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Abstract

Variational interpolants are an indispensable tool for the construction of gradient-flow solutions via the Minimizing Movement Scheme. De Giorgi's lemma provides the associated discrete energy-dissipation inequality. It was originally developed for metric gradient systems. Drawing from this theory we study the case of generalized gradient systems in Banach spaces, where a refined theory allows us to extend the validity of the discrete energy-dissipation inequality and to establish it as an equality. For the latter we have to impose the condition of radial differentiability of the dissipation potential. Several examples are discussed to show how sharp the results are.

1 Introduction

The Minimizing Movement Scheme (MMS) was introduced by De Giorgi in [De93] for constructing solutions for gradient flows in abstract spaces. Since then, the MMS has developed into a versatile tool for analyzing gradient systems in Hilbert spaces, Banach space, and metric spaces. In this paper, we address the specific tool called "variational interpolant", also called "De Giorgi's interpolant" that was first introduced in [Amb95, Lem. 2.5] and further developed in [AGS05]. A generalization to the Banach spaces was done in [MRS13a, Lem. 6.1]. Variational interpolants generalize the idea of piecewise affine interpolants in linear spaces, or geodesic interpolants in geodesic spaces, such that they are applicable in more general situations, namely in general metric spaces. However, even in the cases of geodesic spaces, including Banach and Hilbert spaces, they are useful if the energy functional is not geodesically semiconvex. In general, variational interpolants are no longer continuous in time and hence, the desired discrete energy-dissipation estimate is more difficult to obtain. It is exactly this estimate, which is established in the so-called "*De Giorgi's lemma*". The purpose of this paper is twofold: (i) we generalize the validity of the lemma in the Banach setup and (ii) we discuss the question why and when the discrete energy-dissipation estimate is an *equality*.

To be more precise, we now introduce our approach in more detail by comparing the theory in metric spaces $(M,\mathcal{D})$ and in Banach spaces $(X;\|\cdot\|)$ in parallel. Following [RMS08] (see also [Mie23]) we consider a generalized metric gradient system $(M,\mathcal{E},\mathcal{D},\psi)$, subsequently abbreviated by gMGS; see Definition 2.1 for the precise definition. For a given initial value $u^o\in M$ with $\mathcal{E}(u^o)<\infty$, it is the aim to construct a curve $u:[0,\infty[\,\to M$ of maximal slope emanating from u^o, i.e. u must satisfy for all $t>0$

$$\mathcal{E}(u(t))+\int_0^t\Big(\psi\big(|u'|(s)\big)+\psi^*\big(|\partial\mathcal{E}|(u(s))\big)\Big)\,\mathrm{d}s=\mathcal{E}(u(0))\quad\text{and}\quad u(0)=u^o, \tag{1.1a}$$

where $|u'|\geq 0$ denotes the metric speed of u and $|\partial\mathcal{E}|(u)\geq 0$ denotes the metric slope, see [AGS05]. The case of general dissipation functions $\psi:[0,\infty[\,\to[0,\infty[$ (lower semicontinuous, convex, $\psi(0)=0$), and superlinear) was introduced in [RMS08], the choice $\psi(r)=\frac{1}{2}r^2$ gives the

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classical notion of curve of maximal slope of [Amb95], while $\phi(r) = \frac{1}{p}r^p$ leads to p-curves of maximal slopes as in [AGS05].

For a generalized Banach-space gradient system $(X, \mathcal{E}, \mathcal{R})$, subsequently abbreviated by gBGS and precisely defined in Section 3.1, the aim is to find *energy-dissipation balance (EDB) solutions* $u : [0,\infty[\to X$, which are defined via the following identity

$$\mathcal{E}(u(t)) + \int_0^t \Big(\mathcal{R}\big(u'(s)\big) + \mathcal{R}^*(-\xi(s))\Big)\,\mathrm{d}s = \mathcal{E}(u(0)) \text{ for all } t>0, \qquad \text{and } \xi(s) \in \mathrm{D}\mathcal{E}(u(s)) \text{ for a.a. } s \geq 0, \tag{1.1b}$$

where u' is the distributional derivative of $u \in \mathrm{AC}([0,T];X)$, and $\xi :]0,\infty[\to X^*$ is a selection in the multivalued Fréchet subdifferential $\mathrm{D}\mathcal{E}(u) \subset X^*$ of $\mathcal{E}$, see Section 3.1.

With an initial value $u^o \in M$ and a time step $\tau > 0$ the metric and the Banach-space MMS are defined via $u_\tau^0 = u^o$ and

$$u_\tau^k \quad \text{minimizes} \quad M \ni u \;\mapsto\; \tau\,\psi\big(\tfrac{1}{\tau}\mathcal{D}(u, u_\tau^{k-1})\big) + \mathcal{E}(u) \qquad \text{for all } k \in \mathbb{N}; \tag{1.2a}$$

$$u_\tau^k \quad \text{minimizes} \quad X \ni u \;\mapsto\; \tau\,\mathcal{R}\big(\tfrac{1}{\tau}(u - u_\tau^{k-1})\big) + \mathcal{E}(u) \qquad \text{for all } k \in \mathbb{N}. \tag{1.2b}$$

Variational interpolants $\widetilde{u}^\tau$ are defined for all $t \in [0,\infty[$, satisfy $\widetilde{u}^\tau(k\tau) = u_\tau^k$, and are determined by a variational condition: for all $k \in \mathbb{N}_0$ and $\sigma \in]0,\tau[$, we ask for

$$\widetilde{u}^\tau(k\tau{+}\sigma) \quad \text{minimizes} \quad M \ni u \;\mapsto\; \sigma\psi\big(\tfrac{1}{\sigma}\mathcal{D}(u, u_\tau^{k-1})\big) + \mathcal{E}(u); \tag{1.3a}$$

$$\widetilde{u}^\tau(k\tau{+}\sigma) \quad \text{minimizes} \quad X \ni u \;\mapsto\; \sigma\,\mathcal{R}\big(\tfrac{1}{\sigma}(u - u_\tau^{k-1})\big) + \mathcal{E}(u). \tag{1.3b}$$

In general, one cannot hope to choose the variational interpolant $t \mapsto \widetilde{u}^\tau(t)$ as a continuous function. However, by classical selection theorems for measurable multivalued mappings, it is possible to choose a measurable selection, see Section 3.1.

De Giorgi's lemma, which was first published in [Amb95, Lem. 2.5], now provides a discrete counterpart to the energy-dissipation balances in (1.1a) and (1.1b), namely for all $\sigma \in]0,\tau]$ we have the so-called *De Giorgi's estimates*

$$\mathcal{E}\big(\widetilde{u}^\tau(k\tau{+}\sigma)\big) + \sigma\,\psi\big(\tfrac{1}{\sigma}\mathcal{D}(\widetilde{u}^\tau(k\tau{+}\sigma), u_\tau^k)\big) + \int_0^\sigma \psi^*\big(|\partial\mathcal{E}|(\widetilde{u}^\tau(k\tau{+}\rho)\big)\,\mathrm{d}\rho \leq \mathcal{E}(u_\tau^k); \tag{1.4a}$$

$$\mathcal{E}\big(\widetilde{u}^\tau(k\tau{+}\sigma)\big) + \sigma\,\mathcal{R}\big(\tfrac{1}{\sigma}(\widetilde{u}^\tau(k\tau{+}\sigma) - u_\tau^k)\big) + \int_0^\sigma \mathcal{R}^*\big(-\xi(k\tau{+}\rho))\big)\,\mathrm{d}\rho \leq \mathcal{E}(u_\tau^k) \qquad \text{for some } \xi(t) \in \mathrm{D}\mathcal{E}(\widetilde{u}^\tau(t)) \text{ for a.a. } t>0. \tag{1.4b}$$

For establishing (1.1) via a suitable limit passage, it would be enough to have (1.4) for $\sigma = \tau$, and then adding the results over all subintervals, but we will see that it is very instructive to keep $\sigma \in]0,\tau]$ on the left-hand side as an independent variable.

For general differentiable dissipation functions ψ, De Giorgi's estimate (1.4a) was first established in [RMS08, Lem. 4.5], while $\psi(r) = \frac{1}{p}r^p$ is treated in [AGS05]. The Banach-space case (1.4b) appears first in [MRS13a, Lem. 6.1], but the result therein relies on the *condition of radial differentiability* of $\mathcal{R}$, namely

$$\forall\, v \in X: \quad \text{the function }]0,\infty[\ni \lambda \mapsto \mathcal{R}(\lambda v) \text{ is differentiable.} \tag{1.5}$$

This condition is equivalent to the fact that for all $\xi_1, \xi_2 \in \partial\mathcal{R}(v)$ (with $\partial\mathcal{R} : X \rightrightarrows X^*$ the convex subdifferential of $\mathcal{R}$), there holds $\mathcal{R}^*(\xi_1) = \mathcal{R}^*(\xi_2)$, cf. Proposition 3.8.

So far, our general overview and introduction shows a complete analogy between the metric case and the Banach-space setting. Even the condition of radial differentiability of $\mathcal{R}$ corresponds to the condition of differentiability of ψ. However, the methods for establishing the so-called *De Giorgi's estimates* (1.4a) and (1.4b) involve quite different techniques. In particular, for gBGS we can exploit the linear structure of X and thus obtain an Euler-Lagrange equation for the minimizers $\widetilde{u}_\sigma := \widetilde{u}^\tau(k\tau{+}\sigma)$ (keep k fixed, w.l.o.g. $k=0$), namely

$$0 \in \partial\mathcal{R}\big(\tfrac{1}{\sigma}(\widetilde{u}_\sigma{-}u^k_\tau)\big) + \widetilde{\xi}_\sigma \quad \text{and} \quad \widetilde{\xi}_\sigma \in \mathrm{D}\mathcal{E}(\widetilde{u}_\sigma). \tag{1.6}$$

Indeed, to see a first nontrivial fact, we may assume that $\sigma \mapsto \widetilde{u}_\sigma$ and $\sigma \mapsto \mathcal{E}(\widetilde{u}_\sigma)$ are absolutely continuous and such that the chain rule relation $\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{E}(\widetilde{u}_\sigma) = \langle \widetilde{\xi}_\sigma, \frac{\mathrm{d}}{\mathrm{d}\sigma}\widetilde{u}_\sigma\rangle$ holds. Then, the Euler-Lagrange equation (1.6) gives the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{R}\big(\tfrac{1}{\sigma}(\widetilde{u}_\sigma{-}u^k_\tau)\big) = \big\langle -\widetilde{\xi}_\sigma, \frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}\widetilde{u}_\sigma - \frac{1}{\sigma^2}(\widetilde{u}_\sigma{-}u^k_\tau)\big\rangle.$$

Thus, differentiating the right-hand side of (1.4b) with respect to σ gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\sigma}\,\text{RHS(1.4b)} &= \big\langle \widetilde{\xi}_\sigma, \frac{\mathrm{d}}{\mathrm{d}\sigma}\widetilde{u}_\sigma\big\rangle + \mathcal{R}\big(\tfrac{1}{\sigma}(\widetilde{u}_\sigma{-}u^k_\tau)\big)\\ &\quad + \big\langle -\widetilde{\xi}_\sigma, \frac{\mathrm{d}}{\mathrm{d}\sigma}\widetilde{u}_\sigma - \tfrac{1}{\sigma}(\widetilde{u}_\sigma{-}u^k_\tau)\big\rangle + \mathcal{R}^*\big(-\widetilde{\xi}_\sigma\big)\\ &= \mathcal{R}\big(\tfrac{1}{\sigma}(\widetilde{u}_\sigma{-}u^k_\tau)\big) + \mathcal{R}^*\big(-\widetilde{\xi}_\sigma\big) - \big\langle -\widetilde{\xi}_\sigma, \tfrac{1}{\sigma}(\widetilde{u}_\sigma{-}u^k_\tau)\big\rangle \overset{\text{(1.6)}}{=} 0,\end{aligned}$$

by the Fenchel equivalence $\mu \in \partial\mathcal{R}(v) \;\Leftrightarrow\; \mathcal{R}(v) + \mathcal{R}^*(\mu) = \langle \mu, v\rangle$.

This observation (which we shall revisit in Section 4.4), motivates our first main result, see Theorem 4.1, that De Giorgi's estimate (1.4b) is indeed an equality, then called *De Giorgi's identity*. This result is established for all measurable variational interpolants, under the sole additional assumption of radial differentiability of $\mathcal{R}$, cf. (1.5). However, for this version of De Giorgi's identity, it is essential to be more specific with the choice of $\widetilde{\xi}_\sigma \in \mathrm{D}\mathcal{E}(\widetilde{u}_\sigma)$: one has to restrict to those $\xi \in \mathrm{D}\mathcal{E}(\widetilde{u}_\sigma)$ that minimize $\mathcal{R}^*(-\xi)$ subject to the constraint of satisfying the Euler-Lagrange equation (1.6), see (3.14) for the precise definition. We refer to Example 3.6 for a very simple case, where this restriction is essential for the validity of De Giorgi's estimate as an identity.

For the case of a general $\mathcal{R}$, dropping radial differentiability we are able to establish De Giorgi's estimate (1.4b) if X is a reflexive Banach space, which is our second main result, see Theorem 4.12. For this, we use a Yosida-Moreau regularization $\mathcal{R}_\eta$ of $\mathcal{R}$ with an equivalent norm $\|\cdot\|$ such that $u \mapsto \|u\|^2$ is differentiable. Then, $\mathcal{R}_\eta$ is differentiable, in particular also radially differentiable, and for the corresponding gBGS $(X, \mathcal{E}, \mathcal{R}_\eta)$ De Giorgi's identity holds thanks to Theorem 4.1. It can be shown that in the limit passage $\eta \to 0^+$ De Giorgi's estimate survives.

While Section 3 introduces the definitions and conditions for the case of gradient systems in Banach space that will then be the focus of Section 4, we start in Section 2 with the metric case. The missing Euler-Lagrange equation is replaced by a purely metric identity, not involving the slope $|\partial\mathcal{E}|$ but rather the functions d^+ or d^- defined via

$$\begin{aligned}\mathsf{d}^-_\rho(u^o) &= \inf\big\{\, \mathcal{D}(u^o,u) \,\big|\, u \in \mathrm{Argmin}\big(\mathcal{D}(u^o,\cdot)^2/(2\rho) + \mathcal{E}(\cdot)\big)\,\big\},\\ \mathsf{d}^+_\rho(u^o) &= \sup\big\{\, \mathcal{D}(u^o,u) \,\big|\, u \in \mathrm{Argmin}\big(\mathcal{D}(u^o,\cdot)^2/(2\rho) + \mathcal{E}(\cdot)\big)\,\big\}.\end{aligned}$$

For gMGS with differentiable ψ, the *metric energy identity* takes the form

$$\mathcal{E}(\widetilde{u}_\sigma) + \sigma\,\psi\big(\tfrac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big) + \int_{\rho=0}^{\sigma} \psi^*\Big(\psi'\big(\tfrac{1}{\rho}\mathsf{d}^\pm_\rho(u^o)\big)\Big)\,\mathrm{d}\rho = \mathcal{E}(u^o). \tag{1.7}$$

For $\psi(r) = \frac{1}{p}r^p$ the identity was established in [AGS05, Thm. 3.1.4], whereas the general case is contained in [Mie23, Sec. 4.2].

Clearly, the metric De Giorgi's estimate (1.4a) follows easily from (1.7) by inserting the slope inequality

$$|\partial\mathcal{E}|(\widetilde{u}_\sigma) \leq \psi'\big(\tfrac{1}{\sigma}\mathcal{D}(\widetilde{u}_\sigma, u_\tau^k)\big)\,\big|\partial(-\mathcal{D}(u^o,\cdot)\big|(\widetilde{u}_\sigma) \leq \psi'\big(\tfrac{1}{\sigma}\mathcal{D}(\widetilde{u}_\sigma, u_\tau^k)\big), \tag{1.8}$$

see Proposition 2.9. The latter can be seen as a metric counterpart of the Euler-Lagrange equation. If one of the two inequalities in (1.8) is strict, then De Giorgi's identity is lost. The last inequality is strict, if $\mathcal{D}$ is not a length distance, which means that the metric De Giorgi estimate can only hold in geodesic spaces. However, even there the first inequality may be strict. In Theorem 2.14 we show that full continuity and a uniform slope estimate are sufficient to establish De Giorgi's identity in geodesic metric spaces.

2 The metric case

Following [Mie23], we specify in the following definition the notion of metric gradient system we will be working with hereafter.

Definition 2.1 *We call a quadruple $(M,\mathcal{E},\mathcal{D},\psi)$ a* generalized metric gradient system *(most often abbreviated to gMGS), if*

1 $(M,\mathcal{D})$ is a complete metric space;
2 $\mathcal{E}: M \to (-\infty,\infty]$ is a proper (i.e. with non-empty domain $\mathrm{dom}(\mathcal{E})$) lsc functional;
3 $\psi:\mathbb{R}\to[0,\infty)$ is proper, convex, with $\psi(0)=0$ and $\lim_{r\to\infty}\frac{\psi(r)}{r}=\infty$.

For later use, we introduce the energy sublevels

$$S_E := \{u \in M \,:\, \mathcal{E}(u) \leq E\}, \qquad E > 0. \tag{2.1}$$

Remark 2.2 *Most often, a gMGS is in fact individuated by a* quintuple *$(M,\mathscr{T},\mathcal{E},\mathcal{D},\psi)$, where, mimicking the setup considered in [AGS05], in addition to the topology induced by the metric $\mathcal{D}$, a second (Hausdorff) topology $\mathscr{T}$ is considered on M. Typically, $\mathscr{T}$ is related to 'coercivity' properties of the energy functional, as it turns out to be the topology w.r.t. which the sublevel sets S_E, or the sublevels of a perturbation of $\mathcal{E}$, are compact. Although weaker than the topology induced by $\mathcal{D}$, $\mathscr{T}$ is related to it by the following compatibility condition (here $\overset{\mathscr{T}}{\rightharpoonup}$ denotes convergence with respect to $\mathscr{T}$):*

$$(u_n,v_n) \overset{\mathscr{T}}{\rightharpoonup} (u,v) \implies \varliminf_{n\to\infty} \mathcal{D}(u_n,v_n) \geq \mathcal{D}(u,v).$$

Nonetheless, we have opted for omitting the role of $\mathscr{T}$ in the discussion of the metric case in order to avoid overburdening it, on the one hand, and to highlight the purely metric flavour of the arguments, on the other hand. Instead, in the Banach setup it will be convenient to encompass the weak topology in the picture.

In the setup of a gMGS $(M,\mathcal{E},\mathcal{D},\psi)$, the classical notion of curve of maximal slope is extended by the following definition (cf. [Mie23, Def. 4.8]). To simplify the arguments, we fix an arbitrary $T>0$ and confine the discussion to evolutions on the compact time interval $[0,T]$.

Definition 2.3 *Given a generalized metric gradient system* $(M, \mathcal{E}, \mathcal{D}, \psi)$*, we say that* $u : [0,T] \to M$ *is a* curve of maximal slope *if* $u \in \mathrm{AC}([0,T]; M)$ *and it satisfies for every* $0 \le s \le t \le T$

$$\mathcal{E}(u(t)) + \int_s^t \Big(\psi\big(|u'|(r)\big) + \psi^*\big(|\partial\mathcal{E}|(u(r))\big)\Big)\,\mathrm{d}r = \mathcal{E}(u(s))\,. \tag{2.2}$$

Remark 2.4 *In [RMS08, Prob. 2.6] an alternative definition for the above concept was given, claiming for the curve* $u \in \mathrm{AC}([0,T]; M)$ *the* pointwise estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u(t)) \le -\psi\big(|u'|(t)\big) - \psi^*\big(|\partial\mathcal{E}|(u(t))\big) \qquad \text{for a.a. } t \in (0,T). \tag{2.3}$$

In fact, if the slope $|\partial\mathcal{E}|$ *is a* strong upper gradient *according to the terminology of [AGS05] (namely, if a suitable* chain-rule *inequality holds along* u*), then* (2.3) *is in fact equivalent to the energy-dissipation balance* (2.2)*.*

The Minimizing Movement Scheme for constructing curves of maximal slope fulfilling the initial condition $u(0) = u_0$ for an assigned initial datum $u_0 \in M$ then reads as follows: given a time step $\tau > 0$ inducing a (uniform, without loss of generality) partition $\mathscr{P}_\tau = \{t_\tau^k\}_{k=1}^{K_\tau}$ of the interval $[0,T]$, starting from $u_\tau^0 := u_0$, find $(u_\tau^k)_{k=1}^{K_\tau}$ such that

$$u_\tau^k \quad \text{minimizes} \quad M \ni u \mapsto \left(\tau\psi\big(\frac{1}{\tau}\mathcal{D}(u_\tau^{k-1}, u)\big) + \mathcal{E}(u)\right) \quad \text{for } k \in \{1, \ldots, K_\tau\}\,. \tag{MMS}$$

That is why, from now on we will study the properties of the single-step minimum problem

$$\mathrm{Min}_{u\in M}\Phi_\sigma(u^o; u) \quad \text{with} \quad \Phi_\sigma(u^o; u) := \sigma\psi\big(\frac{1}{\sigma}\mathcal{D}(u^o, u)\big) + \mathcal{E}(u), \quad \sigma > 0, \tag{2.4}$$

for a fixed $u^o \in M$. We will also use the following notation

$$\phi(u^o;\sigma) = \inf\left\{\, \Phi_\sigma(u^o;u) \,\middle|\, u \in M \,\right\} \ \text{and}\ J_\sigma(u^o) = \mathrm{ArgMin}\left\{\, \Phi_\sigma(u^o;u) \,\middle|\, u \in M \,\right\}. \tag{2.5}$$

for the associated value functional and the set of minimizers (which we will assume non-empty, cf. (2.6) below). It is also significant to introduce the following quantities

$$\mathsf{d}_\sigma^-(u^o) := \inf\left\{\, \mathcal{D}(u^o,u) \,\middle|\, u \in J_\sigma(u^o) \,\right\} \quad \text{and} \quad \mathsf{d}_\sigma^+(u^o) := \sup\left\{\, \mathcal{D}(u^o,u) \,\middle|\, u \in J_\sigma(u^o) \,\right\}.$$

Throughout this section, we will work under the following assumptions.

Hypothesis 2.5 (Conditions for generalized metric gradient systems) *We assume that*

- $\psi \in \mathrm{C}^1(\mathbb{R})$ *is strictly convex;*
- $\mathcal{E}$ *is bounded from below by* E_0*, namely* $\inf_{u\in M}\mathcal{E}(u) > E_0 > 0$*;*
- *there exists* $\sigma_* > 0$ *such that*

$$J_\sigma(u^o) \neq \emptyset \ \text{for all } \sigma \in (0,\sigma_*) \ \text{and all } u^o \in \mathrm{dom}(\mathcal{E}). \tag{2.6}$$

Remark 2.6 *Whenever the generalized metric gradient system is individuated by a quintuple* $(M, \mathscr{T}, \mathcal{E}, \mathcal{D}, \psi)$ *such that the topology* $\mathscr{T}$ *is compatible with* $\mathcal{D}$*,* (2.6) *follows by a coercivity property of this type: there exists* $\sigma_* > 0$ *such that for all* $\sigma \in (0,\sigma_*)$ *and all* $u^o \in M$*, for any sequence* $(u_n)_n \subset M$

$$\sup_n \Phi_\sigma(u^o; u_n) < +\infty \implies (u_n)_n \text{ admits a } \mathscr{T}\text{-converging subsequence.} \tag{2.7}$$

Then, the direct method *yields the existence of minimizers for* (2.4)*.*

We can in fact enhance (2.6) by observing that

$$\text{there exists a } \textit{measurable} \text{ selection } (0,\sigma_*) \ni \sigma \mapsto \widetilde{u}_\sigma \in J_\sigma(u^o). \tag{2.8}$$

To show this, let us consider the multi-valued mapping $\Gamma : (0,\infty) \rightrightarrows X, \Gamma(\sigma) := J_\sigma(u^o)$. It is easy to check that Γ is upper semicontinuous from $\mathbb{R}$ to $(M,\mathcal{D})$ in that it fulfills for every $(\sigma_n)_n,\ \sigma \in (0,\infty)$

$$\sigma_n \to \sigma \implies \operatorname*{Ls}_{n\to\infty} \Gamma(\sigma_n) \subset \Gamma(\sigma)\,, \tag{2.9a}$$

where the *Kuratowski upper limit* (cf., e.g., [AmT04, Def. 4.4.13]) of the sequence of closed sets $(\Gamma(\sigma_n))_n$ is defined by

$$u \in \operatorname*{Ls}_{n\to\infty} \Gamma(\sigma_n) \iff \exists\,(\sigma_{n_k})_k,\ (u_k)_k \text{ with } \begin{cases} u_k \in \Gamma(\sigma_{n_k}) \text{ for all } k \in \mathbb{N}\,, \\ \lim_{k\to\infty} \mathcal{D}(u,u_k) = 0\,. \end{cases} \tag{2.9b}$$

Since Γ is upper semicontinuous, its graph is a Borel subset of $(0,\infty)\times X$, and the von Neumann-Aumann selection theorem [CaV77, Thm. 3.22] applies, yielding (2.8).

The variational interpolant. Thanks to condition (2.6) the MMS does admit solutions $(u_\tau^k)_{k=1}^{N_\tau}$. We are then in a position to precisely introduce the notion of interpolant of the values $(u_\tau^k)_{k=1}^{N_\tau}$ we will focus on hereafter.

Definition 2.7 (De Giorgi's variational interpolant) *We denote by $\widetilde{u}_\tau : [0,T] \to M$ any measurable function obtained by setting*

$$\begin{aligned} &\widetilde{u}_\tau(0) := u_\tau^0, \\ &\widetilde{u}_\tau(r) \in J_r(u_\tau^{k-1}) = \operatorname{ArgMin}\big\{\, \Phi_r(u_\tau^{k-1};u) \,\big|\, u \in M \,\big\} \quad \text{if } t = t_\tau^{k-1}{+}r. \end{aligned} \tag{2.10}$$

The cornerstone of the proof that, as $\tau \downarrow 0$, (a subsequence of) the sequence $(\widetilde{u}_\tau)_\tau$ converges to a curve of maximal slope is the discrete estimate obtained by applying De Giorgi's estimate (1.4a) to the interpolant $\widetilde{u}_\tau$. The next section revolves around the validity of (1.4a) as an equality.

2.1 Metric energy identity and De Giorgi's estimate: statements and examples

The following identity was established in [AGS05, Sec. 3.1, eqn. (3.1.27)] for the case $\psi(\delta) = \delta^p/p$ and in [Mie23, Thm. 4.17] for general differentiable scalar dissipation potentials ψ.

Proposition 2.8 (Metric energy identity) *Under Hypothesis 2.5, any measurable selection $(0,\sigma_*) \ni \sigma \to \widetilde{u}_\sigma \in J_\sigma(u^o)$ fulfills*

$$\mathcal{E}(\widetilde{u}_\sigma) + \sigma\,\psi\big(\frac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big) + \int_0^\sigma \psi^*\big(\psi'\big(\frac{1}{\rho}\mathsf{d}_\rho^\pm(u^o)\big)\big)\,\mathrm{d}\rho = \mathcal{E}(u^o). \tag{2.11}$$

There is a straightforward way to relate the metric energy identity to De Giorgi's estimate, and that is throughout the following result in which the slope at $\widetilde{u}_\sigma$ is estimated in terms of the slope of the distance function $\mathcal{D}(u^o,\cdot)$. In fact, (2.12) below extends to the setup of a gMGS $(M,\mathcal{E},\mathcal{D},\psi)$, the slope estimate proved in [AGS05, Lem. 3.1.3] in the quadratic case $\psi(r) = \frac{1}{2}r^2$. For completeness, we also record that a version of (2.12) was proved in [RMS08, Lemma 4.4] for non-differentiable dissipation potentials ψ. Note that this estimate is the metric counterpart of the Euler-Lagrange equation in the Banach-space setting, and hence it is less precise for a general gMGS.

Proposition 2.9 (Slope estimate) *Under Hypothesis 2.5,*

$$\big|\partial\mathcal{E}\big|(\widetilde{u}_\sigma) \le \psi'\big(\frac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big)\ \big|\partial(-\mathcal{D})(u^o,\cdot)\big|(\widetilde{u}_\sigma) \le \psi'\big(\frac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big). \tag{2.12}$$

Proof. We observe that for arbitrary $v \in M$ there holds

$$\begin{aligned}\mathcal{E}(\widetilde{u}_\sigma) - \mathcal{E}(v) &= \Phi_\sigma(u^o,\widetilde{u}_\sigma) - \Phi_\sigma(u^o,v) - \sigma\psi\big(\frac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big) + \psi\big(\frac{1}{\sigma}\mathcal{D}(u^o,v)\big)\\ &\overset{(1)}{\le} \sigma\Big(\psi\big(\frac{1}{\sigma}\mathcal{D}(u^o,v)\big) - \psi\big(\frac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big)\Big)\\ &\overset{(2)}{\le} \psi'\big(\frac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big)\ \Big(-\mathcal{D}(u^o,\widetilde{u}_\sigma) + \mathcal{D}(u^o,v)\Big).\end{aligned}$$

where (1) is due to $\widetilde{u}_\sigma \in J_\sigma(u^o)$, whereas (2) follows by the convexity of ψ. Taking the positive part on both sides (using $\psi' \ge 0$), dividing by $\mathcal{D}(\widetilde{u}_\sigma, v)$, and taking the limsup for $v \to \widetilde{u}_\sigma$ gives the first estimate in (2.12).

By the triangle inequality for the distance $\mathcal{D}$, for any fixed $u^0 \in M$ the functions $\mathcal{D}(u^0,\cdot)$ and $-\mathcal{D}(u^0,\cdot)$ have slope less or equal 1. Therefore, using $|-\partial\mathcal{D}(u^o,\cdot)| \le 1$ the second estimate in (2.12) follows. ∎

We can combine Propositions 2.8 and 2.9 and obtain the De Giorgi's lemma for $\mathrm{gMGS}\ (M,\mathcal{E},\mathcal{D},\psi)$, cf. also [RMS08, Lemma 4.5].

Theorem 2.10 (De Giorgi's lemma for gMGS**)** *Under Hypothesis 2.5, any measurable selection* $(0,\sigma_*) \ni \sigma \mapsto \widetilde{u}_\sigma \in J_\sigma(u^o)$ *fulfills*

$$\mathcal{E}(\widetilde{u}_\sigma) + \sigma\,\psi\big(\frac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big) + \int_0^\sigma \psi^*\big(\big|\partial\mathcal{E}\big|(\widetilde{u}_\rho)\big)\,\mathrm{d}\rho \le \mathcal{E}(u^o). \tag{2.13}$$

With the aim of improving (2.13) to an equality, taking into account the slope estimate, it is thus natural to check for cases in which $|\partial\mathcal{D}(u^o\cdot)| = 1$. This is true for a *geodesic distance*.

Lemma 2.11 *Suppose in addition that the metric space* $(M,\mathcal{D})$ *is a* geodesic space*. Then, for every* $u^o \in M$ *we have* $|\partial\mathcal{D}(u^o,\cdot)|(u) = 1$ *for every* $u \ne u^o$*.*

Proof. Clearly, it suffices to show that $|\partial\mathcal{D}(u^o,\cdot)|(u) \ge 1$. For this, let us consider the constant-speed geodesic γ connecting u^0 to u, such that $\mathcal{D}(\gamma_t,\gamma_s) = (t{-}s)\mathcal{D}(u^o,u)$ for all $0 \le s \le t \le 1$. Then,

$$\begin{aligned}|\partial\mathcal{D}(u^o,\cdot)|(u) &= \limsup_{v\to u} \frac{(\mathcal{D}(u^o,u){-}\mathcal{D}(u^o,v))^+}{\mathcal{D}(u,v)}\\ &\ge \limsup_{t\to 0^+} \frac{(\mathcal{D}(u^o,u){-}\mathcal{D}(u^o,\gamma_t))^+}{\mathcal{D}(u,\gamma_t)} = \lim_{t\to 0^+} \frac{\mathcal{D}(u^o,u) - t\mathcal{D}(u^o,u)}{(1{-}t)\mathcal{D}(u^o,u)} = 1\,.\end{aligned}$$

∎

In contrast, for the non-geodesic distance

$$\mathcal{D}(u,w) := \min\{|u{-}w|_2, R\} \quad \text{on } \mathbb{R}^n \tag{2.14}$$

(with $R > 0$ a given constant), we have $|\partial(\pm\mathcal{D})(u^o,\cdot)|(u) = 0$ whenever $|u{-}u^o| > R$.

We next provide two examples showing that, without a geodesic metric and without a continuous slope, we cannot expect (2.13) to holds as an equality.

Example 2.12 (($(M,\mathcal{D})$ not geodesic) *We consider the* quadratic *metric gradient system* $(M,\mathcal{E},\mathcal{D})$ *with*

$$M=\mathbb{R},\quad \mathcal{E}(u)=\frac{1}{2}\,u^2,\quad \mathcal{D}(u,w)=\min\{|w{-}u|,1\},\quad \psi(\delta)=\delta^2/2.$$

Starting with $u^o>1$ *and setting* $\sigma_*=\big((u^o)^2{-}1\big)^{-1/2}$ *we obtain*

$$J_\sigma(u^o)=\operatorname{argmin}\Big\{\,\frac{1}{\sigma}\mathcal{D}(u^o,u)^2+\mathcal{E}(u)\;\Big|\;u\in\mathbb{R}\,\Big\}=\begin{cases}\frac{u^o}{1+\sigma} & \text{for } \sigma<\sigma_*,\\ \big\{\frac{u^o}{1+\sigma},0\big\} & \text{for } \sigma=\sigma_*,\\ 0 & \text{for } \sigma>\sigma_*.\end{cases}$$

We can calculate all terms in (2.13) *and find equality as long as* $\sigma\le\sigma_*$*; but strict inequality holds for* $\sigma>\sigma_*$.

Note that $|\partial\mathcal{E}|(\widetilde{u}_\sigma)=|\partial\mathcal{E}|(0)=0$ *for* $\sigma>\sigma_*$ *but* $\psi'\big(\frac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big)=\frac{1}{\sigma}\mathcal{D}(u^o,0)=1/\sigma\gneqq 0$. *Hence, the first estimate in Proposition 2.9 holds as an equality, but the second estimate is strict because of* $\big|\partial(-\mathcal{D})(u^o,\cdot)\big|(0)=0<1$.

Our second counterexample involves a *discontinuous* slope functional.

Example 2.13 (Slope of $\mathcal{E}$ is not continuous) *We consider the (again, quadratic) metric gradient system*

$$M=\mathbb{R},\quad \mathcal{E}(u)=\max\{u,0\},\quad \mathcal{D}(u,w)=|u{-}w|,\quad \psi(\delta)=\delta^2/2.$$

Starting at $u^o=1$ *we find the unique variational interpolant* $\widetilde{u}_\sigma=\max\{1{-}\sigma,0\}$. *The curve* $\sigma\mapsto\widetilde{u}_\sigma$ *is absolutely continuous but the slope along the curve is discontinuous, namely*

$$|\partial\mathcal{E}|(\widetilde{u}_\sigma)=1\ \text{ for } \sigma\in[0,1[\quad\text{ and }\quad |\partial\mathcal{E}|(\widetilde{u}_\sigma)=0\ \text{ for } \sigma\ge 1.$$

This time the first estimate in Proposition 2.9 is strict for $\sigma\ge1$, *and hence* (2.13) *is also a strict inequality for* $\sigma>1$.

2.2 Equality in De Giorgi's estimate

The discussion in Section 2.1 has highlighted the link between two properties (one related to the geometry of the space, the other to the driving energy) and equality in the De Giorgi's Lemma. With Theorem 2.14 we now prove that the joint validity of such properties is a sufficient, albeit rather strong, condition for a gMGS to guarantee that all measurable variational interpolants satisfy estimate (2.13) with equality.

Theorem 2.14 *Consider a* gMGS $(M,\mathcal{E},\mathcal{D},\psi)$ *satisfying Hypothesis 2.5 such that, additionally,*

1. $(M,\mathcal{D})$ *is a geodesic space*
2. *the mapping* $u\mapsto|\partial\mathcal{E}|(u)$ *is continuous and that there exists a continuous function* $\omega: M{\times}M\to[0,1]$ *with* $\omega(u,u)=0$ *for all* $u\in M$ *such that the following uniform slope estimate holds:*

$$\forall\,u,w\in M:\quad \mathcal{E}(w)\ge\mathcal{E}(u)-|\partial\mathcal{E}|(u)\,\mathcal{D}(u,w)-\omega(u,w)\mathcal{D}(u,w). \tag{2.15}$$

Then, for any measurable selection $(0,\sigma_*) \ni \sigma \mapsto \widetilde{u}_\sigma \in J_\sigma(u^o)$ *the relation* (2.13) *holds with equality.*

Proof. It suffices to show that the slope estimate from Proposition 2.9 holds with equality and combine this with Lemma 2.11. Since the upper estimate $|\partial\mathcal{E}|(\widetilde{u}_\sigma) \le \psi'\big(\frac{1}{\tau}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big)$ is clear, it remains to show $|\partial\mathcal{E}|(\widetilde{u}_\sigma) \ge \psi'\big(\frac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big)$.

We consider the geodesic curve $[0,1] \ni \theta \mapsto \gamma_\theta$ with $\gamma_0 = u^o$ and $\gamma_1 = \widetilde{u}_\sigma$. Since $\gamma_1 = \widetilde{u}_\sigma$ is a global minimizer for $\Phi_\sigma(u^o;\cdot)$ we can compare with $u = \gamma_\theta$ and obtain

$$\begin{aligned}\mathcal{E}(\gamma_\theta) - \mathcal{E}(\widetilde{u}_\sigma) &\ge \sigma\,\psi\big(\frac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big) - \sigma\,\psi\big(\frac{1}{\sigma}\mathcal{D}(u^o,\gamma_\theta)\big)\\ &\ge \psi'\big(\frac{1}{\sigma}\mathcal{D}(u^o,\gamma_\theta)\big)\,\big(\mathcal{D}(\gamma_0,\gamma_1) - \mathcal{D}(\gamma_0,\gamma_\theta)\big) = \psi'\big(\frac{1}{\sigma}\mathcal{D}(u^o,\gamma_\theta)\big)\,\mathcal{D}(\gamma_\theta,\widetilde{u}_\sigma),\end{aligned}$$

where we used the convexity of ψ and the fact that $(\gamma_\theta)_\theta$ is a geodesic.

Exploiting the slope condition (2.15) with $u = \gamma_\theta$ and $w = \widetilde{u}_\sigma$ we obtain

$$|\partial\mathcal{E}|(\gamma_\theta) \ge \frac{\mathcal{E}(\gamma_\theta) - \mathcal{E}(\widetilde{u}_\sigma)}{\mathcal{D}(\gamma_\theta,\widetilde{u}_\sigma)} - \omega(\gamma_\theta,\widetilde{u}_\sigma) \ge \psi'\big(\frac{1}{\tau}\mathcal{D}(u^o,\gamma_\theta)\big) - \omega(\gamma_\theta,\widetilde{u}_\sigma).$$

We can now take the limit $\theta \nearrow 1$ and the continuity properties of the slope and ψ' yield the desired estimate $|\partial\mathcal{E}|(\widetilde{u}_\sigma) \ge \psi'\big(\frac{1}{\sigma}\mathcal{D}(u^o,\widetilde{u}_\sigma)\big)$.

∎

3 Banach case: examples and preliminary results

3.1 Setup

We consider a generalized Banach-space gradient system (gBGS, for short) $(X,\mathcal{E},\mathcal{R})$ where the state space

$$X \text{ is a reflexive Banach space.} \tag{X}$$

We now collect our working assumptions on $\mathcal{E}$ and $\mathcal{R}$, which partly mirror those collected in Definition 2.3 and Hypothesis 2.5. At the same time, they clearly reflect the underlying Banach setup, involving the weak topology on X, in addition to the norm topology, in conditions (3.3) below (cf. also Remark 2.6). As already mentioned in the Introduction, in the Banach setting we will allow for *nonsmooth* energies, and thus work with the *Fréchet subdiffential* $\partial\mathcal{E}$ of $\mathcal{E}$ in place of its Gâteau derivative $\mathrm{D}\mathcal{E}$. We recall that the multivalued operator $\partial\mathcal{E} : X \rightrightarrows X^*$ is defined at $u \in \mathrm{dom}(\mathcal{E})$ by

$$\xi \in \partial\mathcal{E}(u) \quad \text{if and only if} \quad \mathcal{E}(w) - \mathcal{E}(u) \ge \langle \xi, w{-}u\rangle + \mathrm{o}(\|w{-}u\|_X) \text{ as } w \to u\,. \tag{3.1}$$

Then, in (3.3b) we ask for closedness of the graph of $\partial\mathcal{E}$, w.r.t. the weak topology of $X \times X^*$, along sequences with bounded energy.

Hypothesis 3.1 (Conditions for generalized Banach-space gradient systems) *We assume that*

- *The dissipation potential* $\mathcal{R} : X \to [0, \infty)$ *is lower semicontinuous, convex, and fulfills, along with its convex conjugate* $\mathcal{R}^* : X^* \to [0, +\infty)$*, the following conditions:*

$$\mathcal{R}(0) = 0\,, \qquad \mathcal{R}^*(0) = 0\,, \qquad \lim_{\|v\| \to \infty} \frac{\mathcal{R}(v)}{\|v\|} = \infty, \qquad \lim_{\|\omega\|_* \to \infty} \frac{\mathcal{R}^*(\omega)}{\|\omega\|_*} = \infty. \tag{3.2}$$

- *The energy functional* $\mathcal{E} : X \to (-\infty, \infty]$ *is proper, bounded from below, and weakly-sequentially lower semicontinuous, i.e. for all* $(u_n)_n \subset X$

$$u_n \rightharpoonup u \text{ in } X \implies \liminf_{n \to \infty} \mathcal{E}(u_n) \geq \mathcal{E}(u)\,, \tag{3.3a}$$

and $\partial\mathcal{E} : X \rightrightarrows X^*$ *is closed on energy sublevels (cf. notation* (2.1)*) , i.e.*

$$\forall\, E > 0 : \left\{ \begin{array}{cc} (u_n, \xi_n) \rightharpoonup (u, \xi) & \text{in } X \times X^*, \\ u_n \in S_E,\ \xi_n \in \partial\mathcal{E}(u_n) & \text{for all } n \in \mathbb{N} \end{array} \right\} \implies \xi \in \partial\mathcal{E}(u)\,. \tag{3.3b}$$

It is often significant to consider dissipation potentials that also depend on the state variable, i.e. $\mathcal{R} = \mathcal{R}(u, v)$, but this generalization would be irrelevant for the study of the properties of the single-step minimum problem

$$\mathrm{Min}_{u \in X} \Phi_\sigma(u^o; u) \qquad \text{with } \Phi_\sigma(u^o; u) := \sigma\mathcal{R}\left(\frac{1}{\sigma}(u - u^o)\right) + \mathcal{E}(u), \quad \sigma > 0, \tag{3.4}$$

for a fixed $u^o \in M$. Indeed, in the state-dependent case the dissipation term would be simply replaced by $\mathcal{R}\left(u^o, \frac{1}{\sigma}(u - u^o)\right)$.

We emphasize that the closedness condition (3.3b) assumes only *weak* convergence in X on sequences $(u_n)_n$. However, the additional assumptions $(u_n)_n \subset S_E$ and the existence of a *bounded* sequence $(\xi_n)_n$ such that $\xi_n \in \partial\mathcal{E}(u_n)$ for all $n \in \mathbb{N}$, often grants extra compactness properties to the sequence $(u_n)_n$.

We will stick to notation (2.5), namely

- $\phi = \phi(u^o; \sigma)$ for the value functional associated with the above minimum problem; in fact, we shall also refer to $\phi(u^o; \cdot)$ as *marginal function*. We remark for later use that, in analogy to the metric case in [AGS05], it was proved in [MRS13b, Lem. 6.1] that

$$\lim_{\sigma \downarrow 0} \phi(u^o; \sigma) = \mathcal{E}(u^o)\,. \tag{3.5}$$

- and $J_\sigma(u^o)$ for the set of minimizers.

Additionally, in the Banach setup under Hypothesis 3.1 we have at out disposal that every $\widetilde{u}_\sigma \in J_\sigma(u^o)$ satisfies the Euler-Lagrange equation for (3.4), namely

$$0 \in \partial\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma - u^o)\big) + \widetilde{\xi}_\sigma, \quad \widetilde{\xi}_\sigma \in \partial\mathcal{E}(\widetilde{u}_\sigma). \tag{3.6}$$

Furthermore, observe that, under Hypothesis 3.1 the following holds:

1 We have $J_\sigma(u^o) \neq \emptyset$ for all $u^o \in \mathrm{dom}(\mathcal{E})$ and all $\sigma > 0$.

For this, it suffices to observe that any infimizing sequence for $\Phi_\sigma(u^o;\cdot)$ is bounded in X (thanks to the facts that $\mathcal{E}$ is bounded from below and $\mathcal{R}$ has superlinear growth), and to resort to the weak lower semicontinuity of $\mathcal{E}$ and $\mathcal{R}$.

2 The multi-valued mapping $\Gamma\colon (0,\infty) \rightrightarrows X; \Gamma(\sigma) := J_\sigma(u^o)$ is upper semicontinuous from $\mathbb{R}$ to X, in the sense that inclusion (2.9a) holds (even for the Kuratowski upper limit $\mathrm{Ls}^{\mathrm{weak}}_{n\to\infty}\Gamma(\sigma_n)$ defined in terms of the *weak* topology on X). Therefore, by [CaV77, Thm. 3.22] we may conclude the existence of a (strongly) *measurable* selection $(0,\infty) \ni \sigma \to \widetilde{u}_\sigma \in J_\sigma(u^o)$.

We conclude this section with the example of a gBGS $(X,\mathcal{E},\mathcal{R})$ fulfilling the conditions from Hypothesis 3.1. To keep the exposition simple, we confine the discussion to a Gâteaux differentiable energy $\mathcal{E}$, where $\partial\mathcal{E}(u)$ is always a singleton. This is not a significant restriction when we revisit Example 3.2 later on, because our focus will rather be on the properties of the dissipation potential $\mathcal{R}$. Nonetheless, it would not be difficult to adjust the conditions in such a way as to allow for a nonsmooth, but λ-convex, potential W in (3.8a) below.

Example 3.2 We consider

- the ambient space $X = \mathrm{L}^p(\Omega)$, $p > 1$, with a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$;
- the dissipation potential $\mathcal{R} : \mathrm{L}^p(\Omega) \to [0,\infty)$ defined by $\mathcal{R}(v) = \int_\Omega \mathrm{R}(v(x))\,\mathrm{d}x$ with

$$\begin{aligned} &\mathrm{R} : \mathbb{R} \to [0,\infty) \text{ convex, s.t. } \mathrm{R}(0) = 0 \text{ and} \\ &\exists\,\kappa,\ K > 0\ \forall\, x,y \in \mathbb{R}\ :\ \begin{cases} \mathrm{R}(x) \geq \kappa|x|^p - K, \\ \mathrm{R}^*(y) \geq \kappa|y|^{p'} - K, \end{cases} \end{aligned} \tag{3.7}$$

  where p' is the dual exponent to p.

- the energy functional $\mathcal{E} : \mathrm{L}^p(\Omega) \to (-\infty,\infty]$ features a (possibly) nonconvex, but lower order, perturbation of the Dirichlet integral, i.e. it is defined via

$$\mathcal{E}(u) := \begin{cases} \int_\Omega \frac{1}{2}|\nabla u|^2 + W(u)\,\mathrm{d}x & \text{for } u \in \mathrm{H}^1_0(\Omega) \text{ and } W(u) \in \mathrm{L}^1(\Omega)\,, \\ \infty & \text{otherwise}\,. \end{cases} \tag{3.8a}$$

  Along the footsteps of [RMS08, Sec. 7], we require for the potential energy density $W : \mathbb{R} \to \mathbb{R}$ that $W \in \mathrm{C}^2(\mathbb{R})$ and

$$\exists\, C_W > 0\ \exists\, s_p \in (1, \tfrac{p_d}{p'})\ \forall\, r \in \mathbb{R} :\ \begin{cases} W''(r) \geq -C_W\,, \\ W(r) \geq -C_W\,, \\ |W'(r)| \leq C_W(1+|r|^{s_p})\,. \end{cases} \tag{3.8b}$$

As shown in [RMS08, Sec. 7] (cf. also [MRS23, Sec. 4.2]), $\mathcal{E}$ complies with the lower semicontinuity and closedness conditions (3.3), where $\partial\mathcal{E}(u) = \{\, -\Delta u + W'(u)\,\} \subset \mathrm{H}^{-1}(\Omega) = \mathrm{H}^1_0(\Omega)^*$.

3.2 De Giorgi's estimate and identity in the Banach setup

In order to state the Banach-space versions of the metric energy identity (2.11) and of De Giorgi's estimate (2.13), we are naturally led to introduce the following object, which corresponds to the "slope part of the dissipation" in (2.13).

Definition 3.3 ($\mathcal{R}$-slope of the energy) *Let $u \in \mathrm{dom}(\partial\mathcal{E})$. The quantity*

$$\mathcal{S}_{\mathcal{R}}(u) := \inf \left\{ \mathcal{R}^*(-\xi) \,\middle|\, \xi \in \partial\mathcal{E}(u) \right\} \tag{3.9}$$

is called $\mathcal{R}$-slope *of the energy functional $\mathcal{E}$ at u.*

Indeed, as a consequence of the superlinear growth of $\mathcal{R}^*$, guaranteeing that infimizing sequences for the above minimum problem are bounded in X^*, and of the closedness property (3.3b), the infimum in (3.9) is attained, i.e. we have that

$$\mathcal{S}_{\mathcal{R}}(u) = \min \left\{ \mathcal{R}^*(-\xi) \,\middle|\, \xi \in \partial\mathcal{E}(u) \right\}. \tag{3.10a}$$

Likewise, it is immediate to check that, again by (3.3b), for every $(u_n)_n,\ u \in X$

$$u_n \rightharpoonup u \implies \liminf_{n\to\infty} \mathcal{S}_{\mathcal{R}}(u_n) \geq \mathcal{S}_{\mathcal{R}}(u)\,. \tag{3.10b}$$

Obviously, one may expect that the $\mathcal{R}$-slope $\mathcal{S}_{\mathcal{R}}(\widetilde{u}_\sigma)$, evaluated along a (measurable) selection $\sigma \mapsto \widetilde{u}_\sigma \in J_\sigma(u^o)$, will play the role of the term $\psi^*\big(\big|\partial\mathcal{E}\big|(\widetilde{u}_\sigma)\big)$ in the Banach-space version of (2.13). However, a more careful comparison with the metric identity (2.11), featuring the quantity

$$\tfrac{1}{\sigma}\mathsf{d}_\sigma^-(u^o) = \inf \left\{ \|\tfrac{1}{\sigma}(u - u^o)\| \,\middle|\, u \in J_\sigma(u^o) \right\},$$

suggests that the notion of slope has to be adjusted. In fact, it is expedient to bring into the picture the additional structure available in the Banach setup, namely the fact that every $u \in J_\sigma(u^o)$ fulfills the Euler-Lagrange equation (3.6). We thus introduce a "*conditioned* slope part of the dissipation", defined by the minimization of the dual dissipation potential $\mathcal{R}^*$ over selections $\xi \in \partial\mathcal{E}(u)$ that *additionally* satisfy the Euler-Lagrange equation. Accordingly, we define a multi-valued operator which encodes the validity of (3.6). For brevity, we shall refer to these two objects as *conditioned $\mathcal{R}$-slope* and *conditioned subdifferential*, respectively.

Definition 3.4 (*Conditioned* subdifferential / slope of energy) *The multivalued mapping $\partial_{\mathcal{R}}\mathcal{E} : X \times (0,\infty) \times X \rightrightarrows X^*$ defined by*

$$\partial_{\mathcal{R}}\mathcal{E}(u^o, \sigma; u) := \left\{ \xi \in X^* \,\middle|\, \xi \in \partial\mathcal{E}(u) \text{ and } -\xi \in \partial\mathcal{R}\big(\frac{1}{\sigma}(u - u^o)\big) \right\} \tag{3.11}$$

is called conditioned subdifferential of the energy $\mathcal{E}$. *The quantity*

$$\mathcal{C}_{\mathcal{R}}(u^o, \sigma; u) := \inf \left\{ \mathcal{R}^*(-\xi) \,\middle|\, \xi \in \partial_{\mathcal{R}}\mathcal{E}(u^o, \sigma; u) \right\} \tag{3.12}$$

is called conditioned $\mathcal{R}$-slope *of $\mathcal{E}$ at $u \in \mathrm{dom}(\partial\mathcal{E})$.*

Although with slight abuse, the notation for $\partial_{\mathcal{R}}\mathcal{E}$ highlights the geometry induced by the dissipation potential through the Euler-Lagrange equation (3.6).

Clearly, since every $u \in J_\sigma(u^o)$ fulfills the Euler-Lagrange equation (3.6), we have $\partial_{\mathcal{R}}\mathcal{E}(u^o, \sigma; u) \neq \emptyset$ and thus $\mathcal{C}_{\mathcal{R}}(u^o, \sigma; u) < \infty$. Obviously, in general there holds

$$\mathcal{C}_{\mathcal{R}}(u^o, \sigma; u) \geq \mathcal{S}_{\mathcal{R}}(u) \qquad \text{for all } u \in J_\sigma(u^o),$$

which seems to suggest $\mathcal{C}_{\mathcal{R}}$ as the 'right' object for the attainment of an equality in De Giorgi's estimate, cf. also Example 3.6 below.

We are now in the position to precisely introduce the estimate/identity whose validity we are are going to address hereafter.

Definition 3.5 *We say that a (strongly) measurable selection* $(0, \infty) \ni \sigma \mapsto \widetilde{u}_\sigma \in J_\sigma(u^o)$ *fulfills, on some interval* $(0, \sigma_*)$*,*

De Giorgi's estimate *if there holds for every* $\sigma \in (0, \sigma_*)$

$$\mathcal{E}(\widetilde{u}_\sigma) + \sigma\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma - u^o)\big) + \int_{\rho=0}^{\sigma} \mathcal{C}_{\mathcal{R}}(u^o, \rho; \widetilde{u}_\rho)\,\mathrm{d}\rho \leq \mathcal{E}(u^o); \tag{3.13}$$

De Giorgi's identity *if* (3.13) *holds as an equality, i.e. for every* $\sigma \in (0, \sigma_*)$

$$\mathcal{E}(\widetilde{u}_\sigma) + \sigma\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma - u^o)\big) + \int_{\rho=0}^{\sigma} \mathcal{C}_{\mathcal{R}}(u^o, \rho; \widetilde{u}_\rho)\,\mathrm{d}\rho = \mathcal{E}(u^o)\,. \tag{3.14}$$

Later on, in Section 4.3 we will provide a sufficient condition for the attainment of the $\inf$ in the definition (3.12) of $\mathcal{C}_{\mathcal{R}}$, which is also related to the validity of the analog of the lower semicontinuity properties (3.10b). We will also discuss the existence of measurable selections

$$(0, \infty) \ni \sigma \mapsto \widetilde{\xi}_\sigma \in \partial_{\mathcal{R}}\mathcal{E}(u^o, \sigma; \widetilde{u}_\sigma), \quad \text{with } \sigma \mapsto \widetilde{u}_\sigma \text{ a measurable selection in } J_\sigma(u^o),$$

such that estimate (3.13) rephrases as

$$\mathcal{E}(\widetilde{u}_\sigma) + \sigma\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma - u^o)\big) + \int_0^{\sigma} \mathcal{R}^*(-\widetilde{\xi}_\rho)\,\mathrm{d}\rho \leq \mathcal{E}(u^o)\,,$$

and analogously for (3.14).

We conclude this section by using Example 2.13, revisited in the Banach setup, to convey the idea that the finer information encoded in the conditioned $\mathcal{R}$-slope of the energy may play a key role for the attainment of De Giorgi's identity.

Example 3.6 (Example 2.13 in the Banach setup) *We now treat Example* (2.13) *as a* gBGS *in the Hilbert or Banach space* $X = \mathbb{R}$*:*

$$M = \mathbb{R}, \quad \mathcal{E}(u) = \max\{u, 0\} \doteq u^+, \quad \mathcal{R}(v) = \frac{1}{2}v^2.$$

Since $\mathcal{E}$ *is convex, its Fréchet subdifferential* $\partial\mathcal{E} : \mathbb{R} \rightrightarrows \mathbb{R}$ *coincides with that in the sense of convex analysis, and it is set-valued with* $\partial\mathcal{E}(0) = [0, 1]$*. The* $\mathcal{R}$*-slope of the energy is given by*

$$\mathcal{S}_{\mathcal{R}}(u) = \frac{1}{2} \text{ for } u > 0 \quad \text{and} \quad \mathcal{S}_{\mathcal{R}}(u) = 0 \text{ for } u \leq 0.$$

Starting from $u^o = 1$ *we calculate the variarional interpolant* $\widetilde{u}_\sigma = (1-\sigma)^+ = \max\{1-\sigma, 0\}$*, so that*

$$\partial\mathcal{E}(\widetilde{u}_\sigma) = \begin{cases} \{1\} & \text{if } \sigma \in (0,1), \\ [0,1] & \text{if } \sigma \geq 1. \end{cases}$$

Therefore, for $\sigma > 0$ *we have*

$$\mathcal{E}(u^o) - \mathcal{E}(\widetilde{u}_\sigma) - \sigma\mathcal{R}\big(\tfrac{1}{\sigma}(\widetilde{u}_\sigma - u^o)\big) = 1-(1-\sigma)^+ - \tfrac{1}{2\sigma}|(1-\sigma)^+ - 1|^2 = \begin{cases} \frac{1}{2}\sigma & \text{if } \sigma \leq 1, \\ 1-\frac{1}{2\sigma} & \text{if } \sigma \geq 1. \end{cases}$$

Hence, the estimate

$$\mathcal{E}(u^o) - \mathcal{E}(\widetilde{u}_\sigma) - \sigma\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma - u^o)\big) \geq \int_0^\sigma \mathcal{R}^*\big(-\widetilde{\xi}_s\big)\,\mathrm{d}s \qquad \text{with } \widetilde{\xi}_\sigma \in \partial\mathcal{E}(\widetilde{u}_\sigma) \tag{3.15}$$

holds for $\sigma \in (0,1]$*, while for* $\sigma > 1$ *it only holds for selection* $\sigma \mapsto \widetilde{\xi}_\sigma \in \partial\mathcal{E}(\widetilde{u}_\sigma)$ *with*

$$\int_0^\sigma \mathcal{R}^*\big(-\widetilde{\xi}_s\big)\,\mathrm{d}s = \int_0^1 \frac{1}{2}\mathrm{d}s + \int_1^\sigma \frac{1}{2}\big|-\widetilde{\xi}_s\big|^2\mathrm{d}s \leq 1 - \frac{1}{2\sigma} \quad \Leftrightarrow \quad \int_1^\sigma \big|\widetilde{\xi}_s\big|^2\mathrm{d}s \leq 1 - \frac{1}{\sigma}.$$

Since $\mathcal{S}_\mathcal{R}(\widetilde{u}_s) = 0$ *for* $s \in [1,\sigma]$, (3.15) *is certainly satisfied if we replace the integrand* $\mathcal{R}^*\big(-\widetilde{\xi}_s\big)$ *by* $\mathcal{S}_\mathcal{R}(\widetilde{u}_s)$*. In turn, recalling that* $\sigma \geq 1$ *we have* $\partial\mathcal{E}(\widetilde{u}_\sigma) = [0,1]$*, a wrong choice of* $\widetilde{\xi}_\sigma \in \partial\mathcal{E}(\widetilde{u}_\sigma)$ *can violate* (3.15)*.*

Finally, we consider the Euler-Lagrange equation (3.6) *in the current Banach setting:*

$$0 \in \widetilde{\xi}_\sigma + \partial\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma - u^o)\big) \quad \text{with } \widetilde{\xi}_\sigma \in \partial\mathcal{E}(\widetilde{u}_\sigma).$$

Since $\mathcal{R}$ *is smooth with* $\partial\mathcal{R}(v) = \{v\}$*, there is a unique solution, namely*

$$\widetilde{\xi}_\sigma = -\frac{1}{\sigma}(\widetilde{u}_\sigma - u^o) = \begin{cases} 1 & \text{for } \sigma \in (0,1]\,, \\ 1/\sigma & \text{for } \sigma \geq 1\,. \end{cases}$$

In particular, we have

$$\begin{aligned} \text{for } \sigma \in (0,1]: \quad & \partial_\mathcal{R}\mathcal{E}(u^o,\sigma;\widetilde{u}_\sigma) = \{1\} \quad \text{and} \quad \mathcal{S}_\mathcal{R}(\widetilde{u}_\sigma) = \mathcal{C}_\mathcal{R}(u^o,\sigma;\widetilde{u}_\sigma) = \frac{1}{2}, \\ \text{for } \sigma > 1: \quad & \partial_\mathcal{R}\mathcal{E}(u^o,\sigma;\widetilde{u}_\sigma) = \big\{\frac{1}{\sigma}\big\} \quad \text{and} \quad 0 = \mathcal{S}_\mathcal{R}(\widetilde{u}_\sigma) \lneqq \mathcal{C}_\mathcal{R}(u^o,\sigma;\widetilde{u}_\sigma) = \frac{1}{2\sigma^2}\,. \end{aligned}$$

Moreover, equality in (3.15) *can be achieved only by choosing the special selection*

$$\widetilde{\xi}_\sigma \in \partial_\mathcal{R}\mathcal{E}(u^o,\sigma;\widetilde{u}_\sigma) \qquad \text{for a.a. } \sigma > 0\,.$$

Thus, the 'good selections' of $\partial\mathcal{E}(\widetilde{u}_\sigma)$ *for the purpose of De Giorgi's identity is prescribed by the Euler-Lagrange equation* (3.6)*.*

3.3 Radially differentiable potentials

This section revolves around a structural property for dissipation potentials that will be crucial for obtaining De Giorgi's identity.

Definition 3.7 *We say that a dissipation potential* $\mathcal{R} : X \to [0, \infty)$ *is* radially differentiable *if*

$$\forall\, v \in X : \quad (0, \infty) \ni \lambda \mapsto f(v; \lambda) := \mathcal{R}(\lambda v) \text{ is differentiable at } \lambda = 1. \tag{3.16}$$

Clearly, both differentiability on the one hand, and positive homogeneity (i.e., $\mathcal{R}(\lambda v) = \lambda^p \mathcal{R}(v)$ for some $p \geq 1$) on the other, are sufficient conditions for (3.16). Thus, linear combinations of convex differentiable, or positively homogeneous, potentials are radially differentiable.

We are now going to show that radial differentiability is equivalent to another structural property that was used in [MRS13a] to prove De Giorgi's estimate (3.13).

Proposition 3.8 *A dissipation potential* $\mathcal{R}$ *is radially differentiable if and only if*

$$\xi_1, \xi_2 \in \partial \mathcal{R}(v) \quad \Longrightarrow \quad \mathcal{R}^*(\xi_1) = \mathcal{R}^*(\xi_2). \tag{3.17}$$

In view of the well-known convex-analysis relation

$$\mathcal{R}(v) + \mathcal{R}^*(\xi) = \langle \xi, v \rangle \qquad \text{for all } \xi \in \partial \mathcal{R}(v),$$

condition (3.17) is, in turn, equivalent to the property that

$$\xi_1, \xi_2 \in \partial \mathcal{R}(v) \quad \Longrightarrow \quad \langle \xi_1, v \rangle = \langle \xi_2, v \rangle.$$

This ensures that the (a priori multivalued) mapping

$$\mathfrak{P} : X \rightrightarrows [0, \infty), \qquad \mathfrak{P}(v) := \langle \xi, v \rangle \text{ for all } \xi \in \partial \mathcal{R}(v), \quad \text{is single-valued} \tag{3.18}$$

which will prove useful in Section 4.2 ahead.

Let us now address the proof of the equivalence of (3.17) and (3.16): We start by observing that for every $v \in X$ the mapping $f(v; \cdot)$ is convex. Its subdifferential in the sense of convex analysis $\partial f(v; \cdot) : (0, \infty) \rightrightarrows \mathbb{R}$ is given by

$$\partial f(v; \lambda) = [f'_-(v; \lambda), f'_+(v; \lambda)] \qquad \text{for all } \lambda > 0,\ v \in X\,, \tag{3.19a}$$

where $f'_\pm(v; \cdot)$ are the one-sided derivatives of the mapping $f(v; \cdot)$,

$$\begin{aligned} f'_+(v; \lambda) &:= \lim_{h \to 0^+} \frac{1}{h} \big(f(v; \lambda{+}h) - f(v; \lambda) \big), \\ f'_-(v; \lambda) &:= \lim_{h \to 0^+} \frac{1}{h} \big(f(v; \lambda) - f(v; \lambda{-}h) \big)\,, \end{aligned} \tag{3.19b}$$

(observe that the above limits exist by convexity of $f(v; \cdot)$). Then, we have the following result.

Lemma 3.9 *For every* $v \in X$ *and* $\lambda > 0$

$$f'_+(v; \lambda) = \max_{\xi \in \partial \mathcal{R}(\lambda v)} \langle \xi, v \rangle = \mathcal{R}(\lambda v) - \min_{\xi \in \partial \mathcal{R}(\lambda v)} \mathcal{R}^*(\xi), \tag{3.20a}$$

$$f'_-(v; \lambda) = \min_{\xi \in \partial \mathcal{R}(\lambda v)} \langle \xi, v \rangle = \mathcal{R}(\lambda v) - \max_{\xi \in \partial \mathcal{R}(\lambda v)} \mathcal{R}^*(\xi). \tag{3.20b}$$

Proof. It can be easily checked that, for any $v \in X$ and $\lambda > 0$, there holds

$$\ell \in \partial f(v;\lambda) \text{ if and only if } \exists\, \xi \in \partial\mathcal{R}(\lambda v) \text{ s.t. } \ell = \langle \xi, v\rangle \ .$$

Combining this with (3.19a), we immediately deduce (3.20). ∎

We are now in a position to carry out the

Proof of Proposition 3.8: The mapping $f(v;\cdot)$ from (3.16) is differentiable at $\lambda = 1$ if and only if $f'_-(v;1) = f'_+(v;1)$. This is in turn equivalent, by (3.20), to the fact that

$$\min_{\xi \in \partial\mathcal{R}(v)} \mathcal{R}^*(\xi) = \max_{\xi \in \partial\mathcal{R}(v)} \mathcal{R}^*(\xi)\,,$$

i.e. (3.16). ∎

Remark 3.10 *Let us focus on 'metric-like' dissipation potentials of the form*

$$\mathcal{R}(v) := \psi(\|v\|_X) \qquad \text{for every } v \in X \tag{3.21}$$

with $\psi : [0,\infty) \to [0,\infty)$ convex with superlinear growth at infinity, as in Definition 2.1. For the associated mapping $f(v;\cdot)$, $v \in X$, we have

$$\partial f(v;\lambda) = \|v\|_X \partial\psi(\|\lambda v\|_X) \qquad \text{for all } \lambda > 0\,.$$

Therefore, $\mathcal{R}$ is radially differentiable in the sense of (3.16) *if and only if $\partial\psi$ is single-valued. We thus retrieve the smoothness requirement on ψ from Hyp. 2.5.*

Example 3.2 revisited. *Let us get back to the dissipation potential $\mathcal{R} : \mathrm{L}^p(\Omega) \to [0,\infty)$, $p \geq 1$, from Example 3.2, i.e. $\mathcal{R}(v) = \int_\Omega \mathrm{R}(v(x))\,\mathrm{d}x$, with the dissipation density $\mathrm{R} : \mathbb{R} \to [0,\infty)$ satisfying conditions* (3.7). *In that setting, for a given $v \in \mathrm{dom}(\partial\mathcal{R})$ we have that*

$$\xi \in \partial\mathcal{R}(v) \quad \Longleftrightarrow \quad \xi(x) \in \partial\mathrm{R}(v(x)) \ \text{ for a.a. } x \in \Omega\,.$$

Therefore, it is natural to address the relations between radial differentiability of $\mathcal{R}$ and radial differentiability of R, which is clearly equivalent to differentiability of R in $\mathbb{R}\backslash\{0\}$. We now check that

$$\mathcal{R} \text{ is radially differentiable } \Longleftrightarrow \ \mathrm{R} \text{ is differentiable in } \mathbb{R}\backslash\{0\}. \tag{3.22}$$

Indeed, suppose that $\mathrm{R} : \mathbb{R} \to [0,\infty)$ is radially differentiable.

Then, in view of the characterization provided by Prop. 3.8, for all $v \in \mathrm{dom}(\partial\mathcal{R})$ and $\xi_1,\ \xi_2 \in \partial\mathcal{R}(v)$ we have

$$\begin{aligned} &\mathrm{R}^*(\xi_1(x)) = \mathrm{R}^*(\xi_2(x)) \ \text{ for a.a. } x \in \Omega \quad \text{ and thus} \\ &\mathcal{R}^*(\xi_1) = \int_\Omega \mathrm{R}^*(\xi_1(x))\,\mathrm{d}x = \int_\Omega \mathrm{R}^*(\xi_2(x))\,\mathrm{d}x = \mathcal{R}^*(\xi_2)\,, \end{aligned}$$

hence $\mathcal{R}$ is radially differentiable.

The converse implication holds for, if R is not radially differentiable, then there exist $\hat{v}, \hat{\xi}_1, \hat{\xi}_2 \in \mathbb{R}$ such that $\hat{\xi}_i \in \partial\mathrm{R}(\hat{v})$, $i = 1,2$, and $\mathrm{R}^*(\hat{\xi}_1) \neq \mathrm{R}^*(\hat{\xi}_2)$. Then, defining $v(x) \equiv \hat{v}$ and $\xi_i(x) \equiv \hat{\xi}_i$ for almost all $x \in \Omega$, we obtain a triple $(v,\xi_1,\xi_2) \in \mathrm{L}^p(\Omega)\times\mathrm{L}^{p'}(\Omega)\times\mathrm{L}^{p'}(\Omega)$ for which (3.17) *fails to hold.*

For later use, we point out that, if $\mathcal{R}$ is radially differentiable, then the single-valued mapping $\mathfrak{P}$: $\mathrm{L}^p(\Omega) \to [0,\infty)$ from (3.18) *is indeed given by*

$$\mathfrak{P}(v) = \int_\Omega \mathrm{P}(v(x))\,\mathrm{d}x \qquad \textit{with}\ \mathrm{P}(r) := \begin{cases} r\,\mathrm{R}'(r) & \textit{if}\ r \in \mathbb{R}\backslash\{0\}, \\ 0 & \textit{if}\ r = 0\,. \end{cases} \tag{3.23}$$

■

Ultimately, as a straightforward consequence of Prop. 3.8 we have the following result.

Corollary 3.11 *Suppose that $\mathcal{R} : X \to [0,\infty)$ is* radially differentiable. *Then, for every $u \in J_\sigma(u^o)$ we have that*

$$\mathcal{C}_\mathcal{R}(u^o,\sigma;u) = \mathcal{R}^*(-\xi) \qquad \textit{for all}\ \xi \in \partial_\mathcal{R}\mathcal{E}(u^o,\sigma;\widetilde{u}_\sigma)\,.$$

In particular, as soon as a (measurable) selection $\sigma \mapsto \widetilde{u}_\sigma \in J_\sigma(u^o)$ fulfills De Giorgi's estimate/identity, then (3.13)/(3.14) hold with the slope part of the time integrated dissipation, namely $\int_0^\sigma \mathcal{C}_\mathcal{R}(u^o,\rho;\widetilde{u}_\rho)\,\mathrm{d}\rho$, given by $\int_0^\sigma \mathcal{R}^*(-\widetilde{\xi}_\rho)\,\mathrm{d}\rho$ for *any* (meas.) selection $\sigma \mapsto \widetilde{\xi}_\sigma \in \partial_\mathcal{R}\mathcal{E}(u^o,\sigma;\widetilde{u}_\sigma)$. Furthermore, if De Giorgi's identity holds along an interval $(0,\sigma)$, the corresponding *optimal* integrand $\mathcal{C}_\mathcal{R}(u^o,\rho;\widetilde{u}_\rho)$ fulfills

$$\mathcal{C}_\mathcal{R}(u^o,\rho;\widetilde{u}_\rho) = \frac{\mathrm{d}}{\mathrm{d}s}\big(s\mathcal{R}\big(\frac{1}{s}(\widetilde{u}_\rho - u^0)\big)\big)|_{s=\rho} \qquad \text{for a.a. } \rho \in (0,\sigma)\,. \tag{3.24}$$

In Theorem 4.1 ahead we will indeed prove De Giorgi's identity under the condition that the dissipation potential is radially differentiable.

3.4 Tools

In this section we collect some preliminary results that will be used in the proof of Theorem 4.1.

Radial derivatives of dissipation potentials

Our first result is in the same spirit of Lemma 3.9, in that it provides some information on the left and right derivatives of the function $g(t) = t\mathcal{R}\big(\frac{1}{t}\,v\big)$, $t > 0$, which also features in (3.24).

Lemma 3.12 (Radial derivative) *For fixed $v \in X$ define the function $g(t) = t\mathcal{R}\big(\frac{1}{t}\,v\big)$. Then, g is convex, decreasing, and satisfies $g \in \mathrm{C}^{\mathrm{Lip}}_{\mathrm{loc}}(]0,\infty[)$. For all $t > 0$ the left derivative $g'_-(t)$ and the right derivative $g'_+(t)$ exist, are non-decreasing, and satisfy*

$$\begin{aligned} g'_-(t) &:= \lim_{h\to 0+} \frac{1}{h}\big(g(t) - g(t-h)\big) = -\max\big\{\,\mathcal{R}^*(\eta)\,\big|\,\eta \in \partial\mathcal{R}(\frac{1}{t}v)\,\big\} \\ &\le -\min\big\{\,\mathcal{R}^*(\eta)\,\big|\,\eta \in \partial\mathcal{R}(\frac{1}{t}v)\,\big\} = \lim_{h\to 0+} \frac{1}{h}\big(g(t+h) - g(t)\big) =: g'_+(t) < 0. \end{aligned}$$

Moreover, $t \mapsto g'_-(t)$ is continuous from the left and $t \mapsto g'_+(t)$ is continuous from the right.

In particular, if $\mathcal{R}$ is radially differentiable, then g is continuously differentiable with $g'(t) = g'_\pm(t) = -\mathcal{R}^*(\xi)$ for all $\xi \in \partial\mathcal{R}(\frac{1}{t}\,v)$, in accordance with Lemma 3.9.

Fundamental lemma for marginal functions

For fixed $u^o \in X$, we consider the marginal function $(0,\infty) \ni \sigma \mapsto \phi(u^o;\sigma)$ of $(\sigma,u) \mapsto \Phi_\sigma(u^o;u)$. The point is that one-sided differentiability of Φ with respect to σ provides bounds on the one-sided derivatives of ϕ, for which the behavior of Φ with respect to u is not really important. In the following result, in fact, we do not use the special form of Φ, but only its left and right differentiability with respect to σ.

Proposition 3.13 (Derivatives of marginal functions) *We have $\phi \in \mathrm{C}^{\mathrm{Lip}}_{\mathrm{loc}}(]0,\infty[)$ and for all $\sigma > 0$ the following estimates hold:*

$$\liminf_{h\to 0^+} \frac{1}{h}\big(\phi(u^o;\sigma) - \phi(u^o;\sigma{-}h)\big) \geq \sup\big\{\, \mathrm{D}^-_\sigma \Phi_\sigma(u^o;w) \,\big|\, w \in J_\sigma(u^o) \,\big\} =: \delta^-_\sigma(u^o) \quad (3.25a)$$

$$\limsup_{h\to 0^+} \frac{1}{h}\big(\phi(u^o;\sigma{+}h) - \phi(u^o;\sigma)\big) \leq \inf\big\{\, \mathrm{D}^+_\sigma \Phi_\sigma(u^o;w) \,\big|\, w \in J_\sigma(u^o) \,\big\} =: \delta^+_\sigma(u^o). \quad (3.25b)$$

Proof. For notational simplicity we drop the argument u^o in ϕ and Φ.

Consider $0 < r < \sigma$, then by the marginal property of ϕ we have

$$\Phi_\sigma(\widetilde{u}_r) - \Phi_r(\widetilde{u}_r) \geq \Phi_\sigma(\widetilde{u}_\sigma) - \Phi_r(\widetilde{u}_r) = \phi(\sigma) - \phi(r) \geq \Phi_\sigma(\widetilde{u}_\sigma) - \Phi_r(\widetilde{u}_\sigma).$$

Hence, the local Lipschitz property of ϕ follows from that of $\Phi(\cdot,w)$. Setting $r = t{-}h$ yields

$$\frac{1}{h}\big(\phi(\sigma) - \phi(\sigma{-}h)\big) \geq \frac{1}{h}\big(\Phi_\sigma(\widetilde{u}_\sigma) - \Phi_{\sigma-h}(\widetilde{u}_\sigma)\big).$$

Taking the liminf for $h \to 0^+$ first and then the supremum over $\widetilde{u}_\sigma \in J_\sigma(u^o)$ gives (3.25a).

Similarly, we can replace (r,σ) by $(\sigma,\sigma{+}h)$ to obtain

$$\frac{1}{h}\big(\phi(\sigma{+}h) - \phi(\sigma)\big) \leq \frac{1}{h}\big(\Phi_{\sigma+h}(\widetilde{u}_\sigma) - \Phi_\sigma(\widetilde{u}_\sigma)\big).$$

Taking the limsup $h \to 0^+$ first and then the infimum over $\widetilde{u}_\sigma \in J_\sigma(u^o)$ yields (3.25b). ■

4 The Banach case: results

In the upcoming Section 4.1 we will show that the De Giorgi's estimate previously proved in [MRS13a, Lemma 6.1] in the radially differentiable case, in fact improves to an equality. Then, in Sec. 4.3 we will drop the radial differentiabiilty condition and extend the validity of De Giorgi's estimate to general dissipation potentials.

4.1 Equality in De Giorgi's estimate for radially differentiable potentials

Now we return to the variational integrand where Φ is given in the form

$$\Phi_\sigma(u^o;u) = \sigma\,\mathcal{R}\big(\frac{1}{\sigma}(u{-}u^o)\big) + \mathcal{E}(u).$$

In particular, the left and right derivatives of $\sigma \mapsto \Phi_\sigma(u^o; u)$ exist, see Lemma 3.12, and are independent of the energy $\mathcal{E}$. Moreover, the one-sided derivatives are ordered such that $\mathrm{D}^-_\sigma\Phi(u^o;\sigma,w) \leq \mathrm{D}^+_\sigma\Phi(u^o;\sigma,w) < 0$ for every $w \in J_\sigma(u^o)$.

However, because of the supremum over $\mathrm{D}^-_\sigma\Phi$ and the infimum over $\mathrm{D}^+_\sigma\Phi$, in general one cannot expect to generate one chain of inequalities from the two estimates in (3.25). Even if $J_\sigma(u^o)$ is single-valued, one still has the wrong estimate.

The only case where the marginal estimates (3.25) are useful, is exactly when $\mathcal{R}$ is radially differentiable, cf. (3.16). Then $\mathrm{D}^-_\sigma\Phi = \mathrm{D}^+_\sigma\Phi$ implies $\delta^-_\sigma(u^o) \geq \delta^+_\sigma(u^o)$ and (3.25) leads to

$$\liminf_{h\to 0^+} \frac{1}{h}\big(\phi(u^o;\sigma) - \phi(u^o;\sigma{-}h)\big) \geq \delta^-_\sigma(u^o) \geq \delta^+_\sigma(u^o) \geq \limsup_{h\to 0^+} \frac{1}{h}\big(\phi(u^o;\sigma{+}h) - \phi(u^o;\sigma)\big).$$

From this, De Giorgi's identity follows easily, and a fortiori (4.1b) below for any measurable selection $\widetilde{\xi}_\sigma \in \partial_{\mathcal{R}}\mathcal{E}(u^o,\sigma;\widetilde{u}_\sigma)$. The *existence* of such selections will be addressed in the upcoming Lemma 4.11.

Theorem 4.1 (De Giorgi's identity with radial differentiability) *Consider the* gBGS $(X,\mathcal{E},\mathcal{R})$ *safisfying Hypothesis 3.1. Moreover, assume that $\mathcal{R}$ is radially differentiable, i.e.* (3.17) *or equivalently* (3.16) *holds. Fix $u^o \in \mathrm{dom}(\mathcal{E})$.*

Then, every measurable variational interpolant $(0,\infty) \ni \sigma \mapsto \widetilde{u}_\sigma \in X$ fulfills De Giorgi's identity

$$\mathcal{E}(\widetilde{u}_\sigma) + \sigma\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma{-}u^o)\big) + \int_0^\sigma \mathcal{C}_{\mathcal{R}}(u^o,\rho;\widetilde{u}_\rho)\,\mathrm{d}\rho = \mathcal{E}(u^o)\,, \tag{4.1a}$$

and in particular there holds

$$\forall\,\sigma > 0: \quad \mathcal{E}(\widetilde{u}_\sigma) + \sigma\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma{-}u^o)\big) + \int_0^\sigma \mathcal{R}^*(-\widetilde{\xi}_\rho)\,\mathrm{d}\rho = \mathcal{E}(u^o) \tag{4.1b}$$

for every measurable selection $\sigma \mapsto \widetilde{\xi}_\sigma \in \partial_{\mathcal{R}}\mathcal{E}(u^o,\sigma;\widetilde{u}_\sigma)$.

Proof. We fix a small $r > 0$ and consider the marginal function $[r,\sigma] \ni \rho \mapsto \phi(u^o;\rho)$. According to Proposition 3.13 ϕ is Lipschitz and hence differentiable almost everywhere in $[r,\sigma]$. This implies that ϕ'_- and ϕ'_+ exist and coincide almost everywhere.

We now fix any measurable variational interpolant $\sigma \mapsto \widetilde{u}_\sigma$. Proposition 3.13 and Lemma 3.12 imply, via the radial differentiability of $\mathcal{R}$, that for almost all $\sigma \in (0,\infty)$ there holds

$$\phi'(\sigma) = -\mathcal{R}^*(\eta) \qquad \text{for all } \eta \in \partial\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma{-}u^o)\big)\,.$$

Thus,

$$\phi'(\sigma) = -\mathcal{C}_{\mathcal{R}}(u^o,\sigma;\widetilde{u}_\sigma) \quad \text{for a.a. } \sigma \in (0,\infty)\,.$$

Hence, $\phi(\sigma) = \phi(r) + \int_r^\sigma \phi'(\rho)\,\mathrm{d}\rho$ can be rewritten as

$$\phi(u^o;\sigma) = \mathcal{E}(\widetilde{u}_\sigma) + \sigma\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma{-}u^o)\big) = \phi(u^o;r) - \int_r^\sigma \mathcal{C}_{\mathcal{R}}(u^o,\rho;\widetilde{u}_\rho)\,\mathrm{d}\rho\,.$$

The superlinearity of $\mathcal{R}$ implies $\phi(u^o;r) \to \mathcal{E}(u^o)$ as $r \to 0^+$, see (3.5). Therefore, taking the limit $r \to 0^+$ gives the desired De Giorgi's identity (4.1). ∎

We conclude this section with an example in which $\mathcal{R}$ is not radially differentiable but De Giorgi's identity still holds.

Example 4.2 We consider the $\mathrm{gBGS}\ (X, \mathcal{E}, \mathcal{R})$ with

$$X = \mathbb{R}, \quad \mathcal{E}(u) = \frac{1}{2}\,u^2, \quad \mathcal{R}(v) = \begin{cases} \frac{1}{2}\,v^2 & \text{for } |v| \leq 1, \\ 2v^2 - \frac{3}{2} & \text{for } |v| \geq 1. \end{cases}$$

Clearly, $\mathcal{R}$ is not radially differentiable, hence (3.17) does not hold.

For $u^o = 6$ we obtain the unique variational interpolant

$$\widetilde{u}_\sigma = \begin{cases} \frac{24}{4+\sigma} & \text{for } \sigma \in [0, 2], \\ 6 - \sigma & \text{for } \sigma \in [2, 5], \\ \frac{6}{1+\sigma} & \text{for } \sigma \geq 5. \end{cases}$$

It can easily be checked that identity (3.13) holds for all $\sigma > 0$, where the choice $\widetilde{\xi}_\sigma = \mathrm{D}\mathcal{E}(\widetilde{u}_\sigma) = \widetilde{u}_\sigma$ is mandatory because the Fréchet subdifferential of $\mathcal{E}$ is single-valued. For $\sigma \in [2, 5]$ we have $-\widetilde{\xi}(\sigma) = -6{+}\sigma \in \partial\mathcal{R}(-1) = [-4, -1]$, where we used $\frac{1}{\sigma}(\widetilde{u}_\sigma{-}6) = -1$.

4.2 Regularity of the variational interpolant

In this section we take a slight detour from the main theme of the paper and provide some sufficient conditions for gaining extra time regularity of the variational interpolant $\sigma \mapsto \widetilde{u}_\sigma$. This deviation will only be useful if we employ a slightly strengthened version of radial differentiability.

Indeed, on the one hand, we will require uniform convexity of the mapping $u \mapsto \Phi_\sigma(u^o; \cdot)$, a sufficient condition for which is, of course, uniform convexity of the energy functional $\mathcal{E}$. On the other hand, we will need to reinforce radial differentiability. Indeed, while the latter property is equivalent to the fact that the composed, a priori multi-valued, mapping $\mathcal{R}^*{\circ}\partial\mathcal{R} : X \rightrightarrows X^*$ is single-valued (cf. Proposition 3.8), we will now further require that $\mathcal{R}^*{\circ}\partial\mathcal{R}$ is Lipschitz continuous (on bounded subsets of X).

We collect these conditions in the following

Hypothesis 4.3 *We assume that*

1 *There exist $\overline{\lambda} > 0$ and $\sigma_* > 0$ such that for all $\sigma > 0$ the mapping $u \mapsto \Phi_\sigma(u^o; u)$ is $\overline{\lambda}$-convex, namely*

$$\begin{aligned} &\exists \overline{\lambda} > 0 \ \ \forall\,\sigma > 0\,,\ \forall\,u_0, u_1 \in X\ ,\ \forall\,\theta \in [0,1]\ : \\ &\Phi_\sigma(u^o; (1{-}\theta)u_0{+}\theta u_1) \leq (1{-}\theta)\Phi_\sigma(u^o; u_0) + \theta\Phi_\sigma(u^o; u_1) - \frac{\overline{\lambda}}{2}\theta(1{-}\theta)\|u_0{-}u_1\|^2\,. \end{aligned} \tag{4.2}$$

2 *The mapping $\mathcal{R}^*{\circ}\partial\mathcal{R} : X \rightrightarrows X^*$ (locally) Lipschitz continuous:*

$$\begin{aligned} &\forall\,M > 0\,\exists\,C_M > 0\,\forall\,(u_1, \xi_1),\,(u_2, \xi_2) \in X{\times}X^*,\ \max_{i=1,2}\|u_i\| \leq M,\ \xi_i \in \partial\mathcal{R}(u_i)\ : \\ &\qquad |\mathcal{R}^*(\xi_1){-}\mathcal{R}^*(\xi_2)| \leq C_M\|u_1{-}u_2\|\,. \end{aligned} \tag{4.3}$$

Before stating our result, let us pin down two key consequences of Hypothesis 4.3:

- It follows from (4.2) (cf., e.g., [MRS13b, Prop. 2.4]) that the Fréchet subdifferential $\partial\Phi_\sigma(u^o;\cdot): X \rightrightarrows X^*$ can be characterized in terms of the following *global* estimate: for all $\sigma > 0$ and for every $u \in \mathrm{dom}(\partial\Phi_\sigma(u^o;\cdot))$ we have that

$$\begin{gathered}\omega \in \partial\Phi_\sigma(u^o;u) \text{ if and only if} \\ \Phi_\sigma(u^o;v) - \Phi_\sigma(u^o;u) \geq \langle \omega, v{-}u\rangle + \frac{\overline{\lambda}}{2}\|v{-}u\|^2 \quad \text{for all } v \in X\,. \end{gathered} \tag{4.4}$$

- By Prop. 3.8, (4.3) in particular implies that for every $M > 0$ the restriction of $\mathcal{R}$ to the ball $\overline{B}_M(0)$ is radially differentiable. Therefore, recalling Lemma 3.9 we have that for every $v \in \overline{B}_M(0)$ the function

$$\begin{gathered} g(v;\rho) = \rho\mathcal{R}\left(\frac{1}{\rho}v\right) \text{ is differentiable at every } \rho > 0, \text{ with} \\ \frac{\mathrm{d}}{\mathrm{d}\rho}g(v;\rho) = \mathcal{R}\left(\frac{1}{\rho}v\right) - \langle \eta, \frac{1}{\rho}v\rangle = \mathcal{R}^*(\eta) \qquad \text{for all } \eta \in \partial\mathcal{R}\left(\frac{1}{\rho}v\right)\,. \end{gathered} \tag{4.5}$$

We are now in a position to state the main result of this section.

Theorem 4.4 *Let the* gBGS $(X,\mathcal{E},\mathcal{R})$ *satisfy Hypotheses 3.1 and 4.3; and let* $u^o \in \mathrm{dom}(\mathcal{E})$ *be fixed. Then, for all measurable selection* $(0,\infty) \ni \sigma \mapsto \widetilde{u}_\sigma \in J_\sigma(u^o)$ *we have* $\widetilde{u} \in \mathrm{Lip}_{\mathrm{loc}}(]0,\infty[;X)$.

Proof. Preliminarily, we observe that, because of $\sup_{\sigma>0}\sigma\mathcal{R}((\widetilde{u}_\sigma{-}u^o)/\sigma) \leq C$, we have

$$\exists\, M > 0\; \forall\, \sigma \in (0,\infty)\;:\quad \|\widetilde{u}_\sigma\| \leq M. \tag{4.6}$$

Let us fix $\sigma_\# > 0$ and let $[\sigma_1,\sigma_2] \subset [\sigma_\#,\infty)$ be arbitrary. We apply (4.4) with $\sigma = \sigma_1$, $u = \widetilde{u}_{\sigma_1}$, $v = \widetilde{u}_{\sigma_2}$ and $\omega = 0$ as, indeed, $0 \in \partial\Phi_{\sigma_1}(u^o;\widetilde{u}_{\sigma_1})$ is the Euler-Lagrange equation for $\widetilde{u}_{\sigma_1} \in J_{\sigma_1}(u^o)$. Thus, we obtain

$$\begin{aligned} \frac{\overline{\lambda}}{2}\|\widetilde{u}_{\sigma_2}{-}\widetilde{u}_{\sigma_1}\|^2 &\leq \Phi_{\sigma_1}(u^o;\widetilde{u}_{\sigma_2}) - \Phi_{\sigma_1}(u^o;\widetilde{u}_{\sigma_1}) \\ &= \mathcal{E}(\widetilde{u}_{\sigma_2}) - \mathcal{E}(\widetilde{u}_{\sigma_1}) + \sigma_1\mathcal{R}\left(\frac{1}{\sigma_1}(\widetilde{u}_{\sigma_2}{-}u^o)\right) - \sigma_1\mathcal{R}\left(\frac{1}{\sigma_1}(\widetilde{u}_{\sigma_1}{-}u^o)\right)\,. \end{aligned}$$

Interchanging σ_1 and σ_2 and adding the inequalities gives

$$\begin{aligned} &\overline{\lambda}\|\widetilde{u}_{\sigma_2}{-}\widetilde{u}_{\sigma_1}\|^2 \\ &\leq \sigma_2\mathcal{R}\big(\frac{1}{\sigma_2}(\widetilde{u}_{\sigma_1}{-}u^o)\big) - \sigma_1\mathcal{R}\big(\frac{1}{\sigma_1}(\widetilde{u}_{\sigma_1}{-}u^o)\big) + \sigma_1\mathcal{R}\big(\frac{1}{\sigma_1}(\widetilde{u}_{\sigma_2}{-}u^o)\big) - \sigma_2\mathcal{R}\big(\frac{1}{\sigma_2}(\widetilde{u}_{\sigma_2}{-}u^o)\big) \\ &= \int_{\sigma_1}^{\sigma_2}\left[\frac{\mathrm{d}}{\mathrm{d}\rho}g(\widetilde{u}_{\sigma_1};\rho) - \frac{\mathrm{d}}{\mathrm{d}\rho}g(\widetilde{u}_{\sigma_2};\rho)\right]\mathrm{d}\rho \overset{(4.5)}{=} \int_{\sigma_1}^{\sigma_2}\left(\mathcal{R}^*(\eta_1^\rho){-}\mathcal{R}^*(\eta_2^\rho)\right)\mathrm{d}\rho \end{aligned}$$

for every $\eta_i^\rho \in \partial\mathcal{R}(\frac{1}{\rho}\widetilde{u}_{\sigma_i})$, $i = 1,2$. Now using the Lipschitz property, we find

$$\overline{\lambda}\|\widetilde{u}_{\sigma_2}{-}\widetilde{u}_{\sigma_1}\|^2 \leq \int_{\sigma_1}^{\sigma_2} C_M\|\tfrac{1}{\rho}\widetilde{u}_{\sigma_2}{-}\tfrac{1}{\rho}\widetilde{u}_{\sigma_1}\|\,\mathrm{d}\rho$$

due to (4.3). All in all, using that $\sigma_1 \geq \sigma_\#$ we conclude that

$$\overline{\lambda}\|\widetilde{u}_{\sigma_2}{-}\widetilde{u}_{\sigma_1}\|^2 \leq \frac{C_M}{\sigma_\#}(\sigma_2{-}\sigma_1)\|\widetilde{u}_{\sigma_2}{-}\widetilde{u}_{\sigma_1}\|\,, \tag{4.7}$$

hence $\sigma \mapsto \widetilde{u}_\sigma$ is Lipschitz continuous on $[\sigma_\#,\infty)$. By the arbitrariness of $\sigma_\#$, we conclude $\widetilde{u} \in \mathrm{Lip}_{\mathrm{loc}}(]0,\infty[;X)$. ∎

More on condition (4.3)**.**

We conclude this section by gaining further insight into property (4.3). The key observation is that it can be reformulated in terms of the mapping $\mathfrak{P}$ from (3.18).

Lemma 4.5 *Let $\mathcal{R} : X \to [0,\infty)$ satisfy Hypothesis 3.1* and *be radially differentiable. Then, $\mathcal{R}^*\circ\partial\mathcal{R} : X \to X^*$ fulfills* (4.3) *if and only if $\mathfrak{P} : X \to [0,\infty)$ is locally Lipschitz:*

$$\forall\, M>0\ \exists\, C_M>0\ \forall\, u_1,u_2\in X \text{ with } \max_{i=1,2}\|u_i\|\le M\colon\ |\mathfrak{P}(u_1)-\mathfrak{P}(u_2)|\le C_M\|u_1-u_2\|\,. \tag{4.8}$$

Proof. It suffices to observe that $\mathfrak{P}(u) = \mathcal{R}(u) + (\mathcal{R}^*\circ\partial\mathcal{R})(u_1)$ and to and use that $\mathcal{R}$ is itself locally Lipschitz since $\mathcal{R}^*$ has superlinear growth at infinity. ∎

As as immediate corollary of the above characterization, we have that property (4.3) is stable under the sum of dissipation potentials.

Corollary 4.6 *Let $\mathcal{R}_i : X \to [0,\infty)$, $i = 1,2$, satisfy Hypothesis 3.1 and be radially differentiable. If* (4.3) *is valid for $\mathcal{R}_i$, $i = 1,2$, then it is valid for $\mathcal{R}_1 + \mathcal{R}_2$ as well.*

Proof. It suffices to remark (with slight abuse of notation) that

$$\mathfrak{P}_{\mathcal{R}_1+\mathcal{R}_2}(v) = \langle\partial(\mathcal{R}_1{+}\mathcal{R}_2)(v),v\rangle = \langle\partial\mathcal{R}_1(v){+}\partial\mathcal{R}_2(v),v\rangle = \mathfrak{P}_{\mathcal{R}_1}(v){+}\mathfrak{P}_{\mathcal{R}_2}(v) \quad \text{for all } v\in X\,,$$

where the validity of the sum rule $\partial(\mathcal{R}_1{+}\mathcal{R}_2)(v) = \partial\mathcal{R}_1(v){+}\partial\mathcal{R}_2(v)$ is guaranteed, under the present assumptions, by, e.g., [IoT79, Thm. 1, p. 211]. ∎

Ultimately, p-homogeneous dissipation potentials (including the degenerate case $p = 1$, which however falls outside the scope of this paper), provide examples for the validity of (4.3).

Corollary 4.7 *Let $\mathcal{R} : X \to [0,\infty)$ be positively homogeneous of degree p for some $p > 1$. Then, $\mathcal{R}^*\circ\partial\mathcal{R}$ fulfills* (4.3)*.*

Proof. Firstly, we claim that

$$\mathfrak{P}(v) = p\mathcal{R}(v) \qquad \text{for all } v\in X\,. \tag{4.9}$$

To show (4.9), it suffices to observe that, by definition of $\partial\mathcal{R}(v)$ and p-homogeneity of $\mathcal{R}$, we have for all $\xi\in\partial\mathcal{R}(v)$

$$(\lambda^p{-}1)\mathcal{R}(v) = \mathcal{R}(\lambda v) - \mathcal{R}(v) \ge \langle\xi,\lambda v{-}v\rangle = (\lambda{-}1)\mathfrak{P}(v)\,.$$

Then, we divide the above estimate by $(\lambda{-}1)$ for all $\lambda > 1$, and take the limit as $\lambda\to 1^+$, thus obtaining $p\mathcal{R}(v)\ge\mathfrak{P}(v)$. The converse inequality follows by dividing by $(\lambda{-}1)$ for all $0<\lambda<1$ and sending $\lambda\to 1^-$.

From (4.9) and the local Lipschitz continuity of $\mathcal{R}$ we deduce that $\mathfrak{P}$ satisfies (4.8), hence (4.3) holds thanks to Lemma 4.5. ∎

Eventually, by Corollary 4.6 property (4.3) holds for linear combinations of homogeneous potentials of possibly different degrees, as well.

Remark 4.8 (Condition (4.3) **in the framework of Example 3.2)** *Getting back to Example 3.2, we see that a sufficient condition for the dissipation potential $\mathcal{R}(v) = \int_\Omega \mathrm{R}(v(x))\,\mathrm{d}x$ to satisfy property* (4.3) *is that the function $\mathrm{P} : \mathbb{R}\to[0,\infty)$, defined by $\mathrm{P}(r) = r\,\mathrm{R}'(r)$ for $r\neq 0$ and by $\mathrm{P}(0)=0$ such that $\mathfrak{P}(v) = \int_\Omega \mathrm{P}(v(x))\,\mathrm{d}x$ (cf.* (3.23)*), is, itself, locally Lipschitz continuous.*

4.3 De Giorgi's estimate for general dissipation potentials

We now show that, if we drop the radial differentiability condition (3.17) on $\mathcal{R}$, De Giorgi's estimate can be still retrieved for $\mathrm{gBGS}\,(X,\mathcal{E},\mathcal{R})$ featuring general dissipation potentials $\mathcal{R}$. Our strategy consists in bringing into play the Yosida approximations $(\mathcal{R}_\eta)_\eta$ of $\mathcal{R}$, which clearly satisfy (3.17) guaranteeing the validity of De Giorgi's identity for the corresponding $\mathrm{gBGS}\,(X,\mathcal{E},\mathcal{R}_\eta)$. We will then study the limit $\eta\to 0^+$ in (4.1b).

Remark 4.9 *An approach based on Yosida regularization was also developed in [Bac21, Chap. 3] in order to extend the existence result for* gBGS *provided in [MRS13a],* without *the radial differentiability condition that was adopted therein (recall the discussion prior to Prop. 3.8). In [Bac21] the limit* $\eta\to 0^+$ *was performed on the level of the evolution equation. Here we resort to Yosida approximation to get rid of radial differentiability for the very proof of De Giorgi's estimate. Hence, the limiting process* $\eta\to 0^+$ *will be more delicate than in [Bac21], because variational interpolants are not absolutely continuous in* $[0,\sigma_0]$*, in contrast to solutions of the regularized* gBGS*.*

For taking the limit as $\eta\to 0^+$, we will rely on the following closedness condition, tailored to the multi-valued operator $\partial_{\mathcal{R}}\mathcal{E} : X\times(0,\infty)\times X \rightrightarrows X^*$ from (3.11).

Hypothesis 4.10 (Closedness of $\partial_{\mathcal{R}}\mathcal{E}$) *For every fixed* $u^o\in\mathrm{dom}(\mathcal{E})$ *the conditioned subdifferential* $\partial_{\mathcal{R}}\mathcal{E}(u^o,\cdot;\cdot) : (0,\infty)\times X \rightrightarrows X^*$ *is closed on energy sublevels, i.e.*

$$\forall\, E>0: \left\{ \begin{array}{c} (\sigma_n,u_n,\xi_n) \rightharpoonup (\sigma,u,\xi) \ \text{ in } (0,\infty)\times X\times X^*, \\ u_n\in S_E \text{ and } \xi_n\in\partial_{\mathcal{R}}\mathcal{E}(u^o,\sigma_n;u_n) \ \text{ for all } n\in\mathbb{N} \end{array}\right\} \implies \xi\in\partial_{\mathcal{R}}\mathcal{E}(u^o,\sigma;u)\,. \tag{4.10}$$

Again, we highlight that, although condition (4.10) is required on sequences $(u_n)_n$ *weakly* converging in X, some extra compactness may be derived from the information that the sequence $(u_n)_n$ lies in the fixed energy sublevel S_E and that the sequence $(\xi_n)_n$ with $\xi_n\in\partial_{\mathcal{R}}\mathcal{E}(u^o,\sigma;u_n)$ is bounded.

We point out two crucial consequences of Hypothesis 4.10.

Lemma 4.11 *Let the* $\mathrm{gBGS}\,(X,\mathcal{E},\mathcal{R})$ *satisfy Hypotheses 3.1 and 4.10. Then,*

1 *The* infimum *in the definition of* $\partial_{\mathcal{R}}\mathcal{E}$ *is attained, i.e.*

$$\mathcal{C}_{\mathcal{R}}(u^o,\sigma;u) := \min\big\{\,\mathcal{R}^*(-\xi)\,\big|\,\xi\in\partial_{\mathcal{R}}\mathcal{E}(u^o,\sigma;u)\,\big\} \quad \text{for all } u\in J_\sigma(u^o); \tag{4.11}$$

2 *For every measurable selection* $(0,\infty)\ni\sigma\mapsto\widetilde{u}_\sigma\in J_\sigma(u^o)$ *there exists a measurable selection*

$$\sigma\mapsto\widetilde{\xi}_\sigma\in\mathfrak{A}_{\mathcal{R}}(\sigma,\widetilde{u}_\sigma) := \mathrm{argmin}\big\{\,\mathcal{R}^*(-\xi)\,\big|\,\xi\in\partial_{\mathcal{R}}\mathcal{E}(u^o,\sigma;\widetilde{u}_\sigma)\,\big\} \tag{4.12a}$$

such that

$$\mathcal{C}_{\mathcal{R}}(u^o,\sigma;u) = \mathcal{R}^*(-\widetilde{\xi}_\sigma) \quad \text{for all } \sigma>0\,. \tag{4.12b}$$

Proof. Property (4.11) is easily checked via the *direct method* by relying on (4.10). For (4.12b), observe that, as a consequence of Hypothesis 4.10, the multivalued mapping

$$\mathfrak{A}_{\mathcal{R}} : (0,\infty)\times X \rightrightarrows X^*;\quad \mathfrak{A}_{\mathcal{R}}(\sigma,u) := \mathrm{argmin}\big\{\,\mathcal{R}^*(-\xi)\,\big|\,\xi\in\partial_{\mathcal{R}}\mathcal{E}(u^o,\sigma;u)\,\big\} \tag{4.13}$$

is upper semicontinuous with respect to convergence in $\mathbb{R}$ for σ and weak convergence for u. Therefore, for every measurable selection $(0,\infty) \ni \sigma \mapsto \widetilde{u}_\sigma \in J_\sigma(u^o)$ the multivalued mapping $\sigma \mapsto \mathfrak{A}_{\mathcal{R}}(\sigma,\widetilde{u}_\sigma)$ is measurable, as it is given by the composition of measurable mappings. Then, [CaV77, Thm. 3.22] grants the existence of a *measurable* selection $\sigma \mapsto \widetilde{\xi}_\sigma \in \mathfrak{A}_{\mathcal{R}}(\sigma,\widetilde{u}_\sigma)$. ■

We are now in a position to state our second main result for Banach GS. The following result only states the existence of *at least one* measurable variational interpolant satisfying De Giorgi's estimate. So far, we are not able to show that all measurable interpolants have this property.

Theorem 4.12 (De Giorgi's estimate for general potentials) *Let* $\mathrm{gBGS}\,(X,\mathcal{E},\mathcal{R})$ *satisfy Hypotheses 3.1 and 4.10, and let* $\sigma_* > 0$.

Then, for every $u^o \in \mathrm{dom}(\mathcal{E})$ *there exists a measurable variational interpolant* $(0,\infty) \ni \sigma \mapsto \widetilde{u}_\sigma \in X$ *fulfilling De Giorgi's estimate, and in particular there exists a measurable selection* $\sigma \mapsto \widetilde{\xi}_\sigma \in \partial_{\mathcal{R}}\mathcal{E}(u^o,\sigma;\widetilde{u}_\sigma)$ *along which there holds*

$$\forall\,\sigma \in (0,\sigma_*]: \quad \mathcal{E}(\widetilde{u}_\sigma) + \sigma\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma{-}u^o)\big) + \int_0^\sigma \mathcal{R}^*(-\widetilde{\xi}_\rho)\,\mathrm{d}\rho \leq \mathcal{E}(u^o). \tag{4.14}$$

Prior to carrying out the proof, let us introduce the main tool we are going to resort to: Since X is reflexive, Asplund's renorming theorem [Asp68], see also [BaP12, Thm. 1.105], ensures that there exists an equivalent norm on X (still denoted by $\|\cdot\|_X$), such that, correspondingly, both X and X^* are strictly convex (namely, the mappings $x \mapsto \|x\|_X^2$ and $\xi \mapsto \|\xi\|_{X^*}^2$ are strictly convex). With this, the Moreau-Yosida approximations $(\mathcal{R}_\eta)_{\eta>0}$ of $\mathcal{R}$ are defined via

$$\mathcal{R}_\eta\colon X \to [0,\infty) \qquad \mathcal{R}_\eta(v) := \inf_{w\in X}\left(\frac{\|w{-}v\|^2}{2\eta}{+}\mathcal{R}(w)\right).$$

The following result collects the properties of the functionals $(\mathcal{R}_\eta)_{\eta>0}$ we are going to use.

Lemma 4.13 *For all* $\eta > 0$ *the functional* $\mathcal{R}_\eta$ *is differentiable on* X *and has the convex conjugate* $\mathcal{R}_\eta^*(\xi) = \mathcal{R}^*(\xi) + \frac{\eta}{2}\|\xi\|_{X^*}^2$ *for* $\xi \in X^*$*. Moreover,*

1. *the family* $\big(\mathcal{R}_\eta\big)_{\eta\in(0,1]}$ *is equi-coercive, i.e.*

$$\forall\,S>0\;\exists\,C_S>0\;\forall\,\eta\in(0,1]\;\forall\,v\in X: \quad \mathcal{R}_\eta(v) \leq S \implies \|v\| \leq C_S\,; \tag{4.15}$$

2. *the family* $(\mathcal{R}_\eta)_\eta$ MOSCO*-converge to* $\mathcal{R}$ *as* $\eta \downarrow 0$.

Proof. By strict convexity of X^*, the space X is *smooth* (cf., e.g., [BaP12, Thm. 1.101]), which is equivalent to the property that the duality mapping $\mathrm{J}_X = \partial\big(\frac{1}{2}\|\cdot\|_X^2\big)\colon X \rightrightarrows X^*$ is *single-valued* and one-to-one, see [BaP12, Rmk. 1.100]. Hence, the differentiability of the functionals $\mathcal{R}_\eta$ follows, cf. [Att84, Sec. 3.4.1]; in particular, we have

$$\mathcal{R}_\eta(v) = \mathcal{R}\big(W_\eta(v)\big) + \frac{\|W_\eta(v){-}v\|^2}{2\eta} \;\text{ and }\; \mathrm{D}\mathcal{R}_\eta(v) = \frac{1}{\eta}\mathrm{J}_X\big(v{-}W_\eta(v)\big) \quad \text{for } v \in X, \tag{4.16}$$

with $W_\eta(v) = \big(\mathrm{Id}{+}\eta\mathrm{J}_X{\circ}\partial\mathcal{R}\big)^{-1}(v)$.

To show (4.15), we use (4.16) and that $\mathcal{R}$ has superlinear growth. This gives

$$\exists\,\overline{C}>0\ \ \forall\,\eta>0\ \ \forall\,v\in X: \quad \|W_\eta(v)\| \le \overline{C}+\mathcal{R}(W_\eta(v)) \le \overline{C}+\mathcal{R}_\eta(v)\,.$$

Moreover, using $\mathcal{R}\ge 0$ we find

$$\|W_\eta(v)-v\| \le (2\eta\mathcal{R}_\eta(v))^{1/2} \le (2\mathcal{R}_\eta(v))^{1/2} \quad \text{for all } \eta\in(0,1]\,.$$

Thus, $\|v\| \le \|W_\eta(v)\| + \|W_\eta(v)-v\| \le \overline{C}+\mathcal{R}_\eta(v)+(2\mathcal{R}_\eta(v))^{1/2}$, and (4.15) follows.

Finally, the MOSCO convergence follows from the results in [Att84, Sec. 3.4]. ■

We are now ready to perform the proof of Theorem, which will follow the usual approach of looking at the case $\eta>0$, deriving suitable a priori bounds that are independent of η, and then performing the limit $\eta\to 0^+$.

Proof of Theorem 4.12: We bring into play the Moreau-Yosida approximations $(\mathcal{R}_\eta)_{\eta>0}$. Since for every $\eta>0$ each $\mathcal{R}_\eta$ is radially differentiable, Theorem 4.1 provides the De Giorgi identity

$$\forall\,\sigma>0: \quad \mathcal{E}(\widetilde{u}_\sigma^\eta)+\sigma\mathcal{R}_\eta\big(\frac{1}{\sigma}(\widetilde{u}_\sigma^\eta-u^o)\big)+\int_0^\sigma \mathcal{C}_{\mathcal{R}_\eta}(u^o,\rho;\widetilde{u}_\rho^\eta)\,\mathrm{d}\rho=\mathcal{E}(u^o)\,. \tag{4.17}$$

for every measurable variational interpolant $(0,\infty)\ni\sigma\to\widetilde{u}_\sigma^\eta\in J_\sigma(u^o)$.

Step 1: A priori bounds. Clearly, we have that $\mathcal{E}(\widetilde{u}_\sigma^\eta)\le\mathcal{E}(u^o)$ for every $\sigma,\,\eta>0$. We will now use the place-holder $v_\sigma^\eta:=\frac{1}{\sigma}(u_{\sigma,\eta}-u^o)$. Using that $\mathcal{E}$ is bounded below by some constant E_0, we deduce from (4.17) that

$$\sigma\mathcal{R}_\eta(v_\sigma^\eta)+E_0 \le \sigma\mathcal{R}_\eta(v_\sigma^\eta)+\mathcal{E}(\widetilde{u}_\sigma^\eta) \overset{(1)}{\le} \mathcal{E}(u^o) \quad \text{for all } \sigma,\ \eta>0\,, \tag{4.18}$$

where (1) derives from the minimality of $\widetilde{u}_\sigma^\eta$. Thus, it follows from (4.15) applied with $S:=\mathcal{E}(u^o)$ that there exists a constant $C_S>0$ such that $\sigma\|v_\sigma^\eta\|\le C_S$. Now, recalling that $v_\sigma^\eta=\frac{1}{\sigma}(\widetilde{u}_\sigma^\eta-u^o)$ we infer that $\|\widetilde{u}_\sigma^\eta\|\le\sigma\|v_\sigma^\eta\|\ +\|u^o\|\le C_S+\|u^o\|$. All in all, we proved that

$$\exists\,\overline{E}>0\ \ \forall\,\eta\in(0,1]\ \ \forall\,\sigma\in(0,\sigma_*]: \qquad \mathcal{E}(\widetilde{u}_\sigma^\eta)+\|\widetilde{u}_\sigma^\eta\|\le\overline{E}. \tag{4.19}$$

Step 2: Compactness. Let us fix a null sequence $\eta_n\downarrow 0$. By (4.19), we have $\widetilde{u}_\sigma^{\eta_n}\in\overline{B_{\overline{E}}(0)}$ for every $\sigma\in(0,\sigma_*]$ and $n\in\mathbb{N}$. Therefore, the multi-valued mapping

$$\mathscr{U}:(0,\sigma_*]\rightrightarrows X, \qquad \mathscr{U}(\sigma):=\operatorname*{Ls^w}_{n\to\infty}\{\widetilde{u}_\sigma^{\eta_n}\} \tag{4.20}$$

is well defined with non-empty weakly closed values, and $\mathscr{U}(\sigma)\subset\overline{B_{\overline{E}}(0)}$ for all $\sigma\in(0,\sigma_*]$. Here, the Kuratowski upper limit of the sets $\{\widetilde{u}_\sigma^{\eta_n}\}$ w.r.t. the *weak topology* of X is defined by

$$\forall\,u\in\operatorname*{Ls^w}_{n\to\infty}\{\widetilde{u}_\sigma^{\eta_n}\} \text{ if } \exists\,(\eta_{n_k})_{k\in\mathbb{N}}: \quad \widetilde{u}_\sigma^{\eta_{n_k}}\rightharpoonup u \quad \text{in } X\,. \tag{4.21}$$

The results of [AuF09, Chap. 8] (see also [Mai05, Thm. A.5.4]) imply that $\mathscr{U}$ is measurable.

Recall $\mathcal{C}_{\mathcal{R}_{\eta_n}}(u^o,\sigma;u)$ from (4.11) and introduce the function $\widehat{\mathcal{C}}_{\mathcal{R}}:(0,\sigma_*]\to[0,\infty)$ defined by

$$\widehat{\mathcal{C}}_{\mathcal{R}}(\sigma):=\liminf_{n\to\infty}\mathcal{C}_{\mathcal{R}_{\eta_n}}(u^o,\sigma;\widetilde{u}_\sigma^{\eta_n})\,.$$

It follows from the closedness condition (4.10) that for each fixed $n \in \mathbb{N}$ the functionals $(\sigma, u) \mapsto \mathcal{C}_{\mathcal{R}_{\eta_n}}(u^o, \sigma; u)$ are sequentially weakly lower semicontinuous on $(0, \sigma_*] \times X$. Therefore, the mappings $(0, \sigma_*] \ni \sigma \mapsto \mathcal{C}_{\mathcal{R}_{\eta_n}}(u^o, \sigma; \widetilde{u}_\sigma^{\eta_n})$ are measurable, since they are given by the composition of the measurable curves $\sigma \mapsto (\sigma, \widetilde{u}_\sigma^{\eta_n})$ and of the lsc functionals $\mathcal{C}_{\mathcal{R}_{\eta_n}}(u^o, \cdot; \cdot)$. Hence, their lower limit $\widehat{\mathcal{C}}_{\mathcal{R}}$ is measurable.

Additionally, we define the value function $\widehat{\mathscr{C}}_{\mathcal{R}} \colon (0, \sigma_*] \to [0, \infty)$ (observe the different script font $\mathscr{C}$) via

$$\widehat{\mathscr{C}}_{\mathcal{R}}(\sigma) := \min \left\{ \mathcal{C}_{\mathcal{R}}(u^o, \sigma; u_\sigma) \,\middle|\, u_\sigma \in \mathscr{U}(\sigma) \right\}.$$

Indeed, the $\inf$ is attained for every $\sigma > 0$ since $\mathscr{U}(\sigma)$ is a weakly closed and bounded subset of X, and then one can resort to the closedness property (4.10). We may again check the measurability of $\widehat{\mathscr{C}}_{\mathcal{R}}$ by representing it as the composition of measurable mappings: namely, of the measurable multifuction $\sigma \mapsto (\sigma, \mathscr{U}(\sigma))$ and of the mapping defined by

$$(\sigma, U) \mapsto \mathcal{V}_{\mathcal{R}}(\sigma, U) := \min \left\{ \mathcal{C}_{\mathcal{R}}(u^o, \sigma; v) \,\middle|\, v \in U \right\}$$
$$\text{for all } \sigma \in (0, \sigma_*] \text{ and } U \ \textit{ weakly closed} \text{ subset of } \overline{B_{\overline{E}}(0)} \subset X\,.$$

In turn, the latter value function is measurable as it is lower semicontinuous w.r.t. convergence in $(0, \infty)$ for σ, and Kuratowski convergence (induced by the weak topology of X) for U; we leave the details to the reader, as the argument is analogous to those in the previous and following lines.

We observe that

$$\widehat{\mathcal{C}}_{\mathcal{R}}(\sigma) \geq \widehat{\mathscr{C}}_{\mathcal{R}}(\sigma) \qquad \text{for all } \sigma \in (0, \sigma_*]\,. \tag{4.22}$$

Indeed, fix $\sigma \in (0, \sigma_*]$, let $(\eta_n)_n$ be a (not relabeled) subsequence such that

$$\widehat{\mathcal{C}}_{\mathcal{R}}(\sigma) = \lim_{n\to\infty} \mathcal{C}_{\mathcal{R}_{\eta_n}}(u^o, \sigma; \widetilde{u}_\sigma^{\eta_n}) = \lim_{n\to\infty} \mathcal{R}^*_{\eta_n}(-\xi_n)$$

where $\xi_n \in \partial_{\mathcal{R}} \mathcal{E}(u^o, \sigma; \widetilde{u}_\sigma^{\eta_n})$ attains the optimal value of $\mathcal{R}^*_{\eta_n}(-\xi)$, cf. Lemma 4.11. Thus, we have $\sup_n \mathcal{R}^*_{\eta_n}(-\xi_n) < \infty$, so that by the coercivity of $\mathcal{R}^*$ the sequence $(\xi_n)_n$ is bounded in X^* and hence it admits a not relabeled subsequence weakly converging as $n \to \infty$ to some $\overline{\xi}$. Since the dual dissipation potentials $\mathcal{R}^*_{\eta_n}$ MOSCO-converge to $\mathcal{R}^*$, we have that $\liminf_{n\to\infty} \mathcal{R}^*_{\eta_n}(-\xi_n) \geq \mathcal{R}^*(-\overline{\xi})$. Now, $(\widetilde{u}_\sigma^{\eta_n})_n$ is bounded in X, hence there exists $\overline{u}$ such that, along a further subsequence, $\widetilde{u}_\sigma^{\eta_n} \rightharpoonup \overline{u}$. Therefore, $\overline{u} \in \mathscr{U}(\sigma)$ and, by the closedness property (4.10), we find that $\overline{\xi} \in \partial_{\mathcal{R}} \mathcal{E}(u^o, \sigma; \overline{u})$. All in all, we have found that

$$\lim_{n\to\infty} \mathcal{R}^*_{\eta_n}(-\xi_n) \geq \mathcal{R}^*(-\overline{\xi}) \geq \min \left\{ \mathcal{R}^*(-\xi) \,\middle|\, \xi \in \partial_{\mathcal{R}} \mathcal{E}(u^o, \sigma; u),\ u \in \mathscr{U}(\sigma) \right\},$$

and (4.22) follows.

Step 3: Passing to the limit in (4.17)**.** Let us fix $\sigma \in (0, \sigma_*]$ and an arbitrary element $u \in \mathscr{U}(\sigma)$. By (4.21), there exists a sequence $(\eta_{n_k})_k$ (whose dependence on σ and u is not highlighted), such that (4.21) holds. Therefore, by the weak lower semicontinuity of $\mathcal{E}$ and by the MOSCO-convergence of $\mathcal{R}_{\eta_{n_k}}$ to $\mathcal{R}$, we have

$$\begin{aligned} &\mathcal{E}(u) \leq \liminf_{k\to\infty} \mathcal{E}(\widetilde{u}_\sigma^{\eta_{n_k}})\,, \\ &\sigma \mathcal{R}\big(\frac{1}{\sigma}(u - u^o)\big) \leq \liminf_{k\to\infty} \sigma \mathcal{R}_{\eta_{n_k}}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma^{\eta_{n_k}} - u^o)\big)\,. \end{aligned} \tag{4.23}$$

Finally, we have

$$\int_0^\sigma \widehat{\mathscr{C}}_{\mathcal{R}}(\rho)\,\mathrm{d}\rho \overset{(4.22)}{\leq} \int_0^\sigma \widehat{\mathcal{C}}_{\mathcal{R}}(\rho)\,\mathrm{d}\rho \leq \int_0^\sigma \liminf_{k\to\infty} \mathcal{C}_{\mathcal{R}_{\eta_{n_k}}}(u^o,\rho;\widetilde{u}_\rho^{\eta_{n_k}})\,\mathrm{d}\rho \leq \liminf_{k\to\infty} \int_0^\sigma \mathcal{R}^*_{\eta_{n_k}}(-\widetilde{\xi}_\rho^{\eta_{n_k}})\,\mathrm{d}\rho\,,$$

where the last estimate follows from Fatou's Lemma. Since the lower semicontinuity estimates (4.23) hold for every $u \in \mathscr{U}(\sigma)$, we arrive at the following estimate:

$$\forall\,\sigma \in (0,\sigma_*]\,: \quad \sup_{u\in\mathscr{U}(\sigma)} \left(\mathcal{E}(u){+}\sigma\mathcal{R}\big(\frac{1}{\sigma}(u{-}u^o)\big)\right) + \int_0^\sigma \widehat{\mathscr{C}}_{\mathcal{R}}(\rho)\,\mathrm{d}\rho \leq \mathcal{E}(u^o)\,. \tag{4.24}$$

Step 4: Measurable selections and conclusion of the proof. We now resort to the multivalued mapping $\mathfrak{U}_{\mathcal{R}} : (0,\sigma_*]{\times}X \rightrightarrows X{\times}X^*$ defined by

$$\mathfrak{U}_{\mathcal{R}}(\sigma,u) := \{u\} \times \mathfrak{A}_{\mathcal{R}}(\sigma,u) = \{u\} \times \mathrm{argmin}\big\{\,\mathcal{R}^*(-\xi)\,\big|\,\xi \in \partial_{\mathcal{R}}\mathcal{E}(u^o,\sigma;u)\,\big\}$$

(cf. (4.13)). Clearly, $\mathfrak{U}_{\mathcal{R}}$ is also upper semicontinuous w.r.t. convergence in $\mathbb{R}$ and weak convergence in X, thus the composed multivalued mapping

$$(0,\sigma_*] \ni \sigma \mapsto \mathfrak{U}_{\mathcal{R}}(\sigma,\mathscr{U}(\sigma)) \subset X{\times}X^*$$

is measurable. Then, applying [CaV77, Thm. 3.22] we conclude that for every $\sigma \in (0,\sigma_*]$ there exists a measurable selection

$$\begin{aligned}&(0,\sigma] \ni \rho \mapsto (\widetilde{u}_\rho,\widetilde{\xi}_\rho) \in \mathfrak{U}_{\mathcal{R}}(\rho,\widetilde{u}_\rho) \quad \text{such that}\\ &\forall\,\rho \in (0,\sigma]\,: \quad \mathcal{R}^*(-\widetilde{\xi}_\rho) = \min\big\{\,\mathcal{R}^*(-\xi)\,\big|\,\xi \in \partial_{\mathcal{R}}\mathcal{E}(u^o,\rho;\widetilde{u}_\rho)\,\big\} = \widehat{\mathscr{C}}_{\mathcal{R}}(\rho)\,.\end{aligned} \tag{4.25}$$

Obviously, we have

$$\mathcal{E}(\widetilde{u}_\sigma) + \sigma\mathcal{R}\big(\frac{1}{\sigma}(\widetilde{u}_\sigma{-}u^o)\big) \leq \sup_{u\in\mathscr{U}(\sigma)} \left(\mathcal{E}(u){+}\sigma\mathcal{R}\big(\frac{1}{\sigma}(u{-}u^o)\big)\right)\,.$$

Thus, from (4.24) we obtain the desired (4.14) for the selection fulfilling (4.25). This concludes the proof of Theorem 4.12. ■

4.4 An alternative route to equality via the chain rule

In this section, extending an argument sketched in the introduction we show that, even without radial differentiability for the governing dissipation potential $\mathcal{R}$, De Giorgi's estimate (4.14) can be improved to an identity along a curve $\sigma \mapsto \widetilde{u}_\sigma$

1 with suitable regularity

2 such that the chain rule holds for $\sigma \mapsto \mathcal{E}(\widetilde{u}_\sigma)$.

Thus, the motivation for this section is to support our conjecture that De Giorgi's identity is valid in more general situation than those understood by now.

We substantiate the second requirements in the following hypothesis and highlight that, for the variational interpolant it is in fact sufficient to have (piecewise) absolute continuity, whereas the results from Section 4.2 even granted, under additional assumptions, (local) Lipschitz continuity of the curve $\sigma \mapsto \widetilde{u}_\sigma$.

Hypothesis 4.14 (Regularity of variational interpolant & chain rule) *Let $\sigma_* > 0$. We consider a curve $(0,\sigma_*] \ni \sigma \mapsto \widetilde{u}_\sigma \in J_\sigma(u^o)$ such that*

1. *$\widetilde{u}$ is* piecewise absolutely continuous *on $(0,\sigma_*]$, namely*

$$\exists\, p \geq 1\ \exists \text{ a partition } \{\tau_j\}_{j=0}^J \text{ of } [0,\sigma_*]\ \forall\, j \in \{1,..,J\}:\ \widetilde{u} \in \mathrm{AC}^p\big((\tau_{j-1},\tau_j);X\big)\,. \tag{4.26}$$

As a consequence, for all $\{1,\ldots,J\}$ the one-sided limits $\widetilde{u}_{\tau_j}^- := \lim_{\sigma\to\tau_j^-} \widetilde{u}_\sigma$ and $\widetilde{u}_{\tau_{j-1}}^+ := \lim_{\sigma\to\tau_{j-1}^+} \widetilde{u}_\sigma$ exist;

2. *there exists a measurable selection $\widetilde{\xi} : (0,\sigma_*] \to X^*$ with $\widetilde{\xi}_\sigma \in \partial\mathcal{E}(\widetilde{u}_\sigma)$ for a.a. $\sigma \in (0,\sigma_*)$ such that*

$$\begin{aligned} &\forall\, j \in \{1,\ldots,J\}:\ \sigma \mapsto \mathcal{E}(\widetilde{u}_\sigma) \text{ is absolutely continuous on } (\tau_{j-1},\tau_j), \\ &\qquad \text{and } \frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{E}(\widetilde{u}_\sigma) = \langle \widetilde{\xi}_\sigma, \widetilde{u}'_\sigma\rangle \quad \text{for a.a. } \sigma \in (\tau_{j-1},\tau_j)\,. \end{aligned} \tag{4.27}$$

In Hypothesis 4.14, only the chain rule for $\mathcal{E}$ *evaluated along* the (assumedly) absolutely continuous curve $\sigma \mapsto \widetilde{u}_\sigma$ is required. Nonetheless, it is natural to wonder for which classes of energies the chain rule holds *in general*. Some sufficient conditions for its validity, among which λ-convexity of $\mathcal{E}$ for some $\lambda \in \mathbb{R}$, were provided in [MRS13a, Prop. 2.4], [MiR23, Prop. A.1]. There, it was shown that for *any* pair $(\mathsf{u},\xi) \in \mathrm{AC}([a,b];X)\times \mathrm{L}^1([a,b];X^*)$ such that

$$\sup_{s\in[a,b]} \mathcal{E}(\mathsf{u}(s)) < \infty \text{ and } \int_a^b \|\xi(s)\|_* \,\|\mathsf{u}'(s)\|\,\mathrm{d}s < \infty\,, \tag{4.28}$$

then $s \mapsto \mathcal{E}(\mathsf{u}(s))$ is absolutely continuous and the chain rule formula $\frac{\mathrm{d}}{\mathrm{d}s}(\mathcal{E}\circ\mathsf{u}) = \langle \xi, \mathsf{u}'\rangle$ holds a.e. in (a,b). For instance, the above chain rule holds for the energy functional from Example 3.2, which is indeed λ-convex for $\lambda = -C_W$, cf. [MRS23, Sec. 4.2].

Now, in the upcoming Theorem 4.15 we will consider curves $\sigma \mapsto \widetilde{u}_\sigma \in J_\sigma(u^o)$ and $\sigma \mapsto \widetilde{\xi}_\sigma \in \partial\mathcal{E}(\widetilde{u}_\sigma)$ as in Hypothesis 4.14 *and* satisfying De Giorgi's estimate (4.14). Then, we will clearly have the energy bound $\sup_{\sigma\in]0,\sigma_*]} \mathcal{E}(\widetilde{u}_\sigma) < \infty$, as well as

$$\int_0^{\sigma_*} \mathcal{R}^*(-\widetilde{\xi}_\rho)\,\mathrm{d}\rho < \infty\,.$$

Therefore, *if* the dual dissipation potential $\xi \mapsto \mathcal{R}^*(\xi)$ controls $\|\xi\|_*^{p'}$ with p' conjugate to p, then combining the above estimate with the condition $u \in \mathrm{AC}^p\big((\tau_j,\tau_{j+1});X\big)$, we conclude that for the pair $(\tilde{u},\tilde{\xi})$ the second estimate in (4.28) holds.

We are now in a position to state the our last result that De Giorgi's identity holds even without radial differentiability, if we have some regularity of the variational interpolant $\sigma \mapsto (\widetilde{u}_\sigma, \widetilde{\xi}_\sigma)$.

Theorem 4.15 (From De Giorgi's estimate to De Giorgi's identity) *Let the gBGS $(X,\mathcal{E},\mathcal{R})$ satisfy Hypotheses 3.1. and let $u^o \in \mathrm{dom}(\mathcal{E})$ be fixed. Let $(0,\sigma_*] \ni \sigma \mapsto \widetilde{u}_\sigma \in J_\sigma(u^o)$ and $(0,\sigma_*] \ni \sigma \mapsto \widetilde{\xi}_\sigma \in \partial_{\mathcal{R}}\mathcal{E}(u^o,\sigma;\widetilde{u}_\sigma)$ be as in Hypothesis 4.14. Suppose that they satisfy De Giorgi's estimate* (4.14) *on $(0,\sigma_*]$.*

Then, (4.14) *holds as an equality, i.e.*

$$\mathcal{E}(\widetilde{u}_{\sigma_*}) + \sigma_*\mathcal{R}\big(\frac{1}{\sigma_*}(\widetilde{u}_{\sigma_*}-u^o)\big) + \int_0^{\sigma_*} \mathcal{R}^*(-\widetilde{\xi}_\rho)\,\mathrm{d}\rho = \mathcal{E}(u^o) \tag{4.29}$$

and, a fortiori, we have

$$\widetilde{\xi}_\sigma \in \mathfrak{A}_{\mathcal{R}}(\sigma, \widetilde{u}_\sigma) = \operatorname{argmin}\big\{ \mathcal{R}^*(-\xi) \,\big|\, \xi \in \partial_{\mathcal{R}}\mathcal{E}(u^o, \sigma; \widetilde{u}_\sigma) \big\} \ \textit{for a.a.}\ \sigma \in (0, \sigma_*)\,, \tag{4.30}$$

hence De Giorgi's identity (3.14) *holds.*

Proof. Step 1**:** Let us fix $j \in \{1, \ldots, J{-}1\}$. It is immediate to check that $\widetilde{u}^{\pm}_{\tau_j} \in J_{\tau_j}(u^o)$, and thus $\Phi_{\tau_j}(u^o; \widetilde{u}^-_{\tau_j}) = \phi(u^o; \tau_j) = \Phi_{\tau_j}(u^o; \widetilde{u}^+_{\tau_j})$, i.e.

$$\mathcal{E}(\widetilde{u}^-_{\tau_j}) + \tau_j \mathcal{R}\Big(\frac{1}{\tau_j}(\widetilde{u}^-_{\tau_j} {-} u^o)\Big) = \mathcal{E}(\widetilde{u}^+_{\tau_j}) + \tau_j \mathcal{R}\Big(\frac{1}{\tau_j}(\widetilde{u}^+_{\tau_j} {-} u^o)\Big)\,. \tag{4.31}$$

Let now $\sigma \mapsto \widetilde{\xi}_\sigma \in \partial_{\mathcal{R}}\mathcal{E}(u^o, \sigma; \widetilde{u}_\sigma)$ be as in the statement. Hypothesis 4.14 enables us to apply the chain rule on any interval $[s_*, s^*] \subset (\tau_j, \tau_{j+1})$, thus concluding that

$$\lim_{\sigma \to \tau_{j+1}^-} \mathcal{E}(\widetilde{u}_\sigma) - \lim_{\sigma \to \tau_j^+} \mathcal{E}(\widetilde{u}_\sigma) = \int_{\tau_j}^{\tau_{j+1}} \langle \widetilde{\xi}_\rho, \widetilde{u}'_\rho \rangle \,\mathrm{d}\rho\,.$$

Now, we have that

$$\lim_{\sigma \to \tau_j^+} \mathcal{E}(\widetilde{u}_\sigma) = \lim_{\sigma \to \tau_j^+} \bigg(\phi(u^o; \sigma) {-} \sigma \mathcal{R}\Big(\frac{1}{\sigma}(\widetilde{u}_\sigma {-} u^o)\Big)\bigg) = \phi(u^o; \tau_j) {-} \tau_j \mathcal{R}\Big(\frac{1}{\tau_j}(\widetilde{u}_{\tau_j} {-} u^o)\Big) = \mathcal{E}(\widetilde{u}^+_{\tau_j})$$

by the continuity of ϕ (recall Proposition 3.13) and of $\mathcal{R}$. Analogously, $\lim_{\sigma \to \tau_{j+1}^-} \mathcal{E}(\widetilde{u}_\sigma) = \mathcal{E}(\widetilde{u}^-_{\tau_{j+1}})$. Therefore,

$$\mathcal{E}(\widetilde{u}^-_{\tau_{j+1}}) - \mathcal{E}(\widetilde{u}^+_{\tau_j}) = \int_{\tau_j}^{\tau_{j+1}} \langle \widetilde{\xi}_\rho, \widetilde{u}'_\rho \rangle \,\mathrm{d}\rho\,. \tag{4.32}$$

On the other hand, since $-\widetilde{\xi}_\sigma \in \partial\mathcal{R}\left(\frac{1}{\sigma}(\widetilde{u}_\sigma {-} u^o)\right)$, by the chain rule for $\sigma \mapsto \sigma\mathcal{R}\left(\frac{1}{\sigma}(\widetilde{u}_\sigma {-} u^o)\right)$ we find that

$$\begin{aligned} &\tau_{j+1}\mathcal{R}\Big(\frac{1}{\tau_{j+1}}(\widetilde{u}^-_{\tau_{j+1}} {-} u^o)\Big) - \tau_j \mathcal{R}\Big(\frac{1}{\tau_j}(\widetilde{u}^+_{\tau_j} {-} u^o)\Big) \\ &= \int_{\tau_j}^{\tau_{j+1}} \Big(\mathcal{R}\big(\frac{1}{\rho}(\widetilde{u}_\rho {-} u^o)\big) + \rho \big\langle -\widetilde{\xi}_\rho, \frac{1}{\rho}\widetilde{u}'_\rho {-} \frac{1}{\rho^2}(\widetilde{u}_\rho {-} u^o) \big\rangle \Big) \,\mathrm{d}\rho\,. \end{aligned} \tag{4.33}$$

Adding (4.32) and (4.33), observing the cancellation of the term $\int_{\tau_j}^{\tau_{j+1}} \langle \widetilde{\xi}_\rho, \widetilde{u}'_\rho \rangle \,\mathrm{d}\rho$ and rearranging the remaining integral terms, we find

$$\begin{aligned} &\mathcal{E}(\widetilde{u}^-_{\tau_{j+1}}) + \tau_{j+1}\mathcal{R}\Big(\frac{1}{\tau_{j+1}}(\widetilde{u}^-_{\tau_{j+1}} {-} u^o)\Big) + \int_{\tau_j}^{\tau_{j+1}} \Big(\big\langle -\widetilde{\xi}_\rho, \tfrac{1}{\rho}(\widetilde{u}_\rho {-} u^o) \big\rangle - \mathcal{R}\big(\frac{1}{\rho}(\widetilde{u}_\rho {-} u^o)\big) \Big) \,\mathrm{d}\rho \\ &= \mathcal{E}(\widetilde{u}^+_{\tau_j}) + \tau_j \mathcal{R}\Big(\frac{1}{\tau_j}(\widetilde{u}^+_{\tau_j} {-} u^o)\Big)\,. \end{aligned}$$

Now, since $-\widetilde{\xi}_\rho \in \partial\mathcal{R}\big(\frac{1}{\rho}(\widetilde{u}_\rho {-} u^o)\big)$, the integrand in the third term on the left-hand side equals $\mathcal{R}^*(-\widetilde{\xi}_\rho)$. In turn, by (4.31), the right-hand side equals $\Phi_{\tau_j}(u^o; \widetilde{u}^-_{\tau_j})$. All in all, we conclude that

$$\begin{aligned} &\mathcal{E}(\widetilde{u}^-_{\tau_{j+1}}) + \tau_{j+1}\mathcal{R}\Big(\frac{1}{\tau_{j+1}}(\widetilde{u}^-_{\tau_j} {-} u^o)\Big) + \int_{\tau_j}^{\tau_{j+1}} \mathcal{R}^*(-\widetilde{\xi}_\rho) \,\mathrm{d}\rho \\ &\quad = \mathcal{E}(\widetilde{u}^-_{\tau_j}) + \tau_j \mathcal{R}\Big(\frac{1}{\tau_j}(\widetilde{u}^-_{\tau_j} {-} u^o)\Big) \quad \text{for all } j \in \{1, \ldots, J{-}1\}\,. \end{aligned} \tag{4.34}$$

Step 2**:** It remains to derive the analogue of (4.34) for $j = 0$. With this aim, we consider the first interval $[0, \tau_1]$ of the partition and fix $\mu \in (0, \tau_1]$. Repeating the arguments from Step 1 we find

$$\begin{aligned} &\mathcal{E}(\widetilde{u}^-_{\tau_1}) + \tau_1 \mathcal{R}\Big(\frac{1}{\tau_1}(\widetilde{u}^-_{\tau_1} - u^o)\Big) + \int_\mu^{\tau_{j+1}} \mathcal{R}^*(-\widetilde{\xi}_\rho)\,\mathrm{d}\rho \\ &= \mathcal{E}(\widetilde{u}_\mu) + \mu \mathcal{R}\Big(\frac{1}{\mu}(\widetilde{u}_\mu - u^o)\Big) = \Phi_\mu(u^o; \widetilde{u}_\mu) = \phi(u^o; \mu) \to \mathcal{E}(u^o)\,, \end{aligned}$$

where the last identity follows because $\widetilde{u}_\mu$ is a minimizer and ϕ is the value function, and the convergence stems from (3.5). We thus conclude

$$\mathcal{E}(\widetilde{u}^-_{\tau_1}) + \tau_1 \mathcal{R}\Big(\frac{1}{\tau_1}(\widetilde{u}^-_{\tau_1} - u^o)\Big) + \int_0^{\tau_{j+1}} \mathcal{R}^*(-\widetilde{\xi}_\rho)\,\mathrm{d}\rho = \mathcal{E}(u^o)\,. \tag{4.35}$$

Conclusion of the proof: Adding (4.34) for $j = 1, \ldots, J{-}1$ and (4.35) we obtain that

$$\mathcal{E}(\widetilde{u}^-_{\sigma_*}) + \sigma_* \mathcal{R}\Big(\frac{1}{\sigma_*}(\widetilde{u}^-_{\sigma_*} - u^o)\Big) + \int_0^{\sigma_*} \mathcal{R}^*(-\widetilde{\xi}_\rho)\,\mathrm{d}\rho = \mathcal{E}(u^o)\,.$$

In analogy to (4.31) for $\tau_J = \sigma_*$ we also have $\Phi_{\sigma_*}(u^o; \widetilde{u}^-_{\tau_J}) = \phi(u^o; \sigma_*) = \Phi_{\sigma_*}(u^o; \widetilde{u}_{\sigma_*})$, such that (4.29) is established. ■

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