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Abstract

We introduce a directed spatial random graph model aimed at modelling certain aspects of social media networks. We provide two variants of the model: an infinite version and an increasing sequence of finite graphs that locally converge to the infinite model. Both variants have in common that each vertex is placed into Euclidean space and carries a birth time. Given locations and birth times of two vertices, an arc is formed from younger to older vertex with a probability depending on both birth times and the spatial distance of the vertices. If such an arc is formed, a reverse arc is formed with probability depending on the ratio of the endpoints' birth times. Aside from the local limit result connecting the models, we investigate degree distributions, two different clustering metrics and directed percolation.

1 Motivation and background

A well established paradigm for the modelling of large, sparse networks occurring in the context of the world wide web or social media is the preferential attachment (PA) mechanism [3, 26]: networks are built recursively by adding nodes and links in such a way that new nodes prefer to be connected to existing nodes if they have a high degree. PA networks typically are *scale-free* [8], i.e. have power law degree distributions. They also tend to be *robust under random attack* if the degree distribution is sufficiently heavy-tailed [7]. If their nodes are embedded into space and the attachment mechanism interacts with the spatial geometry, then they can further be shown to display *clustering* effects [29]. However, an obvious but important fact that was discussed very early in the context of webgraph modelling, see e.g. [10], is that the link structure of typical real world networks is intrinsically directed. Therefore such networks ought to be represented as directed graphs (*digraphs*). The currently available mathematical literature on scale-free digraph models is surprisingly sparse. There are essentially only two models for which rigorous mathematical results answering most of the basic questions of interest for network science are available: inhomogeneous random digraphs [6, 12] and the directed configuration model [13, 34, 17]. Most preferential attachment models are intrinsically directed as well, albeit in a deterministic way as an artefact of the recursive modelling scheme. Using solely the 'arrow of time' to direct a preferential attachment network is in general a poor modelling choice, unless some recursive effect dominates the real world network one intends to model[1]. Therefore PA networks are typically studied as undirected graphs. An exception is the recent paper [14] in which a non-spatial preferential attachment model with *arc reciprocity* is introduced.

Here, we propose a *spatial* PA-style directed network with arc reciprocity. One can think of the construction mechanism as a recursive attachment scheme in which each preferentially established arc

[1] One example that comes to mind are scientific citation networks in which a publication sends an arc to every publication it references.

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triggers the potential creation of the corresponding reverse arc. Because degree-based PA mechanisms lead to relatively complicated dependencies among the edges, we build our model from the simplified *age-based* PA-scheme introduced in [21]. The approach of translating degree into age [19] relies on the intuitive fact that preferential attachment creates a strong positive correlation between age and high degree: only vertices that have been in the system for a long time acquire a sufficiently high degree to attract many new connections. Conversely, the first few vertices of the PA-scheme tend to have very high degrees [8, 20].

We provide two variants of the model in the following section, an infinite version that is analytically easier to treat and a finite version which is more appealing from a modelling point of view and related to the infinite model by a local limit theorem. Section 3 then investigates local properties of the model such as degree distribution and clustering metrics and Section 4 is devoted to the existence of large weakly connected components.

2 Model introduction

2.1 The directed Age-dependent Random Connection Model

We first describe our model ad hoc as an infinite digraph. In Section 2.2 we detail how this digraph arises as the weak local limit [4, 2] of a directed PA-type sequence of growing networks. The vertex set of the *directed age-dependent random connection model* (DARCM) is given by a unit intensity Poisson process $\mathcal{X}$ on $\mathbb{R}^d \times (0,1)$. We denote the vertices by $\mathbf{x} = (x, t_x) \in \mathcal{X}$ and call $x \in \mathbb{R}^d$ the vertex' *location* and $t_x \in (0,1)$ the vertex' *birth time*. For two vertices $\mathbf{x} = (x, t_x)$ and $\mathbf{y} = (y, t_x)$ with $t_y < t_x$, we refer to $\mathbf{y}$ as being *older* than $\mathbf{x}$ and to $\mathbf{x}$ as being younger than $\mathbf{y}$, respectively. Almost surely, there are no vertices born at the same time. Our choice of identifying the second vertex coordinate as birth time is rooted in the local limit representation of Section 2.2. To define the distribution of arcs in the graph we introduce the following parameters:

(i) the *spatial decay exponent* $\delta > 1$, defining the *spatial profile* $\rho(x) = 1 \wedge x^{-\delta}$ for $x > 0$;

(ii) the *power-law parameter* $\gamma \in (0,1)$ tuning the tail decay of the degree distribution;

(iii) the *edge intensity* $\beta > 0$;

(iv) the *reciprocity exponent* $\Gamma > 0$.

The digraph $\mathscr{D} = \mathscr{D}[\beta, \gamma, \delta, \Gamma]$ is built using these parameters via the following procedure:

(A) Given $\mathcal{X}$, each vertex $\mathbf{x} = (x, t_x)$ forms an arc to each vertex $\mathbf{y} = (y, t_y)$ with $t_y < t_x$ independently of all other potential arcs with probability

$$\rho\big(\beta^{-1} t_y^{\gamma} t_x^{1-\gamma} |x-y|^d\big). \tag{1}$$

If an arc is formed during this step, we denote this by $\mathbf{x} \to \mathbf{y}$ or $\mathbf{y} \leftarrow \mathbf{x}$.

(B) Given $\mathcal{X}$ and all arcs $\mathbf{x} \to \mathbf{y}$ created in (A), each vertex $\mathbf{y} = (y, t_y)$ sends a *reverse arc* to each $\mathbf{x} = (x, t_x)$ with $\mathbf{x} \to \mathbf{y}$ independently of all other potential reverse arcs with probability $(t_y/t_x)^{\Gamma}$. If such a reciprocal connection is made, we denote this event by $\mathbf{x} \leftrightarrow \mathbf{y}$.

Observe that δ controls the spatial decay of connection probabilities such that vertices in mutually distant locations have a low probability of connecting. This effects gets stronger the larger the value of δ. The strongest spatial restrictions are modelled by the case $\delta \to \infty$ which we identify with the indicator function $\rho = \mathbb{1}_{[0,1]}$. In contrast, the likelihood of a reciprocal connection does not depend on the spatial locations and takes only birth times and the reciprocity exponent into account. The assumptions $\delta > 1$ and $\gamma < 1$ ensure that the expected number of arcs incident to any vertex remains bounded.

For the choice of $\Gamma = 0$ each edge is oriented into both directions, thus $\Gamma = 0$ corresponds to constructing an undirected graph denoted by $\mathscr{G} = \mathscr{D}[\beta, \gamma, \delta, 0]$. The graph $\mathscr{G}$ is the *age-dependent random connection model* introduced in [21]. Clearly, $\mathscr{D}$ can be generated from $\mathscr{G}$ by first pointing all edges in $\mathscr{G}$ from younger end-vertex to older end-vertex and then adding reverse arcs by way of (B).

Let us very briefly motivate our construction. The locations of the vertices describe intrinsic connection affinities and two vertices are more likely to connect if they are spatially close to each other. The age of a vertex indirectly models its performative attractiveness in the graph, the older a vertex the more arcs it attracts. These two sources of connections are intended to model the formation of social media networks: users tend to follow either a friend, i.e. someone with intrinsic affinity to them, or 'influencers', i.e. users who have accumulated a lot of followers already, this effect will become even more apparent in the finite domain version of the model described in the next subsection.

2.2 A generative model on finite domains and a local limit procedure

We modify our construction to obtain a generative version of the model on a finite domain with periodic boundary conditions, akin to the age-based preferential attachment model of [21]. Chose $\beta > 0$,$\gamma \in (0,1)$, $\delta > 1$, and $\Gamma \geq 0$ to build a growing sequence of directed graphs $(\mathscr{D}_t : t \geq 0)$ in continuous time as follows: At time $t = 0$, the graph $\mathscr{D}_0$ is the empty graph consisting of neither vertices nor edges. Then

- vertices are born successively after independent standard exponential waiting times and are placed uniformly at random and independent of all other locations on the d-dimensional unit torus $[-1/2, 1/2)^d$.

- Let τ_n be the birth time of the n-th vertex placed a location x_n, which we denote by $\mathbf{x}_n = (x_n, \tau_n)$. Given the graph $\mathscr{D}_{\tau_n -}$, the graph build up to time τ_n consisting of the $n-1$ previously arrived vertices, the new vertex
  x_n forms an arc to each already existing vertex $\mathbf{x}_j = (x_j, \tau_j)$, $j = 1, \dots, n-1$, born at time $\tau_j < \tau_n$ and located at x_j, independently with probability

$$\rho\Big(\frac{\tau_n\,\mathrm{d}_1(x_n, x_j)^d}{\beta(\tau_n/\tau_j)^\gamma}\Big), \tag{2}$$

  where

$$\mathrm{d}_t(x,y) := \min\big\{|x-y+u| : u \in \{-\sqrt[d]{t}, 0, \sqrt[d]{t}\}^{\times d}\big\}$$

  denotes the standard metric on the d-dimensional torus of volume t.

- Whenever an arc $\mathbf{x}_n \to \mathbf{x}_j$ is formed, the older vertex $\mathbf{x}_j$ forms an arc to $\mathbf{x}_n$ independently with probability $(\tau_j/\tau_n)^\Gamma$.

We think of $(\mathscr{D}_t : t \geq 0)$ as a growing network of social media users and of arcs representing a 'follow'-type relation. Users/accounts arrive over time and establish arcs to already existing accounts based on general popularity (modelled by age) and preexisting preference (modelled by spatial closeness). If an established account receives a new follower, they may decide if they wish to reciprocate the relation, which happens with probability 'birth time quotient to the Γ'. Since $1^\Gamma = 1$, if the two accounts are approximately of the same age, the occurrence of a reverse arc is quite probable. However, the older the old vertex is compared to the younger one, the less likely the presence of the reverse arc becomes. Hence, it is a rare event for a popular (typically old) account to follow one of their many younger followers, which captures the dynamics of real world social media networks.

Since the precise ordering of the birth times becomes irrelevant in the following, we go back to the previously used notation and denote vertices by $\mathbf{x} = (x, t_x)$, where $x \in [-1/2, 1/2)^d$ denotes the location and $t_x \in (0, \infty)$ the birth time of the vertex $\mathbf{x}$. To relate $(\mathscr{D}_t : t \geq 0)$ to $\mathscr{D}$ we need a two step procedure of *rescaling* and *localisation*.

Rescaling. Define for any given $t \in (0, \infty)$ the rescaling map

$$\begin{aligned} h_t : \quad \big[-\tfrac{1}{2}, \tfrac{1}{2}\big)^d \times (0,t) &\to \big[-\tfrac{t^{1/d}}{2}, \tfrac{t^{1/d}}{2}\big)^d \times (0,1) \\ (y,s) &\mapsto \big(t^{1/d} y, \tfrac{s}{t}\big). \end{aligned}$$

For fixed $t > 0$, denote the vertex set of $\mathscr{D}_t$ by $\mathcal{X}_t$. The rescaling map h_t acts on the point set $\mathcal{X}_t$ and is extended canonically to the respective geometric digraphs graphs by defining $h_t(\mathscr{D}_t)$ to be the graph with vertex set $h_t(\mathcal{X}_t)$ where an arc $h_t(\mathbf{x}) \to h_t(\mathbf{y})$ is present if and only if $\mathbf{x} \to \mathbf{y}$ is present in $\mathscr{D}_t$. Note that $h_t(\mathcal{X}_t)$ is distributed as a unit intensity Poisson point process on $[-\sqrt[d]{t}/2, \sqrt[d]{t}/2)^d \times (0,1)$, i.e. its points are located on the d-dimensional torus of volume t and carry birth times in $(0,1)$. Moreover, for each $t_y < t_x < t$

$$\rho\left(\frac{\frac{t_x}{t}\,\mathrm{d}_t(t^{1/d}x, t^{1/d}y)^d}{\beta\big(\frac{t_x/t}{t_y/t}\big)^\gamma}\right) = \rho\Big(\frac{t_x\,\mathrm{d}_1(x,y)^d}{\beta(t_x/t_y)^\gamma}\Big) \text{ as well as } \Big(\frac{t_y/t}{t_x/t}\Big)^\Gamma = \Big(\frac{t_y}{t_x}\Big)^\Gamma,$$

hence $h_t(\mathscr{D}_t)$ agrees in law with a restriction of $\mathscr{D}$ to vertices located in the spatial domain $[-\sqrt[d]{t}/2, \sqrt[d]{t}/2)^d$ with periodic boundary conditions. Note that this equality in law only holds for fixed t and does not extend to the process level. For instance, every given vertex in $\mathscr{D}$ has a fixed finite indegree, but the indegree of a fixed vertex in $\mathscr{D}_t$ diverges, as more and more vertices arrive cf. [21].

Localisation. We are interested in the long time behaviour of the graphs $(\mathscr{D}_t : t \geq 0)$ as seen from a *typical* vertex. Almost surely, there will be at least one vertex in the system if t is sufficiently large, hence from now on we work conditionally on the event that $\mathscr{D}_t$ is not the empty graph. For each[2] sufficiently large t, we choose a root vertex $\mathbf{o}_t$ uniformly at random and perform a shift of spatial coordinates such that $\mathbf{o}_t$ is located at the origin $0 \in [1/2, 1/2)^d$. We denote the resulting rooted geometric digraph by $(\mathscr{D}_t, \mathbf{o}_t)$. Extending the map h_t to rooted digraphs in the obvious way, it is not difficult to see that $h_t(\mathscr{D}_t, \mathbf{o}_t)$ corresponds to a version of $h_t(\mathscr{D}_t)$ with an additional vertex located at the origin $0 \in [-\sqrt[d]{t}/2, \sqrt[d]{t}/2)^d$.

The weak convergence theory of point processes, see e.g. [15], now strongly suggests that the rescaled and localised family of random geometric digraphs $\big(h_t(\mathscr{D}_t, \mathbf{o}_t) : t > 0\big)$ converges in

[2] Clearly, it suffices to update the root vertex only at the birth times of new vertices, since we are merely interested in the one dimensional marginal distributions of the resulting family of rooted graphs.

distribution to a variant $(\mathscr{D}, \mathbf{o})$ of $\mathscr{D}$ with an additional vertex $\mathbf{o} = (0, t_0)$ at the origin, cf. [28] for related general results for finite domains with non-periodic boundary conditions in the undirected setup. A formal proof of this result can be given along the same lines as [29, Proposition 5]. As a corollary, we obtain a distributional limit theorem in D_*, the space of rooted simple digraphs[3] with metric

$$d_*\big((G_1, o_1), (G_2, o_2)\big)\big) = \frac{1}{1 + \sup\{r : (G_1, o_1) \overset{r}{\simeq} (G_2, o_2)\}},$$

where we write $(G_1, o_1) \overset{r}{\simeq} (G_2, o_2)$ if there exists a digraph isomorphism mapping the r-neighbourhood of the root o_1 in G_1 to the r-neighbourhood of the root o_2 in G_2.

Theorem 2.1 (Local limit)**.** *For $t \in \mathbb{N}$, let $(\mathcal{D}_t, \mathbf{o}_t) = h_t(\mathscr{D}_t, \mathbf{o}_t) \in D_*$ denote the rooted simple digraphs obtained from the rooted geometric graphs $(\mathscr{D}_t, \mathbf{o}_t)$ via rescaling. Let further denote $(\mathscr{D}, \mathbf{o})$ the rooted digraph obtained from generating $\mathscr{D}$ with an additional (root) vertex located at the origin. Then*

$$(\mathcal{D}_t, \mathbf{o}_t) \underset{t\to\infty}{\longrightarrow} (\mathscr{D}, \mathbf{o}) \textit{ in distribution.}$$

In the following section, we calculate local metrics for $\mathscr{D}$. By virtue of Theorem 2.1, the corresponding metrics for $(\mathscr{D}_t : t \geq 0)$ converge to the corresponding limit values for (the rooted version of) $\mathscr{D}$.

3 Local properties

In this section, we establish important local properties of $(\mathscr{D}, \mathbf{o})$. Here, $\mathbf{o}$ denotes the additionally added root vertex which can be seen as a typical vertex in $\mathscr{D}$ as explained in the previous paragraph. We call a property local if it only depends on a bounded graph neighbourhood of $\mathbf{o}$. Formally, such properties have representations via continuous functionals on the space D_*. We begin by identifying the degree distribution of $\mathbf{o}$. Afterwards, we discuss clustering metrics.

In the following, we use the established notation $f \asymp g$ for non-negative functions to indicated that $f(x)/g(x)$ is bounded from zero and infinity.

3.1 Degree distribution

For a given vertex $\mathbf{x}$ we denote by

$$\mathscr{N}^{\textsf{in}}(\mathbf{x}) := \{\mathbf{y} \in \mathcal{X} : \mathbf{y} \to \mathbf{x}\}$$

the vertices sending arcs to $\mathbf{x}$ in $\mathscr{D}$ and by $\sharp\mathscr{N}^{\textsf{in}}(\mathbf{x})$ its indegree. If $\mathbf{x} = \mathbf{o}$, we simply write $\mathscr{N}^{\textsf{in}}$. For outgoing arcs and outdegree, we use the analogous notations $\mathscr{N}^{\textsf{out}}(\mathbf{x})$, resp. $\mathscr{N}^{\textsf{out}}$, and $\sharp\mathscr{N}^{\textsf{out}}(\mathbf{x})$, resp. $\sharp\mathscr{N}^{\textsf{out}}$.

[3] In fact, D_* is the quotient space of equivalence classes of rooted graphs up to graph isomorphism, but we do not distinguish between a graph and its isomorphism class here. For more background on D_* and local weak convergence see [1].

Lemma 3.1 (Degree distribution)**.**

(a) For the indegree of $\mathbf{o}$ *in* $\mathscr{D} = \mathscr{D}[\beta, \gamma, \delta, \Gamma]$*, we have for all* $\gamma \in (0,1)$ *and* $\Gamma > 0$ *that*

$$\mathbb{P}_0(\sharp\mathscr{N}^{\text{in}} = k) = k^{-1-1/\gamma+o(1)}, \quad \textit{as } k \uparrow \infty, \textit{ and}$$

(b) for the outdegree of $\mathbf{o}$ *in* $\mathscr{D} = \mathscr{D}[\beta, \gamma, \delta, \Gamma]$*, we have for all* $\gamma \in (0,1)$ *that*

(i) if $\Gamma > \gamma$*, then* $\sharp\mathscr{N}^{\text{out}}$ *is Poisson distributed with parameter* $\beta/(\Gamma-\gamma)$*,*

(ii) if $0 < \Gamma < \gamma$*, then* $\sharp\mathscr{N}^{\text{out}}$ *is mixed Poisson distributed with mixing density*

$$f^{\text{out}}(\lambda) = (\gamma-\Gamma)^{-(2+1/(\gamma-\Gamma))}\big(\tfrac{\beta}{\lambda}\big)^{1+1/(\gamma-\Gamma)}\mathbb{1}_{(\beta/(\gamma-\Gamma),\infty)}(\lambda)$$

and therefore

$$\mathbb{P}_0(\sharp\mathscr{N}^{\text{out}} = k) = k^{-1-1/(\gamma-\Gamma)+o(1)}, \quad \textit{as } k \uparrow \infty.$$

Proof. The incoming edges of $\mathbf{o}$ are all edges to younger neighbours of $\mathbf{o}$ in $\mathscr{G}$ plus the edges to older neighbours where a reciprocal arc has been added. From [21, Proposition 4.1(d)], the number of younger neighbours in $\mathscr{G}$ from $(0,u)$ (i.e. the root's mark is given by $U_0 = u$) is mixed Poisson with mixing density

$$f^{\text{in}}(\lambda) \asymp \lambda^{-1-1/\gamma}.$$

The older incoming neighbours of $(0,u)$ form a Poisson process on $\mathbb{R}^d \times (0,u)$ with intensity

$$(\tfrac{s}{u})^{\Gamma}\rho(\beta^{-1}s^{\gamma}u^{1-\gamma}|x|^d)\mathrm{d}x\mathrm{d}s.$$

Since $(s/u)^{\Gamma} \le 1$, the number of such neighbours is at most Poisson distributed with parameter $\beta/(1-\gamma)$ [21, Proposition 4.1(c)]. Hence, the indegree of $\mathbf{o}$ in $\mathscr{D}$ is bounded from below by the number of younger neighbours of $\mathbf{o}$ in $\mathscr{G}$ and from above by the number of younger neighbours of $\mathbf{o}$ in $\mathscr{G}$ plus an independent Poisson distributed random variable. As the number of younger neighbours of $\mathbf{o}$ in $\mathscr{G}$ is heavy tailed with power-law exponent $\tau = 1 + 1/\gamma$, cf. [21, Lemma 4.4] both bounds are of the same order, proving (a).

Similarly, the outgoing neighbours of $\mathbf{o}$ in $\mathscr{D}$ are the older neighbours of $\mathbf{o}$ in $\mathscr{G}$ plus the younger ones where a reciprocal arc has been added. The number of the first type is again Poisson distributed independently of the root's mark. For fixed mark $U_0 = u$, the latter form a Poisson process on $\mathbb{R}^d \times (u,1)$ with intensity

$$(\tfrac{u}{s})^{\Gamma}\rho\big(\beta^{-1}u^{\gamma}s^{1-\gamma}|x|^d\big)\,\mathrm{d}x\,\mathrm{d}s.$$

The expected number of such vertices is

$$\begin{aligned}&\int_u^1 \mathrm{d}s\,(\tfrac{u}{s})^{\Gamma}\int_{\mathbb{R}^d}\mathrm{d}x\,\rho(\beta^{-1}u^{\gamma}s^{1-\gamma}|x|^d)\\ &\quad\asymp \beta u^{-\gamma}\int_u^1 (\tfrac{u}{s})^{\Gamma}s^{\gamma-1}\mathrm{d}s \asymp \beta u^{\Gamma-\gamma}\int_u^1 s^{\gamma-\Gamma-1}\,\mathrm{d}s \asymp \frac{\beta}{|\gamma-\Gamma|}(1\vee u^{-\gamma+\Gamma}).\end{aligned}$$

Hence, if $\Gamma > \gamma$, the outdegree is Poisson distributed with a parameter independent of u. If $\Gamma < \gamma$, then the outdegree is mixed Poisson distributed and

$$\mathbb{P}_0(\sharp\mathscr{N}^{\text{out}} = k) = \int_0^1 e^{\frac{\beta u^{\Gamma-\gamma}}{\gamma-\Gamma}}\frac{\big(\frac{\beta u^{\Gamma-\gamma}}{\gamma-\Gamma}\big)^k}{k!}\mathrm{d}u \asymp \int_{\beta/(\gamma-\Gamma)}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}f^{\text{out}}(\lambda)\mathrm{d}\lambda.$$

Hence, by Stirling's formula, we have

$$\mathbb{P}_0(\sharp\mathscr{N}^{\mathsf{out}} = k) \asymp \frac{1}{\Gamma(k+1)} \int_0^\infty e^{-\lambda} \lambda^{k-1/(\gamma-\Gamma)-1} \mathrm{d}\lambda \asymp k^{-1-1/(\gamma-\Gamma)},$$

as $k \to \infty$, concluding the proof. $\square$

3.2 Clustering

In this section, we discuss two clustering measures. Firstly, the *local friend clustering coefficient* and secondly the *interest clustering number*. The idea of the friend clustering coefficient is that two friends of a typical vertex are more likely to be friends with each other than two general typical vertices. Here, a 'friendship' denotes a reciprocal connection, and the corresponding clustering coefficient measures the density of strongly connected triangles. This coincides with our motivation of the model that the network consists mainly of typical (young) vertices and some very influential (old) vertices. While the influential vertices are those with much larger in than outdegree, the typical vertices should tend to form triangles, just like in the age-dependent random connection model in the undirected case. The friend clustering coefficient is a straightforward adaptation from the undirected setting, cf. [21].

The idea of interest clustering was proposed in Trolliet et al. [35] explicitly for directed social networks. Our localised adaptation of their coefficient appears here for the first time. It is based on the idea that in an (online) social network, the clustering is also driven by common interests. Whereas the friend clustering coefficient is a metric intended to capture purely the social aspect of network formation, the interest clustering number combines the social with the informational aspect.

We first define both clustering metrics only in the infinite limit model $(\mathscr{D}, \mathbf{o})$ and provide integral representations for them and then consider the finite models $(\mathscr{D}_t : t \geq 0)$.

The friend clustering coefficient. In this section, we call two given vertices $\mathbf{x}$ and $\mathbf{y}$ *friends* in $\mathscr{D}$ if $\mathbf{x} \leftrightarrow \mathbf{y}$. Let $\overset{\leftrightarrow}{\mathscr{V}}_2$ the set of all vertices having at least two friends in $\mathscr{D}$. For $\mathbf{x} \in \overset{\leftrightarrow}{\mathscr{V}}_2$, we define the *friend clustering coefficient* as

$$c^{\mathsf{fc}}(\mathbf{x}) := \frac{\sum\limits_{\mathbf{y},\mathbf{z} \in \mathcal{X}:\, t_y > t_z} \mathbb{1}_{\{\mathbf{x}\leftrightarrow\mathbf{y}\}} \mathbb{1}_{\{\mathbf{x}\leftrightarrow\mathbf{z}\}} \mathbb{1}_{\{\mathbf{y}\leftrightarrow\mathbf{z}\}}}{\binom{\sharp(\mathscr{N}^{\mathsf{out}}(\mathbf{x}) \cap \mathscr{N}^{\mathsf{in}}(\mathbf{x}))}{2}}.$$

If $\mathbf{x} \notin \overset{\leftrightarrow}{\mathscr{V}}_2$, we set its friend clustering coefficient to be zero.

Lemma 3.2 (Friend clustering)**.** *For all $\beta > 0, \gamma \in (0,1), \delta > 1,$ and $\Gamma \geq 0$, we have*

$$\mathbb{E}_0 c^{\mathsf{fc}}(\mathbf{o}) = \int_0^1 du\, \mathbb{P}(\mathbf{Y}^{(u)} \leftrightarrow \mathbf{X}^{(u)}) \mathbb{P}_{(0,u)}\Big(\bigcup_{k\geq 2} F_{(0,u)}(k) \Big) > 0,$$

where $F_{(o,u(k)}$ is the event that the root (o,u) has k friends and $\mathbf{X}^{(u)}$ and $\mathbf{Y}^{(u)}$ are two independent random variables distributed according to the normalised measure $\lambda_u^f / \lambda_u^f(\mathbb{R}^d)$ with

$$\lambda_u^{\mathsf{f}} = \Big((\tfrac{s}{u})^\Gamma \rho\big(\beta^{-1} s^\gamma u^{1-\gamma} |x|^d\big) \mathbb{1}_{\{s<u\}} + (\tfrac{u}{s})^\Gamma \rho\big(\beta^{-1} s^{1-\gamma} u^\gamma |x|^d\big) \mathbb{1}_{\{s\geq u\}} \Big)\, ds\, dx.$$

We do not give the proof here as it works analogously to the undirected ARCM [21, Theorem 5.1].

The interest clustering number. Consider two vertices $\mathbf{x},\mathbf{y}\in\vec{\mathscr{V}}_2$ and define the quantity

$$c^{\mathsf{ic}}(\mathbf{y}|\mathbf{x})=\begin{cases}0, & \text{if } \sharp(\vec{N}(\mathbf{y})\cap\vec{N}(\mathbf{x}))<2,\\ \frac{\sum_{\mathbf{u},\mathbf{v}\in\mathcal{X}}\mathbb{1}_{\{\mathbf{y}\to\mathbf{u},\mathbf{y}\to\mathbf{v}\}}\mathbb{1}_{\{\mathbf{x}\to\mathbf{u},\mathbf{x}\to\mathbf{v}\}}}{\sum_{\mathbf{u},\mathbf{v}\in\mathcal{X}}\mathbb{1}_{\{\mathbf{y}\to\mathbf{u}\text{ or }\mathbf{y}\to\mathbf{v}\}}\mathbb{1}_{\{\mathbf{x}\to\mathbf{u},\mathbf{x}\to\mathbf{v}\}}}, & \text{otherwise.}\end{cases}$$

Note that $c^{\mathsf{ic}}(\mathbf{y}|\mathbf{x})$ is at most 1. Let $\mathbb{P}_{0,x}$ denote the law of the digraph $\mathscr{D}$ constructed from a vertex set with two additional vertices $\mathbf{o},\mathbf{x}$ located at 0 and $x\in\mathbb{R}^d$, respectively. Then we call

$$n^{\mathsf{ic}}(\mathbf{o})=\int_{\mathbb{R}^d}\mathbb{E}_{0,x}c^{\mathsf{ic}}(\mathbf{x}|\mathbf{o})\,\mathrm{d}x$$

the *interest clustering number* of $\mathscr{D}$. The number $n^{\mathsf{ic}}(\mathbf{o})$ can be interpreted as a localised version of the 'interest clustering coefficient' proposed in [35] for directed graphs derived from social and information network data. An important difference is that $n^{\mathsf{ic}}(\mathbf{o})\in(0,\infty)$ is not a normalised quantity. A large value of $n^{\mathsf{ic}}(\mathbf{o})$ implies that typical vertices who share a common interest (i.e. both send an arc to a third vertex) are likely to have further common interests, whereas a small value of $n^{\mathsf{ic}}(\mathbf{o})$ indicates that interests are formed more or less independently of each other. To formulate our result regarding interest clustering in $\mathscr{D}$, we need the two numbers $\mu_{\mathsf{II}}(x)$ and $\mu_{\mathsf{I0}}(x)$ given by

$$\begin{aligned}\mu_{\mathsf{II}}(x)=\int_{\mathbb{R}^d}\int_0^1\int_0^1\int_0^{u\wedge s}&\big(\tfrac{t}{u}\big)^{\Gamma}\rho\big(\beta^{-1}t^{\gamma}u^{1-\gamma}|y|^d\big)\big(\tfrac{t}{s}\big)^{\Gamma}\rho\big(\beta^{-1}t^{\gamma}s^{1-\gamma}|x-y|^d\big)\,\mathrm{d}t\\ &+\mathbb{1}_{\{u<s\}}\int_u^s\rho\big(\beta^{-1}u^{\gamma}t^{1-\gamma}|y|^d\big)\big(\tfrac{t}{s}\big)^{\Gamma}\rho\big(\beta^{-1}t^{\gamma}s^{1-\gamma}|x-y|^d\big)\,\mathrm{d}t\\ &+\mathbb{1}_{\{s<u\}}\int_s^u\big(\tfrac{t}{u}\big)^{\Gamma}\rho\big(\beta^{-1}t^{\gamma}u^{1-\gamma}|y|^d\big)\rho\big(\beta^{-1}s^{\gamma}t^{1-\gamma}|x-y|^d\big)\,\mathrm{d}t\\ &+\int_{s\vee u}^1\rho\big(\beta^{-1}u^{\gamma}t^{1-\gamma}|y|^d\big)\rho\big(\beta^{-1}s^{\gamma}t^{1-\gamma}|x-y|^d\big)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}y,\end{aligned}$$

and

$$\begin{aligned}\mu_{\mathsf{I0}}(x)=\int_{\mathbb{R}^d}\int_0^1\int_0^1\int_0^{u\wedge s}&\big(\tfrac{t}{u}\big)^{\Gamma}\rho\big(\beta^{-1}t^{\gamma}u^{1-\gamma}|y|^d\big)\big(\tfrac{t}{s}\big)^{\Gamma}\rho\big(\beta^{-1}t^{\gamma}s^{1-\gamma}|x-y|^d\big)\,\mathrm{d}t\\ +\mathbb{1}_{\{u<s\}}&\int_u^s\rho\big(\beta^{-1}u^{\gamma}t^{1-\gamma}|y|^d\big)\Big(1-\big(\tfrac{t}{s}\big)^{\Gamma}\rho\big(\beta^{-1}t^{\gamma}s^{1-\gamma}|x-y|^d\big)\Big)\,\mathrm{d}t\\ +\mathbb{1}_{\{s<u\}}&\int_s^u\big(\tfrac{t}{u}\big)^{\Gamma}\rho\big(\beta^{-1}t^{\gamma}u^{1-\gamma}|y|^d\big)\Big(1-\rho\big(\beta^{-1}s^{\gamma}t^{1-\gamma}|x-y|^d\big)\Big)\,\mathrm{d}t\\ +&\int_{s\vee u}^1\rho\big(\beta^{-1}u^{\gamma}t^{1-\gamma}|y|^d\big)\Big(1-\rho\big(\beta^{-1}s^{\gamma}t^{1-\gamma}|x-y|^d\big)\Big)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}y.\end{aligned}$$

Lemma 3.3 (Interest clustering)**.** *For all $\beta>0,\gamma\in(0,1),\delta>1,$ and $\Gamma\geq 0$, we have*

$$\begin{aligned}n^{\mathsf{ic}}(\mathbf{o})=&\int_{\mathbb{R}^d}\mathrm{d}x\sum_{k=2}^{\infty}\mathrm{e}^{-\mu_{\mathsf{II}}(x)}\frac{\mu_{\mathsf{II}}(x)^k}{k!}\\ &-2\mu_{\mathsf{I0}}(x)\int_0^{\infty}\mathrm{d}s\,\mathrm{e}^{\mu_{\mathsf{I0}}(x)(\mathrm{e}^{-2s}-1)}\Big(\mathrm{e}^{\mu_{\mathsf{II}}(x)(\mathrm{e}^{-s}-1)-s}-\mathrm{e}^{-\mu_{\mathsf{II}}(x)-s}-\mathrm{e}^{-\mu_{\mathsf{II}}(x)-2s}\mu_{\mathsf{II}}(x)\Big).\end{aligned}$$

Proof. Under $\mathbb{P}_{0,x}$, the number Y_{II} of vertices connected to both $\mathbf{o}$ and $\mathbf{x}$ is Poisson distributed with parameter $\mu_{\mathsf{II}}=\mu_{\mathsf{II}}(x)$ and the number Y_{I0} of vertices connected to $\mathbf{o}$ but not to $\mathbf{x}$ is Poisson distributed with parameter $\mu_{\mathsf{lo}} = \mu_{\mathsf{lo}}(x)$ and independent of Y_{ll}. On the event $\{Y_{\mathsf{ll}} > 1\}$, we have

$$\frac{\sum_{\mathbf{u},\mathbf{v}\in\mathcal{X}} \mathbb{1}_{\{\mathbf{y}\to\mathbf{u},\mathbf{y}\to\mathbf{v}\}}\mathbb{1}_{\{\mathbf{x}\to\mathbf{u},\mathbf{x}\to\mathbf{v}\}}}{\sum_{\mathbf{u},\mathbf{v}\in\mathcal{X}} \mathbb{1}_{\{\mathbf{y}\to\mathbf{u}\text{ or }\mathbf{y}\to\mathbf{v}\}}\mathbb{1}_{\{\mathbf{x}\to\mathbf{u},\mathbf{x}\to\mathbf{v}\}}} = \frac{Y_{\mathsf{ll}}(Y_{\mathsf{ll}}-1)}{Y_{\mathsf{ll}}(Y_{\mathsf{ll}}-1)+2Y_{\mathsf{ll}}Y_{\mathsf{lo}}} = \frac{Y_{\mathsf{ll}}-1}{Y_{\mathsf{ll}}-1+2Y_{\mathsf{lo}}}.$$

Hence,

$$\begin{aligned}
\mathbb{E}_{0,x}c^{\mathsf{ic}}(\mathbf{x}|\mathbf{o}) &= \sum_{k=2}^{\infty} \mathsf{e}^{-\mu_{\mathsf{ll}}}\frac{\mu_{\mathsf{ll}}^k}{k!}(k-1)\mathbb{E}_{0,x}\Big[\frac{1}{k-1+2Y_{\mathsf{lo}}}\Big] \\
&= \sum_{k=2}^{\infty} \mathsf{e}^{-\mu_{\mathsf{ll}}}\frac{\mu_{\mathsf{ll}}^k}{k!}(k-1)\int_0^\infty \mathbb{E}_{0,x}\mathsf{e}^{-s(k-1+2Y_{\mathsf{lo}})}\,\mathrm{d}s \\
&= \sum_{k=2}^{\infty} \mathsf{e}^{-\mu_{\mathsf{ll}}}\frac{\mu_{\mathsf{ll}}^k}{k!}(k-1)\int_0^\infty \mathsf{e}^{\mu_{\mathsf{lo}}(\mathsf{e}^{-2s}-1)-s(k-1)}\,\mathrm{d}s \\
&= \sum_{k=2}^{\infty} \mathsf{e}^{-\mu_{\mathsf{ll}}}\frac{\mu_{\mathsf{ll}}^k}{k!}\Big(1+\int_0^\infty \mathsf{e}^{-s(k-1)}\frac{\mathrm{d}}{\mathrm{d}s}\mathsf{e}^{\mu_{\mathsf{lo}}(\mathsf{e}^{-2s}-1)}\,\mathrm{d}s\Big).
\end{aligned}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathsf{e}^{\mu_{\mathsf{lo}}(\mathsf{e}^{-2s}-1)} = -2\mu_{\mathsf{lo}}\mathsf{e}^{\mu_{\mathsf{lo}}(\mathsf{e}^{-2s}-1)-2s},$$

we have

$$\begin{aligned}
\mathbb{E}_{0,x}c^{\mathsf{ic}}(\mathbf{x}|\mathbf{o}) &= \sum_{k=2}^{\infty} \mathsf{e}^{-\mu_{\mathsf{ll}}}\frac{\mu_{\mathsf{ll}}^k}{k!}\Big(1-2\mu_{\mathsf{lo}}\int_0^\infty \mathsf{e}^{\mu_{\mathsf{lo}}(\mathsf{e}^{-2s}-1)-s(k+1)}\,\mathrm{d}s\Big) \\
&= \sum_{k=2}^{\infty} \mathsf{e}^{-\mu_{\mathsf{ll}}}\frac{\mu_{\mathsf{ll}}^k}{k!} - 2\mu_{\mathsf{lo}}\int_0^\infty \mathsf{e}^{\mu_{\mathsf{lo}}(\mathsf{e}^{-2s}-1)}\sum_{k=2}^{\infty}\mathsf{e}^{-\mu_{\mathsf{ll}}}\frac{\mu_{\mathsf{ll}}^k}{k!}\mathsf{e}^{-s(k+1)}\,\mathrm{d}s.
\end{aligned}$$

The last sum can be rewritten as

$$\sum_{k=2}^{\infty}\mathsf{e}^{-\mu_{\mathsf{ll}}}\frac{\mu_{\mathsf{ll}}^k}{k!}\mathsf{e}^{-s(k+1)} = \mathsf{e}^{\mu_{\mathsf{ll}}(\mathsf{e}^{-s}-1)-s} - \mathsf{e}^{-\mu_{\mathsf{ll}}-s} - \mathsf{e}^{-\mu_{\mathsf{ll}}-2s}\mu_{\mathsf{ll}},$$

and integrating the whole expression for $\mathbb{E}_{0,x}c^{\mathsf{ic}}(\mathbf{x}|\mathbf{o})$ over $x \in \mathbb{R}^d$ now yields the representation given in the lemma. □

Clustering in $(\mathscr{D}_t : t \geq 0)$. It is straightforward to see, that both metrics are positive and *local*, i.e. continuous with respect to d_*. Hence Theorem 2.1 implies that the finite systems $(\mathscr{D}_t)$ asymptotically display both forms of clustering, in the sense that the corresponding statistically averaged metrics converge to the metrics of the limit system $\mathscr{D}$. To arrive at the same conclusion by defining the average clustering metrics ad hoc for the finite model and evaluate suitable limiting expressions analytically would require a more careful mathematical treatment. The use of the general local convergence framework allows us to avoid these complications.

Theorem 3.4 (Asymptotics of average clustering metrics)**.** *We have that*

$$\frac{1}{\sharp V(\mathscr{D}_t)}\sum_{\mathbf{x}\in V(\mathscr{D}_t)} c^{\mathsf{fc}}(\mathbf{x}) \to \mathbb{E}_0 c^{\mathsf{fc}}(\mathbf{o}) \text{ in probability,}$$

and that

$$\frac{1}{\sharp V(\mathscr{D}_t)}\sum_{\mathbf{x}\in V(\mathscr{D}_t)}\sum_{\mathbf{y}\in V(\mathscr{D}_t)} c^{\mathsf{ic}}(\mathbf{y}|\mathbf{x}) \to n^{\mathsf{ic}}(\mathbf{o}) \text{ in probability.}$$

Sketch. The first statement is immediate from Theorem 2.1, upon noticing that the averaging corresponds to choosing a uniform root at which to evaluate $c^{\mathsf{fc}}(\cdot)$ and taking expectations. The same holds true for the outer averaging in the second expression; an application of Campbell's formula [31] then reduces the inner summation to an integral as in the definition of $n^{\mathsf{ic}}(\mathbf{o})$. □

4 Directed percolation

Originally introduced by Broadbent and Hammersley in 1957 [9], percolation has drawn a lot of attention from the mathematical community and is widely studied until today. The most fundamental question in percolation theory is whether an infinite connected component (*cluster*) exists. If such a component exists, the graph can be seen as well connected whereas if no such component exists the graph decomposes into a collection of disjoint finite clusters. Since the existence of an infinite cluster is monotone in the edge density β and a 0-1-event, one needs to only establish the existence of a critical intensity $\beta_c \in (0,\infty)$ such that an infinite component is almost surely present in the graph if $\beta > \beta_c$ but almost surely absent if $\beta < \beta_c$. For undirected translation invariant models in $\mathbb{R}^d$ it is well established that there is at most one unique infinite component present [11, 30]. If $\beta_c = 0$, the graph is also referred to being *robust*. Percolation was studied for various models since its introduction, see e.g. [24, 32, 16, 18, 22, 23]. A related but typically more difficult question is whether or not a growing sequence of graph contains a (unique) component of linear size. Whether in general percolation in the local limit has a bearing on the existence of linear components in the approximating graph sequence has recently been investigated in [27].

In our directed setting, there are now two types of connected components, weak and strong ones. Let us denote by $\mathbf{x} \longrightarrow \mathbf{y}$ the event that there exists a directed path from $\mathbf{x}$ to $\mathbf{y}$ in $\mathscr{D}$. Then, $\mathbf{x}$ and $\mathbf{y}$ belong to the same *weakly connected component* if either $\mathbf{x} \longrightarrow \mathbf{y}$ or $\mathbf{y} \longrightarrow \mathbf{x}$. On the contrary, $\mathbf{x}$ and $\mathbf{y}$ belong to a *strongly connected component* if $\mathbf{x} \longrightarrow \mathbf{y}$ and $\mathbf{y} \longrightarrow \mathbf{x}$ which we denote from now on as $\mathbf{x} \longleftrightarrow \mathbf{y}$. This gives rise to three components of the root (resp. a given vertex) to consider:

$$\overrightarrow{\mathcal{C}} = \{\mathbf{x}\in\mathcal{X} : \mathbf{o}\longrightarrow \mathbf{x}\},\ \overleftarrow{\mathcal{C}} = \{\mathbf{x}\in\mathcal{X} : \mathbf{x}\longrightarrow \mathbf{o}\},\ \text{and } \overleftrightarrow{\mathcal{C}} = \{\mathbf{x}\in\mathcal{X} : \mathbf{o}\longleftrightarrow \mathbf{x}\}.$$

In this article, we focus on the weak-connectedness event

$$\{\sharp\overrightarrow{\mathcal{C}} = \infty\} := \{\mathbf{o}\longrightarrow\infty\}$$

only, other percolation questions are left for future work. The event $\mathbf{o} \longrightarrow \infty$ can be interpreted as the situation that news spread through the networks $(\mathscr{D}_t)$ by the most influential vertices can reach a positive proportion of the network. In order to make this rigorous however, one would need a convergence result for a weak giant component similar to the undirected case. Since this is not a local event, Theorem 2.1 cannot be applied and an extension of [27] for the directed case is needed. Note however, that in digraphs the existence of a large weakly connected component can occur even if the local limit does not weakly percolate, see e.g. [33], which cannot happen in undirected graphs, hence the situation is more complex as in the undirected setting.

Similar to the undirected case, we are interested in the critical intensity

$$\vec{\beta}_c := \vec{\beta}_c(\gamma,\delta,\Gamma) = \sup\big\{\beta>0 : \mathbb{P}_0\{\mathbf{o}\longrightarrow\infty\} = 0\big\}.$$

If one restricts the graph to edges that point in both directions and vertices with birth time larger than $1/2$, it is easy to see that this graph can be compared with a undirected (long-range) random connection model for which the existence of an infinite component in dimensions $d \geq 2$ and for $\delta \in (1,2)$ in $d = 1$ is well-known [32]. Hence, we immediately infer $\vec{\beta}_c < \infty$ in these cases. Therefore, we deal with the question of positivity of the critical intensity here.

Theorem 4.1 (Positivity of critical intensity)**.** *Consider the DARCM $\mathscr{D} = \mathscr{D}[\beta, \gamma, \delta, \Gamma]$ with $\beta > 0$, $\gamma \in (0,1)$, $\delta > 1$, and $\Gamma \geq 0$.*

(i) If $\gamma < (\delta+\Gamma)/(\delta+1)$, then $\vec{\beta}_c > 0$ and

(ii) if $\gamma > (\delta+\Gamma)/(\delta+1)$, then $\vec{\beta}_c = 0$.

The proof further elaborates the ideas for the undirected case in [22]. There, it was shown that the most promising strategy for building a long path is to use young intermediate 'connectors' to connect old vertices and if $\gamma > \delta/(\delta+1)$ the age's influence is strong enough compared to the geometric restrictions such that this strategy can be repeated indefinitely with positive probability regardless of the edge intensity. We will adapt the strategy of [22] to the directed setting. Let us write

$$\mathbf{x} \xrightarrow[\mathbf{x},\mathbf{y}]{2} \mathbf{y}$$

for the event that $\mathbf{x}$ is connected to $\mathbf{y}$ by a directed path of length 2 where the intermediate vertex is younger than both $\mathbf{x}$ and $\mathbf{y}$. Key to the proof of Theorem 4.1 is the following lemma.

Lemma 4.2 (Two-connection-lemma)**.** *Consider $\mathscr{D} = \mathscr{D}[\beta, \gamma, \delta, \Gamma]$ with $\beta > 0$, $\gamma \in (0,1)$, $\delta > 1$, and $\Gamma \geq 0$.*

(a) Assume $\gamma < (\delta+\Gamma)/(\delta+1)$. Let $\mathbf{x} = (x,t)$ and $\mathbf{y} = (y,s)$ be two given vertices that satisfy $|x-y|^d \geq \beta(t \wedge s)^{-\gamma}(s \vee t)^{\gamma-1}$. Then we have

$$\begin{aligned} \mathbb{P}_{\mathbf{x},\mathbf{y}}\Big(\mathbf{x} \xrightarrow[\mathbf{x},\mathbf{y}]{2} \mathbf{y}\Big) &\leq \mathbb{E}_{\mathbf{x},\mathbf{y}}\big[\sharp\{\mathbf{z} = (z,u) : u > t \text{ and } \mathbf{x} \to \mathbf{z} \to \mathbf{y}\}\big] \\ &\leq \beta C \, \mathbb{P}_{\mathbf{x},\mathbf{y}}(\mathbf{x} \to \mathbf{y}). \end{aligned} \tag{3}$$

for some $C > 0$ only depending on γ, δ, Γ, and the dimension d.

(b) Let $\gamma > (\delta+\Gamma)/(\delta+1)$ and fix two interdependent constants

$$\alpha_1 \in \big(1, \tfrac{\gamma-\Gamma}{\delta(1-\gamma)}\big) \text{ and } \alpha_2 \in \big(\alpha_1, \tfrac{\alpha_1(\gamma\delta-1)+\gamma-\Gamma}{\delta-1}\big).$$

Let $\mathbf{x} = (x,t)$ be a given vertex with $t < 1/2$ and define the event

$$E(\mathbf{x}) = \Big\{\exists \mathbf{y} = (y,s) : s < t^{\alpha_1}, |x-y|^d < t^{-\alpha_2} \text{ and } \mathbf{x} \xrightarrow[\mathbf{x},\mathbf{y}]{2} \mathbf{y}\Big\}.$$

For each $\beta > 0$ there exists some $a < 0$ such that $\mathbb{P}_{\mathbf{x}}\big(E(\mathbf{x})\big) \geq 1 - e^{-t^a}$.

Proof. We start by proving (a) and focus on the case $s < t$. The other case works analogously. Observe that the first inequality in (3) is simply a moment bound, and we hence focus on the second inequality. By our assumptions on the distance of $\mathbf{x}$ and $\mathbf{y}$ we have

$$\begin{aligned}
&\mathbb{E}_{\mathbf{x},\mathbf{y}}\big[\sharp\{\mathbf{z}=(z,u) : u>t \text{ and } \mathbf{x}\to\mathbf{z}\to\mathbf{y}\}\big] \\
&\leq \int_{\mathbb{R}^d} \mathrm{d}z \int_t^1 \mathrm{d}u \, \big(\tfrac{t}{u}\big)^{\Gamma} \rho\big(\tfrac{1}{\beta} t^{\gamma} u^{1-\gamma}|x-z|^d\big)\rho\big(\tfrac{1}{2^d\beta} s^{\gamma} u^{1-\gamma}|x-y|^d\big)\mathbb{1}_{\{|y-z|>\frac{|x-y|}{2}\}} \\
&+ \int_{\mathbb{R}^d} \mathrm{d}z \int_t^1 \mathrm{d}u \, \big(\tfrac{t}{u}\big)^{\Gamma} \rho\big(\tfrac{1}{2^d\beta} t^{\gamma} u^{1-\gamma}|x-y|^d\big)\rho\big(\tfrac{1}{\beta} s^{\gamma} u^{1-\gamma}|y-z|^d\big)\mathbb{1}_{\{|x-z|\geq\frac{|x-y|}{2}\}} \\
&\leq \beta C\big(\tfrac{1}{\beta} s^{\gamma} t^{1-\gamma}|x-y|^d\big)^{-\delta} = \beta C\, \rho\big(\tfrac{1}{\beta} s^{\gamma} t^{1-\gamma}|x-y|^d\big),
\end{aligned}$$

where we integrated both integrals separately and have used that indicator functions are bounded by 1, a change of variables together with the integrability of ρ, as well as $\gamma < {}^{(\delta+\Gamma)}/_{(\delta+1)}$ and our distance assumption.

We also start the proof of (b) by calculating the expected number of vertices $\mathbf{y}$ being part of the event $E(\mathbf{x})$. Using similar arguments as above, it is straight forward to deduce that this expectation is lower bounded by

$$ct^{\Gamma-\gamma+\alpha_2(\delta-1)-\alpha_1(\gamma\delta-1)},$$

for some constant $c>0$. The proof finishes with the observation that due to our choices of α_1 and α_2, we have $a := \Gamma-\gamma+\alpha_2(\delta-1)-\alpha_1(\gamma\delta-1)<0$ and therefore $\mathbb{P}_{\mathbf{x}}(E(\mathbf{x}))\geq 1-e^{-ct^a}$. □

of Theorem 4.1. The proof works similarly to the proof carried out in [22] and shall hence only be sketched. Observe first, that the origin starts a path $\mathbf{o}\longrightarrow\infty$ using young connectors to connect to older and older vertices with strictly positive probability if the root itself is old enough by Lemma 4.2 (b), proving Part (ii).

To prove Part (i), we want to bound the probability that the root $\mathbf{o}$ starts a directed, short-cut free path of length n by a term that goes to 0 as $n\to\infty$ for small enough β. Here, a directed path $P=(\mathbf{x}_0,\dots,\mathbf{x}_n)$ is called *shortcut-free*, if $\mathscr{N}^{\text{in}}(\mathbf{x}_j)\cap P=\mathbf{x}_{j-1}$ and $\mathscr{N}^{\text{out}}(\mathbf{x}_j)\cap P=\mathbf{x}_{j+1}$ for all $j=1,\dots,n-1$. Note that there always exists a directed shortcut-free path to infinity if there exists a directed path to infinity at all. To make use of the previous lemma, we work from now on in the graph $\widehat{\mathscr{D}}$ defined by taking $\mathscr{D}$ and adding each bi-directed arc $\mathbf{x}\leftrightarrow\mathbf{y}$ (if not already there) whenever $\mathbf{x}$ and $\mathbf{y}$ do not fulfil the distance condition of Lemma 4.2 (a). By definition, in a short-cut free path in $\widehat{\mathscr{D}}$ we always have

$$|x_i-x_j|^d > \beta(t_i\wedge t_j)^{-\gamma}(t_i\vee t_j)^{\gamma-1}, \quad \text{whenever} \quad |i-j|\geq 2.$$

From Lemma 4.2 (a), we infer that for all vertices fulfilling this distance condition, it is more probable to be connected by a direct arc than through a single connector. To make use of this fact, let us introduce the notion of a path's *skeleton*.

For a path P we call the collection of vertices with running minimum age from both sides the *skeleton* of P. That is, we start from the initial vertex (x_0,t_0) and search for the first vertex (x_{j_1},t_{j_1}) that has birth time $t_{j_1}<t_0$. Starting from this vertex again we search for the next vertex with smaller birth time still until we reach the oldest vertex of the path. Afterwards we do the same but starting from the last vertex of the path (x_n,t_n) and going backwards across the indices. Another possibility to identify the path's skeleton is the following: We call a vertex $\mathbf{x}_j\in P\setminus\{\mathbf{x}_0,\mathbf{x}_n\}$ local maximum if $t_j>t_{j-1}$ and $t_j>t_{j+1}$. Put differently, $\mathbf{x}_j$ is younger than its preceding and subsequent vertex. We now successively remove all local maxima from P as follows: First, take the local maximum in P with the greatest birth time, remove it from P and connect its former neighbours by a directed edge

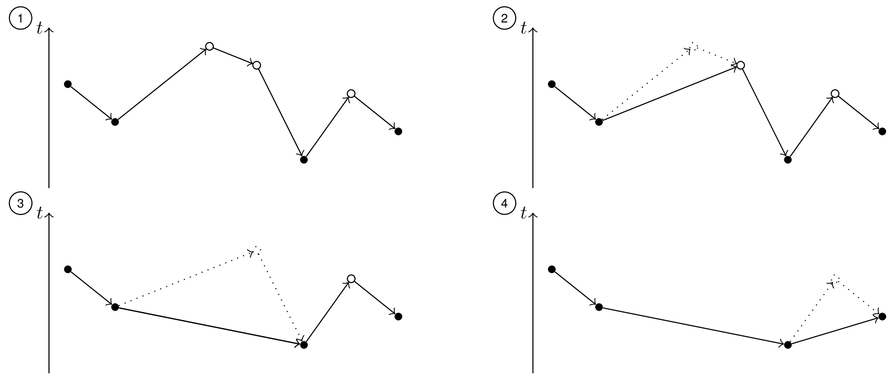

Figure 1: Methodical sketch of the outlined step by step removal of local maxima to observe the skeleton path (the black vertices).

oriented from preceding to subsequent vertex. In the resulting path, we take the local maximum of greatest birth time and remove it, repeating until there is no local maximum left, see Figure 1.

The idea is now the following. Between each two skeleton vertices, we remove all local maxima step by step and replace them by direct arcs. The probabilistic costs for each such replacement is the probability of an arc times βC. Since all combinatorics involved are also of exponential order in the subpath's length, cf. [22, Lemma 2.3], we infer for two given vertices $\mathbf{x}$ and $\mathbf{y}$ satisfying the distance condition

$$\mathbb{P}_{\mathbf{x},\mathbf{y}}\Big(\mathbf{x} \xrightarrow[\mathbf{x},\mathbf{y}]{k} \mathbf{y}\Big) \leq (\beta \cdot 4C)^{k-1}\mathbb{P}_{\mathbf{x},\mathbf{y}}(\mathbf{x} \to \mathbf{y}),$$

where $\mathbf{x} \xrightarrow[\mathbf{x},\mathbf{y}]{k} \mathbf{y}$ denotes the event that x is connected to $\mathbf{y}$ by a direct path of length k where all intermediate vertices are younger than $\mathbf{x}$ and $\mathbf{y}$. Let now $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k = \mathbf{y}$ be a given skeleton and let us write $\mathbf{x} \xrightarrow[\mathbf{x}_0,\dots,\mathbf{x}_k]{n} \mathbf{x}_k$ for the event that there is a directed path from $\mathbf{x}_0$ to $\mathbf{x}_k$ with skeleton $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k$. We can then use the BK-inequality [5] in a version of [25] as outlined in [22, Eq.(11)] to deduce

$$\mathbb{P}_{\mathbf{x}_0,\mathbf{x}_1,\dots,\mathbf{x}_k}\big(\mathbf{x} \xrightarrow[\mathbf{x}_0,\dots,\mathbf{x}_k]{n} \mathbf{y}\big) \quad \leq (\beta\, C)^{n-k}\binom{n}{k}\prod_{j=1}^{k}\mathbb{P}_{\mathbf{x}_{j-1},\mathbf{x}_j}(\mathbf{x}_{j-1} \to \mathbf{x}_j) \tag{4}$$

for some new constant $C > 1$. Define now A_n to be the event that $\mathbf{o}$ starts a directed path of length n that ends in the oldest vertex of the path. In other words, the path has a skeleton with strictly decreasing birth times. Using (4) and Mecke's equation [31], we obtain

$$\mathbb{P}(A_n) \leq \sum_{k=1}^{n}\left[(\beta\, C)^{n-k}\binom{n}{k}\int\limits_{\substack{x_0=0,x_1,\dots x_k\in\mathbb{R}^d\\ 1>t_0>t_1>\dots>t_k}}\bigotimes_{j=0}^{k}\mathrm{d}\mathbf{x}_j\prod_{j=1}^{k}\mathbb{P}_{\mathbf{x}_{j-1},\mathbf{x}_j}(\mathbf{x}_{j-1} \to \mathbf{x}_j)\right]$$

The last equation can be easily calculated similarly as done in [22, Lemma 2.4] from which we infer $\mathbb{P}_0(A_n) \leq (\beta\, C)^n$ for some $C > 1$. Choosing $\beta < 1/C$, we infer from the Borel-Cantelli Lemma that almost surely there is a finite N such that A_n does not occur for all $n > N$. This however implies that every infinite path has bounded from below birth time because a path with birth times approaching zero contains sub paths ending in its oldest vertex of arbitrary length. This concludes the proof as it is easy to see that no such infinite paths can exist for small intensities β. □

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