_0^t+\int_0^t\int_\Omega \big(|\chi_t|^2-a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u}_t)\big)\,\mathrm{d}x\,\mathrm{d}s$$
$$\leq -\int_0^t I_{(-\infty,0]}(\chi_t)\,\mathrm{d}x\,\mathrm{d}s\,.$$

Now, by the chain rule (which holds since η and χ_t are in a duality pairing thanks to (2.23)), the above left-hand side equals $\int_\Omega(-\eta)\chi_t\,\mathrm{d}x$. Thus, we deduce

$$\int_0^t\int_\Omega \eta\chi_t\,\mathrm{d}x\,\mathrm{d}s \geq \int_0^t\int_\Omega I_{(-\infty,0]}(\chi_t)\,\mathrm{d}x\,\mathrm{d}s \tag{2.24}$$

for a.e. $t \in (0,T)$. In turn, inequality (2.22) and a density argument (again relying on (2.23)) implies that

$$\int_\Omega \eta\psi \,\mathrm{d}x \leq \int_\Omega I_{(-\infty,0]}(\psi)\,\mathrm{d}x \qquad \text{for all } \psi \in L^2(\Omega)\,, \tag{2.25}$$

where we note that the inequality becomes empty in the case that $\psi > 0$ on a set of positive measure in Ω. Combining (2.24) and (2.25) we deduce $\eta \in \partial I_{(-\infty,0]}(\chi_t)$ a.e. in Q, *i.e.*, (2.16). □

2.3 Weak-strong uniqueness

We will prove the weak-strong uniqueness property for the Cauchy problem for system (1.1), confining the discussion to the case of the homogeneous Neumann boundary conditions (2.15). We will work under the following conditions.

Hypothesis E. *We assume that* $\mathbb{C} = \mathbb{V}$ *complies with* (2.1)*, and that the nonlinear functions* a*,* b*, and* W *satisfy*

$$a \in \mathrm{C}^2(\mathbb{R}) \text{ is convex and non-decreasing}, \tag{2.26a}$$
$$b \in \mathrm{C}^1(\mathbb{R}) \text{ and } \exists\, b_0 > 0\ \forall\, r \in \mathbb{R}: \quad b(r) \geq b_0, \tag{2.26b}$$
$$W \in \mathrm{C}^2(\mathbb{R}) \text{ and } \exists\, \ell \geq 0 \ : \ \text{the mapping } r \mapsto W(r) + \frac{\ell}{2}|r|^2 \text{ is convex.} \tag{2.26c}$$

Theorem 2.12 (Weak-strong uniqueness)**.** *Under Hypothesis **E**, let* $(\boldsymbol{u},\chi)$ *be a weak solution in the sense of Definition 2.2 and* $(\tilde{\boldsymbol{u}},\tilde{\chi})$ *a strong solution in the sense of Definition 2.5 emanating from the same initial data, with forcing term* $\boldsymbol{f}$ *as in* (2.5a)*. Then it holds*

$$\boldsymbol{u}(t) = \tilde{\boldsymbol{u}}(t) \quad \chi(t) = \tilde{\chi}(t) \quad \text{for all } t \in [0,\hat{T}]\,.$$

The *proof* will be carried out in Section 5 ahead.

3 Proof of Theorem 2.3

We will prove the existence of weak solutions by resorting to a suitable time-discretization scheme. Let $\tau = T/K$ be the time step size of an equidistant partition $\{0 = t_\tau^0 < t_\tau^1 < \ldots < t_\tau^k < \ldots < t_\tau^K = T\}$ of $[0,T]$ into K subintervals. We will approximate the volume and surface forces by local means on the intervals $[t_\tau^{k-1}, t_\tau^k]$, by setting

$$\boldsymbol{f}_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \boldsymbol{f}(s)\,\mathrm{d}s\,, \qquad \boldsymbol{g}_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \boldsymbol{g}(s)\,\mathrm{d}s\,. \tag{3.1}$$

Hence, the time discretization scheme for system (1.1) reads, in its strong formulation,

$$\frac{\mathbf{u}_\tau^k - 2\mathbf{u}_\tau^{k-1} + \mathbf{u}_\tau^{k-2}}{\tau^2} - \operatorname{div}\Big(b(\chi_\tau^k)\mathbb{V}\varepsilon\Big(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\Big) + a(\chi_\tau^k)\mathbb{C}\varepsilon(\mathbf{u}_\tau^k)\Big) = \boldsymbol{f}_\tau^k \quad \text{in } \Omega, \tag{3.2a}$$

$$\left.\begin{aligned} &\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau} + \partial I_{(-\infty,0]}\Big(\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\Big) - \Delta\chi_\tau^k \\ &\quad + \frac{a'(\chi_\tau^k)}{2}\mathbb{C}\varepsilon(\mathbf{u}_\tau^{k-1}){:}\varepsilon(\mathbf{u}_\tau^{k-1}) + \breve{W}'(\chi_\tau^k) + \hat{W}'(\chi_\tau^{k-1}) \ni 0 \end{aligned}\right\} \quad \text{in } \Omega, \tag{3.2b}$$

$$\gamma_0 \boldsymbol{n}\cdot \left(a(\chi_\tau^k)\mathbb{C}\varepsilon(\mathbf{u}_\tau^k) + b(\chi_\tau^k)\mathbb{V}\varepsilon\left(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\right)\right) + \gamma_1 \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} + \gamma_2 \mathbf{u}_\tau^k = \boldsymbol{g}_\tau^k \quad \text{on } \partial\Omega, \tag{3.2c}$$

$$\partial_{\boldsymbol{n}} \chi_\tau^k = 0 \quad \text{on } \partial\Omega, \tag{3.2d}$$

supplemented with the initial conditions $\boldsymbol{u}_0$, $\boldsymbol{u}_\tau^{-1} := \boldsymbol{u}_0 - \tau \boldsymbol{v}_0$, and χ_0. In the above scheme, the convex/concave splitting $W = \breve{W} + \hat{W}$ from (2.6a) has been carefully combined with the choice of implicit/explicit terms in such a way as to yield the validity of a discrete energy-dissipation inequality, cf. Lemma 3.2 ahead.

3.1 Existence and a priori estimates for time-discrete solutions

With our first result, we establish the existence of solutions to the weak formulation of system (3.2). Additionally, we prove the positivity property $\chi_\tau^k \geq 0$ a.e. in Ω via a maximum principle argument mimicking that from [25, Prop. 4.2].

Lemma 3.1 (Existence of time-discrete solutions)**.** *Starting from* $\boldsymbol{u}_\tau^{-1} = \boldsymbol{u}_0 - \tau \boldsymbol{v}_0$, $u_\tau^0 := u_0$, *and* $\chi_\tau^0 := \chi_0$, *there exists* $\overline{\tau} > 0$ *such that for all* $0 < \tau < \overline{\tau}$ *and for every* $k = 1, \ldots, K$ *there exists a* weak *solution* $\mathbf{u}_\tau^k \in H^1(\Omega;\mathbb{R}^d)$ *and* $\chi_\tau^k \in H^1(\Omega) \cap L^\infty(\Omega)$ *to the time-discrete system* (3.2)*, fulfilling*

$$\begin{aligned} &\int_\Omega \left\{ \frac{\mathbf{u}_\tau^k - 2\mathbf{u}_\tau^{k-1} + \mathbf{u}_\tau^{k-2}}{\tau^2} \cdot \boldsymbol{\varphi} + b(\chi_\tau^k)\mathbb{V}\varepsilon\Big(\frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\Big) : \varepsilon(\boldsymbol{\varphi}) + a(\chi_\tau^k)\varepsilon(\mathbf{u}_\tau^k) : \varepsilon(\boldsymbol{\varphi}) \right\} \mathrm{d}x \\ &\quad + \int_{\partial\Omega} \left(\gamma_1 \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau} + \gamma_2 \mathbf{u}_\tau^k \right) \mathrm{d}S = \langle \boldsymbol{f}_\tau^k, \boldsymbol{\varphi} \rangle_{H^1(\Omega)} + \langle \boldsymbol{g}_\tau^k, \boldsymbol{\varphi} \rangle_{H^{1/2}(\partial\Omega)} \end{aligned} \tag{3.3a}$$

for all $\boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^d)$,

$$\begin{aligned} &\int_\Omega \left\{ \frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}(\psi - \chi_\tau^k) + \nabla \chi_\tau^k \cdot \nabla(\psi - \chi_\tau^k) + \frac{1}{2} a'(\chi_\tau^k)\mathbb{C}\varepsilon(\mathbf{u}_\tau^{k-1}) : \varepsilon(\mathbf{u}_\tau^{k-1})(\psi - \chi_\tau^k) \right\} \mathrm{d}x \\ &\qquad + \int_\Omega \left\{ \breve{W}'(\chi_\tau^k)(\psi - \chi_\tau^k) + \hat{W}'(\chi_\tau^{k-1})(\psi - \chi_\tau^k) \right\} \mathrm{d}x \geq 0 \end{aligned} \tag{3.3b}$$

for all $\psi \in X_\tau^{k-1} := \big\{ v \in H^1(\Omega) \cap L^\infty(\Omega) : \ v \leq \chi_\tau^{k-1}$ *a.e. in* $\Omega \big\}$*, as well as the constraints*

$$0 \leq \chi_\tau^k \leq \chi_\tau^{k-1} \leq 1 \qquad \text{a.e. in } \Omega\,. \tag{3.3c}$$

Proof. Let $k = 1, \ldots, K$ be given and, accordingly, $\mathbf{u}_\tau^{k-1} \in H_0^1(\Omega;\mathbb{R}^d)$ and $\chi_\tau^{k-1} \in H^1(\Omega) \cap L^\infty(\Omega)$. We first construct a solution to (3.3b) by finding a minimizer $\chi_\tau^k \in H^1(\Omega)$ for of the convex potential $\mathcal{P} : H^1(\Omega) \cap L^\infty(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{P}(\chi) = \mathcal{P}_1(\chi) + \mathcal{P}_2(\chi) \qquad \text{with}$$

$$\begin{cases} \mathcal{P}_1(\chi) = \displaystyle\int_\Omega \left\{ \tfrac{\tau}{2} \left| \frac{\chi - \chi_\tau^{k-1}}{\tau} \right|^2 + \frac{1}{2}|\nabla \chi|^2 + \breve{W}(\chi) + \hat{W}'(\chi_\tau^{k-1})\chi \right\} \mathrm{d}x \\ \mathcal{P}_2(\chi) = \displaystyle\int_\Omega \tfrac{1}{2} a(\chi)\mathbb{C}\varepsilon(\mathbf{u}_\tau^{k-1}) : \varepsilon(\mathbf{u}_\tau^{k-1})\, \mathrm{d}x + I_{\tilde{X}_\tau^{k-1}}(\chi)\,, \end{cases} \tag{3.4}$$

with the set $\tilde{X}_\tau^{k-1} := \{ v \in H^1(\Omega) : v \leq \chi^{k-1}$ a.e. in $\Omega\}$. We thus address the minimum problem

$$\min_{\chi \in \tilde{X}_\tau^{k-1}} \mathcal{P}(\chi). \tag{3.5}$$

First of all, observe that, for sufficiently small τ the functional $\mathcal{P}$ is bounded from below and suitably coercive. To check this, we recall that, since $\breve{W}$ is convex, it is is bounded from below by an affine function; combining this with the information that $\hat{W}'(\chi_\tau^{k-1}) \in L^\infty(\Omega)$ - since $0 \le \chi_\tau^{k-1} \le \chi_0 \le 1$ a.e. in Ω - we ultimately conclude that there exist positive constants c_W, C_W, only depending on W, such that

$$\mathcal{P}_1(\chi) \ge \int_\Omega \left\{ \frac{\tau}{2}\left|\frac{\chi-\chi_\tau^{k-1}}{\tau}\right|^2 + \frac{1}{2}|\nabla\chi|^2 - c_W|\chi| - C_W \right\} \mathrm{d}x \overset{(1)}{\ge} c_W'\|\chi\|_{H^1(\Omega)}^2 - C_W' , \tag{3.6}$$

where (1) follows from absorbing $-c_W\|\chi\|_{L^1(\Omega)}$ into $\frac{1}{2\tau}\|\chi-\chi_\tau^{k-1}\|_{L^2(\Omega)}^2$, for sufficiently small τ. In turn, we observe that if $\mathcal{P}_2(\chi) < +\infty$, then $\chi^+ \in L^\infty(\Omega)$ and, a fortiori,

$$a(\chi)\mathbb{C}\varepsilon(\mathbf{u}_\tau^{k-1}):\varepsilon(\mathbf{u}_\tau^{k-1}) = a((\chi)^+)\mathbb{C}\varepsilon(\mathbf{u}_\tau^{k-1}):\varepsilon(\mathbf{u}_\tau^{k-1}) \in L^1(\Omega)$$

(the above equality holds because $a \equiv 0$ on $(-\infty, 0]$ by (2.2b)). All in all, we may conclude that

$$\forall S > 0\, \exists C_S > 0\, \forall \chi \in H^1(\Omega) \,: \qquad \mathcal{P}(\chi) \le S \implies \begin{cases} \|\chi\|_{H^1(\Omega)} \le C_S \\ \int_\Omega a'(\chi)\mathbb{C}\varepsilon(\mathbf{u}_\tau^{k-1}):\varepsilon(\mathbf{u}_\tau^{k-1})\,\mathrm{d}x \le C_S \end{cases} .$$

Therefore, any minimizing sequence for $\mathcal{P}$ is bounded in $H^1(\Omega)$ and thus weakly converges, up to a subsequence, to some $\bar{\chi} \in H^1(\Omega) \cap \tilde{X}_\tau^{k-1}$; by standard lower semicontinuity arguments we conclude that $\bar{\chi}$ is a minimizer for $\mathcal{P}$ and we set $\chi_\tau^k := \bar{\chi}$.

We now show that *any* solution χ_τ^k for the minimum problem (3.5) fulfills $\chi_\tau^k \ge 0$ a.e. in Ω. With this aim, we observe that the truncated function $(\chi_\tau^k)^+ := \max\{\chi_\tau^k, 0\}$ fulfills a.e. in Ω

$$\begin{cases} \left\|\frac{(\chi_\tau^k)^+ - \chi_\tau^{k-1}}{\tau}\right\|_{L^2}^2 & \le \left\|\frac{\chi_\tau^k - \chi_\tau^{k-1}}{\tau}\right\|_{L^2}^2, \\ \|\nabla(\chi_\tau^k)^+\|_{L^2}^2 & \le \|\nabla\chi_\tau^k\|_{L^2}^2, \\ a((\chi_\tau^k)^+) & \le a(\chi_\tau^k), \end{cases}$$

where the latter estimate is again due to (2.2b). Furthermore the splitting $W = \breve{W} + \hat{W}$ is constructed in a way such that, by (2.6b),

$$\begin{aligned} \breve{W}((\chi_\tau^k)^+) &\le \breve{W}(\chi_\tau^k) && \text{a.e. in } \Omega, \\ \hat{W}'(\chi^{k-1})(\chi_\tau^k)^+ &\le \hat{W}'(\chi^{k-1})\chi_\tau^k && \text{a.e. in } \Omega, \qquad \text{since } \hat{W}'(\chi^{k-1}) \le 0. \end{aligned}$$

Therefore, $\mathcal{P}((\chi_\tau^k)^+) \le \mathcal{P}(\chi_\tau^k)$. Due to the strict convexity of $\mathcal{P}$, minimizers are unique and thus $\chi_\tau^k = (\chi_\tau^k)^+$. All in all, we have obtained (3.3c).

A fortiori, we have that any solution χ_τ^k of (3.5) is indeed in $L^\infty(\Omega)$. Therefore,

$$\chi_\tau^k \in \mathrm{Argmin}_{\chi \le \chi_\tau^{k-1}} \widetilde{\mathcal{P}}(\chi), \tag{3.7}$$

with $\widetilde{\mathcal{P}} : H^1(\Omega) \cap L^\infty(\Omega) \to \mathbb{R}$ the Gâteaux-differentiable functional defined by

$$\widetilde{\mathcal{P}}(\chi) := \mathcal{P}_1(\chi) + \widetilde{\mathcal{P}}_2(\chi) \text{ and } \widetilde{\mathcal{P}}_2(\chi) = \int_\Omega \tfrac{1}{2}a(\chi)\mathbb{C}\varepsilon(\mathbf{u}_\tau^{k-1}):\varepsilon(\mathbf{u}_\tau^{k-1})\,\mathrm{d}x\,.$$

We may apply, e.g., [34, Lemma 2.21, p. 63] to the auxiliary minimum problem (3.7), and the variational inequality (3.3b) then follows as first-order necessary condition.

Finally, equation (3.3a), with χ_τ^k given as a datum, can be solved for $\mathbf{u}_\tau^k$ by the Lax-Milgram lemma. □

Lemma 3.2 (Time-discrete energy-dissipation inequality)**.** *It holds for all* $0 \le \ell \le k \le K$

$$\begin{aligned} &\mathcal{E}\Big(\mathbf{u}_\tau^k, \chi_\tau^k, \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau}\Big) + \tau \sum_{j=\ell}^{k} \mathcal{D}\Big(\chi_\tau^j, \frac{\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}}{\tau}, \frac{\chi_\tau^j - \chi_\tau^{j-1}}{\tau}\Big) \\ &\le \mathcal{E}\Big(\mathbf{u}_\tau^\ell, \chi_\tau^\ell, \frac{\mathbf{u}_\tau^\ell - \mathbf{u}_\tau^{\ell-1}}{\tau}\Big) + \tau \sum_{j=\ell}^{k} \langle \boldsymbol{f}_j, \mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1} \rangle_{H^1(\Omega)} \,. \end{aligned} \tag{3.8}$$

Proof. Testing (3.3a) with $\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}$ and (3.3b) with χ_τ^{j-1} and using standard convexity estimates yield

$$\left.\begin{aligned} &\frac{1}{2}\|\mathbf{u}_\tau^j\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\mathbf{u}_\tau^{j-1}\|_{L^2(\Omega)}^2 + \int_\Omega a(\chi_\tau^j)\mathbb{C}\varepsilon(\mathbf{u}_\tau^j):\varepsilon(\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1})\,\mathrm{d}x \\ &\quad + \tau \int_\Omega b(\chi_\tau^j)\mathbb{V}\frac{\varepsilon(\mathbf{u}_\tau^j) - \varepsilon(\mathbf{u}_\tau^{j-1})}{\tau} : \frac{\varepsilon(\mathbf{u}_\tau^j) - \varepsilon(\mathbf{u}_\tau^{j-1})}{\tau}\,\mathrm{d}x \\ &\quad + \tau \int_{\partial\Omega} \gamma_1 \left| \frac{\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1}}{\tau} \right|^2 \mathrm{d}S + \frac{\gamma_2}{2}\|\mathbf{u}_\tau^j\|_{L^2(\partial\Omega)}^2 - \frac{\gamma_2}{2}\|\mathbf{u}_\tau^{j-1}\|_{L^2(\partial\Omega)}^2 \\ &\le \langle \boldsymbol{f}_\tau^j, \mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1} \rangle_{H^1(\Omega)} + \langle \boldsymbol{g}_\tau^j, \mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1} \rangle_{H^{1/2}(\partial\Omega)}, \end{aligned}\right\} \tag{3.9a}$$

$$\left.\begin{aligned} &\tau \Big\| \frac{\chi_\tau^j - \chi_\tau^{j-1}}{\tau} \Big\|_{L^2}^2 + \frac{1}{2}\|\nabla\chi_\tau^j\|_{L^2}^2 - \frac{1}{2}\|\nabla\chi_\tau^{j-1}\|_{L^2}^2 \\ &+ \int_\Omega \Big[\tfrac{a'(\chi_\tau^j)}{2}\mathbb{C}\varepsilon(\mathbf{u}_\tau^{j-1}):\varepsilon(\mathbf{u}_\tau^{j-1}) + \breve{W}'(\chi_\tau^j) + \hat{W}'(\chi_\tau^{j-1}) \Big] (\chi_\tau^j - \chi_\tau^{j-1})\,\mathrm{d}x \le 0. \end{aligned}\right\} \tag{3.9b}$$

By the convexity of $\boldsymbol{u} \mapsto \frac{1}{2}\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u})$ and a we have

$$\begin{aligned} \int_\Omega a(\chi_\tau^j)\mathbb{C}\varepsilon(\mathbf{u}_\tau^j):\varepsilon(\mathbf{u}_\tau^j - \mathbf{u}_\tau^{j-1})\,\mathrm{d}x &\ge \int_\Omega \tfrac{1}{2}a(\chi_\tau^j)\mathbb{C}\varepsilon(\mathbf{u}_\tau^j):\varepsilon(\mathbf{u}_\tau^j)\,\mathrm{d}x \\ &\qquad - \int_\Omega \tfrac{1}{2}a(\chi_\tau^j)\mathbb{C}\varepsilon(\mathbf{u}_\tau^{j-1}):\varepsilon(\mathbf{u}_\tau^{j-1})\,\mathrm{d}x\,, \\ \int_\Omega \tfrac{a'(\chi_\tau^j)}{2}\mathbb{C}\varepsilon(\mathbf{u}_\tau^{j-1}):\varepsilon(\mathbf{u}_\tau^{j-1})(\chi_\tau^j - \chi_\tau^{j-1})\,\mathrm{d}x &\ge \int_\Omega \tfrac{1}{2}(a(\chi_\tau^j) - a(\chi_\tau^{j-1}))\mathbb{C}\varepsilon(\mathbf{u}_\tau^{j-1}):\varepsilon(\mathbf{u}_\tau^{j-1})\,\mathrm{d}x\,. \end{aligned}$$

In the same spirit, convexity of $\breve{W}$ and concavity of $\hat{W}$ yield

$$\begin{aligned} \int_\Omega \breve{W}'(\chi_\tau^j)(\chi_\tau^j - \chi_\tau^{j-1})\,\mathrm{d}x &\ge \int_\Omega [\breve{W}(\chi_\tau^j) - \breve{W}(\chi_\tau^{j-1})]\,\mathrm{d}x, \\ \int_\Omega \hat{W}'(\chi_\tau^{j-1})(\chi_\tau^j - \chi_\tau^{j-1})\,\mathrm{d}x &\ge \int_\Omega]\hat{W}(\chi_\tau^j) - \hat{W}(\chi_\tau^{j-1})]\,\mathrm{d}x\,, \end{aligned}$$

so that

$$\int_\Omega \big(\breve{W}'(\chi_\tau^j) + \hat{W}'(\chi_\tau^{j-1})\big)(\chi_\tau^j - \chi_\tau^{j-1})\,\mathrm{d}x \ge \int_\Omega W(\chi_\tau^j)\,\mathrm{d}x - \int_\Omega W(\chi_\tau^{j-1})\,\mathrm{d}x\,.$$

All in all, adding the inequalities (3.9a) and (3.9b), applying the above estimates and summing over index $j \in \{\ell+1, \dots, k\}$ proves the assertion. □

From Lemma 3.2 we deduce the basic a priori estimates for the families $(\overline{\mathbf{u}}_\tau)_\tau$, $(\underline{\mathbf{u}}_\tau)_\tau$, $(\overline{\chi}_\tau)_\tau$, $(\underline{\chi}_\tau)$ of the (left and right continuous) piecewise constant interpolants of the discrete solutions, as well as for their piecewise linear interpolants $(\boldsymbol{u}_\tau)_\tau$, $(\chi_\tau)_\tau$. Furthermore, we will consider the interpolants $(\overline{\mathbf{v}}_\tau)_\tau$, $(\underline{\mathbf{v}}_\tau)_\tau$

and $(\boldsymbol{v}_\tau)_\tau$ of the difference quotients $(\boldsymbol{v}^k := \frac{\mathbf{u}_\tau^k - \mathbf{u}_\tau^{k-1}}{\tau})_{k=1}^K$, and the (left continuous) piecewise constant interpolants $\overline{\boldsymbol{f}}_\tau : [0,T] \to H^1(\Omega;\mathbb{R}^d)^*$ and $\overline{\boldsymbol{g}}_\tau : [0,T] \to H^{-1/2}(\partial\Omega;\mathbb{R}^d)$ of the values $(\boldsymbol{f}_\tau^k)_{k=1}^K$ and $(\boldsymbol{g}_\tau^k)_{k=1}^K$, respectively. We record for later use that, as $\tau \to 0$, we have

$$\overline{\boldsymbol{f}}_\tau \to \boldsymbol{f} \qquad \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^d)^*), \qquad \overline{\boldsymbol{g}}_\tau \to \boldsymbol{g} \qquad \text{in } L^2(0,T;H^{-1/2}(\partial\Omega;\mathbb{R}^d)). \tag{3.10}$$

Proposition 3.3 (A priori estimates)**.** *There exists a constant $S > 0$ such that the following estimates hold for all $0 < \tau < \bar\tau$*

$$\|\boldsymbol{u}_\tau\|_{H^1(0,T;H^1(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))} \leq S, \tag{3.11a}$$
$$\|\overline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))} \leq S, \tag{3.11b}$$
$$\|\underline{\mathbf{u}}_\tau\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))} \leq S, \tag{3.11c}$$
$$\|\chi_\tau\|_{L^\infty(0,T;H^1(\Omega))\cap L^\infty(0,T;L^\infty(\Omega))\cap H^1(0,T;L^2(\Omega))} \leq S, \tag{3.11d}$$
$$\|\overline{\chi}_\tau\|_{L^\infty(0,T;H^1(\Omega))\cap L^\infty(0,T;L^\infty(\Omega))} \leq S, \tag{3.11e}$$
$$\|\underline{\chi}_\tau\|_{L^\infty(0,T;H^1(\Omega))\cap L^\infty(0,T;L^\infty(\Omega))} \leq S, \tag{3.11f}$$
$$\|\overline{\mathbf{v}}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \leq S, \tag{3.11g}$$
$$\|\underline{\mathbf{v}}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \leq S, \tag{3.11h}$$
$$\|\boldsymbol{v}_\tau\|_{H^1(0,T;H^{-1}(\Omega;\mathbb{R}^d))} \leq S. \tag{3.11i}$$
(3.11j)

Proof. Clearly, the discrete energy-dissipation inequality (3.8) rephrases as

$$\begin{aligned}
&\mathcal{E}(\overline{\mathbf{u}}_\tau(t),\overline{\chi}_\tau(t),\overline{\mathbf{v}}_\tau(t)) + \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)} \mathcal{D}\left(\overline{\chi}_\tau(r),\overline{\mathbf{v}}_\tau(r),\chi_\tau'(r)\right)\,\mathrm{d}r \\
&\leq \mathcal{E}(\overline{\mathbf{u}}_\tau(s),\overline{\chi}_\tau(s),\overline{\mathbf{v}}_\tau(s)) + \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)} \langle \overline{\boldsymbol{f}}_\tau(r),\overline{\mathbf{v}}_\tau(r)\rangle_{H^1(\Omega)}\,\mathrm{d}r + \int_{\bar{\mathsf{t}}_\tau(s)}^{\bar{\mathsf{t}}_\tau(t)} \langle \overline{\boldsymbol{g}}_\tau(r),\overline{\mathbf{v}}_\tau(r)\rangle_{H^{1/2}(\partial\Omega)}\,\mathrm{d}r
\end{aligned} \tag{3.12}$$

for all $0 \leq s \leq t \leq T$, where $\bar{\mathsf{t}}_\tau : [0,T] \to [0,T]$ is the (left-continuous) piecewise constant interpolant of the nodes of the partition $(t_\tau^k)_{k=1}^K$, with $\bar{\mathsf{t}}_\tau(0) := 0$. Taking into account the coercivity properties of $\mathcal{E}$ and $\mathcal{D}$ (based on the positive definiteness of the tensors $\mathbb{C}$ and $\mathbb{V}$, on Korn's inequality, on the positivity properties $a \geq 0$, $b \geq b_0 > 0$, and on the fact that W is bounded from below by an affine function (3.6)), from (3.12) we immediately deduce

$$\begin{aligned}
&\frac12\|\overline{\mathbf{v}}_\tau(t)\|_{L^2}^2 + \frac12\|\nabla\overline{\chi}_\tau(t)\|_{L^2}^2 + \int_0^{\bar{\mathsf{t}}_\tau(t)} \left(\|\chi_\tau'(r)\|_{L^2}^2 + c\|\overline{\mathbf{v}}_\tau(r)\|_{H^1}^2\right)\,\mathrm{d}r \\
&\leq \mathcal{E}(\boldsymbol{u}_0,\chi_0,\boldsymbol{v}_0) + \int_0^{\bar{\mathsf{t}}_\tau(t)} \langle \overline{\boldsymbol{f}}_\tau(r),\overline{\mathbf{v}}_\tau(r)\rangle_{H^1(\Omega)}\,\mathrm{d}r + \int_0^{\bar{\mathsf{t}}_\tau(t)} \langle \overline{\boldsymbol{g}}_\tau(r),\overline{\mathbf{v}}_\tau(r)\rangle_{H^{1/2}(\partial\Omega)}\,\mathrm{d}r \\
&\quad + c_W\|\overline{\chi}_\tau(t)\|_{L^1} + C_W.
\end{aligned}$$

Now, by (2.5a) we gather that $|\mathcal{E}(\boldsymbol{u}_0,\chi_0,\boldsymbol{v}_0)| \leq C$; in turn, we have

$$\begin{cases}
\|\overline{\boldsymbol{f}}_\tau\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^d))^*} \leq \|\boldsymbol{f}\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^d))^*} \\
\|\overline{\boldsymbol{g}}_\tau\|_{L^2(0,T;H^{-1/2}(\partial\Omega;\mathbb{R}^d))} \leq \|\boldsymbol{g}\|_{L^2(0,T;H^{-1/2}(\partial\Omega;\mathbb{R}^d))}
\end{cases}$$

for every $\tau > 0$, so that we may immediately absorb the second integral term on the right-hand side into the left-hand side. Finally, since by construction $0 \le \overline{\chi}_\tau \le 1$ a.e. in Q, we clearly have $c_W \|\overline{\chi}_\tau(t)\|_{L^1} \le c_W|\Omega|$. All in all, we conclude estimates (3.11a)–(3.11f) and (3.11g)–(3.11h).

Finally, (3.11i) follows from a comparison argument in equation (3.3a). □

3.2 Conclusion of the proof of Theorem 2.3

Let us a consider a null sequence $(\tau_j)_j$ of time steps. By well known compactness theorems we find a pair $(\boldsymbol{u}, \chi)$ as in (2.9) and a (not relabeled) subsequence of $(\tau_j)_j$ such that the following convergences hold as $j \to \infty$

$$\boldsymbol{u}_{\tau_j} \overset{\star}{\rightharpoonup} \boldsymbol{u} \quad \text{weakly-star in } H^1(0,T;H^1(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{3.13a}$$

$$\overline{\boldsymbol{u}}_{\tau_j}, \underline{\boldsymbol{u}}_{\tau_j} \overset{\star}{\rightharpoonup} \boldsymbol{u} \quad \text{weakly-star in } L^\infty(0,T;H^1(\Omega;\mathbb{R}^d)), \tag{3.13b}$$

$$\boldsymbol{v}_{\tau_j} \rightharpoonup \partial_t \boldsymbol{u} \quad \text{weakly in } H^1(0,T;H^1(\Omega;\mathbb{R}^d)^*), \tag{3.13c}$$

$$\overline{\boldsymbol{v}}_{\tau_j}, \underline{\boldsymbol{v}}_{\tau_j}, \boldsymbol{v}_{\tau_j} \overset{\star}{\rightharpoonup} \partial_t \boldsymbol{u} \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{3.13d}$$

$$\chi_{\tau_j} \overset{\star}{\rightharpoonup} \chi \quad \text{weakly-star in } L^\infty(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)) \cap H^1(0,T;L^2(\Omega)), \tag{3.13e}$$

$$\overline{\chi}_{\tau_j}, \underline{\chi}_{\tau_j} \overset{\star}{\rightharpoonup} \chi \quad \text{weakly-star in } L^\infty(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))\,. \tag{3.13f}$$

From Aubin-Lions type compactness results we see that

$$\begin{aligned} &\chi_{\tau_j}, \overline{\chi}_{\tau_j}, \underline{\chi}_{\tau_j} \to \chi \quad \text{strongly in } L^\infty(0,T;L^p(\Omega)) \text{ for all } p \in (0,2^*), \\ &\chi_{\tau_j}, \overline{\chi}_{\tau_j}, \underline{\chi}_{\tau_j} \to \chi \quad \text{a.e. in } Q. \end{aligned} \tag{3.14}$$

Finally, combining (3.13c) & (3.13d) we also gather

$$\boldsymbol{v}_{\tau_j}(t) \rightharpoonup \boldsymbol{v}(t) = \partial_t \boldsymbol{u}(t) \qquad \text{in } L^2(\Omega;\mathbb{R}^d) \qquad \text{for all } t \in [0,T]\,. \tag{3.15}$$

Obviously, the pair $(\boldsymbol{u}, \chi)$ complies with initial conditions (1.1c), constraints (1.1d). In order to check the validity of (2.10), let us write the discrete weak momentum balance (3.3a) in a time-integrated version:

$$\begin{aligned} &\int_0^t \langle \partial_t \boldsymbol{v}_{\tau_j}, \boldsymbol{\varphi} \rangle_{H^1_0} \, \mathrm{d}t + \int_0^t \int_\Omega \Big\{ b(\overline{\chi}_{\tau_j}) \mathbb{V} \varepsilon(\partial_t \boldsymbol{u}_{\tau_j}) : \varepsilon(\boldsymbol{\varphi}) + a(\overline{\chi}_{\tau_j}) \mathbb{C} \varepsilon(\overline{\boldsymbol{u}}_{\tau_j}) : \varepsilon(\boldsymbol{\varphi}) \Big\} \, \mathrm{d}x \, \mathrm{d}r \\ &\quad + \int_0^t \int_{\partial\Omega} \big\{ \gamma_1 \partial_t \boldsymbol{u}_{\tau_j} \cdot \boldsymbol{\varphi} + \gamma_2 \overline{\boldsymbol{u}}_{\tau_j} \cdot \boldsymbol{\varphi} \big\} \, \mathrm{d}S \, \mathrm{d}r \\ &= \int_0^t \langle \overline{\boldsymbol{f}}_{\tau_j}, \partial_t \boldsymbol{u}_{\tau_j} \rangle_{H^1(\Omega)} \, \mathrm{d}r + \int_0^t \langle \overline{\boldsymbol{g}}_{\tau_j}, \partial_t \boldsymbol{u}_{\tau_j} \rangle_{H^{1/2}(\partial\Omega)} \, \mathrm{d}r. \end{aligned}$$

for all $\boldsymbol{\varphi} \in L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))$. Convergences (3.13), (3.14), and (3.10) allow us to take the limit as $\tau_j \to 0$, and conclude the time-integrated version of (2.10). Hence, the weak momentum balance is shown.

In the next step we aim to obtain the integral inequality (2.11). For this, we observe that, choosing the admissible test-function $\psi = \overline{\chi}_{\tau_j}(t) + \widehat{\psi}$ with $\widehat{\psi} \in H^1_-(\Omega) \cap L^\infty(\Omega)$ (where $H^1_-(\Omega)$ is the cone of negative functions in $H^1(\Omega)$), (3.3b) rewrites for almost all $t \in (0,T)$ as

$$\begin{aligned} &\int_\Omega \Big\{ \chi'_{\tau_j}(t) \widehat{\psi} + \nabla \overline{\chi}_{\tau_j}(t) \cdot \nabla \widehat{\psi} + \frac{1}{2} a'(\chi_{\tau_j}(t)) \mathbb{C} \varepsilon(\underline{\boldsymbol{u}}_{\tau_j}(t)) : \varepsilon(\underline{u}_{\tau_j}(t)) \widehat{\psi} \Big\} \, \mathrm{d}x \\ &\qquad + \int_\Omega \Big\{ \breve{W}'(\overline{\chi}_{\tau_j}(t)) \widehat{\psi} + \hat{W}'(\underline{\chi}_{\tau_j}(t)) \widehat{\psi} \Big\} \, \mathrm{d}x \ge 0\,. \end{aligned}$$

Thus, integrating in time we obtain

$$\iint_Q \left[\partial_t \chi_{\tau_j} \widehat{\psi} + \nabla \overline{\chi}_{\tau_j} \cdot \nabla \widehat{\psi} + \frac{a'(\overline{\chi}_{\tau_j})}{2} \mathbb{C}\varepsilon(\underline{\boldsymbol{u}}_{\tau_j}) : \varepsilon(\underline{\boldsymbol{u}}_{\tau_j}) \widehat{\psi} + \breve{W}'(\overline{\chi}_{\tau_j}) \widehat{\psi} + \hat{W}'(\underline{\chi}_{\tau_j}) \widehat{\psi}\right] \mathrm{d}x\, \mathrm{d}t \geq 0$$

for all $\widehat{\psi} \in L^\infty(0,T;H^1_-(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$. In order to take the limit as $j \to \infty$ we rely on convergences (3.13) observe that, by (3.14) and the fact that $W' \in \mathrm{C}^0(\mathbb{R})$, we immediately have, for instance, that

$$\breve{W}'(\overline{\chi}_{\tau_j}) + \hat{W}'(\underline{\chi}_{\tau_j}) \to \breve{W}'(\chi) + \hat{W}'(\chi) \qquad \text{in } L^\infty(0,T;L^2(\Omega)) .$$

Moreover, we combine the information that

$$\varepsilon(\underline{\boldsymbol{u}}_{\tau_j}) \rightharpoonup \varepsilon(\boldsymbol{u}) \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$$

with the fact that $a'(\overline{\chi}_{\tau_j}) \to a'(\chi)$, e.g. in $L^\infty(0,T;L^2(\Omega))$. Then, well-known lower semicontinuity results (cf. the Ioffe theorem [24]) give

$$\liminf_{j\to\infty} -\iint_Q \frac{a'(\overline{\chi}_{\tau_j})}{2} \mathbb{C}\varepsilon(\underline{\boldsymbol{u}}_{\tau_j}) : \varepsilon(\underline{\boldsymbol{u}}_{\tau_j}) \widehat{\psi}\, \mathrm{d}x\, \mathrm{d}t \geq \int_Q \frac{a'(\chi)}{2} \mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})(-\widehat{\psi})\, \mathrm{d}x\, \mathrm{d}t .$$

In this way, we conclude the time-integrated version of the one-sided variational inequality (2.11), tested by an arbitrary $\widehat{\psi} \in L^\infty(0,T;H^1_-(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$. Ultimately, we conclude (2.11).

Eventually, also relying on the pointwise-in-time convergence (3.15), we are in a position to take the limit as $\tau_j \downarrow 0$ in the discrete energy-dissipation inequality (3.12), written for $s = 0$ and arbitrary $t \in [0,T]$. We thus conclude the time-continuous energy inequality (2.12). □

3.3 Outlook to nonsmooth potential energies

In this section, we provide a possible extension of the existence result for weak solutions, to the case in which the convex part $\breve{W}$ of potential energy W is nonsmooth. A prototypical example will be provided by $\breve{W} = I_{[0,\infty)}$, cf. Remark 3.5 below, but we will indeed allow for more general potentials. Our standing assumptions are collected in the following

Hypothesis F. *The elasticity and the viscosity tensors, and the constitutive functions a and b, comply with* (2.1)*,* (2.2)*, and* (2.3)*, respectively. We suppose that W is ℓ-convex, i.e.,* (2.4b) *holds, and W fulfills* (2.4c)*.*

Note that in comparison to the assumption (2.4) of Hypothesis **A**, we relaxed the smoothness assumptions (2.4a) on the convex part $\breve{W}$. To handle nonsmoothness of $\breve{W}$, we propose a novel generalized formulation which turns out to be consistent with that of Definition 2.2. In fact, in Def. 3.4 below inequality (2.11) is replaced by another one-sided variational inequality, (3.16) below, featuring the derivative of the concave part, only. On the other hand, (3.16) is in the same spirit as the one-sided variational inequality proposed for the damage flow rule in [20, 21] in the specific case $W = \breve{W} = I_{[0,\infty)}$.

Definition 3.4 (Weak solution for nonsmooth potential)**.** *In the setting of Hypothesis **F**, we call a pair $(\boldsymbol{u}, \chi)$ as in* (2.9) *a weak solution to* (1.1)*, if it satisfies initial conditions* (1.1c)*, constraints* (1.1d)*, the weak formulation* (2.10) *of the momentum balance, the upper energy-dissipation inequality* (2.12)*, and*

$$\iint_Q \Big(\chi_t \varphi + \nabla\chi \cdot \nabla\varphi + \tfrac{1}{2} a'(\chi)\varphi \mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}) + \breve{W}(\chi+\varphi) - \breve{W}(\chi) + \hat{W}'(\chi)\varphi\Big)\, \mathrm{d}x\, \mathrm{d}t \geq 0 \tag{3.16}$$

for all $\varphi \in L^\infty(0,T;H^1(\Omega)) \cap L^\infty(Q)$ with $\varphi \leq 0$ a.e. in Q.

Remark 3.5. In the specific case in which $W = \breve{W} = I_{[0,\infty)}$, (3.16) reduces to

$$\iint_Q \left(\chi_t\varphi + \nabla\chi\cdot\nabla\varphi + \tfrac{1}{2}a'(\chi)\varphi\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u})\right)\,\mathrm{d}x\,\mathrm{d}t \geq 0 \tag{3.17}$$

for all $\varphi \in L^\infty(0,T;H^1(\Omega))$ with $-\chi \leq \varphi \leq 0$ a.e. in Q (where the constraint $\varphi \geq -\chi$ ensures that $\iint_Q \breve{W}(\chi+\varphi)\,\mathrm{d}x\,\mathrm{d}t < +\infty$ and thus the inequality is non-trivial). In fact, (3.17) is in accord with the one-sided variational inequality from [20, Def. 4.5], [21, Def. 2.3].

Our first result shows that, as soon as also the convex part of W is smooth, any weak solution in the sense of Definition 3.4 is also a weak solution according to Definition 2.2.

Proposition 3.6. *In addition to Hypothesis F, suppose that $\breve{W} \in \mathrm{C}^1(\mathbb{R})$. Then, any $(\boldsymbol{u},\chi)$ as in* (2.9) *fulfilling* (3.16) *also complies with* (2.11).

Proof. We choose the test function for (3.16) in the form

$$\varphi = \phi\zeta \in L^\infty(0,T;H^1(\Omega)) \quad \text{with } \begin{cases} \phi \in L^\infty(0,T), \quad 0 \leq \phi \leq 1 \text{ a.e. in } (0,T), \\ \zeta \in H^1(\Omega), \quad \zeta \leq 0 \text{ a.e. in } \Omega. \end{cases}$$

We now estimate from above the term $\breve{W}(\chi+\phi\zeta) - \breve{W}(\chi)$ that features on the left-hand side of (3.16) by

$$\begin{aligned} \breve{W}(\chi+\phi\zeta) - \breve{W}(\chi) &= \breve{W}(\phi(\chi+\zeta)+(1-\phi)\chi) - \breve{W}(\chi) \\ &\overset{(1)}{\leq} \phi\breve{W}(\chi+\zeta) + (1-\phi)\breve{W}(\chi) - \breve{W}(\chi) \\ &= \phi\left(\breve{W}(\chi+\zeta) - \breve{W}(\chi)\right) \qquad \text{a.e. in } Q\,, \end{aligned}$$

where (1) follows from the convexity of $\breve{W}$. We use the above inequality to estimate from above the left-hand side of (3.16), thus obtaining

$$\int_0^T \phi\left(\int_\Omega \chi_t\zeta + \nabla\chi\cdot\nabla\zeta + \tfrac{1}{2}a'(\chi)\zeta\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u}) + \breve{W}(\chi+\zeta) - \breve{W}(\chi) + \hat{W}'(\chi)\zeta\,\mathrm{d}x\right)\,\mathrm{d}t \geq 0\,.$$

By the arbitrariness of ϕ we thus infer the pointwise in time formulation

$$\begin{aligned} &\int_\Omega \left(\chi_t(t)\zeta + \nabla\chi(t)\cdot\nabla\zeta + \tfrac{1}{2}a'(\chi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)):\varepsilon(\boldsymbol{u}(t))\zeta + \hat{W}'(\chi(t))\zeta\right)\,\mathrm{d}x \\ &+ \int_\Omega (\breve{W}(\chi(t)+\zeta) - \breve{W}(\chi(t)))\,\mathrm{d}x \geq 0 \end{aligned} \tag{3.18}$$

for all $\zeta \in H^1_-(\Omega) \cap L^\infty(\Omega)$ and for almost all $t \in (0,T)$. Let us now choose $\zeta = h\psi$ with an arbitrary $\psi \in H^1_-(\Omega) \cap L^\infty(\Omega)$. Dividing the resulting inequality by h and sending $h \to 0$ we obtain (2.11). □

We now show that, with minor changes, the argument developed in Secs. 3.1–3.2 also yields the existence of solutions in the sense of Def. 3.4.

Theorem 3.7 (Existence of weak solutions for nonsmooth potentials)**.** *Assume Hypotheses **B** & **F**. Then, there exists a weak solution in the sense of Definition 3.4 to the Cauchy problem for system* (1.1).

Proof. The proof is very similar to the argument for Theorem 2.3, hence we only comment on the relevant changes. We construct time-discrete solutions $(\mathbf{u}_\tau^k,\chi_\tau^k)_{k=1}^K$ as in Lemma 3.1. From the information that χ_τ^k is a minimizer for the functional $\mathcal{P}$, cf. (3.4), as a first order optimality condition we gather that

$$\int_\Omega \left\{ \frac{\chi_\tau^k-\chi_\tau^{k-1}}{\tau}(\psi-\chi_\tau^k)+\nabla\chi_\tau^k\cdot\nabla(\psi-\chi_\tau^k)+\frac{1}{2}a'(\chi_\tau^k)\mathbb{C}\varepsilon(\mathbf{u}_\tau^{k-1}):\varepsilon(\mathbf{u}_\tau^{k-1})(\psi-\chi_\tau^k)\right\}\,\mathrm{d}x \\ +\int_\Omega\left\{\breve{W}(\psi)-\breve{W}(\chi_\tau^k)+\hat{W}'(\chi_\tau^{k-1})(\psi-\chi_\tau^k)\right\}\,\mathrm{d}x\geq 0\,, \tag{3.19}$$

for all $\psi\in X_\tau^{k-1}:=\left\{v\in H^1(\Omega)\cap L^\infty(\Omega)\ :\ \ v\leq\chi_\tau^{k-1}\text{ a.e. in }\Omega\right\}$, as well as the constraints $0\leq\chi_\tau^k\leq\chi_\tau^{k-1}\leq 1$ a.e. in Ω. The discrete energy inequality of Lemma 3.2 is then obtained by testing (3.3a) with $\mathbf{u}_\tau^j-\mathbf{u}_\tau^{j-1}$ and (3.19) with χ_τ^{j-1}; from (3.8) there stem the a priori estimates of Proposition 3.3 and, a fortiori, convergences (3.13)–(3.15).

In order to prove (3.16), first of all we sum (3.19) over all the time intervals induced by the partition, thus obtaining

$$\iint_Q \partial_t\chi_{\tau_j}(\widehat{\psi}-\overline{\chi}_{\tau_j})+\nabla\overline{\chi}_{\tau_j}\cdot\nabla(\widehat{\psi}-\overline{\chi}_{\tau_j})+\frac{a'(\overline{\chi}_{\tau_j})}{2}\mathbb{C}\varepsilon(\underline{\boldsymbol{u}}_{\tau_j}):\varepsilon(\underline{\boldsymbol{u}}_{\tau_j})(\widehat{\psi}-\overline{\chi}_{\tau_j})\,\mathrm{d}x\,\mathrm{d}t \\ +\iint_Q \breve{W}(\widehat{\psi})-\breve{W}(\overline{\chi}_{\tau_j})+\hat{W}'(\underline{\chi}_{\tau_j})(\widehat{\psi}-\overline{\chi}_{\tau_j})\,\mathrm{d}x\,\mathrm{d}t\geq 0\,, \tag{3.20}$$

for all test functions $\widehat{\psi}\in L^\infty(0,T;H^1(\Omega))$ with $\widehat{\psi}\leq\overline{\chi}_{\tau_j}$ a.e. in Q. With the convergences inferred in (3.13), we may pass to the limit is the above formulation.

For any $\psi\in L^\infty(0,T;H^1(\Omega))\cap L^\infty(Q)$ with $\psi\leq\chi$ we construct a sequence of recovery test functions in the following way. For any $j\in\mathbb{N}$ we define

$$\psi_j(x,t):=\min\{\psi(x,t),\overline{\chi}_{\tau_j}(x,t)\}\quad\text{and}\quad A_j:=\{(x,t)\in Q\mid\psi(x,t)\leq\overline{\chi}_{\tau_j}(x,t)\}\,.$$

With the same arguments as in [33, Thm. 3.14] we can prove that

$$(\psi_j)_j\text{ is bounded in }L^\infty(0,T;H^1(\Omega))\cap L^\infty(Q)\text{ and }\psi_j(t)\rightharpoonup\psi(t)\text{ in }H^1(\Omega)\text{ for a.a. }t\in(0,T)\,,$$

so that it is not difficult to deduce that

$$\psi_j\to\psi\quad\text{in }L^r(Q)\text{ for all }r\in[1,\infty)\,,\qquad\psi_j\overset{\star}{\rightharpoonup}\psi\quad\text{weakly-star in }L^\infty(0,T;H^1(\Omega))\cap L^\infty(Q)\,.$$

We now choose $\widehat{\psi}=\psi_j$ in (3.20). Since $\psi_j=\chi_{\tau_j}$ on $Q\backslash A_j$, we thus obtain

$$\iint_Q \mathbb{1}_{A_j}\left(\partial_t\chi_{\tau_j}(\psi-\overline{\chi}_{\tau_j})+\nabla\overline{\chi}_{\tau_j}\cdot\nabla(\psi-\overline{\chi}_{\tau_j})+\frac{a'(\overline{\chi}_{\tau_j})}{2}\mathbb{C}\varepsilon(\underline{\boldsymbol{u}}_{\tau_j}):\varepsilon(\underline{\boldsymbol{u}}_{\tau_j})(\psi-\overline{\chi}_{\tau_j})\right)\,\mathrm{d}x\,\mathrm{d}t \\ +\iint_Q \mathbb{1}_{A_j}\left(\breve{W}(\psi)-\breve{W}(\overline{\chi}_{\tau_j})+\hat{W}'(\underline{\chi}_{\tau_j})(\psi-\overline{\chi}_{\tau_j})\right)\,\mathrm{d}x\,\mathrm{d}t\geq 0\,,$$

and then send $j\to\infty$. We use that, since $\overline{\chi}_{\tau_j}\to\chi$ and $\psi\leq\chi$ a.e. in Q, the sequence $(\mathbb{1}_{A_j})_j$ of the characteristic functions of the sets A_j converges a.e. in Q and strongly in $L^1(Q)$ to the function identically equally to 1. Therefore,

$$\begin{aligned}
&\iint_Q \mathbb{1}_{A_j}\partial_t\chi_{\tau_j}(\psi-\overline{\chi}_{\tau_j})\,\mathrm{d}x\,\mathrm{d}t &&\longrightarrow\quad \iint_Q\partial_t\chi(\psi-\chi)\,\mathrm{d}x\,\mathrm{d}t,\\
&\iint_Q \mathbb{1}_{A_j}\nabla\overline{\chi}_{\tau_j}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t &&\longrightarrow\quad \iint_Q\nabla\chi\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t,\\
&\iint_Q \mathbb{1}_{A_j}\left(\breve{W}(\psi)+\hat{W}'(\underline{\chi}_{\tau_j})(\psi-\overline{\chi}_{\tau_j})\right)\,\mathrm{d}x\,\mathrm{d}t &&\longrightarrow\quad \iint_Q\left(\breve{W}(\psi)+\hat{W}'(\chi)(\psi-\chi)\right)\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}$$

where we have also used that $\hat{W}(r) = -\frac{\ell}{2}r^2$. To handle the remaining terms, we again resort to the Ioffe theorem [24], which gives

$$\limsup_{j\to\infty}\left(-\iint_Q \mathbb{1}_{A_j}|\nabla\overline{\chi}_{\tau_j}|^2\,\mathrm{d}x\,\mathrm{d}t\right) = -\liminf_{j\to\infty}\iint_Q \mathbb{1}_{A_j}|\nabla\overline{\chi}_{\tau_j}|^2\,\mathrm{d}x\,\mathrm{d}t \le -\iint_Q |\nabla\chi|^2\,\mathrm{d}x\,\mathrm{d}t\,,$$
$$\limsup_{j\to\infty}\left(-\iint_Q \mathbb{1}_{A_j}\breve{W}(\overline{\chi}_{\tau_j})\,\mathrm{d}x\,\mathrm{d}t\right) \le -\iint_Q \breve{W}(\chi)\,\mathrm{d}x\,\mathrm{d}t\,,$$

and, likewise

$$\begin{aligned}
&\limsup_{j\to\infty}\iint_Q \mathbb{1}_{A_j}\frac{1}{2}a'(\overline{\chi}_{\tau_j})\mathbb{C}\varepsilon(\underline{\boldsymbol{u}}_{\tau_j}){:}\varepsilon(\underline{\boldsymbol{u}}_{\tau_j})(\psi-\overline{\chi}_{\tau_j})\,\mathrm{d}x\,\mathrm{d}t\\
&= -\liminf_{j\to\infty}\iint_Q \mathbb{1}_{A_j}\frac{1}{2}a'(\overline{\chi}_{\tau_j})\mathbb{C}\varepsilon(\underline{\boldsymbol{u}}_{\tau_j}){:}\varepsilon(\underline{\boldsymbol{u}}_{\tau_j})(\overline{\chi}_{\tau_j}-\psi)\,\mathrm{d}x\,\mathrm{d}t\\
&\le -\iint_Q \mathbb{1}_{A_j}\frac{1}{2}a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u})(\chi-\psi)\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}$$

To apply Ioffe's theorem, here, we have also relied on the fact that $(\breve{W}(\overline{\chi}_{\tau_j}))_j$ is bounded from below and $a'(\overline{\chi}_{\tau_j})(\overline{\chi}_{\tau_j}-\psi) \ge 0$ a.e. in Q. Combining all these convergences, we arrive at

$$\begin{aligned}
&\iint_Q \left(\chi_t(\psi-\chi)+\nabla\chi\cdot\nabla(\psi-\chi)+\tfrac{1}{2}a'(\chi)(\psi-\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u})+\hat{W}'(\chi)(\psi-\chi)\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \iint_Q (\breve{W}(\psi)-\breve{W}(\chi))\,\mathrm{d}x\,\mathrm{d}t \ge 0
\end{aligned} \tag{3.21}$$

for all $\psi \in L^\infty(0,T;H^1(\Omega))$ with $\psi \le \chi$ a.e. in Q. Choosing $\varphi = \psi-\chi$, we get (3.16). □

4 Proof of Theorem 2.9

Our proof of the enhanced regularity (2.14) will be based on estimates that have a local-in-time character only and rely on a Gronwall-type argument. Since there is, apparently, no time-discrete version of local-in-time Gronwall estimates, we will not resort to time discretization for proving the existence of strong solutions, but instead

- devise a suitable approximation of system (1.1), namely system (4.29) ahead,
- prove existence of solutions to (4.29) via the Schauder fixed point theorem,
- perform on it the rigorous regularity estimates.

Such regularity estimates will be at first *formally* performed on the original system (1.1) in Sec. 4.1 below. This will allow us to pinpoint how system (1.1) needs to be approximated in such a way that the calculations of Proposition 4.2 can be rendered rigorous. Hence, in Sec. 4.2 we will set up the approximate system (4.29) by combining Galerkin discretization and Yosida regularization. In Sections 4.3 and 4.4 we will address the existence of local-in-time solutions to the associated Cauchy problem, and rigorously perform the, previously formal, enhanced regularity estimates. Finally, in Sec. 4.5 we will conclude the proof of Thm. 2.9 by taking the limit in system (4.29).

4.1 Formal a priori estimates

Before carrying out the enhanced a priori estimates, it is convenient to rewrite the flow rule (1.1b) as

$$\begin{gathered}\chi_t + I'_{(-\infty,0]}(\chi_t) + \omega = \chi - \hat{W}'(\chi) - \frac{1}{2}a'(\chi)\varepsilon(\boldsymbol{u})\mathbb{C}\varepsilon(\boldsymbol{u}) \qquad \text{a.e. in } Q \\ \text{with } \omega = -\Delta\chi + \breve{W}'(\chi) + \chi\,, \end{gathered} \tag{4.1}$$

where we have formally replaced $\partial I_{(-\infty,0]}(\chi_t)$ by $I'_{(-\infty,0]}(\chi_t)$, hereafter abbreviated as $I'(\chi_t)$, and resorted to the convex/concave decomposition (2.6a) of W. Although in the present setting the convex contribution $\breve{W}$ may be nonsmooth, for notational simplicity we will formally write $\breve{W}'(\chi)$, $\breve{W}''(\chi)$. In fact, in Section 4.3 ahead we will make all estimates rigorous by replacing $\breve{W}$ by a (version of) its Yosida regularization.

The following result (with the caveat that all calculations can be rendered rigorously when $\breve{W}$ is suitably regularized) collects elementary estimates that will nonetheless have a key role in the ensuing calculations; note that (4.2a) is in the spirit of the well-known Brezis-Strauss result [6]. In the proof we will use that $\breve{W}'(0) = 0$, which we can always suppose up to a translation.

Lemma 4.1. *There exists $S_0 > 0$ such that for almost all $t \in (0,T)$*

$$\Big(\|\chi(t)\|_{H^2(\Omega)} + \|\breve{W}'(\chi(t))\|_{L^2(\Omega)}\Big) \leq S_0\|\omega(t)\|_{L^2(\Omega)}\,, \tag{4.2a}$$

$$\|\chi_t(t)\|_{H^1(\Omega)} \leq S_0\|\omega_t(t)\|_{L^2(\Omega)}\,, \tag{4.2b}$$

$$\|\chi_{tt}(t)\|_{H^1(\Omega)} \leq S_0\left(\|\omega_{tt}(t)\|_{L^2(\Omega)} + \|\breve{W}'''(\chi(t))\|_{L^\infty(\Omega)}\|\chi_t(t)\|^2_{L^3(\Omega)}\right)\,. \tag{4.2c}$$

Proof. $\triangleright$ (4.2a) : We calculate

$$\begin{aligned}\|\omega(t)\|^2_{L^2(\Omega)} &= \|\Delta\chi(t)\|^2_{L^2(\Omega)} + \|\breve{W}'(\chi(t))\|^2_{L^2(\Omega)} + \|\chi(t)\|^2_{L^2(\Omega)} \\ &\qquad + 2\int_\Omega \breve{W}'(\chi(t))\chi(t)\,\mathrm{d}x - 2\int_\Omega \Delta\chi(t)(\breve{W}'(\chi(t)) + \chi(t))\,\mathrm{d}x \\ &\overset{(1)}{\geq} c\|\chi(t)\|^2_{H^2(\Omega)} + \|\breve{W}'(\chi(t))\|^2_{L^2(\Omega)} + 2\int_\Omega (\breve{W}''(\chi(t)) + 1)|\nabla\chi(t)|^2\,\mathrm{d}x \\ &\overset{(2)}{\geq} c\|\chi(t)\|^2_{H^2(\Omega)} + \|\breve{W}'(\chi(t))\|^2_{L^2(\Omega)}\,,\end{aligned}$$

where (1) follows the fact that $\breve{W}'(0) = 0$ and (2) from the convexity of $\breve{W}$.

$\triangleright$ (4.2b) : Differentiating in time $\omega = -\Delta\chi + \breve{W}'(\chi) + \chi$ and testing it by χ_t we obtain

$$\|\chi_t\|^2_{H^1(\Omega)} \leq \int_\Omega \left(|\chi_t|^2 + |\nabla\chi_t|^2\right)\,\mathrm{d}x + \int_\Omega \breve{W}''(\chi)|\chi_t|^2\,\mathrm{d}x = \int_\Omega \omega_t\chi_t\,\mathrm{d}x \leq \frac{1}{2}\|\omega_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\chi_t\|^2_{L^2(\Omega)}\,,$$

whence (4.2b).

$\triangleright$ (4.2c) : We differentiate $\omega = -\Delta\chi + \breve{W}'(\chi) + \chi$ twice in time and test it by χ_{tt}, thus obtaining

$$\begin{aligned}\|\chi_{tt}\|^2_{H^1(\Omega)} &\leq \int_\Omega \left(|\chi_{tt}|^2 + |\nabla\chi_{tt}|^2\right)\,\mathrm{d}x + \int_\Omega \breve{W}''(\chi)|\chi_{tt}|^2\,\mathrm{d}x \\ &= \int_\Omega \omega_{tt}\chi_{tt}\,\mathrm{d}x + \int_\Omega \breve{W}'''(\chi)|\chi_t|^2\chi_{tt}\,\mathrm{d}x \\ &\leq \|\omega_t\|^2_{L^2(\Omega)} + \|\breve{W}'''(\chi)\|_{L^\infty(\Omega)}\||\chi_t|^2\|^2_{L^{3/2}(\Omega)} + \frac{1}{2}\|\chi_{tt}\|^2_{L^3(\Omega)}\,,\end{aligned}$$

whence (4.2c). □

Proposition 4.2. *Assume Hypotheses* **C** *&* **D**, *let* Ω *fulfill* (H_Ω). *Then, there exists a time* $\widehat{T} \in (0,T]$ *and a constant* $S_1 > 0$ *such that for all* $t \in (0,\widehat{T})$

$$\begin{aligned} &\|\boldsymbol{u}_t(t)\|^2_{H^2(\Omega)} + \|\chi(t)\|^2_{H^2(\Omega)} + \int_0^t \Big(\|\boldsymbol{u}_t(s)\|^2_{H^3(\Omega)} + \|\chi_t(s)\|^2_{H^1(\Omega)}\Big)\,\mathrm{d}s \\ &\leq S_1\Big(1+\|\boldsymbol{v}_0\|^2_{H^2(\Omega)} + \|\boldsymbol{u}_0\|^2_{H^3(\Omega)} + \|\chi_0\|^2_{H^2(\Omega)} + \||\partial\breve{W}^\circ|(\chi_0)\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_0\|^8_{H^2(\Omega)}\Big)\,. \end{aligned} \tag{4.3}$$

Furthermore, there exists a constant $\widehat{S}_1 > 0$ *such that*

$$\|\boldsymbol{u}_{tt}\|_{L^2(0,\widehat{T};H^1(\Omega))} \leq \widehat{S}_1\,. \tag{4.4}$$

Throughout the proof, we will repeatedly use the following estimate for all $t \in [0,T]$

$$\|\boldsymbol{z}(t)\|^p_{\boldsymbol{X}} = \left\|\boldsymbol{z}(0) + \int_0^t \boldsymbol{z}_t\,\mathrm{d}s\right\|^p_{\boldsymbol{X}} \leq 2^{p-1}\|\boldsymbol{z}(0)\|^p_{\boldsymbol{X}} + (2t)^{p-1}\int_0^t \|\boldsymbol{z}_t(s)\|^p_{\boldsymbol{X}}\,\mathrm{d}s\,, \tag{4.5}$$

which follows, by Jensen's inequality and the elementary inequality $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for all $a, b \in [0,+\infty)$, for every $\boldsymbol{z} \in W^{1,p}(0,T;\boldsymbol{X})$, $p \geq 1$ (where $\boldsymbol{X}$ is a Banach space with the Radon-Nikodým property).

Proof. We split the argument in the following claims.

Claim 0***:*** *The evolution of the mean of* $\boldsymbol{u}$ *is only determined by the given data, i.e.,*

$$\int_\Omega \boldsymbol{u}(t)\,\mathrm{d}x = \int_\Omega \boldsymbol{u}_0\,\mathrm{d}x + t\int_\Omega \boldsymbol{v}_0\,\mathrm{d}x + \int_0^t\int_\Omega (t-r)\boldsymbol{f}(r)\,\mathrm{d}r\,.$$

Integrating in space the momentum balance (2.10) we infer $\partial_t \int_\Omega \boldsymbol{u}_t\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\,\mathrm{d}x$, so that, integrating in time we get

$$\int_\Omega \boldsymbol{u}_t\,\mathrm{d}x(t) = \int_\Omega \boldsymbol{v}_0\,\mathrm{d}x + \int_0^t\int_\Omega \boldsymbol{f}\,\mathrm{d}x\,\mathrm{d}s \tag{4.6}$$

and thus, integrating again over $(0,t)$, we obtain

$$\int_\Omega \boldsymbol{u}(t)\,\mathrm{d}x = \int_\Omega \boldsymbol{u}_0\,\mathrm{d}x + t\int_\Omega \boldsymbol{v}_0\,\mathrm{d}x + \int_0^t\int_0^s\int_\Omega \boldsymbol{f}\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}s\,.$$

Via Fubini's theorem, we find $\int_0^t\int_0^s\int_\Omega \boldsymbol{f}\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}s = \int_0^t\int_\Omega \boldsymbol{f}(r)\,\mathrm{d}x\int_{t-r}^t\,\mathrm{d}s\,\mathrm{d}r = \int_0^t\int_\Omega (t-r)\boldsymbol{f}(r)\,\mathrm{d}r\,.$

Claim 1***:*** *There exists a constant* $S_{1,1} > 0$ *such that for almost all* $t \in (0,T)$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_t(t)\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_t(t)\|^2_{H^1(\Omega)} \\ &\leq S_{1,1}\left(\|\boldsymbol{v}_0\|^2_{L^2(\Omega)} + \|\boldsymbol{f}\|^2_{L^2(Q)} + \Big(\|\chi(t)\|^{2\rho+4}_{L^\infty(\Omega)}+1\Big)\Big(\|\varepsilon(\boldsymbol{u}_0)\|^2_{L^2(\Omega)} + \int_0^t \|\varepsilon(\boldsymbol{u}_t)\|^2_{L^2(\Omega)}\,\mathrm{d}s\Big)\right)\,. \end{aligned} \tag{4.7}$$

We test (2.10) by $\boldsymbol{u}_t$. Taking into account (2.1) and (2.3), by the Poincaré-Korn inequality we find for almost all $t \in (0,T)$

$$\int_\Omega b(\chi_t(t))\mathbb{V}\varepsilon(\boldsymbol{u}_t(t)):\varepsilon(\boldsymbol{u}_t(t))\,\mathrm{d}x \geq c\|\boldsymbol{u}_t(t)\|^2_{H^1(\Omega)} - C\left|\int_\Omega \boldsymbol{u}_t(t)\,\mathrm{d}x\right|^2 \stackrel{(*)}{\geq} c\|\boldsymbol{u}_t(t)\|^2_{H^1(\Omega)} - C\left|\int_\Omega \boldsymbol{v}_0\,\mathrm{d}x\right|^2 - C\|\boldsymbol{f}\|^2_{L^1(Q)}\,,$$

where (*) follows from (4.6). We thus obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\boldsymbol{u}_t(t)\|^2_{L^2(\Omega)} + c\|\boldsymbol{u}_t(t)\|^2_{H^1(\Omega)}\\
&\leq C\|\boldsymbol{v}_0\|^2_{L^1(\Omega)} + C\|\boldsymbol{f}\|^2_{L^2(Q)} + \left|\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_t\,\mathrm{d}x\right| + \left|\int_\Omega a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u}_t)\,\mathrm{d}x\right|\\
&\leq \frac{c}{2}\|\boldsymbol{u}_t(t)\|^2_{H^1(\Omega)} + C\|\boldsymbol{v}_0\|^2_{L^1(\Omega)} + C'\|\boldsymbol{f}\|^2_{L^2(Q)} + \|a(\chi(t))\|_{L^\infty(\Omega)}\|\varepsilon(\boldsymbol{u}(t))\|_{L^2(\Omega)}\|\varepsilon(\boldsymbol{u}_t(t))\|_{L^2(\Omega)}
\end{aligned}$$

and, estimating $\|a(\chi)\|_{L^\infty(\Omega)} \leq C(\|\chi\|^{\rho+2}_{L^\infty(\Omega)}+1)$ via (2.19) and $\|\varepsilon(\boldsymbol{u}(t))\|_{L^2(\Omega)}$ via (4.5), we continue the above chain of inequalities with

$$\begin{aligned}
&\leq \frac{3c}{4}\|\boldsymbol{u}_t(t)\|^2_{H^1(\Omega)} + C\|\boldsymbol{v}_0\|^2_{L^1(\Omega)} + C'\|\boldsymbol{f}\|^2_{L^2(Q)}\\
&\quad + C'(\|\chi(t)\|^{\rho+2}_{L^\infty(\Omega)}+1)^2\left(2\|\varepsilon(\boldsymbol{u}_0)\|^2_{L^2(\Omega)} + 2t\int_0^t \|\varepsilon(\boldsymbol{u}_t(s))\|^2_{L^2(\Omega)}\,\mathrm{d}s\right)\,,
\end{aligned}$$

whence (4.7).

Claim 2***:*** *There exist a constant* $S_{1,2} > 0$ *and* $\overline{\beta} > 1$ *(indeed,* $\overline{\beta} = (4\rho+12)$*, with* $\rho := \max\{p,q\}$ *and* p,q *from* (2.17b) *and* (2.18)*, respectively) such that for every* $t \in [0,T]$

$$\begin{aligned}
&\|\boldsymbol{u}_t(t)\|^2_{H^2(\Omega)} + \int_0^t \|\boldsymbol{u}_t\|^2_{H^3(\Omega)}\,\mathrm{d}s\\
&\leq S_{1,2}\Big(\|\boldsymbol{v}_0\|^2_{H^2(\Omega)}+\|\boldsymbol{u}_0\|^2_{H^3(\Omega)}+\int_0^t \|\boldsymbol{f}\|^2_{H^1(\Omega)}\,\mathrm{d}s\\
&\qquad + \int_0^t \big(\|\chi\|^{\overline{\beta}}_{H^2(\Omega)}+1\big)\times\big(\|\boldsymbol{u}_t\|^2_{H^2(\Omega)}+\int_0^s \|\boldsymbol{u}_t\|^2_{H^3(\Omega)}\,\mathrm{d}\tau+1\big)\,\mathrm{d}s\Big)\,.
\end{aligned}\tag{4.8}$$

We test equation (2.10) by $\nabla\cdot(\mathbb{C}:\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t))))$ and integrate in space, thus obtaining

$$I_1 + I_2 + I_3 = I_4 \qquad \text{with} \begin{cases} I_1 = \int_\Omega \boldsymbol{u}_{tt}\,\nabla\cdot(\mathbb{C}:\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t))))\,\mathrm{d}x,\\ I_2 = -\int_\Omega \nabla\cdot(b(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}_t))\,\nabla\cdot(\mathbb{C}:\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t))))\,\mathrm{d}x,\\ I_3 = -\int_\Omega \nabla\cdot(a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}))\,\nabla\cdot(\mathbb{C}:\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t))))\,\mathrm{d}x,\\ I_4 = \int_\Omega \boldsymbol{f}\,\nabla\cdot(\mathbb{C}:\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t))))\,\mathrm{d}x. \end{cases}\tag{4.9}$$

Then, for the first term we deduce

$$\begin{aligned}
I_1 &= -\int_\Omega \nabla\boldsymbol{u}_{tt}:\mathbb{C}\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}x + \int_{\partial\Omega} \boldsymbol{u}_{tt}\otimes\boldsymbol{n}:\mathbb{C}\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}S\\
&= -\int_\Omega \mathbb{C}\varepsilon(\boldsymbol{u}_{tt}):\nabla\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t))\,\mathrm{d}x + \int_{\partial\Omega} \boldsymbol{u}_{tt}\otimes\boldsymbol{n}:\mathbb{C}\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}S
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_{tt}))\cdot\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))\,\mathrm{d}x + \int_{\partial\Omega} \boldsymbol{u}_{tt}\otimes\boldsymbol{n}{:}\mathbb{C}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}S \\
&\quad - \int_{\partial\Omega} (\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))\otimes\boldsymbol{n})\mathbb{C}\varepsilon(\boldsymbol{u}_{tt})\,\mathrm{d}S\,,
\end{aligned}
$$

integrating by parts and exploiting the symmetry of $\mathbb{C}$. The two boundary terms vanish: this will be proved rigorously for the approximate system (4.29). Thus, we may infer

$$
I_1 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t))|^2\,\mathrm{d}x\,. \tag{4.10}
$$

In order to calculate I_2, we resort to the product rule, yielding

$$
\nabla\cdot(b(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}_t)) = (\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))\nabla\chi b'(\chi) + b(\chi)\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t))\,. \tag{4.11}
$$

Therefore,

$$
\begin{aligned}
I_2 &= -\int_\Omega [(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))\nabla\chi b'(\chi) + b(\chi)\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t))]\cdot\nabla\cdot(\mathbb{C}{:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))))\,\mathrm{d}x \\
&= \int_\Omega b(\chi)\,[\mathbb{C}{:}\varepsilon\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t)){:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))]\,\mathrm{d}x \\
&\quad + \int_\Omega b'(\chi)\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t))\otimes\nabla\chi{:}\mathbb{C}{:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}x \\
&\quad + \int_\Omega \nabla\left((\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))\nabla\chi b'(\chi)\right){:}\mathbb{C}{:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}x \\
&\quad - \int_{\partial\Omega} \nabla\cdot(b(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}_t))\otimes\boldsymbol{n}{:}\mathbb{C}{:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}S \doteq I_{2,1}+I_{2,2}+I_{2,3}+I_{2,4}\,.
\end{aligned} \tag{4.12a}
$$

Now, we remark that, thanks to (2.3)

$$
I_{2,1} \geq b_0\int_\Omega (\mathbb{C}{:}\varepsilon\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t)){:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))))\,\mathrm{d}x \geq b_0\eta_{\mathbb{C}}\int_\Omega |\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))|^2\,\mathrm{d}x, \tag{4.12b}
$$

where the last inequality follows from the positive-definiteness of $\mathbb{C}$, cf. (2.1), whereas we estimate

$$
\begin{aligned}
|I_{2,2}| + |I_{2,3}| \leq{}& C|\mathbb{C}|\,\|\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\|_{L^2(\Omega)}\,\|b'(\chi)\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))\otimes\nabla\chi\|_{L^2(\Omega)} \\
&+ C\,\|\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\|_{L^2(\Omega)}\,\|b'(\chi)(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))\nabla\chi\|_{H^1(\Omega)}\,,
\end{aligned}
$$

where $|\mathbb{C}|$ denotes the tensor norm of $\mathbb{C}$. The boundary term $I_{2,4}$ vanishes again due to the homogeneous Neumann boundary conditions; again, this argument will be made rigorous for our approximation scheme, cf. the proof of Prop. 4.7 later on.

Similarly, by the chain rule and an integration-by-parts we obtain

$$
\begin{aligned}
I_3 = &-\int_\Omega [\mathbb{C}{:}\varepsilon(\boldsymbol{u})\nabla\chi a'(\chi) + a(\chi)\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}))]\cdot\nabla\cdot(\mathbb{C}{:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))))\,\mathrm{d}x \\
= &\int_\Omega a(\chi)\left(\mathbb{C}{:}\varepsilon(\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u})))\right){:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}x \\
&+\int_\Omega a'(\chi)\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}))\otimes\nabla\chi{:}\mathbb{C}{:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}x \\
&+\int_\Omega \nabla\left((\mathbb{C}{:}\varepsilon(\boldsymbol{u}))\nabla\chi a'(\chi)\right){:}\mathbb{C}{:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}x \\
&-\int_{\partial\Omega}\nabla\cdot(a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}))\otimes\boldsymbol{n}{:}\mathbb{C}{:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}S \\
\le &\frac{b_0\eta_{\mathbb{C}}}{2}\int_\Omega|\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))|^2\,\mathrm{d}x + C\|a(\chi)\|^2_{L^\infty(\Omega)}\|\mathbb{C}{:}\varepsilon(\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u})))\|^2_{L^2(\Omega)} \\
&+C\left(\|a'(\chi)\|^2_{L^\infty(\Omega)}\|\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}))\otimes\nabla\chi\|^2_{L^2(\Omega)} + \|\nabla\left((\mathbb{C}{:}\varepsilon(\boldsymbol{u}))\nabla\chi a'(\chi)\right)\|^2_{L^2(\Omega)}\right) \\
&-\int_{\partial\Omega}\nabla\cdot(a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}))\otimes\boldsymbol{n}{:}\mathbb{C}{:}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}S\,,
\end{aligned}
\tag{4.13}
$$

with b_0 and $\eta_{\mathbb{C}}$ the constants from (4.12b). For the boundary term, we observe again that it vanishes, as shown rigorously in the proof of the upcoming Proposition 4.7.

Finally, arguing in the same way as for I_1 we conclude that

$$
\begin{aligned}
I_4 = &-\int_\Omega \mathbb{C}\varepsilon(\boldsymbol{f}) : \nabla\left(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))\right)\,\mathrm{d}x + \int_{\partial\Omega}\boldsymbol{f}\otimes\boldsymbol{n}{:}\mathbb{C}\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\,\mathrm{d}S \\
\le &\frac{b_0\eta_{\mathbb{C}}}{4}\int_\Omega|\varepsilon\left(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))\right)|^2\,\mathrm{d}x + C\|\boldsymbol{f}\|^2_{H^1(\Omega)}\,.
\end{aligned}
\tag{4.14}
$$

Note, again, that the boundary term vanishes cf. the proof of Prop. 4.7.

Combining (4.9) with (4.10), (4.12), (4.1), and (4.14), we infer the estimate

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t))|^2\,\mathrm{d}x + \frac{b_0\eta_{\mathbb{C}}}{4}\int_\Omega|\varepsilon(\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t)))|^2\,\mathrm{d}x \\
&\le C\,\|\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\|_{L^2(\Omega)}\,\|b'(\chi)(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))\nabla\chi\|_{H^1(\Omega)} \\
&\quad+C\,\|\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\|_{L^2(\Omega)}\,\|b'(\chi)\|_{L^\infty(\Omega)}\,\|(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t))\otimes\nabla\chi\|_{L^2(\Omega)} \\
&\quad+C\|a(\chi)\|^2_{L^\infty(\Omega)}\|\varepsilon(\boldsymbol{u})\|^2_{H^2(\Omega)} \\
&\quad+C\left(\|a'(\chi)\|^2_{L^\infty(\Omega)}\|\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}))\otimes\nabla\chi\|^2_{L^2(\Omega)} + \|\nabla\left((\mathbb{C}{:}\varepsilon(\boldsymbol{u}))\nabla\chi a'(\chi)\right)\|^2_{L^2(\Omega)}\right) \\
&\quad+C\,\|\varepsilon(\nabla\cdot(\mathbb{C}{:}\varepsilon(\boldsymbol{u}_t)))\|_{L^2(\Omega)}\,\|a'(\chi)(\mathbb{C}{:}\varepsilon(\boldsymbol{u}))\nabla\chi\|_{H^1(\Omega)} + C\|\boldsymbol{f}\|^2_{H^1(\Omega)} \\
&\overset{(\star)}{\le} C(\|\chi\|^{2(q+1)}_{L^\infty(\Omega)}+1)\left(\|\varepsilon(\boldsymbol{u}_t)\|^2_{W^{1,3}(\Omega)}\|\nabla\chi\|^2_{L^6(\Omega)} + \|\chi\|^2_{H^2(\Omega)}\|\varepsilon(\boldsymbol{u}_t)\|^2_{L^\infty(\Omega)}\right) \\
&\quad+C(\|\chi\|^{2q}_{L^\infty(\Omega)}+1)\|\varepsilon(\boldsymbol{u}_t)\|^2_{L^\infty(\Omega)}\|\nabla\chi\|^4_{L^4(\Omega)} + C(\|\chi\|^{2(p+2)}_{L^\infty(\Omega)}+1)\|\varepsilon(\boldsymbol{u})\|^2_{H^2(\Omega)} \\
&\quad+C(\|\chi\|^{2(p+1)}_{L^\infty(\Omega)}+1)\,\left(\|\varepsilon(\boldsymbol{u})\|^2_{W^{1,3}(\Omega)}\|\nabla\chi\|^2_{L^6(\Omega)} + \|\chi\|^2_{H^2(\Omega)}\|\varepsilon(\boldsymbol{u})\|^2_{L^\infty(\Omega)}\right) \\
&\quad+C(\|\chi\|^{2p}_{L^\infty(\Omega)}+1)\,\|\varepsilon(\boldsymbol{u})\|^2_{L^\infty(\Omega)}\|\nabla\chi\|^4_{L^4(\Omega)} + C\|\boldsymbol{f}\|^2_{H^1(\Omega)}\,,
\end{aligned}
$$

where for $(\star)$ we have also used the growth conditions (2.17b) and (2.18). Integrating in time and inserting the Gagliardo–Nirenberg inequalities in three dimensions

$$\|\zeta\|_{L^\infty(\Omega)} + \|\zeta\|_{W^{1,3}(\Omega)} \leq c\|\zeta\|_{H^2(\Omega)}^{1/2}\|\zeta\|_{H^1(\Omega)}^{1/2}, \qquad \|\zeta\|_{W^{1,4}(\Omega)} \leq c\|\zeta\|_{H^2(\Omega)}^{3/4}\|\zeta\|_{H^1(\Omega)}^{1/4},$$

for all $\zeta \in H^2(\Omega)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t(t)))|^2 \,\mathrm{d}x + \frac{b_0\eta_{\mathbb{C}}}{4}\int_0^t\int_\Omega |\varepsilon(\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t)))|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\leq C\int_0^t (\|\chi\|_{L^\infty}^{2\rho+2}+1)\left(\|\varepsilon(\boldsymbol{u}_t)\|_{H^2(\Omega)}\|\varepsilon(\boldsymbol{u}_t)\|_{H^1(\Omega)} + \|\varepsilon(\boldsymbol{u})\|_{H^2(\Omega)}\|\varepsilon(\boldsymbol{u})\|_{H^1(\Omega)}\right)\|\chi\|_{H^2(\Omega)}^2 \,\mathrm{d}s \\
&\quad + C\int_0^t (\|\chi\|_{L^\infty}^{2\rho}+1)\left(\|\varepsilon(\boldsymbol{u}_t)\|_{H^2(\Omega)}\|\varepsilon(\boldsymbol{u}_t)\|_{H^1(\Omega)} + \|\varepsilon(\boldsymbol{u})\|_{H^2(\Omega)}\|\varepsilon(\boldsymbol{u})\|_{H^1(\Omega)}\right)\|\chi\|_{H^2(\Omega)}^3\|\chi\|_{H^1(\Omega)} \,\mathrm{d}s \\
&\quad + C\int_0^t \left\{\|\boldsymbol{f}\|_{H^1(\Omega)}^2 + (\|\chi\|_{L^\infty}^{2\rho+4}+1)\|\varepsilon(\boldsymbol{u})\|_{H^2(\Omega)}^2\right\}\mathrm{d}s\,, \\
&\overset{(\star\star)}{\leq} \mu\int_0^t \|\boldsymbol{u}_t\|_{H^3(\Omega)}^2 \,\mathrm{d}s + C\|\boldsymbol{v}_0\|_{H^2(\Omega)}^2 + C\int_0^t \left\{\|\boldsymbol{f}\|_{H^1(\Omega)}^2 + (\|\chi\|_{L^\infty}^{2\rho+4}+1)\|\varepsilon(\boldsymbol{u})\|_{H^2(\Omega)}^2\right\}\mathrm{d}s \\
&\quad + C\int_0^t (\|\chi\|_{L^\infty}^{4\rho+4}+1)\left(\|\chi\|_{H^2(\Omega)}^4 + \|\chi\|_{H^2(\Omega)}^8\right)\left(\|\varepsilon(\boldsymbol{u}_t)\|_{H^1(\Omega)}^2 + \|\varepsilon(\boldsymbol{u})\|_{H^1(\Omega)}^2\right)\mathrm{d}s\,,
\end{aligned} \tag{4.15}$$

for a positive constant μ to be specified later, and recalling that $\rho = \max\{p,q\}$. For $(\star\star)$ we have estimated $\|\chi\|_{H^1(\Omega)}$ via $\|\chi\|_{H^2(\Omega)}$, estimated $\|\varepsilon\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t))\|_{L^2(\Omega)}$ via $\|\boldsymbol{u}_t\|_{H^3(\Omega)}$ and $\|\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{v}_0))\|_{L^2(\Omega)}$ via $\|\boldsymbol{v}_0\|_{H^2(\Omega)}$, and used Young's inequality. Again by (4.5), we observe that

$$\|\varepsilon(\boldsymbol{u})\|_{H^j(\Omega)}^2 \leq 2\|\varepsilon(\boldsymbol{u}_0)\|_{H^j(\Omega)}^2 + 2T\int_0^t \|\varepsilon(\boldsymbol{u}_t)\|_{H^j(\Omega)}^2 \,\mathrm{d}s \qquad \text{for } j \in \{1,2\}\,.$$

We now add (4.15) and (4.7) integrated over $(0,t)$, thus obtaining

$$\begin{aligned}
&\|\boldsymbol{u}_t(t)\|_{L^2(\Omega)}^2 + \int_\Omega |\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t(t)))|^2 \,\mathrm{d}x + c\int_0^t \left(\|\boldsymbol{u}_t\|_{H^1(\Omega)}^2 + \int_\Omega |\varepsilon(\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t)))|^2 \,\mathrm{d}x\right)\mathrm{d}s \\
&\leq \mu\int_0^t \|\boldsymbol{u}_t\|_{H^3(\Omega)}^2 \,\mathrm{d}s \\
&\quad + C\left(\|\boldsymbol{v}_0\|_{H^2(\Omega)}^2 + \int_0^t (\|\chi\|_{L^\infty}^{2\rho+4}+1)\left(\|\varepsilon(\boldsymbol{u}_0)\|_{H^2(\Omega)}^2 + \int_0^s \|\varepsilon(\boldsymbol{u}_t)\|_{H^2(\Omega)}^2 \,\mathrm{d}\tau\right) + \|\boldsymbol{f}\|_{H^1(\Omega)}^2 \,\mathrm{d}s\right) \\
&\quad + C\int_0^t (\|\chi\|_{L^\infty(\Omega)}^{4\rho+4}+1)(\|\chi\|_{H^2(\Omega)}^8+1)\left(\|\varepsilon(\boldsymbol{u}_t)\|_{H^1(\Omega)}^2 + \int_0^s \|\varepsilon(\boldsymbol{u}_t)\|_{H^1(\Omega)}^2 \,\mathrm{d}\tau + \|\varepsilon(\boldsymbol{u}_0)\|_{H^1(\Omega)}^2\right)\mathrm{d}s\,.
\end{aligned} \tag{4.16}$$

Now, it follows from the elliptic regularity estimates (A.4) from Corollary A.3 that there exists $\widehat{C}_{\mathrm{ER}} > 0$ such that

$$\|\boldsymbol{u}_t(t)\|_{H^2(\Omega)}^2 \leq \widehat{C}_{\mathrm{ER}}\left(\|\boldsymbol{u}_t(t)\|_{L^2(\Omega)}^2 + \int_\Omega |\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t(t)))|^2 \,\mathrm{d}x\right).$$

Likewise, we have

$$\|\boldsymbol{u}_t(t)\|_{H^3(\Omega)}^2 \leq \widehat{C}_{\mathrm{ER}}\left(\|\boldsymbol{u}_t(t)\|_{H^1(\Omega)}^2 + \int_\Omega |\varepsilon(\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{u}_t)))|^2 \,\mathrm{d}x\right).$$

Hence, we choose μ in (4.16) in such a way as to absorb $\int_0^t \|\boldsymbol{u}_t\|^2_{H^3(\Omega)}\,\mathrm{d}s$ into the left-hand side, thus obtaining

$$\begin{aligned}
&\|\boldsymbol{u}_t(t)\|_{H^2(\Omega)} + c\int_0^t \|\boldsymbol{u}_t\|^2_{H^3(\Omega)}\,\mathrm{d}s\\
&\leq C\left(\|\boldsymbol{v}_0\|^2_{H^2(\Omega)} + \int_0^t (\|\chi\|^{2\rho+4}_{L^\infty(\Omega)}+1)\left(\|\varepsilon(\boldsymbol{u}_0)\|^2_{H^2(\Omega)} + \int_0^s \|\varepsilon(\boldsymbol{u}_t)\|^2_{H^2(\Omega)}\,\mathrm{d}\tau\right) + \|\boldsymbol{f}\|^2_{H^1(\Omega)}\,\mathrm{d}s\right)\\
&\quad + C\int_0^t (\|\chi\|^{4\rho+4}_{L^\infty(\Omega)}+1)(\|\chi\|^8_{H^2(\Omega)}+1)\left(\|\varepsilon(\boldsymbol{u}_t)\|^2_{H^1(\Omega)} + \int_0^s \|\varepsilon(\boldsymbol{u}_t)\|^2_{H^1(\Omega)}\,\mathrm{d}\tau + \|\varepsilon(\boldsymbol{u}_0)\|^2_{H^1(\Omega)}\right)\mathrm{d}s\,.
\end{aligned}$$

Therefore, estimating

$$\begin{cases}\|\chi\|_{L^\infty(\Omega)} \leq C\|\chi\|_{H^2(\Omega)}\\ \|\varepsilon(\boldsymbol{u}_t)\|_{H^2(\Omega)} \leq C\|\boldsymbol{u}_t\|_{H^3(\Omega)},\end{cases} \tag{4.17}$$

we arrive at (4.8).

Claim 3***:*** *there exist a constant* $S_{1,3} > 0$ *and* $\underline{\beta} > 1$ *such that*

$$\begin{aligned}
&\|\omega(t)\|^2_{L^2(\Omega)} + \|\chi(t)\|^2_{H^2(\Omega)} + \int_0^t \left(\|\chi_t\|^2_{L^2(\Omega)} + \|\nabla\chi_t\|^2_{L^2(\Omega)}\right)\mathrm{d}s\\
&\leq S_{1,3}(1+\|\chi_0\|^2_{H^2(\Omega)} + \||\partial\breve{W}^\circ|(\chi_0)\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_0\|^8_{H^2(\Omega)})\\
&\quad + S_{1,3}\int_0^t \left(\|\omega\|^8_{L^2(\Omega)} + \|\boldsymbol{u}_t\|^8_{H^2(\Omega)} + \int_0^s \|\boldsymbol{u}_t\|^8_{H^2(\Omega)}\,\mathrm{d}\tau + \|\chi\|^{\underline{\beta}}_{H^2(\Omega)}\right)\mathrm{d}s\,.
\end{aligned} \tag{4.18}$$

We test (4.1) by ω_t and integrate in space and over the time interval $(0,t)$, $t \in (0,T)$. Thus, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\omega(t)\|^2_{L^2(\Omega)} + \int_0^t \left(\|\chi_t\|^2_{L^2(\Omega)} + \|\nabla\chi_t\|^2_{L^2(\Omega)}\right)\mathrm{d}s\\
&\quad + \underbrace{\int_0^t\int_\Omega \{\breve{W}''(\chi)|\chi_t|^2 + I'(\chi_t)\chi_t + I''(\chi_t)|\nabla\chi_t|^2 + \breve{W}''(\chi)I'(\chi_t)\chi_t\}\,\mathrm{d}x\,\mathrm{d}s}_{I_0\,\geq\,0}\\
&= \int_0^t\int_\Omega \left(\chi - \hat{W}'(\chi) - \frac{1}{2}a'(\chi)\varepsilon(\boldsymbol{u})\mathbb{C}\varepsilon(\boldsymbol{u})\right)\omega_t\,\mathrm{d}x\,\mathrm{d}s \doteq I_1\,.
\end{aligned} \tag{4.19}$$

Indeed, by the convexity of $\breve{W}$ and $I_{(-\infty,0]}$, the first and third contributions to I_0 are non-negative; likewise, the monotonicity of I' and the fact that $I'(0) = 0$ ensure that the second and fourth term in I_0 is positive. We integrate I_1 by parts, thus obtaining

$$\begin{aligned}
I_1 = &\int_0^t \omega\left(\frac{1}{2}a''(\chi)\chi_t\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u}) + a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u}_t) + \hat{W}''(\chi)\chi_t - \chi_t\right)\mathrm{d}x\,\mathrm{d}s\\
&- \int_\Omega \omega(t)\left(\frac{1}{2}a'(\chi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)){:}\varepsilon(\boldsymbol{u}(t)) + \hat{W}'(\chi(t)) - \chi(t)\right)\mathrm{d}x\\
&+ \int_\Omega \omega(0)\left(\frac{1}{2}a'(\chi_0)\mathbb{C}\varepsilon(\boldsymbol{u}_0){:}\varepsilon(\boldsymbol{u}_0) + \hat{W}'(\chi_0) - \chi_0\right)\mathrm{d}x \doteq I_{1,1} + I_{1,2} + I_{1,3}\,.
\end{aligned} \tag{4.20a}$$

By Hölder's and Young's inequalities we have

$$
\begin{aligned}
|I_{1,1}| &\le \frac{1}{2}|\mathbb{C}| \int_0^t \|\omega\|_{L^2(\Omega)} \|a''(\chi)\|_{L^\infty(\Omega)} \|\chi_t\|_{L^6(\Omega)} \|\varepsilon(\boldsymbol{u})\|^2_{L^6(\Omega)} \,\mathrm{d}s \\
&\quad + \int_0^t \|\omega\|_{L^2(\Omega)} \left(\frac{1}{2}|\mathbb{C}| \|a'(\chi)\|_{L^\infty(\Omega)} \|\varepsilon(\boldsymbol{u})\|_{L^4(\Omega)} \|\varepsilon(\boldsymbol{u}_t)\|_{L^4(\Omega)} + \|\hat{W}''(\chi) - 1\|_{L^\infty(\Omega)} \|\chi_t\|_{L^2(\Omega)} \right) \mathrm{d}s \\
&\overset{(1)}{\le} \frac{1}{4} \int_0^t \|\chi_t\|^2_{H^1(\Omega)} \,\mathrm{d}s + C \int_0^t \|\omega\|^2_{L^2(\Omega)} \,\mathrm{d}s + C \int_0^t \|\omega\|^2_{L^2(\Omega)} (\|\chi\|^{2p}_{L^\infty(\Omega)} + 1) \|\varepsilon(\boldsymbol{u})\|^4_{L^6(\Omega)} \,\mathrm{d}s \\
&\qquad + C \int_0^t (\|\chi\|^{2p+2}_{L^\infty(\Omega)} + 1) \|\varepsilon(\boldsymbol{u})\|^2_{L^4(\Omega)} \|\varepsilon(\boldsymbol{u}_t)\|^2_{L^4(\Omega)} \,\mathrm{d}s + \frac{1}{4} \int_0^t \|\chi_t\|^2_{L^2(\Omega)} \,\mathrm{d}s \\
&\overset{(2)}{\le} \frac{1}{2} \int_0^t \|\chi_t\|^2_{H^1(\Omega)} \,\mathrm{d}s + \int_0^t \|\omega\|^2_{L^2(\Omega)} \,\mathrm{d}s \\
&\qquad + C \int_0^t \left(\|\omega\|^8_{L^2(\Omega)} + \|\varepsilon(\boldsymbol{u}_t)\|^8_{L^4(\Omega)} + \|\varepsilon(\boldsymbol{u})\|^8_{L^6(\Omega)} + \|\chi\|^m_{L^\infty(\Omega)} \right) \mathrm{d}s \,,
\end{aligned}
\tag{4.20b}
$$

where for (1) we have resorted to the growth properties (2.17b) and (2.19) of a'' and a', and used that $\hat{W}''(\chi) \equiv -\ell$. Moreover, (2) again follows from Young's inequality; therein, $m = \max\{8p, 4p+4\} = 8p$.

Secondly, we observe via Young's inequality that

$$
|I_{1,2}| \le \frac{1}{4} \|\omega(t)\|^2_{L^2(\Omega)} + \|\hat{W}'(\chi(t)) + \chi(t)\|^2_{L^2(\Omega)} + C \|\varepsilon(\boldsymbol{u}(t))\|^4_{L^4(\Omega)} \,.
$$

From (4.5) we gather $\|\varepsilon(\boldsymbol{u}(t))\|^4_{L^4(\Omega)} \le \|\boldsymbol{u}_0\|^4_{H^2(\Omega)} + C\int_0^t \|\varepsilon(\boldsymbol{u}_t)\|^4_{L^4(\Omega)} \,\mathrm{d}s$. Recalling that $\hat{W}'(\chi) = -\ell\chi$, we find

$$
\begin{aligned}
\|\hat{W}'(\chi(t)) - \chi(t)\|^2_{L^2(\Omega)} \le (\ell+1)^2 \|\chi(t)\|^2_{L^2(\Omega)} &\le 2(\ell+1)^2 \left(\|\chi_0\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \chi_t \chi \,\mathrm{d}x \,\mathrm{d}s \right) \\
&\le \frac{1}{4} \int_0^t \|\chi_t\|^2_{L^2(\Omega)} \,\mathrm{d}s + C \|\chi_0\|^2_{L^2(\Omega)} + C \int_0^t \|\chi\|^2_{L^2(\Omega)} \,.
\end{aligned}
$$

All in all, we conclude

$$
\begin{aligned}
|I_{1,2}| &\le \frac{1}{4} \left(\|\omega(t)\|^2_{L^2(\Omega)} + \int_0^t \|\chi_t\|^2_{L^2(\Omega)} \,\mathrm{d}s \right) \\
&\qquad + C \left(\|\chi_0\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_0\|^4_{H^2(\Omega)} + \int_0^t \|\varepsilon(\boldsymbol{u}_t)\|^4_{L^4(\Omega)} \,\mathrm{d}s + \int_0^t \|\chi\|^2_{L^2(\Omega)} \right) .
\end{aligned}
\tag{4.20c}
$$

Finally, we have

$$
|I_{1,3}| \le \frac{1}{4} \|\omega(0)\|^2_{L^2(\Omega)} + C \left(\|\boldsymbol{u}_0\|^4_{H^2(\Omega)} + \|\hat{W}'(\chi_0)\|^2_{L^2(\Omega)} + \|\chi_0\|^2_{L^2(\Omega)} \right) \le C. \tag{4.20d}
$$

Combining (4.19) with (4.20), and again using that $\|\chi\|_{L^\infty(\Omega)} \le C\|\chi\|_{H^2(\Omega)}$ we obtain

$$
\begin{aligned}
&\|\omega(t)\|^2_{L^2(\Omega)} + \int_0^t \left(\|\chi_t\|^2_{L^2(\Omega)} + \|\nabla \chi_t\|^2_{L^2(\Omega)} \right) \mathrm{d}s \\
&\le C(1 + \|\chi_0\|^2_{L^2(\Omega)} + \|\omega(0)\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_0\|^4_{H^2(\Omega)}) \\
&\quad + C \int_0^t \left(\|\omega\|^8_{L^2(\Omega)} + \|\varepsilon(\boldsymbol{u}_t)\|^8_{L^4(\Omega)} + \|\varepsilon(\boldsymbol{u})\|^8_{L^6(\Omega)} + \|\chi\|^{8p}_{H^2(\Omega)} \right) \mathrm{d}s \,.
\end{aligned}
\tag{4.21}
$$

By (4.5) we have $\|\varepsilon(\boldsymbol{u}(t))\|_{L^6(\Omega)}^8 \leq 2^7\|\varepsilon(\boldsymbol{u}_0)\|_{L^6(\Omega)}^8 + 2^7T^7\int_0^t \|\varepsilon(\boldsymbol{u}_t)\|_{L^6(\Omega)}^8 \,\mathrm{d}s$. Furthermore, we use that $\|\varepsilon(\boldsymbol{u}_t)\|_{H^1(\Omega)} \leq C\|\boldsymbol{u}_t\|_{H^2(\Omega)}$. In order to conclude (4.18), it remains to observe that, by (2.20c),

$$\|\omega(0)\|_{L^2(\Omega)} \leq \|\chi_0\|_{H^2(\Omega)} + \||\partial\breve{W}^\circ|(\chi_0)\|_{L^2(\Omega)} + \|\chi_0\|_{L^2(\Omega)}\,, \tag{4.22}$$

and to remark that the L^2-norm of ω does bound the H^2-norm of χ, cf. (4.2a). All in all, we arrive at (4.18) with $\underline{\beta} = 8p$.

Claim 4***:*** *there exists a constant $S_{1,4} > 0$ such that for $\beta := \max\{\overline{\beta}, \frac{1}{2}\underline{\beta}\}$ there holds*

$$\begin{aligned}
&\|\boldsymbol{u}_t(t)\|_{H^2(\Omega)}^2 + \int_0^t \|\boldsymbol{u}_t\|_{H^3(\Omega)}^2 \,\mathrm{d}s + \|\omega(t)\|_{L^2(\Omega)}^2 + \|\chi(t)\|_{H^2(\Omega)}^2 + \int_0^t \|\chi_t\|_{H^1(\Omega)}^2 \,\mathrm{d}s \\
&\leq S_{1,4}\Big(1+\|\boldsymbol{v}_0\|_{H^2(\Omega)}^2+\|\boldsymbol{u}_0\|_{H^3(\Omega)}^2+\|\chi_0\|_{H^2(\Omega)}^2+\||\partial\breve{W}^\circ|(\chi_0)\|_{L^2(\Omega)}^2+\|\boldsymbol{u}_0\|_{H^2(\Omega)}^8+\int_0^t \|\boldsymbol{f}\|_{H^1(\Omega)}^2 \,\mathrm{d}s\Big) \\
&\quad + S_{1,4}\int_0^t \left\{\|\chi\|_{H^2(\Omega)}^{2\beta}+\|\boldsymbol{u}_t\|_{H^2(\Omega)}^8+\left(\int_0^s \|\boldsymbol{u}_t\|_{H^3(\Omega)}^2 \,\mathrm{d}\tau\right)^2+\|\omega\|_{L^2(\Omega)}^8\right\} \mathrm{d}s\,.
\end{aligned} \tag{4.23}$$

It suffices to add estimates (4.8) and (4.18): as for the left-hand side of (4.8), we use that

$$\begin{aligned}
&\int_0^t \big(\|\chi\|_{H^2(\Omega)}^{\overline{\beta}}+1\big)\times\Big(\|\boldsymbol{u}_t\|_{H^2(\Omega)}^2+\int_0^s \|\boldsymbol{u}_t\|_{H^3(\Omega)}^2 \,\mathrm{d}\tau+1\big)\Big) \,\mathrm{d}s \\
&\leq C\left(T+\int_0^t \left\{\|\chi\|_{H^2(\Omega)}^{2\overline{\beta}}+\|\boldsymbol{u}_t\|_{H^2(\Omega)}^4+\left(\int_0^s \|\boldsymbol{u}_t\|_{H^3(\Omega)}^2 \,\mathrm{d}\tau\right)^2\right\} \mathrm{d}s\right)\,,
\end{aligned}$$

whereas we trivially observe that

$$\int_0^t \int_0^s \|\boldsymbol{u}_t\|_{H^2(\Omega)}^8 \,\mathrm{d}\tau \,\mathrm{d}s \leq T\int_0^t \|\boldsymbol{u}_t\|_{H^2(\Omega)}^8 \,\mathrm{d}s$$

for the corresponding term on the right-hand side of (4.18). Then, (4.23) ensues.

Conclusion of the proof:

With the choice

$$\psi(t) := \|\boldsymbol{u}_t(t)\|_{H^2(\Omega)}^2 + \int_0^t \|\boldsymbol{u}_t\|_{H^3(\Omega)}^2 \,\mathrm{d}s + \|\omega(t)\|_{L^2(\Omega)}^2 + \|\chi(t)\|_{H^2(\Omega)}^2 + 1,$$

we observe that for $\beta > 4$, the estimate (4.23) yields

$$\psi(t) \leq \widetilde{S}_1\psi(0) + \int_0^t \widetilde{S}_1\psi(s)^\beta \,\mathrm{d}s \qquad \text{for all } t \in [0,T^*] \tag{4.24}$$

for some suitable constant $\widetilde{S}_1$ also encompassing $\|\boldsymbol{u}_0\|_{H^2(\Omega)}^8$, $\|\boldsymbol{u}_0\|_{H^3(\Omega)}^2$, $\|\boldsymbol{v}_0\|_{H^2(\Omega)}^2$, $\||\partial\breve{W}^\circ|(\chi_0)\|_{L^2(\Omega)}^2$, $\|\chi_0\|_{H^2(\Omega)}^2$, and $\int_0^T \|\boldsymbol{f}\|_{H^1(\Omega)}^2 \,\mathrm{d}s$. Let us define $\phi(t) := \widetilde{S}_1\psi(0) + \int_0^t \widetilde{S}_1\psi(s)^\beta \,\mathrm{d}s$. Then, taking the inequality (4.23) to the power β one has

$$\phi'(t) = \psi^\beta(t) \leq \left(\widetilde{S}_1\psi(0) + \int_0^t \widetilde{S}_1\psi(s)^\beta \,\mathrm{d}s\right)^\beta = \phi^\beta(t) \qquad \text{for all } t \in [0,T^*]\,.$$

Via the usual comparison arguments for ODEs, from $\phi' \leq \phi^\beta$ we conclude that

$$\psi(t) \leq \phi(t) \leq \left(\frac{1}{\phi^{1-\beta}(0)-(\beta-1)t}\right)^{1/(\beta-1)} \qquad \text{for all } t < \frac{1}{\beta-1}\phi^{1-\beta}(0) = \frac{1}{\beta-1}\psi^{1-\beta}(0)\,.$$

Therefore, we may conclude, e.g., that

$$\psi(t) \leq \frac{1}{2^{\beta-1}}\phi(0) = \frac{\widetilde{S}_1}{2^{\beta-1}}\psi(0) \qquad \text{for } t \in \left(0, \frac{1}{2(\beta-1)}\psi^{1-\beta}(0)\right]\,.$$

In this way, we conclude estimate (4.3).

Eventually, (4.4) follows from (4.3), arguing by comparison in the momentum balance. This concludes the proof. □

The following sections will be devoted to the rigorous justification of Proposition 4.2.

4.2 Regularization and Galerkin approximation

We will approximate system (1.1) by

1 Regularizing the possibly *nonsmooth* (cf. Hypothesis C) convex contribution $\breve{W}$ to W, in order to rigorously carry out the estimates leading to Claim 3 in the proof of Prop. 4.2. In fact, we will need to replace $\breve{W}$ by a regularised version $W_\delta \in \mathrm{C}^3(\mathbb{R})$, $\delta \in (0,1)$, such that

$$\begin{cases} 0 \leq \breve{W}_\delta''(r) \leq \frac{1}{\delta} \\ |W_\delta'''(r)| \leq \frac{C}{\delta^3} \end{cases} \qquad \text{for all } r \in \mathbb{R} \text{ and } \lim_{\delta\to 0} \breve{W}_\delta(r) = \breve{W}(r) \text{ for all } r \in \mathrm{dom}\,\breve{W}\,. \tag{4.25}$$

Likewise, we will replace the the indicator function $I_{(-\infty,0]}$ by its *smoothened* Moreau-Yosida approximation I_δ.

2 Adding an elliptic time-regularizing term to the damage flow rule, tuned by a second parameter $\nu > 0$ that will need to scale suitably w.r.t. δ, cf. (4.41) below.

3 Adopting a Galerkin discretization for the momentum balance, consisting of eigenfunctions of a self-adoint operator, see below.

In order to obtain a smooth approximation $\breve{W}_\delta$ of $\breve{W}$ and I_δ of $I_{(-\infty,0]}$, we shall apply the construction detailed in Section B ahead, and based on the results in [18, Sec. 3], to the operators $\beta = \partial\breve{W}$ and $\beta = \partial I_{(-\infty,0]}$. Let us now delve into the Galerkin discretization of the momentum balance.

Galerkin approximation

We are going to use a Galerkin scheme to discretize the elasticity subsystem in space. Hereafter, we will use the notation

$$L^2(\Omega)_{/\mathbb{R}} := \{\boldsymbol{v} \in L^2(\Omega;\mathbb{R}^d) \, : \, \int_\Omega \boldsymbol{v}\,\mathrm{d}x = 0\}, \qquad H^1(\Omega)_{/\mathbb{R}} = H^1(\Omega;\mathbb{R}^d)\cap L^2(\Omega)_{/\mathbb{R}}\,.$$

For the approximation of the elasticity equation, we use an $L^2(\Omega)$-orthonormal Galerkin basis consisting of eigenfunctions $\boldsymbol{y}_1,\, \boldsymbol{y}_2,\, \ldots$ of the differential operator corresponding to the boundary value problem

$$-\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{y})) = \boldsymbol{h} \quad \text{in } \Omega\,, \quad \int_\Omega \boldsymbol{y}\,\mathrm{d}x = 0\,, \quad \boldsymbol{n}\cdot\mathbb{C}\varepsilon(\boldsymbol{y}) = 0 \quad \text{on } \partial\Omega\,. \tag{4.26}$$

The above problem is a symmetric strongly elliptic system that possesses, by the Lax-Milgram lemma, a unique weak solution $\boldsymbol{y} \in H^1(\Omega)_{/\mathbb{R}}$ for any $\boldsymbol{h} \in (H^1(\Omega)_{/\mathbb{R}})^*$. Its solution operator is thus a compact selfadjoint operator in $L^2(\Omega)_{/\mathbb{R}}$. Hence there exists an orthogonal basis of eigenfunctions $\boldsymbol{y}_1,\, \boldsymbol{y}_2,\, \ldots$ in $L^2(\Omega)_{/\mathbb{R}}$. The regularity result of Proposition A.1 ahead ensures that the eigenfunctions $\boldsymbol{y}_1,\, \boldsymbol{y}_2,\, \ldots$ are, indeed, in $H^3(\Omega;\mathbb{R}^d)$. Therefore, the space spanned by them, and by $\boldsymbol{y}_0 \equiv \mathbf{1}$,

$$V_n := \mathrm{span}\,\{\mathbf{1}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\} \subset H^3(\Omega;\mathbb{R}^d)\,.$$

We will need to consider both the orthogonal projections

$$\mathbb{P}^n_{H^3} : H^3(\Omega;\mathbb{R}^d) \longrightarrow V_n \quad \text{and} \quad \mathbb{P}^n_{H^2} : H^2(\Omega;\mathbb{R}^d) \longrightarrow V_n\,.$$

. With slight abuse, we will drop the subscript H^k, $k \in \{2,3\}$, in their notation.

The approximate system

Combining the regularization for the damage model with the Galerkin-discretization for the elasticity equations, we end up with the regularized–discretized system

$$\int_\Omega \boldsymbol{u}_{tt}\cdot\boldsymbol{z} + (b(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}_t) + a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u})) : \varepsilon(\boldsymbol{z})\,\mathrm{d}x \tag{4.27a}$$

$$= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{z}\,\mathrm{d}x \qquad \text{for all } \boldsymbol{z} \in V^n, \text{ a.e. in } (0,T), \tag{4.27b}$$

$$\nu\omega_{tt} + \omega + \chi_t + I'_\delta(\chi_t) + \tfrac{1}{2}a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u}) + \hat{W}'(\chi) - \chi = 0 \quad \text{a.e. in } Q, \tag{4.27c}$$

$$-\Delta\chi + \breve{W}'_\delta(\chi) + \chi = \omega \quad \text{a.e. in } Q\,, \tag{4.27d}$$

$$\partial_{\boldsymbol{n}}\chi = 0 \quad \text{a.e. on } \Sigma\,. \tag{4.27e}$$

4.3 Existence for the regularized approximate system

First of all, let us show that the Cauchy problem for system (4.27), supplemented with the initial data $(\mathbb{P}^n(\boldsymbol{u}_0), \mathbb{P}^n(\boldsymbol{v}_0), \chi_0)$ and with an additional initial datum for ω_t, does admit a local-in-time *strong* solution (here ‘strong’ refers to the fact that (4.27c)–(4.27e) are satisfied pointwise).

Proposition 4.3. *Let $(\boldsymbol{u}_0, \boldsymbol{v}_0, \chi_0) \in H^3(\Omega;\mathbb{R}^d) \times H^2(\Omega;\mathbb{R}^d) \times H^2(\Omega)$ fulfill Hyp.* ***D****. Let $\varpi_0 \in L^2(\Omega)$ be given.*

For every $\delta,\, \nu > 0$ there exists $\widetilde{T} = \widetilde{T}(\delta,\nu) \in (0,T]$ such that for every $n \in \mathbb{N}$ system (4.27) *admits a solution $(\boldsymbol{u},\chi)$ with the regularity*

$$\begin{aligned} &\boldsymbol{u} \in H^1(0,\widetilde{T};H^3(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,\widetilde{T};H^2(\Omega;\mathbb{R}^d)) \cap H^2(0,\widetilde{T};H^1(\Omega;\mathbb{R}^d)),\\ &\chi \in W^{1,\infty}(0,\widetilde{T};H^2(\Omega)) \cap W^{2,\infty}(0,\widetilde{T};H^1(\Omega))\,, \end{aligned} \tag{4.28}$$

satisfying the initial conditions

$$\begin{cases} \boldsymbol{u}(0) = \mathbb{P}^n(\boldsymbol{u}_0), \\ \boldsymbol{v}(0) = \mathbb{P}^n(\boldsymbol{v}_0), \\ \chi(0) = \chi_0, \\ \omega'(0) = \varpi_0 \end{cases} \qquad \text{a.e. in } \Omega\,.$$

In fact, as a consequence of the a priori estimates from Proposition 4.7, we will improve the above local existence result and show that the final time $\widetilde{T}$ neither depends on δ nor on ν.

In order to prove the existence of solutions for the discretized-regularized system, we will apply Schauder's fixed-point argument. More precisely, for fixed $\bar{\chi} \in L^\infty(Q)$ we will solve the Cauchy problem

$$\int_\Omega \boldsymbol{v}_t \cdot \boldsymbol{z} + (b(\bar{\chi})\mathbb{C}\varepsilon(\boldsymbol{u}_t) + a(\bar{\chi})\mathbb{C}\varepsilon(\boldsymbol{u})) : \varepsilon(\boldsymbol{z})\,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{z}\,\mathrm{d}x \tag{4.29a}$$

$$\text{for all } \boldsymbol{z} \in V^n, \qquad \text{a.e. in } (0,T),$$

$$\boldsymbol{u}_t = \boldsymbol{v} \qquad \text{a.e. in } (0,T), \tag{4.29b}$$

$$\nu\omega_{tt} + \omega + \chi_t + I_\delta'(\chi_t) + \tfrac{1}{2}a'(\bar{\chi})\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u}) + \widehat{W}'(\bar{\chi}) - \bar{\chi} = 0 \qquad \text{a.e. in } Q, \tag{4.29c}$$

$$-\Delta\chi + \breve{W}_\delta'(\chi) + \chi = \omega \qquad \text{a.e. in } Q\,, \tag{4.29d}$$

$$\partial_{\boldsymbol{n}}\chi = 0 \qquad \text{a.e. on } \Sigma\,, \tag{4.29e}$$

and prove that the solution operator $\bar{\chi} \mapsto \chi$ admits a fixed point as soon as $(x,t) \mapsto \bar{\chi}(x,t)$ is defined on a cylinder $\Omega\times(0,\widetilde{T})$ with sufficiently small $\widetilde{T}$.

The fixed point argument: solving the momentum balance

Firstly, we solve the discretized momentum balance (4.29a)–(4.29b) for fixed $\bar{\chi} \in L^\infty(Q)$. For notational simplicity, we will consider as a solution operator the mapping $\bar{\chi} \mapsto \boldsymbol{u}$, disregarding the solution component $\boldsymbol{v}$.

Lemma 4.4. *Let $\bar{\chi} \in L^\infty(Q)$ be fixed. For every pair $(\boldsymbol{u}_0, \boldsymbol{v}_0) \in H^3(\Omega;\mathbb{R}^d)\times H^2(\Omega;\mathbb{R}^d)$ fulfilling* (2.20b) *there exists a unique solution*

$$(\boldsymbol{u}, \boldsymbol{v}) \in H^1(0,T;V^n\times V^n) \tag{4.30}$$

to the Cauchy problem for system (4.29a)–(4.29b)*, supplemented with the initial conditions*

$$(\boldsymbol{u}(0), \boldsymbol{v}(0)) = (\mathbb{P}^n(\boldsymbol{u}_0), \mathbb{P}^n(\boldsymbol{v}_0)).$$

Moreover, there exists a function $\zeta_{\boldsymbol{u}} : [0,\infty)^4 \to [0,\infty)$, monotonously increasing w.r.t. all of its arguments, such that

$$\begin{aligned} &\|\boldsymbol{u}\|_{L^\infty(0,T;V^n)} + \|\boldsymbol{v}\|_{L^\infty(0,T;V^n)} + \|\boldsymbol{v}_t\|_{L^2(0,T;V^n)} \\ &\leq \zeta_{\boldsymbol{u}}\Big(\|\boldsymbol{f}\|_{L^2(0,T;H^1(\Omega))}, \|\bar{\chi}\|_{L^\infty(Q)}, \|\boldsymbol{u}_0\|_{H^3(\Omega)}, \|\boldsymbol{v}_0\|_{H^2(\Omega)}\Big)\,, \end{aligned} \tag{4.31}$$

and the solution operator $\mathcal{T}_{\boldsymbol{u}} : L^\infty(Q) \to H^1(0,T;V^n)$ defined by $\bar{\chi} \mapsto \boldsymbol{u}$, is continuous.

Proof. A classical existence theorem (see [19, Chapter I, Theorem 5.2]) ensures that, for every $n \in \mathbb{N}$, there exists a time T_n^* such that there exists a (unique) maximal solution $(\boldsymbol{u}, \boldsymbol{v})$, in the sense of Carathéodory, to the Cauchy problem for (4.29a)–(4.29b) with

$$(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{AC}([0,\tau]; V^n\times V^n) \qquad \text{for all } 0 < \tau < T_n^*.$$

With straightforward arguments, based on the norm-equivalence of all finite-dimensional norms, we obtain that

$$\begin{aligned}&\|\boldsymbol{u}\|_{L^\infty(0,T_n^*;V^n)} + \|\boldsymbol{v}\|_{L^\infty(0,T_n^*;V^n)} + \|\boldsymbol{v}_t\|_{L^2(0,T_n^*;V^n)}\\ &\le \zeta_{\boldsymbol{u}}\Big(\|\boldsymbol{f}\|_{L^2(0,T;H^1(\Omega)}, \|\bar\chi\|_{L^\infty(Q)}, \|\mathbb{P}^n(\boldsymbol{u}_0)\|_{V^n}, \|\mathbb{P}^n(\boldsymbol{v}_0)\|_{V^n}\Big)\,.\end{aligned}$$

Since

$$\|\mathbb{P}^n(\boldsymbol{u}_0)\|_{V^n} \le c\|\boldsymbol{u}_0\|_{H^3(\Omega)}, \qquad \|\mathbb{P}^n(\boldsymbol{v}_0)\|_{V^n} \le c\|\boldsymbol{v}_0\|_{H^2(\Omega)}, \tag{4.32}$$

the right-hand side in the above estimate does not depend on n and thus the pair $(\boldsymbol{u}_n, \boldsymbol{v}_n)$ extends to the whole interval $[0,T]$. Estimate (4.31) is then a consequence of (4.32) and of the monotonicity of the function $\zeta_{\boldsymbol{u}}$.

The continuity of the solution operator follows from estimate (4.33) below. To prove it, we consider system (4.29a)–(4.29b), corresponding to two given functions $\bar\chi_1$, $\bar\chi_2$, subtract (4.29a) with $\bar\chi = \bar\chi_2$ from (4.29a) for $\bar\chi = \bar\chi_1$, and test the obtained relation by $\widehat{\boldsymbol{v}} := \boldsymbol{v}_1 - \boldsymbol{v}_2 = \partial_t\widehat{\boldsymbol{u}}$, with $\widehat{\boldsymbol{u}} := \boldsymbol{u}_1 - \boldsymbol{u}_2$. Integrating in time and taking into account that $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0)$ and $\boldsymbol{v}_1(0) = \boldsymbol{v}_2(0)$ we obtain for all $t \in [0,T]$

$$\begin{aligned}&\int_\Omega \tfrac12 |\widehat{\boldsymbol{v}}(t)|^2\,\mathrm{d}x + b_0\eta_{\mathbb{C}}\int_0^t\int_\Omega |\varepsilon(\widehat{\boldsymbol{v}})|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\le \int_0^t |\mathbb{C}|\Big\{\|\varepsilon(\boldsymbol{v}_2)\|_{L^2(\Omega)}\|\varepsilon(\widehat{\boldsymbol{v}})\|_{L^2(\Omega)}\|b(\bar\chi_1) - b(\bar\chi_2)\|_{L^\infty(\Omega)}\\
&\qquad + \|a(\bar\chi_1)\|_{L^\infty(\Omega)}\|\varepsilon(\widehat{\boldsymbol{u}})\|_{L^2(\Omega)}\|\varepsilon(\widehat{\boldsymbol{v}})\|_{L^2(\Omega)}\\
&\qquad + \|\varepsilon(\boldsymbol{u}_2)\|_{L^2(\Omega)}\|\varepsilon(\widehat{\boldsymbol{v}})\|_{L^2(\Omega)}\|a(\bar\chi_1) - a(\bar\chi_2)\|_{L^\infty(\Omega)}\Big\}\,\mathrm{d}s\\
&\overset{(1)}{\le} \frac{b_0\eta_{\mathbb{C}}}{2}\|\varepsilon(\widehat{\boldsymbol{v}})\|^2_{L^2(\Omega)} + K_1 T\int_0^t s\left(\int_0^s \|\varepsilon(\widehat{\boldsymbol{v}})\|^2_{L^2(\Omega)}\,\mathrm{d}r\right)\mathrm{d}s + K_2\|\bar\chi_1 - \bar\chi_2\|^2_{L^\infty(\Omega\times(0,T))}\,,\end{aligned}$$

where (1) follows from Young's inequality and from estimating $\|\varepsilon(\widehat{\boldsymbol{u}}(s))\|^2_{L^2(\Omega)} \le s\int_0^s \|\varepsilon(\widehat{\boldsymbol{v}})\|^2_{L^2(\Omega)}\,\mathrm{d}r$. The constant K_2 depends on $|\mathbb{C}|$, on $\max_{|r|\le M}(|a'(r)| + |b'(r)|)$ (with $M = \|\bar\chi_1\|_{L^\infty(Q)} + \|\bar\chi_2\|_{L^\infty(Q)}$), and on $\sup_{t\in[0,T]}\big(\|\varepsilon(\boldsymbol{u}_2(t))\|_{L^2(\Omega)} + \|\varepsilon(\boldsymbol{v}_2(t))\|_{L^2(\Omega)}\big)$, cf. (4.31). Likewise, the constant K_1 also depends on $\|\bar\chi_1\|_{L^\infty(\Omega\times(0,T))}$. All in all, with the Gronwall Lemma we conclude that

$$\int_0^t \|\varepsilon(\widehat{\boldsymbol{v}})\|^2_{L^2(\Omega)}\,\mathrm{d}s \le \kappa_2\|\bar\chi_1 - \bar\chi_2\|^2_{L^\infty(Q)}\exp(\kappa_1 T^2) \tag{4.33a}$$

with $\kappa_i = 2(b_0\eta_{\mathbb{C}})^{-1}K_i$ and, a fortiori, we have for some constant κ_3

$$\sup_{t\in[0,T]}\|\varepsilon(\widehat{\boldsymbol{v}}(t))\|_{L^2(\Omega)} \le \kappa_3\|\bar\chi_1 - \bar\chi_2\|_{L^\infty(Q)}\,. \tag{4.33b}$$

□

The fixed point argument: solving the damage flow rule

We now solve the approximate flow rule (4.29c)–(4.29e) for fixed $\bar\chi \in L^\infty(Q)$ and with $\boldsymbol{u} = \bar{\boldsymbol{u}} := \mathcal{T}_{\boldsymbol{u}}(\bar\chi)$. The statement of Lemma 4.5 mirrors that of Lemma 4.4 and, again with slight abuse, we will consider as a solution operator the map $(\bar\chi, \bar{\boldsymbol{u}}) \mapsto \chi$, disregarding the solution component ω.

Lemma 4.5. *Let $\bar\chi \in L^\infty(Q)$ and $\bar{\boldsymbol{u}} = \mathcal{T}_{\boldsymbol{u}}(\bar\chi) \in L^\infty(0,T;V^n)$. Set*

$$\bar h := \bar\chi - \hat W'(\bar\chi) - \tfrac12 a'(\bar\chi)\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}){:}\varepsilon(\bar{\boldsymbol{u}})\,,$$

and consider the PDE system

$$\begin{aligned} \nu\omega_{tt} + \omega + \chi_t + I_\delta'(\chi_t) &= \bar h \quad \text{a.e. in } Q,\\ -\Delta\chi + \breve W_\delta'(\chi) + \chi &= \omega \quad \text{a.e. in } Q\,, \end{aligned} \tag{4.34}$$

supplemented with the boundary condition (4.29e)*.*

Then, for every $\chi_0 \in H^2(\Omega)$ fulfilling (2.20c) *and $\varpi_0 \in L^2(\Omega)$ there exists a unique solution*

$$\chi \in \boldsymbol{X} := W^{1,\infty}(0,T;H^2(\Omega)) \cap W^{2,\infty}(0,T;H^2(\Omega)), \qquad \omega \in W^{2,\infty}(0,T;L^2(\Omega)),$$

to system (4.34) *satisfying $\chi(0) = \chi_0$ and $\omega_t(0) = \varpi_0$.*

Moreover, there exists a function $\zeta_\chi : [0,\infty)^5 \to [0,\infty)$, monotonously increasing w.r.t. all of its arguments, such that

$$\begin{aligned} &\|\chi\|_{W^{1,\infty}(0,T;H^2(\Omega))\cap W^{2,\infty}(0,T;H^1(\Omega))} + \nu\|\omega\|_{W^{2,\infty}(0,T;L^2(\Omega))}\\ &\le \zeta_\chi\Big(\frac1\delta, \|\bar\chi\|_{L^\infty(Q)}, \|\bar{\boldsymbol{u}}\|_{L^\infty(0,T;V^n)}, \|\chi_0\|_{H^2(\Omega)}, \||\partial\breve W^\circ|(\chi_0)\|_{L^2(\Omega)}, \nu^{1/2}\|\varpi_0\|_{L^2(\Omega)}\Big)\,, \end{aligned} \tag{4.35}$$

and the solution operator $\mathcal{T}_\chi$ mapping $(\bar\chi,\bar u) \mapsto \chi$ is continuous from $L^\infty(Q) \times L^\infty(0,T;V^n)$ to $\boldsymbol{X}$ endowed with the weak topology.*

Proof. It is rather standard to prove the existence of solutions, e.g. by time discretization. That is why, we focus here mainly on deriving the necessary *a priori* estimates to deduce the regularity $\chi \in \boldsymbol{X}$ for the solution, and estimate (4.35). We test equation (4.29d) by ω_t, which provides the estimate

$$\begin{aligned} \frac12\frac{\mathrm{d}}{\mathrm{d}t}\left(\nu\|\omega_t(t)\|^2_{L^2(\Omega)} + \|\omega(t)\|^2_{L^2(\Omega)}\right) &\le \|\chi_t + I_\delta'(\chi_t)\|_{L^2(\Omega)}\|\omega_t\|_{L^2(\Omega)} + \|\bar h\|_{L^2(\Omega)}\|\omega_t\|_{L^2(\Omega)}\\ &\overset{(1)}{\le} \left(1+\frac1\delta\right)\|\chi_t\|_{L^2(\Omega)}\|\omega_t\|_{L^2(\Omega)} + \|\bar h\|_{L^2(\Omega)}\|\omega_t\|_{L^2(\Omega)}\\ &\overset{(2)}{\le} \frac12\|\bar h\|^2_{L^2(\Omega)} + \left(S_0 + \frac{S_0}{\delta} + \frac12\right)\|\omega_t\|^2_{L^2(\Omega)}\,, \end{aligned}$$

where (1) follows from the fact that $\|I_\delta'(\chi_t)\|_{L^2(\mathbb{R})} \le \frac1\delta\|\chi_t\|_{L^2(\Omega)}$ by the Lipschitz continuity of I_δ' and the fact that $I_\delta'(0) = 0$, cf. (B.3a) ahead, while (2) is a consequence of (4.2b). Then, with the Gronwall lemma we obtain that

$$\sup_{t\in[0,T]}\left(\nu^{1/2}\|\omega_t(t)\|_{L^2(\Omega)} + \|\omega(t)\|_{L^2(\Omega)}\right) \le \widetilde\zeta\Big(\frac1\delta, \|\bar h\|_{L^\infty(0,T;L^2(\Omega))}, \|\omega(0)\|_{L^2(\Omega)}, \nu^{1/2}\|\varpi_0\|_{L^2(\Omega)}\Big)$$

for some $\widetilde\zeta : [0,\infty)^4 \to [0,\infty)$, increasing w.r.t. all arguments. Taking into account estimates (4.2), estimating $\|\bar h\|_{L^\infty(0,T;L^2(\Omega))}$ via $\|\bar\chi\|_{L^\infty(Q)}$ and $\|\bar{\boldsymbol{u}}\|_{L^\infty(0,T;H^2(\Omega))}$, and estimating $\|\omega(0)\|_{L^2(\Omega)}$ via (4.22), we find

$$\sup_{t\in[0,T]}\left(\|\chi(t)\|_{H^2(\Omega)} + \|\chi_t(t)\|_{H^1(\Omega)}\right) \le \overline\zeta_\chi^{\delta,\nu}$$

with the constant $\overline{\zeta}_\chi^{\delta,\nu}$ depending on the same quantities as in (4.35). A comparison argument in $\omega_t = -\Delta\chi_t + \breve{W}_\delta''(\chi)\chi_t + \chi_t$ (recalling that $\|\breve{W}_\delta''(\chi)\|_{L^\infty(\Omega\times(0,T))} \leq \frac{1}{\delta}$), then allows us to conclude an estimate for $-\Delta\chi_t$ in $L^\infty(0,T;L^2(\Omega))$. Therefore, χ_t is estimated in $L^\infty(0,T;H^2(\Omega))$. Arguing by comparison in (4.29b), we ultimately deduce an estimate for ω_{tt} in $L^\infty(0,T;L^2(\Omega))$. A fortiori, by (4.2c) and taking into account that $\sup_{t\in[0,T]}\|\breve{W}'''(\chi(t))\|_{L^\infty(\Omega)}\|\chi_t(t)\|^2_{L^3(\Omega)} \leq C$, we infer an estimate for χ_{tt} in $L^\infty(0,T;H^1(\Omega))$. Thus, suitably adapting the right-hand side term $\overline{\zeta}_\chi^{\delta,\nu}$, we conclude estimate (4.35).

In order to have the solution operator $\mathcal{T}_\chi$ well defined, let us verify that, for given $\bar{h}$ and data χ_0, ϖ_0, the initial boundary-value problem for (4.34) admits a unique solution. Indeed, let (χ_i,ω_i), $i = 1,2$, two solution pairs. Set $\widehat{\chi} = \chi_1 - \chi_2$ and t $\widehat{\omega} = \omega_1 - \omega_2$. We subtract system (4.34) for ω_2 from (4.34) for ω_1, thus obtaining

$$\begin{cases} \nu\widehat{\omega}_{tt} + \widehat{\omega} + \widehat{\chi}_t + I_\delta'(\partial_t\chi_1) - I_\delta'(\partial_t\chi_2) &= 0 \\ -\Delta\widehat{\chi} + \breve{W}_\delta'(\chi_1) - \breve{W}_\delta'(\chi_2) + \widehat{\chi} &= \widehat{\omega} \end{cases} \quad \text{a.e. in } Q\,. \tag{4.36}$$

We test the first equation by $\widehat{\omega}_t$, while we differentiate in time the second equation and test it by $\widehat{\chi}_t$. Adding the resulting relations and integrating in time and space, we obtain

$$\frac{\nu}{2}\|\widehat{\omega}_t(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\widehat{\omega}_t(t)\|^2_{L^2(\Omega)} + \int_0^t \|\widehat{\chi}_t\|^2_{H^1(\Omega)}\,\mathrm{d}s \leq I_1 + I_2$$

where, using that I_δ' is Lipschitz continuous with Lipschitz constant $\frac{1}{\delta}$, we estimate

$$I_1 = \int_0^t \|I_\delta'(\partial_t\chi_1){-}I_\delta'(\partial_t\chi_2)\|_{L^2(\Omega)}\|\widehat{\omega}_t\|_{L^2(\Omega)}\,\mathrm{d}s \leq \frac{1}{2}\int_0^t \|\widehat{\chi}_t\|^2_{L^2(\Omega)}\,\mathrm{d}s + \frac{1}{2\delta}\int_0^t \|\widehat{\omega}_t\|^2_{L^2(\Omega)}\,\mathrm{d}s,$$

while we have

$$\begin{aligned} I_2 &= \int_0^t\int_\Omega \Big(-\breve{W}_\delta''(\chi_1)\partial_t\chi_1{+}\breve{W}_\delta''(\chi_2)\partial_t\chi_2\Big)\,\widehat{\chi}_t\,\mathrm{d}x\,\mathrm{d}s \\ &\leq -\int_0^t\int_\Omega \breve{W}_\delta''(\chi_1)|\widehat{\chi}_t|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \|\breve{W}_\delta''(\chi_2){-}\breve{W}_\delta''(\chi_1)\|_{L^2(\Omega)}\,\|\partial_t\chi_2\|_{L^\infty(\Omega)}\|\widehat{\chi}_t\|_{L^2(\Omega)}\,\mathrm{d}s \\ &\stackrel{(1)}{\leq} \frac{1}{4}\int_0^t \|\widehat{\chi}_t\|^2_{L^2(\Omega)}\,\mathrm{d}s + C\int_0^t\int_0^s \|\widehat{\chi}_t\|^2_{L^2(\Omega)}\,\mathrm{d}r\,\mathrm{d}s\,. \end{aligned}$$

Indeed, (1) follows from the convexity of $\breve{W}_\delta$, from Young's inequality (with the constant C depending on $\|\chi_2\|_{W^{1,\infty}(0,T;H^2(\Omega))}$), and from estimating

$$\|\breve{W}_\delta''(\chi_2(s)){-}\breve{W}_\delta''(\chi_1(s))\|_{L^2(\Omega)} \leq C\|\widehat{\chi}(s)\|_{L^2(\Omega)} \leq C\int_0^s \|\widehat{\chi}_t(s)\|_{L^2(\Omega)}\,\mathrm{d}s\,.$$

All in all, we obtain

$$\begin{aligned} &\frac{\nu}{2}\|\widehat{\omega}_t(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\widehat{\omega}_t(t)\|^2_{L^2(\Omega)} + \frac{1}{4}\int_0^t \|\widehat{\chi}_t\|^2_{H^1(\Omega)}\,\mathrm{d}s \\ &\qquad\qquad \leq \frac{1}{2\delta}\int_0^t \|\widehat{\omega}_t\|^2_{L^2(\Omega)}\,\mathrm{d}s + C\int_0^t\int_0^s \|\widehat{\chi}_t\|^2_{L^2(\Omega)}\,\mathrm{d}r\,\mathrm{d}s\,, \end{aligned}$$

and via the Gronwall Lemma we conclude the desired uniqueness $\widehat{\chi} = \widehat{\omega} \equiv 0$ a.e. in Q.

Finally, let us sketch the proof of the continuity of $\mathcal{T}_\chi$. Consider $(\bar{\chi}_n,\bar{\boldsymbol{u}}_n)_n \subset L^\infty(Q)\times L^\infty(0,T;V^n)$ such that $\bar{\chi}_n \to \bar{\chi}_\infty$ in $L^\infty(Q)$ and $\bar{\boldsymbol{u}}_n \to \bar{\boldsymbol{u}}_\infty$ in $L^\infty(0,T;V^n)$. Due to estimate (4.35), the corresponding sequence $(\chi_n = \mathcal{T}_\chi(\bar{\chi}_n,\bar{\boldsymbol{u}}_n))_n$ is bounded in $\boldsymbol{X} = W^{1,\infty}(0,T;H^2(\Omega))\cap W^{2,\infty}(0,T;H^1(\Omega))$.

Likewise, the associated sequence $(\omega_n)_n$ is bounded in $W^{2,\infty}(0,T;L^2(\Omega))$. Hence, there exist a pair $(\chi_\infty,\omega_\infty)$ and a subsequence $(n_k)_k$ such that $\chi_{n_k} \rightharpoonup^* \chi_\infty$ in $\boldsymbol{X}$ and $\omega_{n_k} \rightharpoonup^* \omega_\infty$ in $W^{2,\infty}(0,T;L^2(\Omega))$. We standardly check that $(\omega_\infty,\chi_\infty)$ solve the Cauchy problem for system (4.34) with $\bar{h}_\infty = \bar{\chi}_\infty - \hat{W}'(\bar{\chi}_\infty) - \frac{1}{2}a'(\bar{\chi}_\infty)\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}_\infty):\varepsilon(\bar{\boldsymbol{u}}_\infty)$. Thus, $\chi_\infty = \mathcal{T}_\chi(\bar{\chi}_\infty,\bar{\boldsymbol{u}}_\infty)$. Since the limit is uniquely identified, a posteriori we have convergence along the whole sequence $(\chi_n)_n$. We have thus shown that

$$\bar{\chi}_n \to \bar{\chi}_\infty \text{ in } L^\infty(Q) \text{ and } \bar{\boldsymbol{u}}_n \to \bar{\boldsymbol{u}}_\infty \text{ in } L^\infty(0,T;V^n) \implies \mathcal{T}_\chi(\bar{\chi}_n,\bar{\boldsymbol{u}}_n) \rightharpoonup^* \mathcal{T}_\chi(\bar{\chi}_\infty,\bar{\boldsymbol{u}}_\infty) \text{ in } \boldsymbol{X}\,.$$

This finishes the proof. □

Let us now introduce the operator

$$\mathcal{T}: L^\infty(Q) \to L^\infty(Q), \qquad \overline{\chi} \mapsto \mathcal{T}(\bar{\chi}) := \mathcal{T}_\chi(\mathcal{T}_{\boldsymbol{u}}(\bar{\chi}))\,,$$

and, for given $\mathsf{T} \in (0,T]$ and $\mathsf{R} > 0$, the notation

$$B_{\mathsf{R}}^\infty(\mathsf{T}) := \{\bar{\chi} \in L^\infty(\Omega\times(0,\mathsf{T})) \mid \|\bar{\chi}-\chi_0\|_{L^\infty(\Omega\times(0,\mathsf{T}))} \le \mathsf{R}\}\,.$$

With our next result we will show that there exists $\widetilde{T} \in (0,T]$ such that, if we restrict $\mathcal{T}$ a closed ball in $L^\infty(\Omega\times(0,\widetilde{T}))$, $\mathcal{T}$ maps $B_{\mathsf{R}}^\infty(\mathsf{T})$ into itself and indeed admits a fixed point, which in fact provides a local-in-time solution to the Cauchy problem for system (4.27). This concludes the **proof of Proposition 4.3**.

Lemma 4.6. *Let $(\boldsymbol{u}_0,\boldsymbol{v}_0,\chi_0) \in H^3(\Omega;\mathbb{R}^d)\times H^2(\Omega;\mathbb{R}^d)\times H^2(\Omega)$ fulfill Hypothesis D. Let $\varpi_0 \in L^2(\Omega)$ be given.*

Then, for a suitably chosen $\mathsf{R} > 0$ there exists $\widetilde{T} = \widetilde{T}(\delta,\nu) \in (0,T]$ such that the operator $\mathcal{T}$ admits a fixed point in $B_{\mathsf{R}}^\infty(\widetilde{T})$.

As a consequence, the Cauchy problem for system (4.27) *admits a solution $(\boldsymbol{u},\chi,\omega)$ as in* (4.28).

Proof. Combining the continuity properties of the operator $\mathcal{T}_{\boldsymbol{u}}$ with those of $\mathcal{T}_\chi$, we easily check that $\mathcal{T}: L^\infty(Q) \to L^\infty(Q)$ is continuous.

It follows from estimates (4.31) and (4.35) that there exists a function $\zeta : [0,\infty)^5 \to [0,\infty)$, increasing w.r.t. all arguments, such that for every $\bar{\chi} \in L^\infty(Q)$ there holds

$$\|\mathcal{T}(\chi)\|_{W^{1,\infty}(0,T;H^2(\Omega))\cap W^{2,\infty}(0,T;H^1(\Omega))} \le \zeta\Big(\frac{1}{\delta},\frac{1}{\nu},\|\boldsymbol{f}\|_{L^2(0,T;H^1(\Omega))},\|\bar{\chi}\|_{L^\infty(Q)},m_0\Big)\,,$$

where we have set $m_0 := \|\boldsymbol{u}_0\|_{H^3(\Omega)} + \|\boldsymbol{v}_0\|_{H^2(\Omega)} + \|\chi_0\|_{H^2(\Omega)} + \||\partial\breve{W}^\circ|(\chi_0)\|_{L^2(\Omega)} + \|\varpi_0\|_{L^2(\Omega)}$. Since $W^{1,\infty}(0,T;H^2(\Omega))\cap H^2(0,T;L^2(\Omega))$ compactly embeds in $L^\infty(\Omega\times(0,T))$, we conclude that the operator $\mathcal{T}$ is compact.

Finally, let us choose

$$\mathsf{R} > \zeta_0 := \zeta\Big(\frac{1}{\delta},\frac{1}{\nu},\|\boldsymbol{f}\|_{L^2(0,T;H^1(\Omega))},0,m_0\Big).$$

For any $\widetilde{T} \in (0,T]$, for every $t \in [0,\widetilde{T}]$ and $\bar{\chi} \in B^\infty_R(\widetilde{T})$ we have

$$\begin{aligned}
\|\mathfrak{T}(\bar{\chi})(t) - \chi_0\|_{L^\infty(\Omega)} &= \|\chi(t) - \chi_0\|_{L^\infty(\Omega)} \\
&= \left\| \int_0^t \chi_t \, \mathrm{d}s \right\|_{L^\infty(\Omega)} \\
&\le t\, \|\chi_t\|_{L^\infty(\Omega\times(0,t))} \\
&\le C_{H^2,L^\infty} \widetilde{T} \|\chi\|_{W^{1,\infty}(0,\widetilde{T};H^2(\Omega))} \\
&\le C_{H^2,L^\infty} \widetilde{T} \zeta\Big(\frac{1}{\delta}, \frac{1}{\nu}, \|\boldsymbol{f}\|_{L^2(0,T;H^1(\Omega))}, \|\bar{\chi}\|_{L^\infty(\Omega\times(0,\widetilde{T}))}, m_0\Big) \\
&\overset{(1)}{\le} C_{H^2,L^\infty} \widetilde{T} \zeta\Big(\frac{1}{\delta}, \frac{1}{\nu}, \|\boldsymbol{f}\|_{L^2(0,T;H^1(\Omega))}, R + \|\chi_0\|_{L^\infty(\Omega)}, m_0\Big)\,,
\end{aligned}$$

where C_{H^2,L^∞} is the constant for the continuous embedding $H^2(\Omega) \subset L^\infty(\Omega)$, and (1) follows from the estimate

$$\|\bar{\chi}\|_{L^\infty(\Omega\times(0,\widetilde{T}))} \le \|\bar{\chi} - \chi_0\|_{L^\infty(\Omega\times(0,\widetilde{T}))} + \|\chi_0\|_{L^\infty(\Omega\times(0,\widetilde{T}))} \le R + \|\chi_0\|_{L^\infty(\Omega\times(0,\widetilde{T}))}\,,$$

and the monotonicity of ζ. Hence, upon choosing

$$\widetilde{T} \le R\, C^{-1}_{H^2,L^\infty}\, \zeta\Big(\frac{1}{\delta}, \frac{1}{\nu}, \|\boldsymbol{f}\|_{L^2(0,T;H^1(\Omega))}, R + \|\chi_0\|_{L^\infty(\Omega)}, m_0\Big)^{-1}$$

we have that $\mathfrak{T}(B^\infty_R(\widetilde{T})) \subset B^\infty_R(\widetilde{T})$. Therefore, we are in a position to apply Schauder's fixed point theorem. This concludes the proof. □

4.4 A priori estimates for the regularized approximate system

With the following result we rigorously prove the estimates of Prop. 4.2 for the local-in-time solutions $(\boldsymbol{u}, \chi, \omega)$ of the approximate system (4.29) (for better readability, we choose to omit the dependence on the parameters n and δ in their notation). Since estimate (4.38) below holds for a constant independent of $n \in \mathbb{N}$ and $\delta,\ \nu > 0$, we deduce that the local-in-time solution $(\boldsymbol{u}, \chi)$ found in Prop. 4.3 exists up to a time $\widehat{T}$ independent of such parameters.

Proposition 4.7 (Enhanced local-in-time estimates for the approximate solution)**.** *Assume Hypotheses **C** and **D**, and let Ω fulfill condition (H_Ω). Then, there exist a time $\widehat{T} \in (0,T]$ such that*

1 *for every $n \in \mathbb{N}$ and $\delta,\ \nu > 0$ the solution $(\boldsymbol{u}, \chi)$ from Proposition 4.3 extends to the interval $[0,\widehat{T}]$ with the regularity*

$$\begin{aligned}
&\boldsymbol{u} \in H^1(0,\widehat{T}; H^3(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,\widehat{T}; H^2(\Omega;\mathbb{R}^d)),\\
&\chi \in L^\infty(0,\widehat{T}; H^2(\Omega)) \cap H^{1,\infty}(0,\widehat{T}; H^1(\Omega))
\end{aligned} \tag{4.37}$$

and we have that $\omega = -\Delta\chi + \breve{W}'_\delta(\chi) + \chi \in W^{2,\infty}(0,\widehat{T}; L^2(\Omega))$;

2 *there exists and a function $\boldsymbol{\zeta} : [0,\infty)^5 \to [0,\infty)$, increasing w.r.t. all its arguments, such that for every $n \in \mathbb{N}$ and $\delta,\ \nu > 0$ there holds for all $t \in (0,\widehat{T})$*

$$\begin{aligned}
&\|\boldsymbol{u}_t(t)\|^2_{H^2(\Omega)} + \|\chi(t)\|^2_{H^2(\Omega)} + \|\omega(t)\|^2_{L^2(\Omega)} + \nu\|\omega_t(t)\|^2_{L^2(\Omega)} + \nu\|\chi_t(t)\|^2_{H^1(\Omega)}\\
&\qquad + \int_0^t \Big(\|\boldsymbol{u}_t(s)\|^2_{H^3(\Omega)} + \|\chi_t(s)\|^2_{H^1(\Omega)} \Big)\, \mathrm{d}s\\
&\le \boldsymbol{\zeta}\Big(\|\boldsymbol{u}_0\|_{H^3(\Omega)}, \|\boldsymbol{v}_0\|_{H^2(\Omega)}, \|\chi_0\|_{H^2(\Omega)}, \||\partial\breve{W}^\circ|(\chi_0)\|_{L^2(\Omega)}, \nu^{1/2}\|\varpi_0\|_{L^2(\Omega)} \Big)\,.
\end{aligned} \tag{4.38}$$

Furthermore, there exists a constant C such that for every $n \in \mathbb{N}$ and $\delta,\ \nu > 0$

$$\|\boldsymbol{u}_{tt}\|_{L^2(0,\widehat{T};H^1(\Omega))} + \|\breve{W}'_\delta(\chi)\|_{L^\infty(0,\widehat{T};L^2(\Omega))} \leq C\,. \tag{4.39}$$

Clearly, in view of Lemma 4.1, the regularity $\omega \in W^{2,\infty}(0,\widehat{T};L^2(\Omega))$ leads to additional regularity for χ. However, we shall not emphasize it, as it will not carry over to the limit as $\delta,\ \nu \downarrow 0$.

Proof. Here, we revisit the various claims in the proof of Prop. 4.2 and show how the related calculations can be made rigorous.

Claim 1***:*** *The evolution of the mean of $\boldsymbol{u}$ is only determined by the given data, i.e.,*

$$\int_\Omega \boldsymbol{u}(t)\,\mathrm{d}x = \int_\Omega \boldsymbol{u}_0\,\mathrm{d}x + t\int_\Omega \boldsymbol{v}_0\,\mathrm{d}x + \int_0^t\int_\Omega (t-r)\boldsymbol{f}(r)\,\mathrm{d}r\,.$$

This claim follows exactly as in the proof of Proposition 4.2 by choosing the basis function $\mathbf{1}$ as test function in the Galerkin discretization (4.27a).

Claim 2***:*** *there exists a constant $S_{1,1} > 0$ such that estimate* (4.7) *holds.*
It follows exactly as in the proof of Proposition 4.2.

Claim 3***:*** *there exist a constant $S_{1,2} > 0$ and $\overline{\beta} > 1$ such that estimate* (4.8) *holds.*

In order to rigorously prove this claim, we use the special choice of the Galerkin basis. First of all, we observe that testing (4.27a) by $\nabla\cdot(\mathbb{C}:\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t))))$ is possible, since the choice of our Galerkin basis ensures that $\nabla\cdot(\mathbb{C}:\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t)))) \in V^n$ for $\boldsymbol{u}_t \in V^n$. Moreover, we observe that the following boundary conditions are fulfilled due to the choice of the Galerkin basis

$$\boldsymbol{n}\cdot\mathbb{C}\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{u}_t))) = 0\,, \quad \boldsymbol{n}\cdot\mathbb{C}\varepsilon(\boldsymbol{u}_{tt}) = 0 \quad \text{on } \partial\Omega\times(0,T)\,.$$

This follows from the fact that $\boldsymbol{u}_t$ and $\boldsymbol{u}_{tt}$ are just linear combinations of the basis functions and for all basis functions $\boldsymbol{y}_i \in V^n$ with $i \in \{1,\dots,n\}$ it holds by construction that $\boldsymbol{n}\cdot\mathbb{C}\varepsilon(\boldsymbol{y}_i) = 0$. Moreover, the basis functions are eigenfunctions of the operator (4.26), so that for any $\boldsymbol{y}_i \in V^n$ also $\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{y}_i)) = \lambda_i\boldsymbol{y}_i$ fulfills the associated boundary condition, *i.e.*,

$$\boldsymbol{n}\cdot\mathbb{C}\varepsilon(\nabla\cdot(\mathbb{C}:\varepsilon(\boldsymbol{y}_i))) = \lambda_i\boldsymbol{n}\cdot\mathbb{C}\varepsilon(\boldsymbol{y}_i) = 0 \quad \text{on } \partial\Omega\times(0,T)\,.$$

With this observation, all the formal calculations of ***Claim*** *2* can be performed rigorously and all boundary terms are null.

Claim 4***:*** *there exist a constant $S_{1,3} > 0$ and $\underline{\beta} > 1$ such that for almost all $t \in (0,T)$*

$$\begin{aligned}
&\nu\|\omega_t(t)\|^2_{L^2(\Omega)} + \|\omega(t)\|^2_{L^2(\Omega)} + \|\chi(t)\|^2_{H^2(\Omega)} + \int_0^t \left(\|\chi_t\|^2_{L^2(\Omega)} + \|\nabla\chi_t\|^2_{L^2(\Omega)}\right)\mathrm{d}s \\
&\leq S_{1,3}(1+\|\varpi_0\|^2_{L^2(\Omega)} + \|\chi_0\|^2_{H^2(\Omega)} + \||\partial\breve{W}^\circ|(\chi_0)\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_0\|^8_{H^2(\Omega)}) \\
&\quad + S_{1,3}\int_0^t \left(\|\omega\|^8_{L^2(\Omega)} + \|\boldsymbol{u}_t\|^8_{H^2(\Omega)} + \int_0^s \|\boldsymbol{u}_t\|^8_{H^2(\Omega)}\,\mathrm{d}\tau + \|\chi\|^{\underline{\beta}}_{H^2(\Omega)}\right)\mathrm{d}s\,.
\end{aligned} \tag{4.40}$$

In fact, thanks to Lemma 4.6, for a solution to the system (4.27) we have the regularity property $\omega \in W^{2,\infty}(0,T;L^2(\Omega))$ (although we have $\|\omega\|_{W^{2,\infty}(0,\widehat{T};L^2(\Omega))} \leq C(\delta,\nu)$ for a positive constant $C(\delta,\nu)$, with $C(\delta,\nu) \uparrow +\infty$ as $\delta,\,\nu \downarrow 0$). Hence, ω_t is an admissible test function for equation (4.27c). The same arguments as in the proof of Proposition 4.2 can now be followed step by step in order to derive the estimate (4.40).

Combining the estimates from Claim 2 and 3, we obtain the analogue of inequality (4.23), with the same constants $S_{1,4}$ and β, but with the additional term $\nu\|\omega_t(t)\|^2_{L^2(\Omega)}$ on the left-hand side. Recall that $\nu\|\omega_t(t)\|^2_{L^2(\Omega)} \geq c\nu\|\chi_t(t)\|^2_{H^1(\Omega)}$ by Lemma 4.1. Hence, the very same local-in-time Gronwall-type estimate as in the proof of Proposition 4.2 allows us to deduce estimate (4.38). Estimate (4.39) for $\boldsymbol{u}_{tt}$ then follows in view of (4.31), by the equivalence of all finite-dimensional norms, while the bound for $\breve{W}'_\delta(\chi)$ follows from that for ω in $L^\infty(0,\widehat{T};L^2(\Omega))$, arguing by comparison.

Since the involved constants are independent of $n \in \mathbb{N}$ and of $\delta,\,\nu > 0$, by a standard prolongation argument we obtain that the solution found in Prop. 4.3 extends to an interval $(0,\widehat{T})$ independent of all parameters, on which estimate (4.38) holds. This concludes the proof. □

4.5 Limit passage in the regularized system and conclusion of the proof of Theorem 2.9

We split the argument in some steps. Let us mention in advance that we shall resort to Proposition 2.11: thus, we will show that the limiting pair $(\boldsymbol{u},\chi)$ fulfills the damage flow rule pointwise a.e. in Q by proving the variational inequality (2.22) and the energy-dissipation inequality (2.12).

For the compactness argument below we recall that, for a given reflexive space $\boldsymbol{X}$, convergence in the space $\mathrm{C}^0([0,S];\boldsymbol{X}_{\mathrm{weak}})$ is, by definition, convergence in $\mathrm{C}^0([0,S];(\boldsymbol{X},d_{\mathrm{weak}}))$, where the metric d_{weak} induces the weak topology on a closed bounded set of $\boldsymbol{X}$.

Step* 1*: compactness. Since the *a priori* estimate (4.3) holds independently of the parameters $n \in \mathbb{N}$ and $\delta, \nu > 0$, we may choose two sequences

$$\delta_n \downarrow 0 \text{ and } \nu_n \downarrow 0 \text{ such that } \frac{\nu_n^{1/2}}{\delta_n} \to 0 \text{ as } n \to \infty\,. \tag{4.41}$$

We also consider a sequence $(\varpi_0^n)_n \subset L^2(\Omega)$ of initial data such that

$$\nu_n^{1/2}\|\varpi_0^n\|_{L^2(\Omega)} \longrightarrow 0 \text{ as } n \to \infty\,. \tag{4.42}$$

Correspondingly, by Proposition 4.7 we find a sequence of solutions $(\boldsymbol{u}_{n,\delta_n,\nu_n},\chi_{n,\delta_n,\nu_n})_n$, hereafter simply denoted as $(\boldsymbol{u}_n,\chi_n)_n$, with associated $\omega_n = -\Delta\chi_n + \breve{W}'_{\delta_n}(\chi_n) + \chi_n$, ad a quadruple $(\boldsymbol{u},\chi,\omega,\xi)$, for which, along a (not-relabeled) subsequence, the weak-convergences associated with the bounds (4.38) hold, namely

$$\boldsymbol{u}_n \rightharpoonup^* \boldsymbol{u} \quad \text{weakly-star in } H^2(0,\widehat{T};H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,\widehat{T};H^2(\Omega;\mathbb{R}^d)) \cap H^1(0,\widehat{T};H^3(\Omega;\mathbb{R}^d)), \tag{4.43a}$$

$$\chi_n \rightharpoonup^* \chi \quad \text{weakly-star in } L^\infty(0,T;H^2(\Omega)) \cap H^1(0,T;H^1(\Omega)), \tag{4.43b}$$

$$\omega_n \rightharpoonup^* \omega \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)) \tag{4.43c}$$

$$\breve{W}'_{\delta_n}(\chi_n) \rightharpoonup^* \xi \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega))\,. \tag{4.43d}$$

Furthermore, by well-known compactness results, we gather the strong convergences

$$\partial_t \boldsymbol{u}_n \rightharpoonup^* \partial_t \boldsymbol{u} \qquad \text{strongly in } \mathrm{C}^0([0,\widehat{T}]; H^2(\Omega;\mathbb{R}^d)_{\mathrm{weak}}) \,, \tag{4.44a}$$

$$\chi_n \rightharpoonup^* \chi \qquad \text{strongly in } \mathrm{C}^0([0,\widehat{T}]; H^2(\Omega)_{\mathrm{weak}}) \,. \tag{4.44b}$$

Finally, from (4.38) we also deduce an estimate for $(\nu_n^{1/2}\omega_n)_n \subset W^{1,\infty}(0,\widehat{T};L^2(\Omega))$, so that

$$\nu_n \omega_n \to 0 \qquad \text{strongly in } W^{1,\infty}(0,\widehat{T};L^2(\Omega)) \,. \tag{4.44c}$$

By the weak lower semi continuity of the involved norms, we may take the limit in estimate (4.38) and deduce that its analogue holds at least for almost all $t \in (0,\widehat{T})$. Indeed, since $\boldsymbol{u}_t \in \mathrm{C}^0([0,\widehat{T}];H^2(\Omega;\mathbb{R}^d)_{\mathrm{weak}})$ and $\chi \in \mathrm{C}^0([0,\widehat{T}];H^2(\Omega)_{\mathrm{weak}})$, we ultimately have that the pointwise estimates (4.38) for $\boldsymbol{u}_t$ and χ hold for all times.

Step* 2*: momentum balance. Using the convergences (4.43) and (4.44), it is a standard manner to pass to the limit in the Galerkin approximation (4.27a). In this way, we deduce that the pair $(\boldsymbol{u},\chi)$ satisfies the momentum balance pointwise a.e. in Q

Step* 3*: variational inequality (2.22)***.*** Multiplying the regularized flow rule (4.27c) by a test function $\psi \in \mathrm{C}^1_{\mathrm{c}}(0,\widehat{T})\otimes L^2(\Omega)$ such that $\psi \leq 0$ a.e. in Q and integrating in space and time, we find

$$\begin{aligned}\iint_Q \left(\partial_t\chi_n - \Delta\chi_n + \breve{W}'_\delta(\chi_n) + \hat{W}'(\chi_n) + \frac{1}{2}a'(\chi_n)\mathbb{C}\varepsilon(\boldsymbol{u}_n){:}\varepsilon(\boldsymbol{u}_n)\right)\psi - \nu\partial_t\omega_n\partial_t\psi \,\mathrm{d}x\,\mathrm{d}s = \\ -\iint_Q I'_\delta(\partial_t\chi_n)\psi\,\mathrm{d}x\,\mathrm{d}s \geq 0\,.\end{aligned} \tag{4.45}$$

The last term on the left-hand side vanishes in the limit as $n\to\infty$ due to (4.44c).

$$\nu_n\int_Q \partial_t\omega_n\partial\psi\,\mathrm{d}x\,\mathrm{d}s \leq \nu_n\|\partial_t\omega_n\|_{L^\infty(0,\widehat{T};L^2(\Omega))}\|\partial_t\psi\|_{L^1(0,\widehat{T};L^2(\Omega))} \longrightarrow 0 \text{ as } n\to\infty\,.$$

Passing to the limit in the remaining terms on the left-hand side of inequality (4.45) is now a standard procedure in view of convergences (4.43) and (4.44). In particular, combining the weak convergence of $\breve{W}'_{\delta_n}(\chi_n)$ with the strong convergence (4.44b) for χ_n we conclude that $\xi\in\partial\breve{W}(\chi)$ a.e. in Q. In the limit of the inequality (4.45), we infer that (2.22) is fulfilled.

Step* 4*: energy-dissipation inequality (2.12)***.*** First of all, we observe that an approximate version of (2.12) holds for system (4.27). Indeed, testing (4.27a) by $\boldsymbol{z}=\partial_t\boldsymbol{u}_n$, multiplying (4.27c) multiplied by $\partial_t\chi_n$, adding the obtained relations and integrating in time leads to

$$\begin{aligned}&\mathcal{E}_{\delta_n}(\boldsymbol{u}_n(t),\chi_n(t),\partial_t\boldsymbol{u}_n(t)) + \int_0^t \mathcal{D}(\chi_n(s),\partial_t\boldsymbol{u}_n(s),\partial_t\chi_n(s))\,\mathrm{d}s + \nu_n\int_0^t\int_\Omega \partial_{tt}\omega_n(s)\partial_t\chi_n(s)\,\mathrm{d}s\,\mathrm{d}x \\ &= \mathcal{E}_{\delta_n}(\boldsymbol{u}_n(0),\chi_n(0),\partial_t\boldsymbol{u}_n(0)) + \iint_Q \boldsymbol{f}\cdot\partial_t\boldsymbol{u}_n\,\mathrm{d}x\,\mathrm{d}s\,,\end{aligned} \tag{4.46}$$

featuring the regularized energy and dissipation functionals

$$\mathcal{E}_{\delta_n}(\boldsymbol{u},\chi,\boldsymbol{u}_t) := \int_\Omega \left\{\frac{1}{2}|\boldsymbol{u}_t|^2 + \frac{1}{2}a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u}) + \frac{1}{2}|\nabla\chi|^2 + \breve{W}_{\delta_n}(\chi) + \hat{W}(\chi)\right\}\,\mathrm{d}x\,, \tag{4.47}$$

$$\mathcal{D}(\chi, \boldsymbol{u}_t, \chi_t) := \int_\Omega \left\{ b(\chi)\mathbb{V}\varepsilon(\boldsymbol{u}_t):\varepsilon(\boldsymbol{u}_t)+|\chi_t|^2+I_{\delta_n}(\chi_t)\right\} \,\mathrm{d}x\,. \tag{4.48}$$

For the last term on the left-hand side, we find

$$\begin{aligned}
&\nu_n \int_0^t \int_\Omega \partial_{tt}\omega_n(s)\partial_t\chi_n(s)\,\mathrm{d}s\,\mathrm{d}x \\
&= \nu_n \int_0^t \int_\Omega \partial_t \left(-\Delta\partial_t\chi_n+\breve{W}''_{\delta_n}(\chi_n)\partial_t\chi_n+\partial_t\chi_n\right)\partial_t\chi_n\,\mathrm{d}x\,\mathrm{d}s \\
&= \nu_n \int_0^t \int_\Omega \left(-\Delta\partial_{tt}\chi_n+\breve{W}'''_{\delta_n}(\chi_n)\partial_t\chi_n\partial_t\chi_n+\breve{W}''_{\delta_n}(\chi_n)\partial_{tt}\chi_n+\partial_t^2\chi_n\right)\partial_t\chi_n\,\mathrm{d}x\,\mathrm{d}s \\
&= \nu_n \int_0^t \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left\{|\nabla\partial_t\chi_n|^2+|\partial_t\chi_n|^2\right\}+\breve{W}'''_{\delta_n}(\chi_n)(\partial_t\chi_n)^3+\breve{W}''_{\delta_n}(\chi_n)\partial_{tt}\chi_n\partial_t\chi_n\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\nu_n}{2}\int_\Omega |\nabla\partial_t\chi_n|^2+|\partial_t\chi_n|^2+|\sqrt{\breve{W}''_{\delta_n}(\chi_n)}\partial_t\chi_n|^2\,\mathrm{d}x\right\}\mathrm{d}s \\
&\quad + \frac{\nu_n}{2}\int_0^t \int_\Omega \breve{W}'''_{\delta_n}(\chi_n)|\partial_t\chi_n|^2\partial_t\chi_n\,\mathrm{d}x\,\mathrm{d}s\,,
\end{aligned}$$

where we used the fact that $\breve{W}''_{\delta_n} \geq 0$ and that

$$\partial_t \frac{\nu_n}{2}|\sqrt{\breve{W}''_{\delta_n}(\chi_n)}\partial_t\chi_n|^2 = \frac{\nu_n}{2}\breve{W}'''_{\delta_n}(\chi_n)|\partial_t\chi_n|^2\partial_t\chi_n + \nu_n\breve{W}''_{\delta_n}(\chi_n)\partial_{tt}\chi_n\partial_t\chi_n \qquad \text{a.e. in } Q\,.$$

Thus, (4.46) rephrases as

$$\begin{aligned}
&\mathcal{E}_{\delta_n}(\boldsymbol{u}_n(t), \chi_n(t), \partial_t\boldsymbol{u}_n(t)) + \int_0^t \mathcal{D}(\chi_n(s), \partial_t\boldsymbol{u}_n(s), \partial_t\chi_n(s))\,\mathrm{d}s + \mathcal{V}_n(\chi_n(t), \partial_t\chi_n(t)) \\
&= \mathcal{E}_{\delta_n}(\boldsymbol{u}_n(0), \chi_n(0), \partial_t\boldsymbol{u}_n(0)) + \int_0^t \int_\Omega \boldsymbol{f}\cdot\partial_t\boldsymbol{u}_n\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \mathcal{V}_n(\chi_n(0), \partial_t\chi_n(0)) + \frac{\nu_n}{2}\int_0^t \int_\Omega \breve{W}'''_{\delta_n}(\chi_n)|\partial_t\chi_n|^2\partial_t\chi_n\,\mathrm{d}x\,\mathrm{d}s\,,
\end{aligned}$$

where we have used the place-holder

$$\mathcal{V}_n(\chi_n, \partial_t\chi_n) = \frac{\nu_n}{2}\|\partial_t\chi_n\|^2_{H^1(\Omega)} + \frac{\nu_n}{2}\int_\Omega \left|\sqrt{\breve{W}''_{\delta_n}(\chi_n)}\partial_t\chi_n\right|^2\,\mathrm{d}x\,.$$

Now, observe that

$$\begin{aligned}
\mathcal{V}_n(\chi_n(0), \partial_t\chi_n(0)) &= \frac{\nu_n}{2}\int_\Omega \partial_t\omega_n(0)\partial_t\chi_n(0)\,\mathrm{d}x \\
&\leq \frac{\nu_n}{2}\|\partial_t\omega_n(0)\|_{L^2(\Omega)}\|\partial_t\chi_n(0)\|_{L^2(\Omega)} \overset{(1)}{\leq} \frac{\nu_n S_0}{2}\|\partial_t\omega_n(0)\|^2_{L^2(\Omega)} \overset{(2)}{\longrightarrow} 0
\end{aligned}$$

as $n\to\infty$, where in (1), we have used (4.2b) and in (2), we resorted to condition (4.42) for $\partial_t\omega_n(0) = \varpi_0^n$. Furthermore, by (4.25) we have $|\breve{W}'''_{\delta_n}(\chi_n)| \leq \frac{1}{\delta_n^3}$ a.e. in Q, therefore we infer that

$$\frac{\nu_n}{2}\int_0^t \int_\Omega \breve{W}'''_{\delta_n}(\chi_n)|\partial_t\chi_n|^2\partial_t\chi_n\,\mathrm{d}x\,\mathrm{d}s \leq \frac{1}{2}\nu_n^{1/2}\|\breve{W}'''_{\delta_n}(\chi_n)\|_{L^\infty(Q)}\nu_n^{1/2}\|\partial_t\chi_n\|^3_{L^3((0,\widehat{T})\times\Omega)} \longrightarrow 0$$

as $n \to \infty$, where the last assertion follows from combining the bound for $\nu_n^{1/2} \|\partial_t \chi_n\|_{L^\infty(0,\widehat{T};H^1(\Omega))}$ from (4.38), with the scaling condition (4.41). In turn, we immediately see that for every $t \in (0, \widehat{T}]$

$$
\begin{aligned}
\mathcal{E}(\boldsymbol{u}(t), \chi(t), \partial_t \boldsymbol{u}(t)) &\le \liminf_{n\to\infty} \mathcal{E}_{\delta_n}(\boldsymbol{u}_n(t), \chi_n(t), \partial_t \boldsymbol{u}_n(t)) \,, \\
\int_0^t \mathcal{D}(\chi(s), \partial_t \boldsymbol{u}(s), \partial_t \chi(s)) \,\mathrm{d}s &\le \liminf_{n\to\infty} \int_0^t \mathcal{D}(\chi_n(s), \partial_t \boldsymbol{u}_n(s), \partial_t \chi_n(s)) \,\mathrm{d}s \,, \\
\mathcal{E}_{\delta_n}(\boldsymbol{u}_n(0), \chi_n(0), \partial_t \boldsymbol{u}_n(0)) &\longrightarrow \mathcal{E}(\boldsymbol{u}_0, \chi_0, \boldsymbol{v}_0) \,, \\
\int_0^t \int_\Omega \boldsymbol{f} \cdot \partial_t \boldsymbol{u}_n \,\mathrm{d}x \,\mathrm{d}s &\longrightarrow \int_0^t \int_\Omega \boldsymbol{f} \cdot \partial_t \boldsymbol{u} \,\mathrm{d}x \,\mathrm{d}s \,.
\end{aligned}
$$

All in all, sending $n \to \infty$ in (4.46) we find that the energy inequality (2.12) holds, in the limit, on $[0, t]$ for all $t \in (0, \widehat{T}]$. By Proposition 2.11, this completes the proof of Theorem 2.9. □

5 Relative energy inequality

This Section is devoted to the proof of Theorem 2.12. The key result is Proposition 5.1, where we will compare a weak solution $(\boldsymbol{u}, \chi)$ (to the initial-boundary value problem for system (1.1) with the homogeneous Neumann boundary condition (2.15)), and a strong solution $(\tilde{\boldsymbol{u}}, \tilde{\chi})$ in terms of the following quantities:

- the relative energy

$$
\begin{aligned}
\mathcal{R}(\boldsymbol{u}, \chi, \boldsymbol{u}_t | \tilde{\boldsymbol{u}}, \tilde{\chi}, \tilde{\boldsymbol{u}}_t) := &\int_\Omega \frac{1}{2} |\nabla \chi - \nabla \tilde{\chi}|^2 + W(\chi) - W(\tilde{\chi}) - W'(\tilde{\chi})(\chi - \tilde{\chi}) + \frac{\ell}{2} |\chi - \tilde{\chi}|^2 \,\mathrm{d}x \\
&+ \int_\Omega \frac{1}{2} a(\chi) \mathbb{C} \varepsilon(\boldsymbol{u} - \tilde{\boldsymbol{u}}) : \varepsilon(\boldsymbol{u} - \tilde{\boldsymbol{u}}) + \frac{1}{2} |\boldsymbol{u}_t - \tilde{\boldsymbol{u}}_t|^2 \,\mathrm{d}x \,,
\end{aligned}
\tag{5.1}
$$

  where $\ell \ge 0$ is such that $r \mapsto W(r) + \frac{\ell}{2}|r|^2$ is convex, cf. (2.4b), and

- the relative dissipation

$$
\mathcal{W}(\chi, \boldsymbol{u}_t, \chi_t | \tilde{\boldsymbol{u}}_t, \tilde{\chi}_t) := \int_\Omega |\chi_t - \tilde{\chi}_t|^2 + b(\chi) \mathbb{V} \varepsilon(\boldsymbol{u}_t - \tilde{\boldsymbol{u}}_t) : \varepsilon(\boldsymbol{u}_t - \tilde{\boldsymbol{u}}_t) \,\mathrm{d}x \,, \tag{5.2}
$$

  where we have omitted the terms $\int_\Omega I_{(-\infty,0])}(\chi_t) + I_{(-\infty,0])}(\tilde{\chi}_t)) \,\mathrm{d}x$ as they will be null as soon as they are evaluated along a weak and a strong solution.

Indeed, $\mathcal{R}$ and $\mathcal{W}$ will be involved in the Gronwall-type inequality (REI) below, which will be the core ingredient in the proof of Thm. 2.12.

Proposition 5.1. *Let Hypothesis **E** be fulfilled and let $(\boldsymbol{u}, \chi)$ be a weak solution to the Cauchy problem for system* (1.1) *in the sense of Definition 2.2 and $(\tilde{\boldsymbol{u}}, \tilde{\chi})$ a strong solution in the sense of Definition 2.5. Then the relative energy-inequality*

$$
\begin{aligned}
\mathcal{R}(\boldsymbol{u}, \chi, \boldsymbol{u}_t | \tilde{\boldsymbol{u}}, \tilde{\chi}, \tilde{\boldsymbol{u}}_t)(t) + \int_0^t \left[\mathcal{W}(\chi, \boldsymbol{u}_t, \chi_t | \tilde{\boldsymbol{u}}_t, \tilde{\chi}_t) - \int_\Omega a'(\chi) \tilde{\chi}_t \mathbb{C} \varepsilon(\boldsymbol{u} - \tilde{\boldsymbol{u}}) : \varepsilon(\boldsymbol{u} - \tilde{\boldsymbol{u}}) \,\mathrm{d}x \right] \mathrm{e}^{\int_s^t \mathcal{K}(\tilde{\boldsymbol{u}}, \tilde{\chi}) \,\mathrm{d}\tau} \,\mathrm{d}s \\
\le \mathcal{R}(\boldsymbol{u}(0), \chi(0), \boldsymbol{u}_t(0) | \tilde{\boldsymbol{u}}(0), \tilde{\chi}(0), \tilde{\boldsymbol{u}}_t(0)) \mathrm{e}^{\int_0^t \mathcal{K}(\tilde{\boldsymbol{u}}, \tilde{\chi}) \,\mathrm{d}s}
\end{aligned}
\tag{REI}
$$

holds for a.e. $t \in (0,T)$*, where* $\mathcal{K}$ *is given by*

$$\mathcal{K}(\tilde{\boldsymbol{u}},\tilde{\chi}) := C_{\mathrm{REI}}\left(\|\tilde{\chi}_t\|_{L^{3/2}(\Omega)}+\|\varepsilon(\tilde{\boldsymbol{u}}_t)\|^2_{L^3(\Omega)}+\ell^2+\|\varepsilon(\tilde{\boldsymbol{u}})\|^2_{L^\infty(\Omega)}+\|\varepsilon(\tilde{\boldsymbol{u}})\|^2_{L^3(\Omega)}+\|\varepsilon(\tilde{\boldsymbol{u}})\|^4_{L^6(\Omega)}\right) \tag{5.3}$$

for some positive constant $C_{\mathrm{REI}} > 0$ *only depending on the problem data.*

Proof. For the elastic energy density $\frac{1}{2}a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u})$, we observe by some calculations

$$\begin{aligned}\frac{1}{2}\int_\Omega a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}}){:}\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}})\,\mathrm{d}x = {}&\frac{1}{2}\int_\Omega a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u})+a(\tilde{\chi})\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}})\,\mathrm{d}x\\ &-\frac{1}{2}\int_\Omega 2a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\tilde{\boldsymbol{u}}) - (a(\chi)-a(\tilde{\chi}))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}})\,\mathrm{d}x\,.\end{aligned}$$

We now evaluate the second line between 0 and t. We have

$$\begin{aligned}&-\frac{1}{2}\int_\Omega [2a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\tilde{\boldsymbol{u}})-(a(\chi)-a(\tilde{\chi}))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}})\,\mathrm{d}x]\Big|_0^t\\ &\quad= -\int_0^t\int_\Omega\Big[a'(\chi)\chi_t\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\tilde{\boldsymbol{u}})-\frac{1}{2}\partial_t(a(\chi)-a(\tilde{\chi}))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}})\Big]\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad-\int_0^t\int_\Omega\Big[a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}_t){:}\varepsilon(\tilde{\boldsymbol{u}})+a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\tilde{\boldsymbol{u}}_t)-(a(\chi)-a(\tilde{\chi}))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}}_t)\Big]\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

and with algebraic manipulations we easily obtain

$$\begin{aligned}&= -\int_0^t\int_\Omega\Big[\frac{1}{2}\chi_t a'(\tilde{\chi})\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}})+\frac{1}{2}\tilde{\chi}_t a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u})\Big]\,\mathrm{d}x\,\mathrm{d}s\\ &\quad-\int_0^t\int_\Omega\Big[a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\tilde{\boldsymbol{u}}_t)+a(\tilde{\chi})\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\boldsymbol{u}_t)\Big]\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega\Big[\frac{1}{2}\partial_t(a(\chi)-a(\tilde{\chi}))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}})-a'(\chi)\chi_t\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\tilde{\boldsymbol{u}})\Big]\,\mathrm{d}x\,\mathrm{d}s\\ &\quad-\int_0^t\int_\Omega\Big[a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}_t){:}\varepsilon(\tilde{\boldsymbol{u}})+a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\tilde{\boldsymbol{u}}_t)-(a(\chi)-a(\tilde{\chi}))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}}_t)\Big]\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega\Big[\frac{1}{2}\chi_t a'(\tilde{\chi})\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}})+\frac{1}{2}\tilde{\chi}_t a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u})\Big]\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega\Big[a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\tilde{\boldsymbol{u}}_t)+a(\tilde{\chi})\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\boldsymbol{u}_t)\Big]\,\mathrm{d}x\,\mathrm{d}s\\ &= -\int_0^t\int_\Omega\Big[\frac{1}{2}\chi_t a'(\tilde{\chi})\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}})+\frac{1}{2}\tilde{\chi}_t a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u})\Big]\,\mathrm{d}x\,\mathrm{d}s\\ &\quad-\int_0^t\int_\Omega\Big[a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\tilde{\boldsymbol{u}}_t)+a(\tilde{\chi})\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\boldsymbol{u}_t)\Big]\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega\Big[\frac{1}{2}\partial_t(a(\chi)-a(\tilde{\chi}))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}})-a'(\chi)\chi_t\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\tilde{\boldsymbol{u}})\Big]\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega\Big[a(\tilde{\chi})\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\boldsymbol{u}_t)-a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}_t){:}\varepsilon(\tilde{\boldsymbol{u}})+(a(\chi)-a(\tilde{\chi}))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}}_t)\Big]\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_\Omega\frac{1}{2}\Big[\chi_t a'(\tilde{\chi})\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\tilde{\boldsymbol{u}})+\tilde{\chi}_t a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}){:}\varepsilon(\boldsymbol{u})\Big]\,\mathrm{d}x\,\mathrm{d}s\,.\end{aligned}$$

For the last three lines, we observe

$$
\begin{aligned}
&\frac{1}{2}\partial_t(a(\chi)-a(\tilde\chi))\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u}) - a'(\chi)\chi_t\mathbb{C}\varepsilon(\boldsymbol u):\varepsilon(\tilde{\boldsymbol u})\\
&+ a(\tilde\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\boldsymbol u_t) - a(\chi)\mathbb{C}\varepsilon(\boldsymbol u_t):\varepsilon(\tilde{\boldsymbol u}) + (a(\chi)-a(\tilde\chi))\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u}_t)\\
&+\frac{1}{2}\left(\chi_t a'(\tilde\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u}) + \tilde\chi_t a'(\chi)\mathbb{C}\varepsilon(\boldsymbol u):\varepsilon(\boldsymbol u)\right)\\
&= (a(\chi)-a(\tilde\chi))\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u}_t) - a(\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\boldsymbol u_t) + a(\tilde\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\boldsymbol u_t)\\
&+\frac{1}{2}\left[(a'(\chi)\chi_t - a'(\tilde\chi)\tilde\chi_t)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u}) - 2a'(\chi)\chi_t\mathbb{C}\varepsilon(\boldsymbol u):\varepsilon(\tilde{\boldsymbol u})\right]\\
&+\frac{1}{2}\left[a'(\chi)\tilde\chi_t\mathbb{C}\varepsilon(\boldsymbol u):\varepsilon(\boldsymbol u) + a'(\tilde\chi)\chi_t\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u})\right] .
\end{aligned}
$$

All in all, we have calculated

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega a(\chi)\mathbb{C}\varepsilon(\boldsymbol u-\tilde{\boldsymbol u}):\varepsilon(\boldsymbol u-\tilde{\boldsymbol u})\,\mathrm{d}x\Big|_0^t\\
&=\frac{1}{2}\int_\Omega a(\chi)\mathbb{C}\varepsilon(\boldsymbol u):\varepsilon(\boldsymbol u)+a(\tilde\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u})\,\mathrm{d}x\Big|_0^t\\
&-\int_0^t\int_\Omega\Big[\frac{1}{2}\chi_t a'(\tilde\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u})+\frac{1}{2}\tilde\chi_t a'(\chi)\mathbb{C}\varepsilon(\boldsymbol u):\varepsilon(\boldsymbol u)\Big]\,\mathrm{d}x\,\mathrm{d}s\\
&-\int_0^t\int_\Omega\Big[a(\chi)\mathbb{C}\varepsilon(\boldsymbol u):\varepsilon(\tilde{\boldsymbol u}_t)+a(\tilde\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\boldsymbol u_t)\Big]\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\int_\Omega\Big[(a(\chi)-a(\tilde\chi))\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u}_t)-a(\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\boldsymbol u_t)+a(\tilde\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\boldsymbol u_t)\Big]\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\int_\Omega\frac{1}{2}\left[(a'(\chi)\chi_t-a'(\tilde\chi)\tilde\chi_t)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u})-2a'(\chi)\chi_t\mathbb{C}\varepsilon(\boldsymbol u):\varepsilon(\tilde{\boldsymbol u})\right]\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\int_\Omega\frac{1}{2}\left[a'(\chi)\tilde\chi_t\mathbb{C}\varepsilon(\boldsymbol u):\varepsilon(\boldsymbol u)+a'(\tilde\chi)\chi_t\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u})\right]\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}
$$

Concerning the nonlinear potential W, we find

$$
\begin{aligned}
&\int_\Omega W(\chi)-W(\tilde\chi)-W'(\tilde\chi)(\chi-\tilde\chi)\,\mathrm{d}x\Big|_0^t\\
&\quad=\int_\Omega\big[W(\chi)+W(\tilde\chi)\big]\,\mathrm{d}x\Big|_0^t-\int_\Omega\big[2W(\tilde\chi)+W'(\tilde\chi)(\chi-\tilde\chi)\big]\,\mathrm{d}x\Big|_0^t\\
&\quad=\int_\Omega W(\chi)+W(\tilde\chi)\,\mathrm{d}x\Big|_0^t-\int_0^t\int_\Omega\big[W'(\tilde\chi)\tilde\chi_t+W'(\tilde\chi)\chi_t+W''(\tilde\chi)\tilde\chi_t(\chi-\tilde\chi)\big]\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_\Omega W(\chi)+W(\tilde\chi)\,\mathrm{d}x\Big|_0^t-\int_0^t\int_\Omega\big[W'(\chi)\tilde\chi_t+W'(\tilde\chi)\chi_t\big]\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+\int_0^t\int_\Omega\big[\tilde\chi_t\left(W'(\chi)-W'(\tilde\chi)-W''(\tilde\chi)(\chi-\tilde\chi)\right)\big]\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}
$$

For the remaining quadratic terms in the relative energy, we find

$$
\int_\Omega\big[\frac{1}{2}|\nabla\chi-\nabla\tilde\chi|^2+\frac{1}{2}|\boldsymbol u_t-\tilde{\boldsymbol u}_t|^2+\frac{\ell}{2}|\chi-\tilde\chi|^2\big]\,\mathrm{d}x\Big|_0^t
$$

$$
\begin{aligned}
&= \int_\Omega \Big[\frac{1}{2}|\nabla\chi|^2+\frac{1}{2}|\boldsymbol{u}_t|^2+\frac{1}{2}|\nabla\tilde\chi|^2+\frac{1}{2}|\tilde{\boldsymbol{u}}_t|^2\Big]\,\mathrm{d}x\Big|_0^t \\
&\quad + \int_0^t\int_\Omega \big[\chi_t\Delta\tilde\chi-\nabla\chi\cdot\nabla\tilde\chi_t\big]\,\mathrm{d}x\,\mathrm{d}s - \int_0^t \Big[\langle \boldsymbol{u}_{tt},\tilde{\boldsymbol{u}}_t\rangle+\int_\Omega \tilde{\boldsymbol{u}}_{tt}\cdot\boldsymbol{u}_t\,\mathrm{d}x\Big]\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega \ell(\chi_t-\tilde\chi_t)(\chi-\tilde\chi)\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}
$$

Moreover, we relate the relative dissipation (5.2) to the pseudo-potential $\mathcal{D}$ (2.8) (where γ_1 is set to 0 in view of the boundary condition (2.15)) via

$$
\begin{aligned}
\mathcal{W}(\chi,\boldsymbol{u}_t,\chi_t|\tilde{\boldsymbol{u}}_t,\tilde\chi_t) &= \mathcal{D}(\chi,\boldsymbol{u}_t,\chi_t)+\mathcal{D}(\tilde\chi,\tilde{\boldsymbol{u}}_t,\tilde\chi_t) - \int_\Omega \big[2\chi_t\tilde\chi_t+b(\chi)\mathbb{V}\varepsilon(\boldsymbol{u}_t):\varepsilon(\tilde{\boldsymbol{u}}_t)\big]\,\mathrm{d}x \\
&\quad - \int_\Omega \big[b(\tilde\chi)\mathbb{V}\varepsilon(\tilde{\boldsymbol{u}}_t):\varepsilon(\boldsymbol{u}_t)+(b(\tilde\chi)-b(\chi))\mathbb{V}\varepsilon(\tilde{\boldsymbol{u}}_t-\boldsymbol{u}_t):\varepsilon(\tilde{\boldsymbol{u}}_t)\big]\,\mathrm{d}x\,.
\end{aligned}
$$

Combining all the above calculations, we obtain (note that, we have $\gamma_2=0$ in $\mathcal{E}$ due to (2.15)),

$$
\begin{aligned}
&\mathcal{R}(\boldsymbol{u},\chi,\boldsymbol{u}_t|\tilde{\boldsymbol{u}},\tilde\chi,\tilde{\boldsymbol{u}}_t)\Big|_0^t + \int_0^t \mathcal{W}(\chi,\boldsymbol{u}_t,\chi_t|\tilde{\boldsymbol{u}}_t,\tilde\chi_t)\,\mathrm{d}s \\
&= \mathcal{E}(\boldsymbol{u},\chi,\boldsymbol{u}_t)\Big|_0^t + \int_0^t \mathcal{D}(\chi,\boldsymbol{u}_t,\chi_t)\,\mathrm{d}s + \mathcal{E}(\tilde{\boldsymbol{u}},\tilde\chi,\tilde{\boldsymbol{u}}_t)\Big|_0^t + \int_0^t \mathcal{D}(\tilde\chi,\tilde{\boldsymbol{u}}_t,\tilde\chi_t)\,\mathrm{d}s \\
&\quad - \int_0^t\int_\Omega \chi_t \boxed{\tilde\chi_t-\Delta\tilde\chi+\frac{1}{2}a'(\tilde\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}})+W'(\tilde\chi))}\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_0^t \boxed{\int_\Omega \chi_t\tilde\chi_t+\nabla\tilde\chi_t\cdot\nabla\chi+\frac{1}{2}a'(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u})\tilde\chi_t+W'(\chi)\tilde\chi_t\,\mathrm{d}x}\,\mathrm{d}s \\
&\quad - \int_0^t \boxed{\langle \boldsymbol{u}_{tt},\tilde{\boldsymbol{u}}_t\rangle+\int_\Omega \big[b(\chi)\mathbb{V}\varepsilon(\boldsymbol{u}_t):\varepsilon(\tilde{\boldsymbol{u}}_t)+a(\chi)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\tilde{\boldsymbol{u}}_t)\big]\,\mathrm{d}x}\,\mathrm{d}s \\
&\quad - \int_0^t \boxed{\int_\Omega \tilde{\boldsymbol{u}}_{tt}\cdot\boldsymbol{u}_t+b(\tilde\chi)\mathbb{V}\varepsilon(\tilde{\boldsymbol{u}}_t):\varepsilon(\boldsymbol{u}_t)+a(\tilde\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\boldsymbol{u}_t)\,\mathrm{d}x}\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega (b(\chi)-b(\tilde\chi))\mathbb{V}\varepsilon(\tilde{\boldsymbol{u}}_t):\varepsilon(\boldsymbol{u}_t-\tilde{\boldsymbol{u}}_t)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega \Big[\tilde\chi_t\,(W'(\chi)-W'(\tilde\chi)-W''(\tilde\chi)(\chi-\tilde\chi))+\ell(\chi_t-\tilde\chi_t)(\chi-\tilde\chi)\Big]\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega \Big[(a(\chi)-a(\tilde\chi))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}}_t)-a(\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\boldsymbol{u}_t)+a(\tilde\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\boldsymbol{u}_t)\Big]\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t\int_\Omega \Big[(a'(\chi)\chi_t-a'(\tilde\chi)\tilde\chi_t)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}})-2a'(\chi)\chi_t\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\tilde{\boldsymbol{u}})\Big]\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t\int_\Omega \Big[a'(\chi)\tilde\chi_t\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u})+a'(\tilde\chi)\chi_t\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}})\Big]\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}
$$

On the one hand, since $(\boldsymbol{u},\chi)$ is a weak solution in the sense of Definition 2.2, for the damage flow rule we have the one-sided variational inequality (2.11): thus, since $\tilde\chi_t\le 0$ a.e. in $\Omega\times(0,T)$, we find that the term in the **blue** box is negative. In turn, the terms in the **magenta** box equals $\langle \boldsymbol{f},\tilde{\boldsymbol{u}}_t\rangle_{H^1(\Omega)}$. On the other hand, since $(\tilde{\boldsymbol{u}},\tilde\chi)$ is a strong solution in the sense of Definition 2.5, the term in the **green** box is null a.e.

in $\Omega\times(0,T)$, whereas the term in the **red** box equals $\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x$. Hence, we find

$$
\begin{aligned}
&\mathcal{R}(\boldsymbol{u},\chi,\boldsymbol{u}_t|\tilde{\boldsymbol{u}},\tilde{\chi},\tilde{\boldsymbol{u}}_t)\Big|_0^t + \int_0^t \mathcal{W}(\chi,\boldsymbol{u}_t,\chi_t|\tilde{\boldsymbol{u}}_t,\tilde{\chi}_t)\,\mathrm{d}s\\
&\quad\le \boxed{\mathcal{E}(\boldsymbol{u},\chi,\boldsymbol{u}_t)\Big|_0^t + \int_0^t \mathcal{D}(\chi,\boldsymbol{u}_t,\chi_t)\,\mathrm{d}s - \int_0^t \langle \boldsymbol{f},\tilde{\boldsymbol{u}}_t\rangle_{H^1(\Omega)}\,\mathrm{d}s}\\
&\qquad+ \boxed{\mathcal{E}(\tilde{\boldsymbol{u}},\tilde{\chi},\tilde{\boldsymbol{u}}_t)\Big|_0^t + \int_0^t \mathcal{D}(\tilde{\chi},\tilde{\boldsymbol{u}}_t,\tilde{\chi}_t)\,\mathrm{d}s - \int_0^t\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}s}\\
&\qquad+ \int_0^t\int_\Omega (b(\chi)-b(\tilde{\chi}))\mathbb{V}\varepsilon(\tilde{\boldsymbol{u}}_t):\varepsilon(\boldsymbol{u}_t-\tilde{\boldsymbol{u}}_t)\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+ \int_0^t\int_\Omega \left[\tilde{\chi}_t\,(W'(\chi)-W'(\tilde{\chi})-W''(\tilde{\chi})(\chi-\tilde{\chi})) + \ell(\chi_t-\tilde{\chi}_t)(\chi-\tilde{\chi})\right]\mathrm{d}x\,\mathrm{d}s\\
&\qquad+ \int_0^t\int_\Omega \left[(a(\chi)-a(\tilde{\chi}))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}}_t) - a(\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\boldsymbol{u}_t) + a(\tilde{\chi})\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\boldsymbol{u}_t)\right]\mathrm{d}x\,\mathrm{d}s\\
&\qquad+ \frac{1}{2}\int_0^t\int_\Omega \left[(a'(\chi)\chi_t - a'(\tilde{\chi})\tilde{\chi}_t)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}}) - 2a'(\chi)\chi_t\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\tilde{\boldsymbol{u}})\right]\mathrm{d}x\,\mathrm{d}s\\
&\qquad+ \frac{1}{2}\int_0^t\int_\Omega \left[a'(\chi)\tilde{\chi}_t\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u}) + a'(\tilde{\chi})\chi_t\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}})\right]\mathrm{d}x\,\mathrm{d}s\\
&\quad\doteq I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7\,.
\end{aligned}
$$

Again, we use the fact that $(\boldsymbol{u},\chi)$ is a weak solution, and thus satisfies the energy-dissipation inequality (2.12) (with $g\equiv 0$, as we are confining the discussion to homogeneous Neumann boundary conditions), to conclude that the term in the **dark blue** box is negative. Analogously, since the strong solution $(\tilde{\boldsymbol{u}},\tilde{\chi})$ satisfies the energy-dissipation balance, we have that the term in the **orange** box is null.

We now calculate the integrands of $I_5+I_6+I_7$. Indeed, we have that

$$
\begin{aligned}
&(a(\chi)-a(\tilde{\chi}))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}}_t) - a(\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\boldsymbol{u}_t) + a(\tilde{\chi})\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\boldsymbol{u}_t)\\
&\quad= (a(\chi)-a(\tilde{\chi}))\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}}_t-\boldsymbol{u}_t)
\end{aligned}
$$

as well as

$$
\begin{aligned}
&(a'(\chi)\chi_t - a'(\tilde{\chi})\tilde{\chi}_t)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}}) - 2a'(\chi)\chi_t\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\tilde{\boldsymbol{u}}) + a'(\chi)\tilde{\chi}_t\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{u}) + a'(\tilde{\chi})\chi_t\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}})\\
&= a'(\chi)\tilde{\chi}_t\mathbb{C}\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}}):\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}}) + a'(\chi)\tilde{\chi}_t\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\tilde{\boldsymbol{u}}) + a'(\chi)\tilde{\chi}_t\mathbb{C}\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}})\\
&\quad+ a'(\chi)\chi_t\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}-\boldsymbol{u}):\varepsilon(\tilde{\boldsymbol{u}}) + a'(\tilde{\chi})(\chi_t-\tilde{\chi}_t)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}}) - a'(\chi)\chi_t\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\tilde{\boldsymbol{u}})\\
&= a'(\chi)\tilde{\chi}_t\mathbb{C}\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}}):\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}}) + a'(\chi)(\tilde{\chi}_t-\chi_t)\mathbb{C}\varepsilon(\boldsymbol{u}):\varepsilon(\tilde{\boldsymbol{u}})\\
&\quad+ a'(\chi)(\tilde{\chi}_t-\chi_t)\mathbb{C}\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}}) + a'(\tilde{\chi})(\chi_t-\tilde{\chi}_t)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}})\\
&= a'(\chi)\tilde{\chi}_t\mathbb{C}\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}}):\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}}) + 2a'(\chi)(\chi_t-\tilde{\chi}_t)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}-\boldsymbol{u}):\varepsilon(\tilde{\boldsymbol{u}})\\
&\quad+ (a'(\tilde{\chi})-a'(\chi))(\chi_t-\tilde{\chi}_t)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}):\varepsilon(\tilde{\boldsymbol{u}})\,.
\end{aligned}
$$

Since a is non-decreasing and $\tilde{\chi}_t\le 0$ a.e. in $\Omega\times(0,T)$, the first term on the right-hand side has a negative sign. Inserting everything back into the relative energy inequality, we find

$$
\begin{aligned}
&\mathcal{R}(\boldsymbol{u},\chi,\boldsymbol{u}_t|\tilde{\boldsymbol{u}},\tilde{\chi},\tilde{\boldsymbol{u}}_t)\Big|_0^t + \int_0^t \mathcal{W}(\chi,\boldsymbol{u}_t,\chi_t|\tilde{\boldsymbol{u}}_t,\tilde{\chi}_t) - \frac{1}{2}\int_\Omega a'(\chi)\tilde{\chi}_t\mathbb{C}\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}}):\varepsilon(\boldsymbol{u}-\tilde{\boldsymbol{u}})\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad\le \int_0^t\int_\Omega (b(\chi)-b(\tilde{\chi}))\mathbb{V}\varepsilon(\tilde{\boldsymbol{u}}_t):\varepsilon(\boldsymbol{u}_t-\tilde{\boldsymbol{u}}_t)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t\int_\Omega\left[\tilde\chi_t\left(W'(\chi)-W'(\tilde\chi)-W''(\tilde\chi)(\chi-\tilde\chi)\right)+\ell(\chi_t-\tilde\chi_t)(\chi-\tilde\chi)\right]\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\int_\Omega(a(\chi)-a(\tilde\chi))\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u}_t-\boldsymbol u_t)\,\mathrm{d}x\,\mathrm{d}s\\
&+\frac12\int_0^t\int_\Omega\left[2a'(\chi)(\chi_t-\tilde\chi_t)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}-\boldsymbol u):\varepsilon(\tilde{\boldsymbol u})+(a'(\tilde\chi)-a'(\chi))(\chi_t-\tilde\chi_t)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u})\right]\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}
$$

The right-hand side will be estimated by the relative energy $\mathcal{R}$. Indeed, it holds

$$
\begin{aligned}
&\mathcal{R}(\boldsymbol u,\chi,\boldsymbol u_t|\tilde{\boldsymbol u},\tilde\chi,\tilde{\boldsymbol u}_t)\Big|_0^t+\int_0^t\mathcal{W}(\chi,\boldsymbol u_t,\chi_t|\tilde{\boldsymbol u}_t,\tilde\chi_t)-\frac12\int_\Omega a'(\chi)\tilde\chi_t\mathbb{C}\varepsilon(\boldsymbol u-\tilde{\boldsymbol u}):\varepsilon(\boldsymbol u-\tilde{\boldsymbol u})\,\mathrm{d}x\,\mathrm{d}s\\
&\quad\le\int_0^t\|b(\chi)-b(\tilde\chi)\|_{L^6(\Omega)}\|\mathbb{V}\varepsilon(\tilde{\boldsymbol u}_t)\|_{L^3(\Omega)}\|\varepsilon(\boldsymbol u_t-\tilde{\boldsymbol u}_t)\|_{L^2(\Omega)}\,\mathrm{d}s\\
&\qquad+\int_0^t\left\|\tilde\chi_t\int_0^1W''(\tilde\chi+\rho(\chi-\tilde\chi))\,\mathrm{d}\rho\right\|_{L^{3/2}(\Omega)}\|\chi-\tilde\chi\|^2_{L^6(\Omega)}\,\mathrm{d}s\\
&\qquad+\ell\int_0^t\|\chi_t-\tilde\chi_t\|_{L^2(\Omega)}\|\chi-\tilde\chi\|_{L^2(\Omega)}\,\mathrm{d}s\\
&\qquad+\int_0^t\|\mathbb{C}\varepsilon(\tilde{\boldsymbol u})\|_{L^3(\Omega)}\|a(\chi)-a(\tilde\chi)\|_{L^6(\Omega)}\|\varepsilon(\boldsymbol u_t-\tilde{\boldsymbol u}_t)\|_{L^2(\Omega)}\,\mathrm{d}s\\
&\qquad+\int_0^t\|a'(\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\boldsymbol u-\tilde{\boldsymbol u})\|_{L^2(\Omega)}\,\|\chi_t-\tilde\chi_t\|_{L^2(\Omega)}\,\mathrm{d}s\\
&\qquad+\int_0^t\|a'(\tilde\chi)-a'(\chi)\|_{L^3(\Omega)}\|\chi_t-\tilde\chi_t\|^2_{L^2(\Omega)}\|\mathbb{C}\varepsilon(\tilde{\boldsymbol u}):\varepsilon(\tilde{\boldsymbol u})\|^2_{L^6(\Omega)}\,\mathrm{d}s\\
&\quad\doteq I_8+I_9+I_{10}+I_{11}+I_{12}+I_{13}\,.
\end{aligned}
\tag{5.4}
$$

Now, since $b\in\mathrm{C}^1(\mathbb{R})$ and $\chi,\ \tilde\chi\in L^\infty(\Omega)$, we can estimate

$$
\begin{aligned}
I_8&\le c\int_0^t\|\chi-\tilde\chi\|_{L^6(\Omega)}\|\varepsilon(\tilde{\boldsymbol u}_t)\|_{L^3(\Omega)}\|\varepsilon(\boldsymbol u_t-\tilde{\boldsymbol u}_t)\|_{L^2(\Omega)}\,\mathrm{d}s\\
&\le\frac14\int_0^t\int_\Omega b(\chi)\mathbb{V}\varepsilon(\boldsymbol u_t-\tilde{\boldsymbol u}_t):\varepsilon(\boldsymbol u_t-\tilde{\boldsymbol u}_t)\,\mathrm{d}x\,\mathrm{d}s+c\int_0^t\|\varepsilon(\tilde{\boldsymbol u}_t)\|^2_{L^3(\Omega)}\|\chi-\tilde\chi\|^2_{L^6(\Omega)}\,\mathrm{d}s\,,
\end{aligned}
$$

where we have used the lower bound b, implying

$$
\|\varepsilon(\boldsymbol u_t-\tilde{\boldsymbol u}_t)\|^2_{L^2(\Omega)}\le c\int_\Omega b(\chi)\mathbb{V}\varepsilon(\boldsymbol u_t-\tilde{\boldsymbol u}_t):\varepsilon(\boldsymbol u_t-\tilde{\boldsymbol u}_t)\,\mathrm{d}x\,.
$$

Similarly, relying on the fact that $W\in\mathrm{C}^2(\mathbb{R})$, we check that

$$
I_9\le c\int_0^t\|\tilde\chi_t\|_{L^{3/2}(\Omega)}\,\|\chi-\tilde\chi\|^2_{L^6(\Omega)}\,\mathrm{d}s\,,
$$

while we obviously have

$$
I_{10}\le\frac14\int_0^t\|\chi_t-\tilde\chi_t\|^2_{L^2(\Omega)}\,\mathrm{d}s+c\ell^2\int_0^t\|\chi-\tilde\chi\|^2_{L^2(\Omega)}\,\mathrm{d}s\,.
$$

Relying now on the fact that $a\in\mathrm{C}^1(\mathbb{R})$, we may estimate

$$
\begin{aligned}
I_{11}&\le\int_0^t\|\varepsilon(\tilde{\boldsymbol u})\|_{L^3(\Omega)}\|\chi-\tilde\chi\|_{L^6(\Omega)}\|\varepsilon(\boldsymbol u_t-\tilde{\boldsymbol u}_t)\|_{L^2(\Omega)}\,\mathrm{d}s\\
&\le\frac14\int_0^t\int_\Omega b(\chi)\mathbb{V}\varepsilon(\boldsymbol u_t-\tilde{\boldsymbol u}_t):\varepsilon(\boldsymbol u_t-\tilde{\boldsymbol u}_t)\,\mathrm{d}x\,\mathrm{d}s+c\int_0^t\|\varepsilon(\tilde{\boldsymbol u})\|^2_{L^3(\Omega)}\|\chi-\tilde\chi\|^2_{L^6(\Omega)}\,\mathrm{d}s
\end{aligned}
$$

The assumptions on a' imply the estimate

$$\begin{aligned}
&\int_\Omega \left|a'(\chi)\mathbb{C}\varepsilon(\tilde{\boldsymbol{u}}){:}\varepsilon(\boldsymbol{u}{-}\tilde{\boldsymbol{u}})\right|^2 \,\mathrm{d}x \\
&\le \|a'(\chi)\|^2_{L^\infty(\Omega)}|\mathbb{C}|^2\|\varepsilon(\tilde{\boldsymbol{u}})\|^2_{L^\infty(\Omega)}\int_\Omega |\varepsilon(\boldsymbol{u}{-}\tilde{\boldsymbol{u}})|^2\,\mathrm{d}x \\
&\le c\|a'(\chi)\|^2_{L^\infty(\Omega)}\|\varepsilon(\tilde{\boldsymbol{u}})\|^2_{L^\infty(\Omega)}\|\varepsilon(\boldsymbol{u}(0)){-}\tilde{\boldsymbol{u}}(0)\|^2_{L^2(\Omega)} \\
&\quad + c\|a'(\chi)\|^2_{L^\infty(\Omega)}\|\varepsilon(\tilde{\boldsymbol{u}})\|^2_{L^\infty(\Omega)}\int_0^s \|\varepsilon(\boldsymbol{u}_t{-}\tilde{\boldsymbol{u}}_t)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \\
&\doteq M(\chi,\tilde{\boldsymbol{u}},\boldsymbol{u},\tilde{\boldsymbol{u}}_t)\,,
\end{aligned}$$

and therefore we have

$$I_{12} \le \frac{1}{4}\int_0^t \|\chi_t-\tilde{\chi}_t\|^2_{L^2(\Omega)}\,\mathrm{d}s + M(\chi,\tilde{\boldsymbol{u}},\boldsymbol{u},\tilde{\boldsymbol{u}}_t)\,.$$

Finally, we estimate

$$I_{13} \le \frac{1}{4}\int_0^t \|\chi_t-\tilde{\chi}_t\|^2_{L^2(\Omega)}\,\mathrm{d}s + c\int_0^t \|\varepsilon(\tilde{\boldsymbol{u}})\|^4_{L^6(\Omega)}\|\tilde{\chi}{-}\chi\|^2_{L^3(\Omega)}\,\mathrm{d}s\,.$$

Inserting all the above estimates in (5.4), we ultimately deduce

$$\begin{aligned}
&\mathcal{R}(\boldsymbol{u},\chi,\boldsymbol{u}_t|\tilde{\boldsymbol{u}},\tilde{\chi},\tilde{\boldsymbol{u}}_t)\Big|_0^t + \int_0^t \mathcal{W}(\chi,\boldsymbol{u}_t,\chi_t|\tilde{\boldsymbol{u}}_t,\tilde{\chi}_t) - \int_\Omega a'(\chi)\tilde{\chi}_t\mathbb{C}\varepsilon(\boldsymbol{u}{-}\tilde{\boldsymbol{u}}){:}\varepsilon(\boldsymbol{u}{-}\tilde{\boldsymbol{u}})\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_0^t \left[\frac{b}{2}\int_\Omega \mathbb{V}{:}\varepsilon(\boldsymbol{u}_t{-}\tilde{\boldsymbol{u}}_t)){:}\varepsilon(\boldsymbol{u}_t{-}\tilde{\boldsymbol{u}}_t)\,\mathrm{d}x\,\mathrm{d}s{+}\frac{1}{2}\|\chi_t{-}\tilde{\chi}_t\|^2_{L^2(\Omega)}\right]\mathrm{d}s \\
&\quad + c\int_0^t \left(\|\tilde{\chi}_t\|_{L^{3/2}(\Omega)}{+}\|\varepsilon(\tilde{\boldsymbol{u}}_t)\|^2_{L^3(\Omega)}{+}\ell^2{+}\|\varepsilon(\tilde{\boldsymbol{u}})\|^2_{L^3(\Omega)}{+}\|\varepsilon(\tilde{\boldsymbol{u}})\|^4_{L^6(\Omega)}\right)\|\chi{-}\tilde{\chi}\|^2_{L^6(\Omega)}\,\mathrm{d}s \\
&\quad + c\|a'(\chi)\|^2_{L^\infty(\Omega)}\|\varepsilon(\tilde{\boldsymbol{u}})\|^2_{L^\infty(\Omega)}\|\varepsilon(\boldsymbol{u}(0)){-}\tilde{\boldsymbol{u}}(0)\|^2_{L^2(\Omega)} \\
&\qquad + c\|a'(\chi)\|^2_{L^\infty(\Omega)}\|\varepsilon(\tilde{\boldsymbol{u}})\|^2_{L^\infty(\Omega)}\int_0^t\int_0^s\int_\Omega b(\chi)\mathbb{V}\varepsilon(\boldsymbol{u}_t{-}\tilde{\boldsymbol{u}}_t){:}\varepsilon(\boldsymbol{u}_t{-}\tilde{\boldsymbol{u}}_t)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s\,.
\end{aligned}$$

Then, estimate (REI) follows by Gronwall's inequality. □

Conclusion of the proof of Theorem 2.12: Let $(\boldsymbol{u},\chi)$ and $(\tilde{\boldsymbol{u}},\tilde{\chi})$ be a weak and a strong solution pair, respectively, emanating from the same initial data. Then, the right-hand side of estimate (REI) is null. We thus conclude that $\mathcal{R}(\boldsymbol{u}(t),\chi(t),\boldsymbol{u}_t(t)|\tilde{\boldsymbol{u}}(t),\tilde{\chi}(t),\tilde{\boldsymbol{u}}_t(t)) \equiv 0$ for almost all $t \in (0,T)$, which obviously yields $\boldsymbol{u} \equiv \tilde{\boldsymbol{u}}$ and $\chi \equiv \tilde{\chi}$. □

A Elliptic regularity results

The main result of this section, Corollary A.3 below, collects the two key elliptic regularity estimates for the momentum balance, which are at the core of our analysis of strong solutions. Corollary A.3 follows from the following

Proposition A.1. *Let* $\mathbb{C} \in \mathbb{R}^{d\times d\times d\times d}$ *fulfill the assumptions* (2.1) *of Hypothesis* ***A*** *and let the domain*

$\Omega \subset \mathbb{R}^d$ *fulfill* (H_Ω)*. Then, there exists a constant* $C_{\mathrm{ER}} > 0$ *such that for any* $\boldsymbol{h} \in H^1(\Omega;\mathbb{R}^d)$ *the solution* $\boldsymbol{y}$ *to the boundary-value problem*

$$-\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{y})) = \boldsymbol{h} \quad \text{in } \Omega\,, \qquad \int_\Omega \boldsymbol{y}\,\mathrm{d}x = 0\,, \quad \boldsymbol{n}\cdot\mathbb{C}\varepsilon(\boldsymbol{y}) = 0 \quad \text{on } \partial\Omega\,. \tag{A.1}$$

fulfills $\boldsymbol{y} \in H^3(\Omega)$*, and there holds*

$$\|\boldsymbol{y}\|_{H^2(\Omega)} \le C_{\mathrm{ER}} \left(\|\boldsymbol{h}\|_{L^2(\Omega)} + \|\boldsymbol{y}\|_{H^1(\Omega)}\right) \tag{A.2a}$$

as well as

$$\|\boldsymbol{y}\|_{H^3(\Omega)} \le C_{\mathrm{ER}} \left(\|\boldsymbol{h}\|_{H^1(\Omega)} + \|\boldsymbol{y}\|_{H^1(\Omega)}\right)\,. \tag{A.2b}$$

In fact, the result of this proposition is a consequence of [7, Thm. 3.45], compare also to [7, Sec. 4.3b].

We will also resort to the following abstract version of Poincaré's inequality, see [17].

Lemma A.2. *Let* $\mathsf{V}, \mathsf{H}, \mathsf{W}, \mathsf{Z}$ *be four Hilbert spaces with* $\mathsf{V} \Subset \mathsf{H}$ *compactly. Let* $A : \mathsf{V} \to \mathsf{W}$ *and* $B : \mathsf{V} \to \mathsf{Z}$ *be linear and continuous operators such that*

- $\mathrm{Ker}(A) \cap \mathrm{Ker}(B) = \{0\}$*;*
- *there exists a positive constant* $C > 0$ *such that for all* $v \in \mathsf{V}$ *we have*

$$\|v\|_{\mathsf{V}} \le C\left(\|v\|_{\mathsf{H}} + \|Av\|_{\mathsf{W}}\right)\,. \tag{A.3}$$

Then,

$$\exists\, M > 0 \quad \forall\, v \in \mathsf{V}: \qquad \|v\|_{\mathsf{H}} \le M\left(\|Bv\|_{\mathsf{Z}} + \|Av\|_{\mathsf{W}}\right)\,,$$

so that $\|v\|_{\mathsf{V}}$ *is equivalent to* $\|Bv\|_{\mathsf{Z}} + \|Av\|_{\mathsf{W}}$*.*

We are now in a position to derive the following

Corollary A.3. *Under the assumptions of Proposition A.1, there exists a constant* $C_{\mathrm{ER}} > 0$ *such that for any* $\boldsymbol{h} \in H^1(\Omega;\mathbb{R}^d)$ *the solution* $\boldsymbol{y}$ *to the boundary-value problem* (A.1) *satisfies*

$$\|\boldsymbol{y}\|_{H^2(\Omega)} \le \widehat{C}_{\mathrm{ER}} \left(\|\boldsymbol{h}\|_{L^2(\Omega)} + \|\boldsymbol{y}\|_{L^2(\Omega)}\right)\,, \tag{A.4a}$$

$$\|\boldsymbol{y}\|_{H^3(\Omega)} \le \widehat{C}_{\mathrm{ER}} \left(\|\varepsilon(\boldsymbol{h})\|_{L^2(\Omega)} + \|\boldsymbol{y}\|_{H^1(\Omega)}\right)\,. \tag{A.4b}$$

Proof. $\triangleright$ (A.4a): We apply Lemma A.2 with the following choices: $\mathsf{V} = H^2(\Omega;\mathbb{R}^d)$, $\mathsf{H} = H^1(\Omega;\mathbb{R}^d)$, $\mathsf{W} = L^2(\Omega;\mathbb{R}^d) = \mathsf{Z}$, and $A\boldsymbol{y} = -\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{y}))$, $B\boldsymbol{y} = y$. Observe that (A.3) holds thanks to (A.2a). Then, $\|\boldsymbol{y}\|_{H^2}$ is equivalent to $\|\boldsymbol{y}\|_{L^2} + \|\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{y}))\|_{L^2}$.
$\triangleright$ (A.4b): We now apply Lemma A.2 with the very same choices for H, Z, and B, as in the previous lines, while we set $\mathsf{V} = H^3(\Omega;\mathbb{R}^d)$, $\mathsf{W} = L^2(\Omega;\mathbb{R}^{d\times d})$, and $A\boldsymbol{y} = \varepsilon(\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{y})))$. In this case, (A.3) reads

$$\|\boldsymbol{y}\|_{H^3} \le C\left(\|\boldsymbol{y}\|_{H^1} + \|\varepsilon(\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{y})))\|_{L^2}\right)\,,$$

which holds true thanks to (A.2b), taking into account that, again by a Korn-type inequality, $\|\varepsilon(\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{y})))\|_{L^2}$ controls $\|\nabla\cdot(\mathbb{C}\varepsilon(\boldsymbol{y}))\|_{H^1}$. Then, (A.4b) ensues. □

B Smoothening the Yosida approximation

Following, e.g., the lines of [18, Sec. 3], for a given convex function $\widehat{\beta} : \mathbb{R} \to \mathbb{R}$ with subdifferential $\beta = \partial\widehat{\beta} : \mathbb{R} \rightrightarrows \mathbb{R}$, and for a fixed $\delta \in (0,1)$, we define

$$\beta_\delta := \beta_\delta^{\mathrm{Y}} \star \varrho_\delta \tag{B.1}$$

where $\beta_\delta^{\mathrm{Y}}$ is the Yosida regularization of the maximal monotone operator β (we refer to, e.g., [5]) and

$$\varrho_\delta(x) := \tfrac{1}{\delta^2}\varrho\left(\tfrac{x}{\delta^2}\right) \qquad \text{with} \begin{cases} \varrho \in \mathrm{C}^\infty(\mathbb{R}), \\ \|\varrho\|_{L^1(\mathbb{R})} = 1, \\ \mathrm{supp}(\varrho) \subset [-1,1]. \end{cases} \tag{B.2}$$

Thus, $\beta_\delta \in \mathrm{C}^\infty(\mathbb{R})$ and it has been shown in [18] that

$$\|\beta_\delta'\|_{L^\infty(\mathbb{R})} \le \frac{1}{\delta}, \qquad |\beta_\delta(x) - \beta_\delta^{\mathrm{Y}}(x)| \le \delta \text{ for all } x \in \mathbb{R}\,. \tag{B.3a}$$

Taking into account the properties of the Yosida approximation we also deduce that

$$|\beta_\delta(x)| \le |\beta^o(x)| + \delta \qquad \text{with } |\beta^o(x)| = \inf\{|y| \,:\, y \in \beta(x)\}\,. \tag{B.3b}$$

Furthermore, β_δ admits a convex potential $\widehat{\beta}_\delta$ satisfying, as a consequence of (B.3a), (below $\widehat{\beta}_\delta^{\mathrm{Y}}$ denotes the Yosida approximation of $\widehat{\beta}$):

$$-\delta|x| \le \widehat{\beta}_\delta^{\mathrm{Y}}(x) - \delta|x| \le \widehat{\beta}_\delta(x) \le \widehat{\beta}_\delta^{\mathrm{Y}}(x) + \delta|x| \le \widehat{\beta}(x) + \delta|x| \text{ and } \widehat{\beta}_\delta(x) \to \widehat{\beta}(x) \text{ for all } x \in \mathbb{R}\,. \tag{B.3c}$$

We also point out that the following analogue of Minty's trick holds: given $I \subset \mathbb{R}$ and sequence $(v_\delta)_\delta\, v,\ \beta \in L^2(I;\mathbb{R})$ such that $v_\delta \rightharpoonup v$ and $\beta_\delta(v_\delta) \rightharpoonup \beta$ in $L^2(I)$,

$$\limsup_{\delta\to 0^+} \int_I \beta_\delta(v_\delta)v_\delta \,\mathrm{d}x \le \int_I \beta v \,\mathrm{d}x \quad \Longrightarrow \quad \beta \in \partial\widehat{\beta}(v) \text{ a.e. in } I. \tag{B.3d}$$

We conclude this section with a new result, ensuring an additional estimate for ${\beta_\delta}''$.

Lemma B.1. *The function* β_δ *from* (B.1) *fulfills*

$$|\beta_\delta''(x)| \le \frac{\widehat{C}_\rho}{\delta^3} \qquad \textit{for all } x \in \mathbb{R} \tag{B.4}$$

with $\widehat{C}_\rho = \|\varrho'\|_{L^1(\mathbb{R})}$.

Proof. We have

$$\beta_\delta'(x) = \int_{-\delta^2}^{\delta^2} \varrho_\delta(y)(\beta_\delta^{\mathrm{Y}})'(x-y)\,\mathrm{d}y = -\int_{x-\delta^2}^{x+\delta^2} \varrho_\delta(x-y)(\beta_\delta^{\mathrm{Y}})'(y)\,\mathrm{d}y\,.$$

Therefore, by the first of (B.3a) we have

$$\begin{aligned} \beta_\delta''(x) = -\int_{x-\delta^2}^{x+\delta^2} \varrho_\delta'(x-y)(\beta_\delta^{\mathrm{Y}})'(y)\,\mathrm{d}y &\le \frac{1}{\delta}\left|\int_{\mathbb{R}} \varrho_\delta'(x-y)\,\mathrm{d}y\right| \\ &= \frac{1}{\delta}\left|\int_{\mathbb{R}} \frac{1}{\delta^4}\varrho'\left(\frac{y}{\delta^2}\right)\,\mathrm{d}y\right| = \frac{1}{\delta^3}\left|\int_{\mathbb{R}} \varrho'(z)\,\mathrm{d}z\right| \le \frac{\widehat{C}_\rho}{\delta^3}\,. \end{aligned}$$

□

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