Strange attractors with topologically simple basins

Hans Günther Bothe

November 14, 1997

We shall consider attractors Λ of C^1 diffeomorphisms $f\colon M \to M$, where M is a differentiable manifold without boundary; i.e. Λ is a compact subset of M for which there is an open neighbourhood U in M such that

$$\mathrm{Cl}(f(U)) \subset U, \quad \Lambda = \bigcap_{i=0}^{\infty} f^i(U),$$

where Cl denotes the closure in M. By the basin of Λ we mean the open subset

$$W^s_\Lambda = \left\{x \in M \,\middle|\, \lim_{i \to 0} d(f^i(x), \Lambda) = 0\right\} = \bigcup_{i=0}^{\infty} f^{-i}(U)$$

of M, where d denotes the distance with respect to a Riemannian metric of M. Obviously both attractors and its basins are invariant sets, i.e. $f(\Lambda) = \Lambda$, $f(W^s_\Lambda) = W^s_\Lambda$. Two attractors Λ_i ($i = 1, 2$) of diffeomorphisms $f_i\colon M_i \to M_i$, respectively, will be called intrinsically equivalent (basin equivalent) if there is a homeomorphism $h\colon \Lambda_1 \to \Lambda_2$ ($h\colon W^s_{\Lambda_1} \to W^s_{\Lambda_2}$) such that $hf_1 = f_2 h$ on Λ_1 (on $W^s_{\Lambda_2}$). The class of all attractors which are intrinsically equivalent (basin equivalent) to an attractor Λ will be called the intrinsic type (basin type) of Λ.

If an attractor Λ is not topologically connected, then, by compactness, it is the disjoint union of a finite number of compact connected components $\Lambda_1, \dots, \Lambda_q$ which are permuted by f, and $f^k(\Lambda_i) = \Lambda_i$ holds for some $k \geq 1$ and all Λ_i. Then each Λ_i is an attractor of f^k, and

$$W^s_\Lambda = W^s_{\Lambda_1} \cup \dots \cup W^s_{\Lambda_i}$$

where the basins $W^s_{\Lambda_i}$ of the attractors Λ_i of f^k are disjoint. Therefore it is sufficient to consider the attractors Λ_i; i.e. we may (and shall) assume that attractors are connected.

The following question was the motivation for a previous paper (see [1] and Theorem A below).

(1) *When does the intrinsic type of an attractor determine its basin type?*

In the present paper continuing on this line we try to find, in some cases, an answer to the following second question.

(2) *Provided for a class of attractors Λ the answer to* (1) *is affirmative under some assumptions, how can we describe W^s_Λ if the intrinsic structure of Λ is known?*

In many cases, if an attractor is complicated, so will be its basin. But in this paper, giving some partial answers to question (2), we shall find wild attractors with relatively tame basins. Of course, an answer can be expected only for those attractors whose intrinsic structure is known to a considerable extend. A class of such attractors is given by the 1-*dimensional hyperbolic attractors* which were described intrinsically by R. F. Williams [5], [7], [8], and only these will be considered here. We proceed presenting some facts concerning their intrinsic structure. Since in this paper we are interested in topological questions only we neglect the differential structure of these attractors from which they have their name 'hyperbolic' and restrict our attention to the topological level (i.e. up to homeomorphism for their geometry and up to topological conjugacy for their dynamics: see [5] and [7] for the broader context).

the union of finitely many smooth arcs $A_1, \ldots, A_r$ with the following properties.

(1_Σ) Each two of the arcs A_i are disjoint, or their intersection is a common end point.

(2_Σ) If τ is a common end point of A_i and A_j, then the tangents $T_\tau A_i$, $T_\tau A_j$ of A_i and A_j at τ coincide.

(3_Σ) Each point $\tau \in \Sigma$ lies in the interior of a smooth subarc of Σ.

(4_Σ) If τ is a *branch point* of Σ (i.e. if τ belongs to more than two of the arcs A_i), then all arcs A_i containing τ except one leave τ with the same direction (these arcs are called the *branches of* τ) while the remaining one (called the *stem of* τ) runs into the opposite direction.

Branched manifolds Σ are considered in connection with certain expanding mappings $\varphi\colon \Sigma \to \Sigma$ which will be called *W-mappings*. They are assumed to be C^1 on each smooth subarc of Σ with differential $d\varphi : T_\tau(\Sigma) \to T_{\varphi(\tau)}(\Sigma)$ $(\tau \in \Sigma)$ and to satisfy the following conditions.

(1_φ) $|d_\tau\varphi| > 1$, where this value is given by the metric of $\mathbb{R}^3$ (i.e. φ is expanding).

(2_φ) Each $\tau \in \Sigma$ has a neighborhood Γ in Σ such that $\varphi(\Gamma)$ lies in a smooth arc (i.e. φ is locally flattining at the branch points).

(3_φ) If A is a subarc of Σ then $\varphi^k(A) = \Sigma$ for some positive integer k.

If $\varphi^k(A) = \Sigma$ then $A \subset \varphi^k(A)$ which shows that φ^k has a fixed point in A. Therefore W-mappings have periodic points.

In [7] 1-dimensional hyperbolic attractors are described intrinsically by inverse limits of branched 1-manifolds with W-mappings. Here we prefer an equivalent more geometric description which uses tubular neighbourhoods of branched 1-manifolds.

A *tubular neighbourhood* of a branched 1-manifold Σ is an orientable compact 3-dimensional C^1 manifold N with boundary together with a mapping $\pi\colon N \to \Sigma$ which is C^1 in the obvious sense and has the following properties. For each arc A in Σ without branch points there is C^1 embedding

$$\sigma\colon A \times \mathbb{D}^2 \to N$$

such that $\sigma(A\times\mathbb{D}^2) = \pi^{-1}(A)$, $\pi^{-1}(A)\cap\partial N = \sigma(A\times\partial\mathbb{D}^2)$ and $\pi(\sigma(\tau, x)) = \tau$. (By ∂, Int we denote the boundary or the interior of a manifold.) If τ is a branch point with q branches then $\pi^{-1}(\tau)$ is the union of q disks with a common boundary point, and for a neighbourhood Γ of τ the set $\pi^{-1}(\Gamma)$ looks like Figure 1. As easily seen, tubular neighbourhoods are handlebodies i.e. smooth manifolds which are homeomorphic to the cartesian product of $[0, 1]$ and a closed 2-dimensional disk D with the interiors of a finite number of disjoint closed disks in Int D removed. (For examples of branched 1-manifolds with tubular neighbourhoods see the figures 3 - 5 below.)

In Section 2 we shall consider *tubular neighbourhoods with corners*, which differ from the ordinary tubular neighbhbods near the counterimages of branch points which now are disks with neighbourhoods in N as shown in Figure 2.

Figure 2

Now let N be a tubular neighbourhood (possibly with corners) of a branched 1-manifold Σ, and let $\varphi\colon \Sigma \to \Sigma$ be a W-mapping. Then by an *embedding* $f_\varphi\colon N \to \operatorname{Int} N$ *over* φ we mean an orientation preserving C^1 embedding which has the following properties.

(1_W) The following diagram is commutative.

$$\begin{array}{ccc} N & @> f_\varphi >> & N \\ \pi \downarrow & & \downarrow \pi \\ \Sigma & @>> \varphi > & \Sigma \end{array} \tag{W}$$

(2_W) The restrictions $f_\varphi\colon \pi^{-1}(\tau) \to \pi^{-1}(\varphi(\tau))$ $(\tau \in \Sigma)$ of f_φ are contracting with respect to a metric on N.

Under these assumptions the intersection

$$\Lambda_\varphi = \bigcap_{k=0}^{\infty} f_\varphi^k(N)$$

satisfies $f_\varphi(\Lambda_\varphi) = \Lambda_\varphi$, and Λ_φ with the mapping $f_\varphi\colon \Lambda_\varphi \to \Lambda_\varphi$ is determined up topological conjugation by the lower part $\varphi\colon \Sigma \to \Sigma$ of (W). Properties $(1_\varphi) - (3_\varphi)$ of φ imply that for $\tau \in \Sigma$ the intersection $\Lambda_\varphi \cap \pi^{-1}(\tau)$ is a Cantor set and that each arc component of Λ_φ is a dense subset of Λ_φ which is a smooth curve (i.e. the image of an injective C^1 immersion $\mathbb{R} \to N$) of infinite length in both directions. Williams has proved that for each 1-dimensional hyperbolic attractor Λ of a diffeomorphism $f\colon M \to M$ there is a W-mapping $\varphi\colon \Sigma \to \Sigma$ such that $f\colon \Lambda \to \Lambda$ is topologically conjugate to $f_\varphi\colon \Lambda_\varphi \to \Lambda_\varphi$ and that for any W-mapping φ the mapping $f_\varphi\colon \Lambda_\varphi \to \Lambda_\varphi$ is topologically conjugate to some 1-dimensional hyperbolic attractor Λ. The diagram (W) above will be called a *W-representation* of Λ. In this way the relatively simple W-mappings describe the relatively complicated intrinsic structure of 1-dimensional hyperbolic attractors. In this paper we shall try to find out how, in certain cases, a W-mapping φ corresponding to a 1-dimensional hyperbolic attractor Λ affects its basin W^s_Λ.

manifold M are smooth curves with tangents which depend continuously on the points in Λ. By a *transverse section* of Λ we mean an open $(m-1)$-dimensional C^1 submanifold S of M which intersects Λ in a non empty compact set and is transverse to these curves. Then $S \cap \Lambda$ is a Cantor set. If $m-1 \geq 3$ it can happen that $\Lambda \cap S$ is *wild* in S; i.e. there may be a positive ε such that it is impossible to find a finite family of disjoint closed topological $(m-1)$-balls in S with diameter at most ε whose interiors cover $S \cap \Lambda$ (see [1]). If $S \cap \Lambda$ is wild for one section then this holds for all sections and we say that Λ is *transversely wild.* As shown in [1] there are many types of transversely wild attractors, but nevertheless being transversely wild is a rare phenomenon and the assumption of transverse tameness is a weak restriction for 1-dimensional hyperbolic attractors.

Our theorems below will not depend on whether Λ is regarded as attractor of $f\colon M \to M$ or of $f^k\colon M \to M$ for some $k \geq 2$ (since Λ is connected so is W^s_Λ, and W^s_Λ remains unchanged if f is replaced by f^k). If Λ as attractor of $f\colon M \to M$ for some W-mapping φ is conjugate to Λ_φ, then Λ as attractor of f^k is conjugate to Λ_{φ^k} (obviously with φ the mapping φ^k is a W-mapping too). Therefore, since W-mappings φ have periodic points, we may replace the diffeomorphisms f belonging to an attractor under consideration by some suitable f^k and get the corresponding W-mapping φ^k which has a fixed point τ_0. This justifies our assumptions made below that φ has a fixed point. The construction of Λ_φ shows that f_φ has a fixed point in Λ_φ if and only if φ has a fixed point in Σ. For a fixed point τ_0 of φ we shall consider the fundamental group $\pi_1(\Sigma, \tau_0)$ and the homomorphism $\varphi_*\colon \pi_1(\Sigma, \tau_0) \to \pi_1(\Sigma, \tau_0)$ defined by φ. Since topologically Σ is a finite graph, this group is free and finitely generated. Its rank can be regarded as the number of loops in Σ or the number of handles of a tubular neighbourhood N of Σ.

For a better understanding of the results in this paper it will help looking at the following fact (from [2]) concerning the question (1) asked above.

Theorem A *Let Λ_i $(i = 1, 2)$ be 1-dimensional hyperbolic attractors with the same intrinsic type of diffeomorphisms $f_i\colon M_i \to M_i$, respectively, where $\dim M_i = m_i$. We assume*

1) $m_1 = m_2 = m$,
2) $W^s_{\Lambda_1}, W^s_{\Lambda_2}$ *are orientable,*
3) $m \geq 4$,
4) Λ_1, Λ_2 *are transversely tame.*

Then the basins $W^s_{\Lambda_1}, W^s_{\Lambda_2}$ are homeomorphic. If 4) is omitted from the assumptions, then we get the weaker conclusion that $W^s_{\Lambda_1} \times \mathbb{R}, W^s_{\Lambda_2} \times \mathbb{R}$ are homeomorphic.

Obviously 1) is necessary (since $m_i = \dim W^s_{\Lambda_i}$), and 2) can not be omitted too since simple examples show that basins of attractors with the same intrinsic type can be orientable or non orientable. Condition 3) is necessary since the strings (i.e. the arc components) of a 1-dimensional hyperbolic attractor in a 3-manifold can be plaited in differential ways and this affects the shape of the basins. The case $m = 2$ is simple and will be omitted here. Though not proved, it seems reasonable that the basins of a transversely tame 1-dimensional hyperbolic attractor never can be too.

Now let Λ be a transversely tame 1-dimensional hyperbolic attractor of a diffeomorphism $f\colon M \to M$, where M is an m-dimensional ($m \geq 4$) and W^s_Λ is orientable. The results of this paper are stated in the following three theorems.

Theorem 1 *There is an open subset G of $\mathbb{R}^3$ such that W^s_Λ is homeomorphic to $G \times \mathbb{R}^{m-3}$.*

Let $\varphi\colon \Sigma \to \Sigma$ be the lower part of a W-representation (W) of Λ, i.e. φ is a W-mapping such that $f_\varphi\colon \Lambda_\varphi \to \Lambda_\varphi$ is topolgically conjugate to $f\colon \Lambda \to \Lambda$. According to the remark above we assume that f has a fixed point x_0 in Λ. Then any φ which can appear as lower part of a W-representation of Λ must have a fixed point τ_0. By an m-dimensional open handlebody we mean the cartesian product of an open 2-disk minus a finite number of disjoint closed subdiks with $\mathbb{R}^{m-2}$.

Theorem 2 *The following conditions are equivalent.*

(i) *W^s_Λ is an open handlebody.*

(ii) *There is a W-representation (W) for Λ such that*

$$\varphi_*^{r+1}(\pi_1(\Sigma,\tau_0)) = \varphi_*^r(\pi_1(\Sigma,\tau_0)),$$

where τ_0 is a fixed point of φ_* and r denotes the rank of $\pi_1(\Sigma,\tau_0)$.

(iii) *For any W-representation (W) for Λ*

$$\varphi_*^{r+1}(\pi_1(\Sigma,\tau_0)) = \varphi_*^r(\pi_1(\Sigma,\tau_0)),$$

where τ_0, r are as in (ii).

(iv) *There is a W-representation (W) for Λ such that*

$$\varphi_*\colon \pi_1(\Sigma,\tau_0) \to \pi_1(\Sigma,\tau_0)$$

is an automorphism. (The rank rk $\pi_1(\Sigma,\tau_0)$ *of this group is the rank r' of the group $\varphi_*^r(\pi_1(\Sigma,\tau_0))$ in* (ii)*).*

(v) *The fundamental group of W^s_Λ is finitely generated.*

The number of handles of W^s_Λ equals the rank r' of the group $\varphi_*^r(\pi_1(\Sigma,\tau_0))$ in (ii) *and* (iii) *and the rank of the group $\pi_1(\Sigma,\tau_0)$ in* (iv).

In the following theorem we shall use *loop bundles* L in the 3-dimensional sphere $\mathbb{S}^3$ which are defined to be the union of a finite set of simple closed curves $L_1, \ldots, L_r$, all passing through a point z but being disjoint elsewhere. It is assumed that the open arcs $L_i \backslash \{z\}$ are of class C^1. A bundle L will be called *wild* if it is infinitely knotted or linked near z in the sense that the fundamental group of $\mathbb{S}^3 \backslash L$ is not finitely generated.

the fixed point τ_0, and let $\alpha_1, \dots, \alpha_r$ be a basis of $\pi_1(\Sigma, \tau_0)$. If there is a mapping $\kappa\colon \{1, \dots, r\} \to \{1, \dots, r\}$ such that $\varphi_*(\alpha_i)$ $(i = 1, \dots, r)$ is conjugate to $\alpha_{\kappa(i)}$ in $\pi_1(\Sigma, \tau_0)$, then there is a loop bundle L in $\mathbb{S}^3$ such that W^s_Λ is homeomorphic to $(\mathbb{S}^3 \backslash L) \times \mathbb{R}^{m-3}$. If W^s_Λ is not a handlebody then L is wild. The number r' of loops in L is the number of elements in the set $\kappa^{r-1}(\{1, \dots, r\})$.

Remark As the proof will show, the bundle L can be obtained by the following construction, where $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ is defined for some $\rho \in (0,1)$ by $\phi(x) = \rho x$, and $\mathbb{S}^3$ is regarded as $\mathbb{R}^3$ with a point added at infinity. The unit ball in $\mathbb{R}^3$ is denoted by $\mathbb{D}^3$. We can choose disjoint arcs $A_1, \dots, A_{r'}$ in $\mathbb{R}^3 \backslash \operatorname{Int} \mathbb{D}^3$ with end points $c_1, \dots, c_{2r'}$ on $\partial \mathbb{D}^3$ and disjoint arcs $B_1, \dots, B_{2r'}$ in $\mathbb{D}^3 \backslash \phi(\operatorname{Int} \mathbb{D}^3)$ whose end points are $c_1, \dots, c_{2r'}, \phi(c_1), \dots, \phi(c_{2r'})$ such that $A_1 \cup \dots \cup A_{r'} \cup B_1 \cup \cdots B_{2r'}$ is the union of r' disjoint smooth arcs with end points on $\phi(\partial \mathbb{D}^3)$ and

$$L = \bigcup_{j=1}^{r'} A_j \cup \bigcup_{i=0}^{\infty} \phi^i \left(\bigcup_{l=1}^{2r'} B_l \right) \cup \{o\}.$$

where o denotes the centre of $\mathbb{D}^3$ (See Fig. 4c and Fig. 5c below). Each set

$$L_s = \bigcup_{j=0}^{r'} A_j \cup \bigcup_{i=0}^{s} \phi^i \Big(\bigcup_{l=1}^{2r'} B_l \Big) \cup \phi^{s+1}(\mathbb{D}^3) \qquad (s = 0, 1, \dots)$$

is the union of the ball $\phi^{s+1}(\mathbb{D}^3)$ with r' disjoint arcs whose end points lie on $\phi^{s+1}(\partial \mathbb{D}^3)$ and which are unknotted and unlinked in the sense that there is a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ which maps $\phi^{s+3}(\mathbb{D}^3)$ onto itself and each of the arcs onto a part of a circle lying in the plane $\{(t_1, t_2, t_3) \in \mathbb{R}^3 | t_3 = 0\}$.

To show that Theorem 2 and Theorem 3 are not void we give some examples to which these theorems apply. The intrinsic structure of the attractors in these examples will be described by drawing an embedding $f_\varphi\colon N \to \operatorname{Int} N$ over a W-mapping $\varphi\colon \Sigma \to \Sigma$ belonging to a W-representation of the attractor, and for the description of L we use the remark made above.

1) Here

$$\pi_1(\Sigma, \tau_0) = \langle \alpha_1, \alpha_2 \rangle,$$
$$\varphi_*(\alpha_1) = \alpha_2, \quad \varphi_*(\alpha_2) = \alpha_2 \alpha_1 \alpha_2,$$

Figure 3a, b

2) Here

$$\begin{aligned} \pi_1(\Sigma, \tau_0) &= \langle \alpha_1, \alpha_2 \rangle \\ \varphi_*(\alpha_1) = \alpha_1, \quad \varphi_*(\alpha_2) &= \alpha_2 \alpha_1 \alpha_2^{-1}, \end{aligned}$$

and by Theorem 3 the basin W_Λ^s is homeomorphic to $(\mathbb{S}^3 \backslash L) \times \mathbb{R}^{m-3}$, where L is a loop bundle with one loop, i.e. L is a simple closed curve. To show that L

Figure 4a

Figure 4b

Figure 4c

3) This example is similar to the preceding one, but now

$$\pi_1(\Sigma, \tau_0) = \langle \alpha_1, \alpha_2 \rangle$$
$$\varphi_*(\alpha_1) = \alpha_1, \quad \varphi_*(\alpha_2) = \alpha_2 \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1},$$

Figure 5a

Figure 5b

Figure 5c

2 Proof of Theorem 1

The main part of this section will be a proof of the following proposition.

Proposition 2.1 *If Λ is a 1-dimensional hyperbolic attractor of a diffeomorphism $f\colon M \to M$, then there is an integer $k \geq 1$ and a diffeomorphism $f^{*}\colon \mathbb{S}^3 \to \mathbb{S}^3$*

equivalent to Λ provided Λ is regarded as an attractor of f^k instead of f.

Remark It is well known that each 1-dimensional hyperbolic attractor is intrinsically equivalent to an attractor of a diffeomorphism $f\colon M \to M$, where M is a 3-dimensional manifold; but the general construction of M may yield manifolds which are ravelled at infinity in the sense that they are not embeddable in any compact 3-dimensional manifold. It would be nice to know whether there are attractors for which necessarily $k > 1$ in the proposition.

Before proving this proposition we show how it implies Theorem 1. Let $f\colon M \to M$, Λ, $m = \dim M \geq 4$ be as in the theorem. By the proposition we find a diffeomorphism $f^*\colon \mathbb{S}^3 \to \mathbb{S}^3$ with the attractor Λ^*. We consider the manifold $M' = \mathbb{S}^3 \times \mathbb{R}^{m-3}$ and the diffeomorphism $f'\colon M' \to M'$ which is defined by

$$f'(x,t) = \left(f^*(x), \tfrac{1}{2}t\right).$$

Obviously, $\Lambda' = \Lambda^* \times \{o\}$ (o the origin of $\mathbb{R}^{m-3}$) is a 1-dimensional hyperbolic attractor of f' with the basin $W^s_{\Lambda'} = W^s_{\Lambda^*} \times \mathbb{R}^{m-3}$. If Λ^+ denotes Λ regarded as attractor of f^k, then, by the choice of f^* the attractors $\Lambda', \Lambda^*, \Lambda^+$ are intrinsically equivalent. Since Λ^* is transversely tame, it is not hard to see that Λ' has the same property, and applying Theorem A we see that $W^s_{\Lambda^+}$ and $W^s_{\Lambda'}$ are homeomorphic. Then $W^s_{\Lambda^+} = W^s_{\Lambda}$ proves the theorem with $G = W^s_{\Lambda^*}$.

In the proof of the proposition we shall use *elementary branched 1-manifolds* and *elementary W-mappings* as defined below which apart from a slight modification were introduced in [8]. By an *elementary W-representation* of a 1-dimensional hyperbolic attractor we mean a W-representation (W) whose lower part is an elementary W-mapping.

A branched 1-manifold Σ is called *elementary* if it is a circle, i.e., without branch points, or if it has exactly two branch points with a common stem A, i.e. if Σ looks like Figure 6 or a part of it.

Figure 6

A W-mapping $\varphi\colon \Sigma \to \Sigma$ is called *elementary* provided Σ is elementary, and, if Σ has branch points τ_1, τ_2 with the common stem A, then $\varphi(A)$ is an arc which contains A in its interior.

The following lemma is an easy consequence of Theorem 5.2 in [8].

Λ has an elementary W-representation provided it is regarded as attractor of f^k, for some suitably chosen integer $k \geq 1$.

If Σ is an elementary branched 1-manifold with branch points τ_1, τ_2, then A_Σ will denote the common stem of τ_1, τ_2. If Σ has no branch points, i.e. if Σ is a circle, then let A_Σ be an arbitrarily chosen arc in Σ. In the proof of Proposition 2.1 we shall consider two special kinds of tubular neighbourhoods with cornes of Σ in $\mathbb{R}^3$, the flat and the monotone ones which are defined as follows. If N is a tubular neighbourhood of one of these two classes and $\pi\colon N \to \Sigma$ is the corresponding projection, then it is assumed that each set $\pi^{-1}(\tau)$ $(\tau \in \Sigma)$ is an Euclidean disk in a plane which is orthogonal to the plane $\mathbb{R}^2 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 | t_3 = 0\}$ in $\mathbb{R}^3$. The centre of $\pi^{-1}(\tau)$ will be denoted by $z(\tau)$. We say that N is *flat* if $z(\tau) \in \mathbb{R}^2$ for all $\tau \in \Sigma$. For being *monotone* N must satisfy the following conditions.

(1) For all $\tau \in A_\Sigma$ the disks $\pi^{-1}(\tau)$ have the same radii, and $z(\tau) \in \mathbb{R}^2$.

(2) If Σ has branch points and C_1, C_2 are two components of $\Sigma \backslash A_\Sigma$, then the projections of $\pi^{-1}(C_1)$, $\pi^{-1}(C_2)$ to the third axis of $\mathbb{R}^3$ are disjoint.

(3) If Σ has branch points, τ, τ' are different points in $\Sigma \backslash A_\Sigma$ and $z(\tau) = (t_1, t_2, t_3)$, $z(\tau') = (t_1', t_2', t_3')$, then $t_3 \neq t_3'$. This implies that for each component C of $\Sigma \backslash A_\Sigma$ the central line of the curved cylinder $\pi^{-1}(C)$ is monotone with respect to its third coordinate t_3.

(4) If Σ has no branch point, then for each plane P in $\mathbb{R}^3$ parallel to $\mathbb{R}^2$ there are at most two points $\tau \in \Sigma \backslash A_\Sigma$ such that $z(\tau) \in P$.

Proof of Proposition 2.1 Let Λ be as in the proposition. Using Lemma 2.2 we choose $k \geq 1$ an elementary W-representation

$$\begin{array}{ccc} N & @> f_\varphi >> & N \\ \pi \downarrow & & \downarrow \pi \\ \Sigma & @>> \varphi > & \Sigma \end{array} \qquad (W_k)$$

for Λ regarded as attractor of f^k. It is a simple geometric fact that we may choose (W) so that N is a flat and $f_\varphi(N)$ is a monotone tubular neighbourhood of Σ. Since the handles of a flat or monotone tubular neighbourhood are neither knotted nor linked in $\mathbb{S}^3$, there is a diffeomorphism $f^*\colon \mathbb{S}^3 \to \mathbb{S}^3$ such that $g(N) = f_\varphi(N)$; even more, g can be chosen so that $f^*|_N\colon N \to f_\varphi(N)$ is an embedding over φ. (The difference between $f^*|_N$ and f_φ may be caused by the fact that f_φ possibly twists handles of N while $f^*|_N$ avoids such twists.) Replacing f_φ in (W_k) by $f^*|_N$ we get a new W-representation (W^*) for the attractor Λ of f^k, and $\Lambda^* = \bigcap_{i=0}^\infty f^*(N)$ is a 1-dimensional hyperbolic attractor of f^*. Since Λ as attractor of f^k and Λ^* have the same W-representation (W^*) they are intrinsically equivalent. (That Λ^* is transversely tame follows from the general fact that all 1-dimensional hyperbolic attractors in 3-dimensional manifolds are transversely tame or more directly from our construction of Λ^*.) ■

3 Proof of Theorem 2

We shall prove the implications (v) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i). Since the implications (i) $\Rightarrow$ (v) and (iii) $\Rightarrow$ (ii) are trivial this will prove the theorem.

This proof is an immediate consequence of the following two lemmas.

Lemma 3.1 *If*

$$\begin{array}{ccc} N & @>f>> & N \\ \pi\downarrow & & \downarrow\pi \\ \Sigma & @>\varphi>> & \Sigma \end{array}$$

any W-representation for Λ and τ_0 is a fixed point of φ, then the fundamental group of W^s_Λ is isomorphic to the direct limit of the sequence

$$\pi_1(\Sigma,\tau_0)@>>\varphi_*>\pi_1(\Sigma,\tau_0)@>>\varphi_*>\cdots.$$

Lemma 3.2 *If $\varphi_*\colon G\to G$ is a homomorphism of a finitely generated free group G with rank r and if the direct limit $\tilde G$ of $G@>>\varphi_*>G@>>\varphi_*>\cdots$ is finitely generated, then $\varphi_*^{r+1}(G)=\varphi_*^r(G)$.*

Proof of Lemma 3.1. As proved in [2], there is a topological embedding $h\colon N\times\mathbb{D}^{m-3}\to W^s_\Lambda$ ($m=\dim W^s_\Lambda$, $\mathbb{D}^{m-3}$ the unit ball in $\mathbb{R}^{m-3}$) with $N^m=h(N\times\mathbb{D}^{m-3})$ a C^1 manifold such that $f^k(N^m)\subset\operatorname{Int}N^m$ for some $k\geq 1$ and

$$\Lambda=\bigcap_{i=0}^{\infty}f^{ik}(N^m),\quad W^s_\Lambda=\bigcup_{i=0}^{\infty}f^{-ik}(N^m).$$

Moreover, if the projection $\bar\pi\colon N^m\to\Sigma$ is defined by $\bar\pi h(x,t)=\pi(x)$ the following diagram is commutative

$$\begin{array}{ccc} N^m & @>f^k>> & N^m \\ \bar\pi\downarrow & & \downarrow\bar\pi \\ \Sigma & @>\varphi^k>> & \Sigma \end{array}$$

If τ_0 is our fixed point of φ then $\pi_N^{-1}(\tau_0)$ contains a unique fixed point x_0 of f^k and we consider the projections $\bar\pi f^{ik}\colon f^{-ik}(N)\to\Sigma$ ($i=0,1,2,\ldots$). The homomorphisms $(\bar\pi f^{ik})_*\colon\bar\pi(f^{-ik}(N),x_0)\to\pi_1(\Sigma,\tau_0)$ are isomorphisms, and the diagram

$$\begin{array}{ccccccc} \pi_1(N^m,x_0) & @>>\iota_*>\cdots @>>\iota_*> & \pi_1(f^{-ik}(N^m),x_0) & @>>\iota_*> & \pi_1(f^{(i+1)k}(N^m),x_0) & @>>\iota \\ \downarrow\pi & & \downarrow(\bar\pi f^{ik})_* & & \downarrow(\bar\pi f^{(i+1)k})_* & \\ \pi_1(\Sigma,\tau_0) & @>>\varphi_*^k>\cdots @>>\varphi_*^k> & \pi_1(\Sigma,\tau_0) & @>>\varphi_*^k> & \pi_1(\Sigma,\tau_0) & @>>\varphi \end{array}$$

(with $\iota\colon f^{-ik}(N)\to f^{-(i+1)k}(N)$ the inclusion) is commutative. Therefore the direct limits of the two sequences are isomorphic and isomorphic to the limit in the lemma. Since $f^{-ik}(N)\subset f^{-(i+1)k}(N)$ and $W^s_\Lambda=\bigcup_{i=0}^\infty f^{-ik}(N^m)$, the direct limit of the upper sequence is isomorphic to $\pi_1(W^s_\Lambda,x_0)$. This proves the lemma. ■

Proof of Lemma 3.2. Here we use some facts from the theory of free groups which are proved in [3] e.g. Since subgroups of free groups are free, all groups G, $\varphi_*(G)$, $\varphi_*^2(G),\ldots$ are finitely generated and free. Moreover, concerning the ranks of isomorphy of these groups and $\mathrm{rk}\varphi_*^l(G) = \mathrm{rk}\varphi_*^k(G)$ for all $l \geq k$, we get $\mathrm{rk}\varphi_*^l(G) = \mathrm{rk}\varphi_*^r(G)$ for $l \geq r$. Let $\varphi_*^r(G)$ be denoted by H. Since $\varphi_*\colon H \to \varphi_*(H)$ is an isomorphism we have either $H = \varphi_*(H) = \varphi_*^2(H) = \cdots$ or $H \supsetneq \varphi_*(H) \supsetneq \varphi_*^2(H) \supsetneq \cdots$. If the sesond case would be true, then the direct limit of $H@>>\varphi_*>H@>>\varphi_*>H@>>\varphi_*>\cdots$ could not be finitely generated. But this is impossible since this limit coincides with the direct limit $\tilde{G}$ of $G@>>\varphi_*>G@>>\varphi_*>G@>>\varphi_*>\cdots$ which is finitely generated. So we have $\varphi_*^r(G) = H = \varphi_*(H) = \varphi_*^{r+1}(G)$. ■

3.2 Proof of (ii) $\Rightarrow$ (iv).

Let

$$\begin{array}{ccc} N & @>f>> & N \\ \pi\downarrow & & \downarrow\pi \\ \Sigma & @>\varphi>> & \Sigma \end{array} \qquad (W)$$

be a W-representation for Λ such that

$$\varphi_*^{r+1}(\pi_1(\Sigma,\tau_0)) = \varphi_*^r(\pi_1(\Sigma,\tau_0)).$$

We consider the covering space $\rho\colon \tilde{\Sigma}_0 \to \Sigma$ of Σ corresponding to the subgroup $\varphi_*^r(\pi_1(\Sigma,\tau_0))$ of $\pi_1(\Sigma,\tau_0)$. By the definition of concerning spaces we can fix a point $\tilde{\tau}_0 \in \rho^{-1}(\tau_0)$ such that $\rho_*(\pi_1(\tilde{\Sigma}_0,\tilde{\tau}_0)) = \varphi_*^r(\pi_1(\Sigma,\tau_0))$, and ρ_* is an isomorphism. In this proof we shall denote $\pi_1(\Sigma,\tau_0)$ by G and $\pi_1(\tilde{\Sigma}_0,\tilde{\tau}_0)$ by $\tilde{G}$. The mapping $\varphi\colon \Sigma \to \Sigma$ is covered by a unique mapping $\tilde{\varphi}\colon \tilde{\Sigma}_0 \to \tilde{\Sigma}_0$ satisfying $\rho\tilde{\varphi} = \varphi\rho$ and $\tilde{\varphi}(\tilde{\tau}_0) = \tilde{\tau}_0$ (see [4], Theorem 5.1 on p. 156, e.g.). A mapping $\chi\colon \Sigma \to \tilde{\Sigma}_0$ satisfying $\chi\varphi = \tilde{\varphi}\chi$ and $\chi(\tau_0) = \tilde{\tau}_0$ can be defined as follows. For $\tau \in \Sigma$ choose a path γ in Σ from τ_0 to τ and the lifting $\tilde{\gamma}$ of $\varphi^r(\gamma)$ in $\tilde{\Sigma}_0$ starting at $\tilde{\tau}_0$. Then, as easily seen, the end point $\tilde{\tau}$ of $\tilde{\gamma}$ does not depend on the choice of γ and we define $\chi(\tau) = \tilde{\tau}$.

The image $\chi(\Sigma)$ will be denoted by $\tilde{\Sigma}$. We have $\tilde{\varphi}(\tilde{\Sigma}) = \tilde{\Sigma}$, $\pi_1(\tilde{\Sigma},\tilde{\tau}_0) = \tilde{G}$, i.e. each element of $\tilde{G}$ can be represented by a loop in $\tilde{\Sigma}$. As easily seen, $\tilde{\Sigma}$ is a branched 1-manifold, and $\tilde{\varphi}\colon \tilde{\Sigma} \to \tilde{\Sigma}$ is a W-mapping. With a tubular neighbourhood $\tilde{N}$ of $\tilde{\Sigma}$ we get the W-representation

$$\begin{array}{ccc} \tilde{N} & @>f_{\tilde{\varphi}}>> & \tilde{N} \\ \tilde{\pi}\downarrow & & \downarrow\tilde{\pi} \\ \tilde{\Sigma} & @>\tilde{\varphi}>> & \tilde{\Sigma} \end{array} \qquad (\tilde{W})$$

To show that $(\tilde{W})$ is a new W-represenation for our attractor Λ we remark that

$$\begin{array}{ccc} \tilde{\Sigma} & @>\tilde{\varphi}^r>> & \tilde{\Sigma} \\ \varphi\downarrow & \nearrow\chi & \downarrow\rho \\ \Sigma & @>\varphi^r>> & \Sigma \end{array}$$

is commutative. Indeed, this diagram and $\chi\varphi = \tilde{\varphi}\chi$, $\rho\tilde{\varphi} = \varphi\rho$ are just the conditions under which Williams ([8], Theorem 3.3) shows that (W) and $(\tilde{W})$ represent intrinsically equivalent attractors. Since ρ_* is an isomorphism the groups $\tilde{G}$ and $\varphi_*^r(G)$ have the same rank.

is a finitely generated free group), $\tilde{\varphi}_*(\tilde{G}) = \tilde{G}$. Let $\tilde{\alpha}$ be an arbitrary element of $\tilde{G}$. Since

$$\rho_*(\tilde{\alpha}) \in \rho_*(\tilde{G}) = \varphi_*^r(G) = \varphi_*^{r+1}(G)$$

we can find an element $\beta \in \rho_*(\tilde{G})$ such that $\varphi_*(\beta) = \rho_*(\tilde{\alpha})$. Then for the element $\tilde{\beta} = \rho_*^{-1}(\beta)$ of $\tilde{G}$ we have

$$\tilde{\varphi}_*(\tilde{\beta}) = \tilde{\varphi}_*\rho_*^{-1}(\beta) = \rho_*^{-1}\varphi_*(\beta) = \rho_*^{-1}\rho_*(\tilde{\alpha}) = \tilde{\alpha}.$$

■

3.3 Proof of (iv) ⇒ (i).

Some definitions and facts concerning smooth 3-dimensional handlebodies, here shortly called handlebodies, which will be used here and in Section 4 are collected in the appendix. Especially the concepts of handles and twistings of handles are introduced there. If $f'\colon N \to \mathrm{Int}\, N$ is a C^1 embedding of a handlebody N, we define a diffeomorphism, also denoted f', of an open 3-manifold M' containing N such that the new mapping extends the original embedding. This diffeomorphism $f'\colon M' \to M'$ will be called the *extension* of $f'\colon N \to \mathrm{Int}\, N$. To get M' take the union $N \cup R_1 \cup R_2 \cup \cdots$ of N with disjoint copies $R_1, R_2, \ldots$ of $R_0 = N \setminus \mathrm{Int}\, f'(N)$ and identify each point x in R_i $(i = 0, 1, \ldots)$ corresponding to a point $y \in R_0$ which lies on ∂N with the point in R_{i+1} which corresponds to $f'(y) \in R_0$. If $x \in R_i$ $(i = 1, 2, \ldots)$ corresponds to $y \in N_0$ define $f'(x)$ to be the point in R_{r-1} which corresponds to the same point y in R_0.

According to (iv) let

$$\begin{array}{ccc} N & @>f'>> & N \\ \pi\downarrow & & \downarrow\pi \\ \Sigma & @>>\varphi> & \Sigma \end{array}$$

be a W-representation for Λ in which $\varphi_*\colon \pi_1(\Sigma, \tau_0) \to \pi_1(\Sigma, \tau_0)$ is an automorphism, and let τ_0, x_0 be fixed points of φ, f', respectively, such that $\pi(x_0) = \tau_0$. Looking at Theorem A in Section 1 we see that (i) will be proved if we have found a diffeomorphism $\tilde{f}\colon \tilde{M} \to \tilde{M}$ of an m-dimensional open handlebody $\tilde{M}$ with a transversely tame 1-dimensional hyperbolic attractor $\tilde{\Lambda}$ such that $\tilde{\Lambda}$ is intrinsically equivalent to Λ and $W^s_{\tilde{\Lambda}} = \tilde{M}$.

Our construction of $\tilde{M}$ and $\tilde{f}$ starts with some arbitrarily chosen disjoint arcs $A_1, \ldots, A_{r'}$ in Σ which do not contain branch points or the point τ_0 and for which $\Sigma \setminus (A_1 \cup \cdots \cup A_{r'})$ is connected and simply connected. Then by $H_1 = \pi^{-1}(A_1), \ldots, H_{r'} = \pi^{-1}(A_{r'})$ we get a set of handles of N (see the appendix). By an adjusted twist of H_j we mean a twist of H_j which maps each disk $\pi^{-1}(\tau)$ onto itself. Now we consider a diffeomorphism $\theta\colon N \to N$ which is the composition of adjusted twists which will be defined later. Then $f'_\theta = f'\theta\colon N \to N$ is an embedding over φ, and, if $f'_\theta\colon M' \to M'$ is the extension of $f'_\theta\colon N \to N$ as defined above, by $\Lambda' = \bigcap_{i=0}^\infty f'_\theta(N)$ we get a 1-dimensional hyperbolic attractor Λ' of f'_θ which has the W-representation

$$\begin{array}{ccc} N & @>f'_\theta>> & N \\ \pi\downarrow & & \downarrow\pi \\ \Sigma & @>>\varphi> & \Sigma \end{array}$$

transversely wild Cantor sets in 2-dimensional manifolds (here the transverse sections of Λ' in M') Λ' is transversely tame.

Now we define $\tilde{f}\colon \tilde{M} \to \tilde{M}$ by

$$\tilde{M} = M' \times \mathbb{R}^{m-3}, \quad \tilde{f}(x,t) = (f'_\theta(x), \tfrac{1}{2}t).$$

Obviously $\tilde{M}$ is the basin of the 1-dimensional hyperbolic attractor $\tilde{\Lambda} = \Lambda' \times \{o\}$ (o the origin of $\mathbb{R}^{m-3}$) which is intrinsically equivalent to Λ' and therefore to Λ. Moreover transverse tameness of Λ' implies the same property for $\tilde{\Lambda}$.

To prove (i) it is sufficient to choose θ so that $\tilde{M}$ becomes an open m-dimensional handlebody. But before doing so we define an orientation preserving C^1 embedding $f''\colon N \to \operatorname{Int} N$ such that $f''(x_0) = x_0, f'_* = f''_*\colon \pi_1(N,x_0) \to \pi_1(N,x_0)$ and the manifold M'' in the extension $f''\colon M'' \to M''$ is an open handlebody with r' handles. To this aim let ν be a collar of ∂N in N, i.e., ν is a C^1 embedding $\nu\colon \partial N \times [0,1] \to N$ satisfying $\nu(x,1) = x$, and let $\gamma\colon [0,1] \to [0,\frac{1}{2}]$ be a diffeomorphism which is the identity near 0. Then a C^1 embedding $f''_0\colon N \to \operatorname{Int} N$ is defined by

$$f''_0(x) = \begin{cases} x, & \text{if } x \notin \nu(\partial N \times [0,1]), \\ \nu(y,\gamma(t)), & \text{if } x = \nu(y,t) \in \nu(\partial N \times [0,1]). \end{cases}$$

Obviously, $f''_0 = id\colon \pi_1(N,x_0) \to \pi_1(N,x_0)$. Now we apply Lemma 5.1 to choose an orientation preserving diffeomorphism $f''_1\colon N \to N$ such that $f''_1(x_0) = x_0, f''_{1*} = f'_* = f'_{\theta*}$. Then we define $f'' = f''_0 f''_1$. As easily seen the manifold M'' in the extension of f'' is an open handlebody with r' handles.

Since M'' is an open handlebody with r' handles so is $M'' \times \mathbb{R}^{m-3}$. Therefore, to finish our proof it is sufficient to show that $\tilde{M} = M' \times \mathbb{R}^{m-3}$ is homeomorphic to $M'' \times \mathbb{R}^{m-3}$. This is easily done by a direct application of Lemma 5.2 with handles H_j as chosen above. By this lemma we get the twist map θ in the definition of f'_θ which was not fixed until now. We have assumed that θ is adapted which is not mentioned in the lemma; but it is a simple fact that θ can be chosen with this property. ■

4 Proof of Theorem 3

Let $\Lambda' = \bigcap_{i=0}^{\infty} f^i_\varphi(N)$ in our W-representation (W) of Λ. The mapping $f_\varphi\colon N \to \operatorname{Int} N$ will be denoted by f', and by $f'\colon M' \to M'$ we denote the extension of $f'\colon N \to \operatorname{Int} N$ as defined at the beginning of the proof of (iv) $\Rightarrow$ (i) in the preceeding section. We define $\tilde{M} = M' \times \mathbb{R}^{m-3}$ and $\tilde{f}\colon \tilde{M} \to \tilde{M}$ by $\tilde{f}(x,t) = (f'(x), \frac{1}{2}t)$. Obviously, $\tilde{M}$ is the basin $W^s_{\tilde{\Lambda}}$ of the attractor $\tilde{\Lambda} = \Lambda' \times \{o\}$ of $\tilde{f}$, and this attractor is intrinsically equivalent to Λ. One more using the fact that 1-dimensional hyperbolic attractors in 3-dimensional manifolds are transversely tame, we see that Λ' and $\tilde{\Lambda}$ have this property. Then by Theorem A the basin $\tilde{M}$ of $\tilde{\Lambda}$ is homeomorphic to the basin of Λ, and to prove our theorem it is sufficient to show that $\tilde{M}$ is homeomorphic to $M'' \times \mathbb{R}^{m-3}$, where M'' is the complement of a loop bundle with r' loops in $\mathbb{S}^3$. To find M'' we shall embed N in $\mathbb{S}^3$, and then by a modification of f' we shall get a new C^1 embedding $f''\colon N \to \operatorname{Int} N$ which is not necessarily an embedding over φ but which can be extended to a diffeomorphism $f''\colon \mathbb{S}^3 \to \mathbb{S}^3$. This diffeomorphism will be chosen so that

$$M'' = \bigcup_{i=0}^{\infty} f''^{-i}(N)$$

bundle with r' loops in $\mathbb{S}^3$ (Lemma 4.2).

Moreover, f'' will be defined so that for a common fixed point x_0 of f' and f'' the homomorphisms

$$f'_*, f''_* \colon \pi_1(N, x_0) \to \pi_1(N, x_0)$$

of the fundamental group coincide. Then Lemma 5.2 shows that $M' \times \mathbb{R}^{m-3}$ and $M'' \times \mathbb{R}^{m-3}$ are homeomorphic, and the proof will be finished.

The embedding $N \subset \mathbb{S}^3$ will be chosen unknotted in the sense that $\mathbb{S}^3 \setminus \operatorname{Int} N$ is a handlebody. Moreover, we may assume that f' does not twist the handle of N. This condition means that for each closed curve C on ∂N which bounds (i.e. is null homologous) in $\mathbb{S}^3 \setminus \operatorname{Int} N$ the image $f'(C)$ bounds in $\mathbb{S}^3 \setminus \operatorname{Int} f'(N)$. (We do not know whether f' can always be chosen to be extendable to a diffeomorphism of $\mathbb{S}^3$ which would simplify the proof considerably.)

If $\pi \colon N \to \Sigma$ denotes the projection in (W) and τ_0 is the fixed point of φ, then $\pi^{-1}(\tau_0)$ contains a fixed point of f' which will be denoted by x_0. The inverse π_*^{-1} of the isomorphism $\pi_* \colon \pi_1(N, x_0) \to \pi_1(\Sigma, \tau_0)$ conjugates the homomorphism $\varphi_* \colon \pi(\Sigma, \tau_0) \to \pi_1(\Sigma, \tau_0)$ to $f'_* \colon \pi_1(N, x_0) \to \pi_1(N, x_0)$. Let $\alpha_1, \ldots, \alpha_r \in \pi_1(\Sigma, \tau_0)$ be as in Theorem 3. By $\beta_j = \pi_*^{-1}(\alpha_j)$ $(1 \le j \le r)$ we get free generators $\beta_1, \ldots, \beta_r$ of $\pi_1(N, x_0)$, and by our assumptions $f'_*(\beta_j)$ is conjugate to $\beta_{\kappa(j)}$.

Let $H_1, \ldots, H_r$ be handles of N which correspond to $\beta_1, \ldots, \beta_r$ as described in the appendix.

Now we begin the construction of $f''_i \colon N \to \operatorname{Int} N$. Since N is unknotted, it is not hard to find disks $D'_1, \ldots, D'_r$ in $\mathbb{S}^3$ with boundaries $L'_1, \ldots, L'_r$, respectively, which have the following properties.

$(1'_D)$ $L'_j \subset \operatorname{Int} N$ $(j = 1, \ldots, r)$.

$(2'_D)$ $D_1, \ldots, D_r$ are transverse to ∂N.

$(3'_D)$ $D'_j \cap D'_{j'} = L'_j \cap L'_{j'} = \{x_0\}$ $(j \neq j')$.

$(4'_D)$ $L'_j \cap H_j$ is an arc running from one end of H_j to the other.

$(5'_D)$ $L'_j \cap H_{j'} = \emptyset$ for $j \neq j'$.

$(6'_D)$ $D'_j \cap \mathbb{S}^3 \setminus \operatorname{Int} N$ is a subdisk D'^*_j of $\operatorname{Int} D'_j$.

$(7'_D)$ $\partial D'^*_j \cap H_j$ is an arc running from the one end of H_j to the other and $D'^*_j \cap H_{j'} = \emptyset$, for $j \neq j'$.

By $(4'_D), (5'_D)$ the loop L'_j with some orientation represents β_j.

For $j = 1, \ldots, r$ we consider thickenings of D'^*_j i.e. C^1 embeddings $\vartheta_j \colon \mathbb{D}^2 \times [-1, 1] \to \mathbb{S}^3$ which have the following properties, where the image of ϑ_j is denoted by K_j.

(1_ϑ) $\vartheta_j(\mathbb{D}^2 \times \{0\}) = D'^*_j$.

(2_ϑ) $K_j \cap N = K_j \cap \partial N = \vartheta_j(\partial \mathbb{D}^2 \times [-1, 1])$.

(3_ϑ) $K_j \cap K_{j'} = \emptyset$ if $j \neq j'$.

handlebody $\mathbb{S}^3 \setminus \operatorname{Int} N$. Then, using the fact that $f'_*(\beta_j)$ is conjugate to $\beta_{\kappa(j)}$ by the process which is indicated for $f'_*(\beta_j) = \beta_l \beta_{\kappa(j)} \beta_l^{-1}$ in Figure 7 we find C^1 disks $D_1, \ldots, D_r$ in $\mathbb{S}^3$ with boundaries $L_1, \ldots, L_r$, respectively which have the following properties.

Figure 7

(1_D) $L_j \subset \operatorname{Int} N$ $(j = 1, \ldots, r)$.

(2_D) $D_1, \ldots, D_r$ are transverse to ∂N.

(3_D) $D_j \cap D_{j'} = \{x_0\}$ $(j \neq j')$.

(4_D) L_j with some orientation represents $f'_*(\beta_j)$ in $\pi_1(N, x_0)$.

(5_D) $D_j \setminus \operatorname{Int} N$ is a subdisk D_j^* of $\operatorname{Int} D_j$, and $D_j^* = \vartheta_{\kappa(j)}(\mathbb{D}^2 \times \{t_j\})$ for some $t_j \in [-1, 1]$.

Now we choose a handlebody N'' which is a thin tubular neighbourhood of $L_1 \cup \cdots \cup L_r$ in $\operatorname{Int} N$. More precisely: we start with a small ball B'' in $\operatorname{Int} N$ with centre x_0 for which each set $A_j = L_j \setminus \operatorname{Int} B''$ $(j = 1, \ldots, r)$ is an arc and choose for each arc A_j a thin tubular neighbourhood H''_j such that $N'' = B'' \cup H''_1 \cup \cdots \cup H''_r$ is a handlebody, and $H''_1, \ldots, H''_r$ is a set of handles of N''.

We assume that N'' is chosen so that the disks D_j are transverse to $\partial N''$ and each set $D_j \setminus \operatorname{Int} N''$ is a subdisk D''_j of $\operatorname{Int} D_j$. Later we shall use C^1 thickenings $\vartheta''_j \colon \mathbb{D}^2 \times [-1, 1] \to \mathbb{S}^3$ of these disks which have the following properties, where $G_j = \vartheta''_j(\mathbb{D}^2 \times [-1, 1])$.

$(1''_\vartheta)$ $\vartheta''_j(\mathbb{D}^2 \times \{0\}) = D''_j$.

$(2''_\vartheta)$ $G_j \cap N'' = G_j \cap \partial N'' = \vartheta''_j(\partial \mathbb{D}^2 \times [-1, 1])$.

$(3''_\vartheta)$ $G_j \cap G'_j = \emptyset$, if $j \neq j'$.

$(4''_\vartheta)$ $G_j \setminus \operatorname{Int} N$ is a subhandle of the handle $K_{\kappa(j)}$ of $\mathbb{S}^3 \setminus \operatorname{Int} N$, i.e. a handle which is contained in $K_{\kappa(j)}$ minus the ends of $K_{\kappa(j)}$.

Looking at the disks D''_j we see that the handlebody N'' is unknotted in $\mathbb{S}^3$ and that $G_1, \ldots, G_r$ is a set of handles of the complementary handlebody $\mathbb{S}^3 \setminus \operatorname{Int} N''$.

connects one end of G_j with the other (see Figure 8, where for $j \neq j', \kappa(j) = \kappa(j')$ the handles $G_j, G_{j'}$ and their intersections with $K_{\kappa(j)}$ are sketched).

Figure 8

Since N and N'' are unknotted, we can choose a C^1 diffeomorphism $f''\colon \mathbb{S}^3 \to \mathbb{S}^3$ such that $f''(N) = N''$, $f''(H_j) = H_j''$ and $f''(K_j) = G_j$ $(j = 1, \ldots, r)$. By (4_D) the handle H_j'' of N'' represents the element $f'_*(\beta_j)$ in $\pi_1(N, x_0)$. Therefore we get $f''_*(\beta_j) = f'_*(\beta_j)$ $(j = 1, \ldots, r)$, and since $\beta_1, \ldots, \beta_r$ is a basis of $\pi_1(N, x_0)$ this implies $f''_* = f'_*$. To get an extension $f''\colon M'' \to M''$ of the embeddings $f''\colon N \to \operatorname{Int} N$ we merely have to restrict the diffeomorphism $f''\colon \mathbb{S}^3 \to \mathbb{S}^3$ to

$$M'' = \bigcup_{i=0}^{\infty} f''^{-i}(N).$$

Now, as already mentioned above, what remains to be done are the proofs of the following two lemmas.

Lemma 4.1 *M'' is homeomorphic to $\mathbb{S}^3 \backslash L$, where L is a loop bundle with r' loops in $\mathbb{S}^3$.*

Lemma 4.2 *$\tilde{M} = M' \times \mathbb{R}^{m-3}$ is homeomorphic to $M'' \times \mathbb{R}^{m-3}$.*

Proof of Lemma 4.1. Since $N'' = f''(N)$ is unknotted, the complement

$$P = \mathbb{S}^3 \backslash \operatorname{Int} N''$$

is a handlebody with r handles and $f''^{-1}(P) = \mathbb{S}^3 \backslash \operatorname{Int} N \subset \operatorname{Int} P$. In this proof we shall write h instead of f''^{-1}. We have already defined the handles $G_1, \ldots, G_r$ of P, and by the properties of these handles and the handles K_j of $\mathbb{S}^3 \backslash \operatorname{Int} N = h(P)$ mentioned above we get the following facts.

(1_G) $G_j \cap h(P)$ is a subhandle of $K_{\kappa(j)} = h(G_{\kappa(j)})$ $(j = 1, \ldots, r)$.

(2_G) $G_j \cap h(P)$ is a curved cylinder which runs in G_j from one end to the other, i.e. each end of G_j intersects $h(P)$ in exactly one end of the handle $G_j \cap h(P)$ of $h(P)$ $(j = 1, \ldots, r)$.

the closure of the component containing z of $B\backslash(D_1 \cup \cdots \cup D_s)$, where $D_1, \ldots, D_s$ are disjoint euclidean spanning disks in $B\backslash\{z\}$. (Here $\mathbb{S}^3$ is regarded as $\mathbb{R}^3$ with a point at infinity.) Then we say that a handlebody P in $\mathbb{S}^3$ with handles $G_1, \ldots, G_r$ has the central ball B if $\mathrm{Cl}(P\backslash(G_1 \cup \cdots \cup G_r))$ is a clipped ball belonging to B and the clipping disks of B are the $2r$ ends of these handles G_j. By a parametrization of a handle G_j we mean a diffeomorphism $\eta\colon \mathbb{D}^2 \times [-1,1] \to G_j$. The disks $\eta(\mathbb{D}^2 \times \{t\})$ ($t \in [-1,1]$) are called meridional disks of G_j.

To finish the proof of Lemma 4.1, i.e. to prove that $M'' = \mathbb{S}^3\backslash \bigcap_{i=0}^{\infty} h^i(P)$ is homeomorphic to the complement $\mathbb{S}^3\backslash L$ of a loop bundle L with r' loops we shall choose handlebodies $P_0, P_1, \ldots$ in $\mathbb{S}^3$ such that $P_{i+1} \subset \mathrm{Int}\, P_i$ ($i = 0, 1, \ldots$) and a C^1 homeomorphism

$$g\colon M'' = \mathbb{S}^3\backslash\bigcap_{i=0}^{\infty} h^i(P) \to \mathbb{S}^3\backslash\bigcap_{i=0}^{\infty} P_i,$$

where $\bigcap_{i=0}^{\infty} P_i$ is a bundle with r' loops.

The construction of $P_0, P_1, \ldots$ will proceed as follows. Using the properties of the handlebodies $P \subset h(P) \subset h^2(P), \ldots$ mentioned above it is not hard to find diffeomorphisms $g_i\colon \mathbb{S}^3 \to \mathbb{S}^3$ ($i = 0, 1, \ldots$) such that for $0 \le i \le i'$ we have $g_{i'} = g_i$ on $\mathbb{S}^3\backslash h^i(P)$ and the images $g_i h^i(P)$, which are the handlebodies P_i we are looking for, have the following properties, where $G_{i,j} = g_i h^i(G_j)$ and $\eta_{i,j}\colon \mathbb{D}^2 \times [-1,1] \to G_{i,j}$ are parametrizations of these handles $G_{i,j}$ of P_i ($i = 0, 1, \ldots,; j = 1, \ldots, r$).

(1_P) The handlebody P_i has as central ball the euclidean ball D_i in $\mathbb{R}^3\backslash\mathbb{S}^3$ with centre o (origin of $\mathbb{R}^3$) and radius 2^{-i}. The corresponding handles are the handles $G_{i,j}$ mentioned above.

(2_P) $G_{i,j} \cap P_{i+1}$ is a subhandle of $G_{i+1,\kappa(j)}$ ($i = 0, 1, \ldots;\ j = 1, \ldots, r$).

(3_P) $\mathrm{diam}\, \eta_{i,j}(\mathbb{D}^2(t)) \le 10^{-i}$ ($i = 0, 1, \ldots;\ j = 1, \ldots, r,\ t \in [-1,1]$).

(4_P) If $i = 0, 1, \ldots;\ j = 1, \ldots, r,\ t \in [-1,1]$ then for some $t' \in (-1,1)$

$$\eta_{i,j}\left(\mathbb{D}^2 \times \{t\}\right) \cap P_{i+1} = \eta_{i+1,\kappa(j)}\left(\mathbb{D}^2 \times \{t'\}\right).$$

The position of two consecutive handlebodies P_i, P_{i+1} is illustrated for $r = 3$,

Figure 9

The sequence $g_0, g_1, \dots$ is obviously convergent on M'', and its limit is a homeomorphism

$$M'' = \mathbb{S}^3 \backslash \bigcap_{i=0}^{\infty} h^i(P) \to \mathbb{S}^3 \backslash \bigcap_{i=0}^{\infty} P_i.$$

To prove the lemma it is sufficient to show that

$$L = \bigcap_{i=0}^{\infty} P_i$$

is a loop bundle with r' loops.

Since by (3_P) the handles $G_{i,j}$ become thinner and thinner if $i \to \infty$ and since for $i' > i \geq 0, G_{i',j'} \cap G_{i,j} \neq \emptyset$ each meridional disk of the parametrization $\eta_{i',j'}$ lies in a meridional disk of $\eta_{i,j}$, each set

$$A_{i,j} = G_{i,j} \cap \bigcap_{i'=0}^{\infty} P_{i'}$$

is an arc which runs in $G_{i,j}$ from one end to the other. If the handlebodies P_i were suitable chosen (this can easily be done), then each $A_{i,j}$ is a C^1 arc. Property (1_P) of the handles $G_{i,j}$ implies

$$A_{i,j} \subset \operatorname{Int} A_{i+1,\kappa(j)} \quad (i = 0, 1, \dots; \ j = 1, \dots, r).$$

Therefore each set

$$A_j = \bigcup_{i=0}^{\infty} A_{i,\kappa^i(j)}$$

is the image of a C^1 embedding $(0,1) \to \mathbb{R}^3$, and $A_j = A_{j'}$ holds if and only if $\kappa^l(j) = \kappa^l(j')$ for some $l \geq 1$. This shows that the number of these arcs is r'. Moreover, the following facts are easily checked. The closure $\operatorname{Cl} A_i$ is A_i plus the is a loop centered at the point o and

$$L = \bigcup_{j=1}^{r} \mathrm{Cl}\, A_j = \bigcap_{i=0}^{\infty} P_i$$

is a loop bundle with r' loops. ■

Proof of Lemma 4.2. Since $f'_* = f''_* \colon \pi_1(N, x_0) \to \pi_1(N, x_0)$ and $m - 3 \geq 1$ we can apply Lemma 5.2. The embedding f' does not twist the handles of N, and since f'' is extendable to $\mathbb{S}^3$ the same holds for f''. Therefore θ in this lemma is not necessary, i.e. we can choose $\theta =$id so that $f'_\theta = f'$, and the manifolds $M' \times \mathbb{R}^{m-3}$, $M'' \times \mathbb{R}^{m-3}$ are homeomorphic.

5 Appendix

Here by a *handlebody* N we mean a 3-dimensional C^1 manifold which is homeomorphic to the cartesian product of $[0,1]$ with a set obtained from $\mathbb{D}^2$ by removing the interiors of a finite number of disjoint disks in $\mathrm{Int}\, \mathbb{D}^2$. By a *handle* of N we mean a subset H which is the image of a C^1 embedding $\eta \colon \mathbb{D}^2 \times [-1,1] \to N$ where $H \cap \partial N = \eta(\partial \mathbb{D}^2 \times [-1,1])$ and $N \backslash H$ is connected. We say that η is a *parametrization* of H, and the disks $\eta(\mathbb{D}^2 \times \{e\})$ $(e = \pm 1)$ will be called the *ends* of H. If $H_1, \ldots, H_r$ is a maximal family of disjoint handles in N for which $N \backslash (H_1, \cup \cdots \cup H_r)$ is connected, then $H_1, \ldots, H_r$ will be called a set of handles of N. In this case $\mathrm{Cl}(N \backslash (H_1, \cup \cdots \cup H_r))$ is a topological ball whose boundary with the exception of $2r$ closed curves is smooth.

The fundamental group of a handlebody with r handles is free of rank r, and if $H_1, \ldots, H_r$ is a set of handles, $x_0 \in N \backslash (H_1 \cup \cdots \cup H_r)$, then oriented loops $L_1, \ldots, L_r$ in N centered at x_0 defines a basis of $\pi_1(N, x_0)$ provided $L_i \cup H_j = \emptyset$ for $i \neq j$ and $L_i \cap H_i$ is an arc which rund from one end of H_i to the other.

If $\beta_1, \ldots, \beta_r$ and $H_1, \ldots, H_r$ are related in this way we say that $\beta_1, \ldots, \beta_r$ correspond to $H_1, \ldots, H_r$ and vice versa. So to each set of handles of N there corresponds a free basis of $\pi(N, x_0)$, and it is a well known fact that the converse is true too. This proves the following lemma.

Lemma 5.1 *If $\varphi \colon \pi_1(N, x_0) \to \pi_1(N, x_0)$ is an automorphism, then there is a diffeomorphism $g \colon N \to N$ leaving x_0 fixed such that the homomorphism $g_* \colon \pi_1(N, x_0) \to \pi_1(N, x_0)$ defined by g coincides with φ.*

If $\eta \colon \mathbb{D}^2 \times [-1,1] \to N$ is a parametrization of a handle H of N then by a twist of H we mean a diffeomorphism $\theta_H \colon N \to N$ which maps each disk $\eta(\mathbb{D}^2 \times t)$ $(t \in [-1,1])$ onto itself and is the identity outside H.

Now let N be a handlebody with a set of handles $H_1, \ldots, H_r$ and a point $x_0 \in N \backslash (H_1, \ldots, H_r)$. We consider two orientation preserving C^1 embeddings $f', f'' \colon N \to \mathrm{Int}\, N$ with the common fixed point x_0 and a diffeomorphism $\theta \colon N \to N$ which is the composition of twists θ_{H_j} $(j = 1, \ldots, r)$. By $f'_\theta \colon M' \to M'$, $f'' \colon M'' \to M''$ we denote extensions of the embeddings $f_\theta = f'\theta$, $f'' \colon N \to \mathrm{Int}\, N$, respectively, as defined at the beginning of the proof of (vi) $\to$ (i) in Section 3. The homomorphisms of the fundamental group of N corresponding to f' or f'' are denoted by

$$f'_*,\ f''_* \colon \pi_1(N, x_0) \to \pi_1(N, x_0),$$

4.

Lemma 5.2 *If $f'_* = f''_*$ then there is a diffeomorphism $\theta\colon N \to N$ which is the composition of twists θ_{H_j} such that for any $n \geq 1$ the cartesian products $M' \times \mathbb{R}^n$, $M'' \times \mathbb{R}^n$ are homeomorphic, where M', M'' are the manifolds introduced above.*

Proof. The main step in the proof is the following lemma. Let $\bar{N} = N \times [-1, 1]$, and for an embedding $g\colon N \to \operatorname{Int} N$ let the embedding $\bar{g}\colon \bar{N} \to \operatorname{Int} \bar{N}$ be defined by

$$\bar{g}(x, t) = (g(x), \tfrac{1}{2}t).$$

Obviously $\bar{g}(\bar{N}) = g(N) \times [-\frac{1}{2}, \frac{1}{2}]$.

Lemma 5.3 *If f', f'' are as in* Lemma 5.2 *then there is a composition θ of twistings θ_{H_j} and a homeomorphism $h\colon \bar{N} \to \bar{N}$ such that $h =$id on $\partial\bar{N}$ and $\overline{hf'_\theta} = \overline{f''}$ on $\bar{N}$, where $f'_\theta = f'\theta$.*

Proof. Let $g', g''\colon N \to N$ be diffeomorphisms which will be fixed later. It is assumed that g', g'' are isotopic to the identity by isotopies g'_s, g'_s $(0 \leq s \leq 1)$, respectively, such that $g'_0 = g''_0 = id$, $g'_1 = g'$, $g''_1 = g''$ and $g'_s = g''_s = id$ on ∂N. Then we consider the embeddings $g'f'$, $g''f''\colon N \to \operatorname{Int} N$ and

$$\overline{g'f'}, \overline{g''f''}\colon \bar{N} \to \operatorname{Int} \bar{N}.$$

We shall write ^{0} for $'$ or $''$ and define two diffeomorphisms $\tilde{g}^0\colon \bar{N} \to \bar{N}$ by

$$\tilde{g}^0(x, t) = \begin{cases} (g^0_{2t+2}(x), t) & \text{for } -1 \leq t \leq -\frac{1}{2} \\ (g^0(x), t) & \text{for } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ (g^0_{-2t+2}(x), t) & \text{for } \ \ \frac{1}{2} \leq t \leq 1. \end{cases}$$

These mappings $\tilde{g}^0$ are homeomorphisms and have the following properties

$$\begin{aligned} \tilde{g}^0 &= \text{id on } \partial\bar{N}, \\ \tilde{g}^0\bar{f}^0 &= \overline{g^0 f^0}. \end{aligned}$$

If the lemma is proved for $g'f'$, $g''f''$ instead of f', f'', respectively and $\tilde{h}\colon \bar{N} \to \bar{N}$ is a corresponding homeomorphism, then by $h = \tilde{g}''^{-1}\tilde{h}\tilde{g}'$ we get a homeomorphism $h\colon \bar{N} \to \bar{N}$ which has the properties in the lemma for the original mappings f', f''.

Let $H_1, \ldots, H_r$ be a set of handles of N such that the ball $B = \operatorname{Cl}(N \backslash (H_1, \ldots, H_r))$ contains x_0. Since f', f'' are orientation preserving there are mappings $g', g''\colon N \to N$ with the properties mentioned above such that $g'f' = g''f''$ on B. Moreover, it will help to understand our construction given below if we choose g', g'' so that $g'f'(B) = g''f''(B)$ is a small convex ball and the handles $g'f'(H_j), g''f''(H_j)$ are thin, i.e., they are small tubular neighbourhoods of arcs.

By the remarks at the beginning of this proof it is sufficient to prove the lemma with f', f'' replaced by $g'f', g''f''$, respectively; or, equivalently, we may assume, as we shall do, that f', f'' in addition to the hypotheses of the lemma satisfy $f' = f''$ on B and $f'(B) = f''(B)$ is a small convex ball with two sets of thin handles $f'(H_j), f''(H_j)$ $(j = 1, \ldots, r)$.

starts with $f_0 = f'$ and ends with an embedding f_1 whose image is $f''(N)$ and is constant on B. Since the handles $f'(H_j), f''(H_j)$ $(1 \leq j \leq r)$ are thin it is easy to see that we may chose f_t as an isotopy with a finite number of exceptions at which handles $f_r(H_j)$ undergo crossings as shown in Figure 10.

Figure 10

If for a suitable chosen composition θ of twistings θ_{H_j} we replace f' by $f'\theta = f'_\theta$ we can get instead of $f_1 f'(N) = f''(N)$ the stronger property $f_1 f'_\theta = f''$.

Using the additional fourth dimension in $\bar{N}$ the crossings in $f_t(N)$ can be avoided in a suitably chosen homotopy $\bar{f}_t \colon \bar{N} \to \operatorname{Int} \bar{N},\ 0 \leq t \leq 1$ from $\bar{f}_0 = \bar{f}'_\theta$ to $\bar{f}_1 = \bar{f}''$ which covers f_t, i.e., $\bar{f}_t$ is an isotopy. This isotopy can easily be extended to an isotopy $h_t \colon \bar{N} \to \bar{N}$ with $h_0 = id, h_t = id$ on $\partial \bar{N}$ and $h_t \bar{f}'_\theta = \bar{f}_t \colon \bar{N} \to \bar{f}_t(\bar{N})$. Then $h = h_1 \colon \bar{N} \to \bar{N}$ has the properties which are required in the lemma. ■

Now coing back to the proof of Lemma 5.2 we use the homeomorphism $h \colon \bar{N} \to \bar{N}$ of Lemma 5.3 to define a homeomorphism $\tilde{h}_0 \colon M' \times \mathbb{R} \to M'' \times \mathbb{R}$. To this aim we first extend each of the two embeddings $\bar{f}'_\theta, \bar{f}'' \colon \bar{N} \to \operatorname{Int} \bar{N}$ to a homeomorphism $\bar{f}'_\theta, \bar{f}'' \colon M' \times \mathbb{R} \to M^0 \times \mathbb{R}$, respectively, by setting

$$\begin{aligned} \bar{f}'_\theta(x,t) &= (f'_\theta(x), \tfrac{1}{2}t) \qquad ((x,t) \in M' \times \mathbb{R}), \\ \bar{f}''(x,t) &= (f''(x), \tfrac{1}{2}t) \qquad ((x,t) \in M'' \times \mathbb{R}). \end{aligned}$$

Then

$$\begin{aligned} \bar{f}'^{-i}_\theta(\bar{N}) &= f'^{-i}_\theta(N) \times [-2^i, 2^i]. \\ \bar{f}''^{-i}(\bar{N}) &= f''^{-i}(N) \times [-2^i, 2^i], \end{aligned}$$

and

$$M' \times \mathbb{R} = \bigcup_{i=0}^{\infty} \bar{f}'^{-i}_\theta(\bar{N}), \qquad M'' \times \mathbb{R} = \bigcup_{i=0}^{\infty} \bar{f}''^{-i}(\bar{N}).$$

Now we define $\tilde{h}_0$ piecewise by

$$\begin{aligned} \tilde{h}_0 &= h \qquad\qquad \text{on } \bar{N} \\ \tilde{h}_0 &= \bar{f}''^{-i} h \bar{f}'^{i}_\theta \text{ on } \bar{f}'^{-i}_\theta(\bar{N}) \backslash \bar{f}'^{-(i-1)}(\bar{N}) = \bar{f}'^{-i}_\theta(\bar{N} \backslash \bar{f}'(\bar{N})). \end{aligned}$$

By the properties $h = id$ on $\partial \bar{N}, h \bar{f}'_\theta = \bar{f}''$ on $\bar{N}$ each of these pieces fits its neighbours, and $\tilde{h}_0$ is continuous. Obviously, $\tilde{h}_0$ is one-to-one, and, since $\tilde{h}_0^{-1}$ can be homeomorphism.

The step from $\tilde{h}_0 \colon M' \times \mathbb{R} \to M'' \times \mathbb{R}$ to the final homeomorphism $\tilde{h} \colon M' \times \mathbb{R}^n \to M'' \times \mathbb{R}^n$ is so simple that there is nothing more to be said, and the proof of Lemma 5.2 is finished. ■

References

[1] H. G. Bothe, *Transversely wild expanding attractors*, Math. Nachr. **157** (1992), 25–49.

[2] ______, *How hyperbolic attractors determine their basin*, Nonlinearity **9** (1996), no. 5, 1137–1190.

[3] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Ergebn. d. Math **89** (1977).

[4] W. S. Massey, *Algebraic topology: An introduction*, Hartcourt, Brace & World, Inc., 1967.

[5] C. Robinson and R. Williams, *Classification of expanding attractors: An example.*, Topology **15** (1976), 321–323.

[6] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

[7] R. F. Williams, *One dimensional non wandering sets*, Topology (1967), 473–487.

[8] ______, *Classification of one-dimensional attractors*, Global Analysis. Proc. Symp. Pure Math. **14** (1970), 341–361.