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Three-state p-SOS models on binary Cayley trees

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Abstract

We consider a version of the solid-on-solid model on the Cayley tree of order two in which vertices carry spins of value $0, 1$ or 2 and the pairwise interaction of neighboring vertices is given by their spin difference to the power $p > 0$. We exhibit all translation-invariant splitting Gibbs measures (TISGMs) of the model and demonstrate the existence of up to seven such measures, depending on the parameters. We further establish general conditions for extremality and non-extremality of TISGMs in the set of all Gibbs measures and use them to examine selected TISGMs for a small and a large p. Notably, our analysis reveals that extremality properties are similar for large p compared to the case $p = 1$, a case that has been explored already in previous work. However, for the small p, certain measures that were consistently non-extremal for $p = 1$ do exhibit transitions between extremality and non-extremality.

1 Introduction

In this paper we consider spin-configurations σ which are functions from the vertices of a Cayley tree of order $k \geq 1$ (that is an infinite graph without cycles such that exactly $k + 1$ edges originate from each vertex) to the local configuration space $\Phi = \{0, 1, \dots, m\}$, $m \geq 1$. For most of our analysis we will restrict to the case $m = 2$ and the Cayley tree with $k = 2$. By $\langle x, y \rangle$ we denote a pair of nearest-neighbor vertices. A two-parametric solid-on-solid model (called p-SOS) is a spin system with spins taking values in Φ, and with formal Hamiltonian

$$H(\sigma) = -J \sum_{\langle x,y \rangle} |\sigma(x) - \sigma(y)|^p, \tag{1}$$

where $p > 0$ and $J \in \mathbb{R}$ the coupling constant.

Let us note that, if $m = 1$, i.e., $\Phi = \{0, 1\}$ then, the p-SOS model can be reduced to the classical Ising (i.e., 2-state Potts) model, since in this case, $|\sigma(x) - \sigma(y)|^p \in \{0, 1\}$ for any $p > 0$ and $\langle x, y \rangle$. A complete analysis of Gibbs measures for tree-indexed Ising and Potts models can be found for example in [15, 16]. On the other hand, for $m \geq 2$ and $p = 1$ the model becomes the classical SOS-model which is considered on the cubic lattice in [2, 9] and on Cayley trees in [6, 13, 14, 15]. The case $p = 2$ is known as the discrete Gaussian case, see for example [1, 18] and references therein.

In the recent paper [3] the authors consider p-SOS models with spin values in $\mathbb{Z}$ on Cayley trees. There, a family of extremal tree-automorphism non-invariant Gibbs measures is presented that arises as low temperature perturbations of ground states. Moreover, the extremality of low-temperature states in the set of all Gibbs measures is shown. Further, in [13] the SOS model (for $p = 1$) is treated and a vector-valued functional equation for possible boundary laws of the model is obtained. It is known that each solution to this functional equation determines a splitting Gibbs measure (SGM) and in particular, the vertex-independent boundary laws define translation-invariant (TI) SGMs.

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In this paper, we take a similar approach and present a description of all TISGMs for the three-state ($m = 2$) p-SOS model on the Cayley tree of order two via solutions for the fixed-point equations of the vertex-independent boundary laws. This is a non-trivial extension of the analysis presented in [6] which deals only with the case $p = 1$.

The paper is organized as follows. In Section 2 we introduce the general setup and present the defining functional equations for the p-SOS model. In Section 3 we present the description of all TISGMs and show that their number can be up to seven, for any given parameters $p > 0$ and $\theta = \exp(J)$. Finally, in Section 4 we study the extremality questions for TISGMs and use the methods of [6, 7] based on the Kesten–Stigum's non-extremality condition [5] and the Martinelli–Sinclair–Weitz's extremality condition, see [8].

2 Setup and functional equations

We denote by $\Gamma^k = (V, L)$ the *Cayley tree* of order $k \geq 1$, where V is the set of *vertices* and L the set of *edges*. A collection of nearest-neighbor pairs of vertices $\langle x, x_1\rangle, \langle x_1, x_2\rangle, ..., \langle x_{d-1}, y\rangle$ is called a *path* from x to y. The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y. For a fixed $x^0 \in V$, called the *root*, we set

$$W_n := \{x \in V \mid d(x, x^0) = n\}, \qquad V_n := \bigcup_{m=0}^{n} W_m$$

and denote by

$$S(x) := \{y \in W_{n+1} : d(x, y) = 1\}, \;\; x \in W_n,$$

the set of *direct successors* of x. Next, we consider real vector-valued function from $V \setminus \{x^0\}$ to $\mathbb{R}^{m+1}$ given as

$$h\colon x \mapsto h_x = (h_{0,x}, h_{1,x}, \dots, h_{m,x}),$$

and the corresponding probability distributions $\mu^{(n)}$ on Φ^{V_n}, the set of all configuration given on V_n, $n \in \mathbb{N}$, defined as

$$\mu^{(n)}(\sigma_n) := Z_n^{-1} \exp\Big(- H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\Big). \tag{2}$$

Here, $\sigma_n\colon x \in V_n \mapsto \sigma(x) \in \{0, \dots, m\}$ and Z_n is the corresponding partition function

$$Z_n := \sum_{\widetilde{\sigma}_n \in \Phi^{V_n}} \exp\Big(- H(\widetilde{\sigma}_n) + \sum_{x \in W_n} h_{\widetilde{\sigma}(x),x}\Big). \tag{3}$$

We say that the sequence of probability distributions $(\mu^{(n)})_{n\geq 1}$ are *compatible* if for all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$ we have that

$$\sum_{\omega_n \in \Phi^{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) = \mu^{(n-1)}(\sigma_{n-1}). \tag{4}$$

Here $\sigma_{n-1} \vee \omega_n \in \Phi^{V_n}$ is the concatenation of σ_{n-1} and ω_n. If this is the case, then, by the Kolmogorov extension theorem, there exists a unique measure μ on Φ^V such that, for all n and $\sigma_n \in \Phi^{V_n}$,

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n).$$

Such a measure is called a *splitting Gibbs measure* (SGM) corresponding to the Hamiltonian H and function $x \mapsto h_x$, $x \neq x^0$. Let us note that these measures are also called *Markov chains* for example in [12].

Let us now turn our attention to the p-SOS model with Hamiltonian defined in (1). The following statement presents a system of functional equations whose solutions correspond to (infinite-volume) Gibbs measures of the p-SOS model on Cayley trees. Let us note that every extremal Gibbs measure also arises in this way, but not necessarily every measure which arises in this way is extremal, see [4, Chapter 12]. In other words, the following statement describes conditions on h_x guaranteeing compatibility of distributions $\mu^{(n)}(\sigma_n)$.

Proposition 1. *Probability distributions $\mu^{(n)}$, $n = 1, 2, \dots$, in* (2) *are compatible iff, for any $x \in V \setminus \{x^0\}$, the following equation holds,*

$$h_x^* = \sum_{y \in S(x)} F(h_y^*, m, \theta). \tag{5}$$

Here $\theta := \exp(J)$ and h_x^* stands for the vector $(h_{0,x} - h_{m,x}, h_{1,x} - h_{m,x}, \dots, h_{m-1,x} - h_{m,x})$ and the vector-valued function $F(\cdot, m, \theta)\colon \mathbb{R}^m \to \mathbb{R}^m$ is $F(h, m, \theta) := (F_0(h, m, \theta), \dots, F_{m-1}(h, m, \theta))$, with

$$F_i(h, m, \theta) := \ln \frac{\sum_{j=0}^{m-1} \theta^{|i-j|^p} \exp(h_j) + \theta^{(m-i)^p}}{\sum_{j=0}^{m-1} \theta^{(m-j)^p} \exp(h_j) + 1}, \tag{6}$$

where $h := (h_0, h_1, \dots, h_{m-1})$, $i = 0, \dots, m-1$.

Proof. The proof is similar to the proof of [13, Proposition 2.1.]. □

From Proposition 1 it follows that for any $h = \{h_x \colon x \in V\}$ satisfying (5) there exists a unique SGM μ for the p-SOS model. However, the analysis of solutions to (5) for an arbitrary m is challenging. We therefore restrict our attention to a smaller class of measures, namely the translation-invariant SGMs.

It is natural to begin with translation-invariant solutions where $h_x = h \in \mathbb{R}^m$ is independent of x. In this case (5) becomes

$$z_i = \left(\frac{\sum_{j=0}^{m-1} \theta^{|i-j|^p} z_j + \theta^{(m-i)^p}}{\sum_{j=0}^{m-1} \theta^{(m-j)^p} z_j + 1} \right)^k, \qquad i = 0, \dots, m-1, \tag{7}$$

where $z_i = \exp(h_i)$. The vector $(z_0, \dots, z_{m-1})$ is called a (translation-invariant) *law* and the non-translation-invariant quantities $l_x(i) = \exp(h_{i,x})$ are the *boundary laws*, see [4, pp. 242]. In the present manuscript we present a full analysis of the solutions of the system (7) for the case where $k = 2$, $p > 0$, $m = 2$ and additionally study extremality properties of the corresponding TISGMs. In particular, we extend the results in [6] for general $p > 0$, which were obtained there only for $p = 1$.

3 The case $k = m = 2$: complete analysis of solutions

Assuming $k = m = 2$, the two-dimensional fixed-point equation (7) for the two components of the boundary law can be written in terms of the variables $x = \sqrt{z_0}$ and $y = \sqrt{z_1}$ in the form

$$x = \frac{x^2 + \theta y^2 + \theta^{2^p}}{\theta^{2^p} x^2 + \theta y^2 + 1}, \tag{8}$$

$$y = \frac{\theta x^2 + y^2 + \theta}{\theta^{2^p} x^2 + \theta y^2 + 1}. \tag{9}$$

From (8) we get $x = 1$ or

$$\theta y^2 = (1 - \theta^{2^p})x - \theta^{2^p}(x^2 + 1). \tag{10}$$

Remark 1. *Since $x > 0$ we have that* (10) *can hold iff $\theta < 1$.*

3.1 Case: $x = 1$.

In this case, from (9), we get

$$\theta y^3 - y^2 + (\theta^{2^p} + 1)y - 2\theta = 0. \tag{11}$$

Using Cardano's formula, the discriminant of the cubic equation (11) can be written as

$$\Delta := \Delta(\theta, p) := \frac{1}{27\theta^2}\left[4(1 - 3\theta - 3\theta^{2^p+1})^3 - (2 - 9\theta + 54\theta^3 - 9\theta^{2^p+1})^2\right]. \tag{12}$$

From this we get the following statement.

Lemma 1. *The following assertions hold*

- *If $\Delta > 0$, then* (11) *has three solutions $0 < y_3 < y_2 < y_1$.*
- *If $\Delta = 0$, then* (11) *has two solutions $0 < y_2 < y_1$.*
- *If $\Delta < 0$, then* (11) *has one solution $0 < y_1$.*

Proof. It is well known that the number of real roots for a cubic equation is determined by the value of Δ. It remains to show that all real roots are positive. Descartes' rule of signs is very helpful to count the number of the positive roots of a polynomial. In (11) the sign of the coefficients changes three times, and hence, there might be one positive root or three positive roots. If $\Delta < 0$, then the unique real root is positive. If $\Delta \geq 0$, then the rule of signs applied to the cubic equation, with $-y$ as a substitute, tells us that there are no negative roots, and since we can not have $\theta = 0$, there must be three positive roots. □

3.2 Case: $x \neq 1$ and (10) satisfied.

By Remark 1, we only consider the case $\theta < 1$ and (9) can be written as

$$y^2 = \left(\frac{\theta x^2 + y^2 + \theta}{\theta^{2^p} x^2 + \theta y^2 + 1}\right)^2. \tag{13}$$

In the case, when (10) is satisfied, from (13) we get that

$$((1 - \theta^{2^p})x - \theta^{2^p}(x^2 + 1))\theta = \left(\frac{(\theta^2 - \theta^{2^p})(x^2 + 1) + (1 - \theta^{2^p})x}{(1 - \theta^{2^p})(x + 1)}\right)^2. \tag{14}$$

Let $\xi := x + 1/x$ and note that $\xi > 2$ if $x > 0$. Then, from (14) we get

$$a\xi^2 + b\xi + c = 0, \tag{15}$$

where

$$a := a(\theta,p) := \theta^{2^p+1}(1-\theta^{2^p})^2 + (\theta^2-\theta^{2^p})^2,$$
$$b := b(\theta,p) := (1-\theta^{2^p})[2(\theta^2-\theta^{2^p}) - \theta(1-\theta^{2^p})(1-3\theta^{2^p})],$$
$$c := c(\theta,p) := (1-\theta^{2^p})^2[1-2\theta(1-\theta^{2^p})].$$

This equation has no solution if $D = b^2 - 4ac < 0$, it has a unique solution if $D = 0$ and two solutions if $D > 0$. Thus we have the following statements.

- If $D > 0$, then (15) has two solutions $\xi_1(\theta,p) < \xi_2(\theta,p)$ (below we denote $q = 1-\theta^{2^p}$) given by

$$\xi_{1,2}(\theta,p) := \frac{q}{2}\frac{-3\theta q^2+2(\theta+1)q+2(\theta^2-1)\mp\theta\sqrt{q(q+2\theta-2)[(q-\theta-1)^2+(\theta+1)(3\theta-1)]}}{(q-\theta-1)[\theta q^2+(\theta^2-1)(q+\theta-1)]}. \quad (16)$$

- If $D = 0$, then (15) has a unique solution

$$\xi_{1,2} := \frac{q}{2}\frac{-3\theta q^2+2(\theta+1)q+2(\theta^2-1)}{(q-\theta-1)[\theta q^2+(\theta^2-1)(q+\theta-1)]}.$$

- If $D < 0$, then (15) has no solution.

For $D = 0$ we have

$$D(\theta,p) = \theta^2(\theta^{2^p}-1)^3(\theta^{2^p}-2\theta+1)((\theta^{2^p}+\theta)^2+3\theta^2+2\theta-1) = 0. \quad (17)$$

Define

$$l(\theta,p) := \theta^{2^p} - 2\theta + 1 \qquad \text{and} \qquad q(\theta,p) := (\theta^{2^p}+\theta)^2 + 3\theta^2 + 2\theta - 1.$$

Since $0 < \theta < 1$ we see that $D > 0$ if (see Fig. 1)

$$l(\theta,p)q(\theta,p) < 0. \quad (18)$$

Next, define

$$m(\theta) = \frac{1}{\ln 2}\ln\left(\frac{\ln(-\theta+\sqrt{(\theta+1)(1-3\theta)})}{\ln\theta}\right), \quad \theta \in \left(0, \frac{\sqrt{5}-1}{4}\right), \text{ and}$$
$$M(\theta) = \frac{1}{\ln 2}\ln\left(\frac{\ln(2\theta-1)}{\ln\theta}\right), \quad \theta \in \left(\frac{1}{2}, 1\right).$$

We solve (18) with respect to p and get the following solution,

$$\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2 := \left\{(\theta,p) : 0 < \theta < \tfrac{\sqrt{5}-1}{4}, p > m(\theta)\right\} \bigcup \left\{(\theta,p) : \tfrac{1}{2} < \theta < 1, p > M(\theta)\right\}. \quad (19)$$

Introduce

$$\mathcal{Q}_- = \{(\theta,p) : \Delta(\theta,p) < 0\}, \quad \mathcal{Q}_0 = \{(\theta,p) : \Delta(\theta,p) = 0\} \quad \text{and} \quad \mathcal{Q}_+ = \{(\theta,p) : \Delta(\theta,p) > 0\},$$

and note that $\mathcal{Q}_+ \subset \mathcal{P}_1$. Next, for all θ, p with $D \geq 0$ and

$$2 < \xi_1(\theta,p) \leq \xi_2(\theta,p) \quad (20)$$

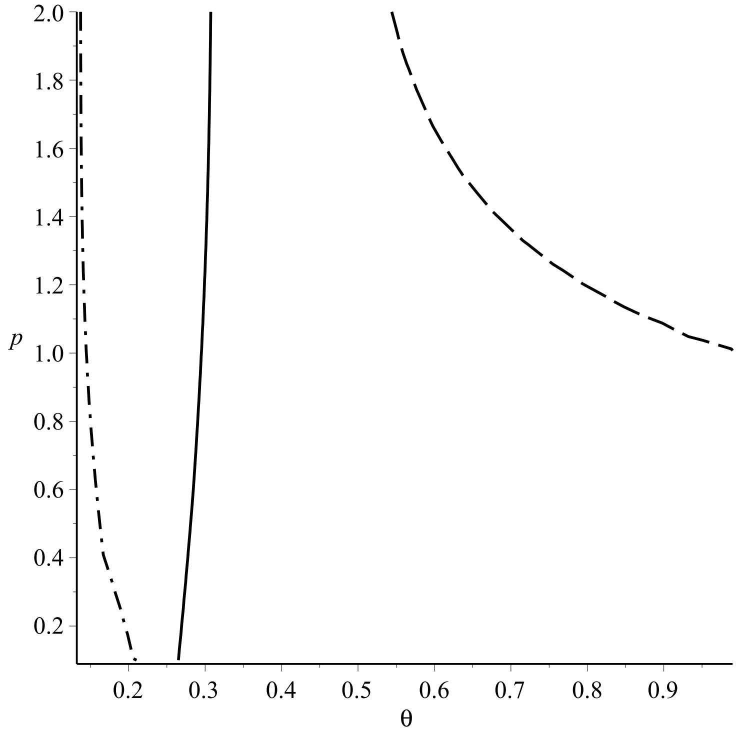

Figure 1: The dash-dot curve defined by $\Delta = 0$ with (12) and the solid curve is $q(\theta, p) = 0$ and the dash curve is $l(\theta, p) = 0$ defined by (17).

we find all four positive solutions to (14) explicitly, i.e., we have

$$\begin{aligned} x_4(\theta,p) &:= \tfrac{1}{2}(\xi_2 - \sqrt{\xi_2^2 - 4}), \quad x_5(\theta,p) = \tfrac{1}{2}(\xi_1 - \sqrt{\xi_1^2 - 4}), \\ x_6(\theta,p) &:= \tfrac{1}{2}(\xi_1 + \sqrt{\xi_1^2 - 4}), \quad x_7(\theta,p) = \tfrac{1}{2}(\xi_2 + \sqrt{\xi_2^2 - 4}). \end{aligned} \tag{21}$$

Fig. 2 presents the graphs of x_i, $i = 4, 5, 6, 7$.

Now, for each $x_i(\theta, p)$, using the following condition on the parameters θ, p,

$$(1 - \theta^{2^p})x_i - \theta^{2^p}(x_i^2 + 1) \geq 0, \tag{22}$$

we define

$$y_i(\theta, p) = \sqrt{\theta}^{-1}\sqrt{(1 - \theta^{2^p})x_i - \theta^{2^p}(x_i^2 + 1)}, \quad i = 4, 5, 6, 7. \tag{23}$$

Remark 2. *In [6], for $p = 1$ it is proven that the Conditions* (20) *and* (22) *are satisfied for all values of θ where the solutions exist. But in case $p \neq 1$ we do not have such a result since the solutions have a very bulky form. Below, we consider two concrete values $p = 0.1$ and $p = 10$ for any $\theta \in (0, 1)$. We note that for these values of p, the Conditions* (20) *and* (22) *are satisfied too, see Fig. 4 and 5.*

Note that $(2\sqrt{2} - 1)/7$ is the unique positive solution of $q(\theta, 0) = 0$. Summarizing, we exhibit the full characterization of the solutions as follows.

Proposition 2. *Assume that the Conditions* (20) *and* (22) *are satisfied. Then, the set $S(\theta, p)$ of solutions to the system* (8), (9) *changes under variations of the parameters θ and p as follows:*

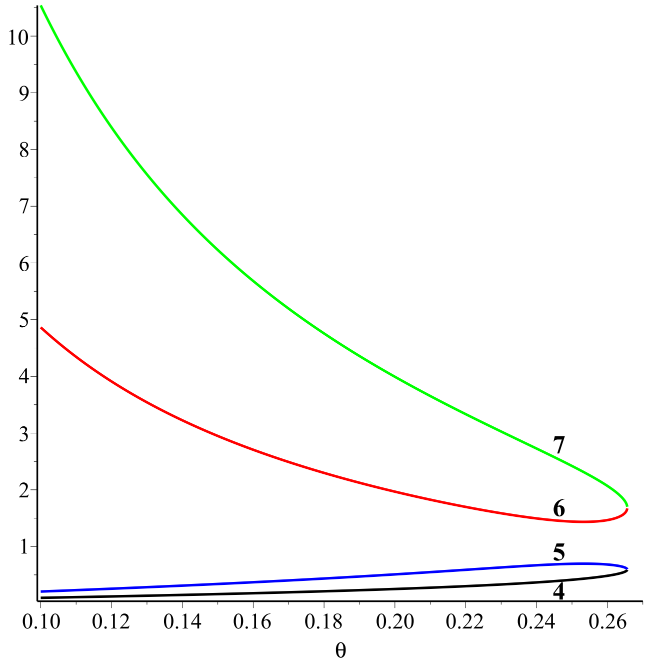

Figure 2: The graphs of the functions $x_i = x_i(\theta, 0.1)$, $i = 4, 5, 6, 7$. The coloring represents $x_4 = black$, $x_5 = blue$, $x_6 = red$ and $x_7 = green$.

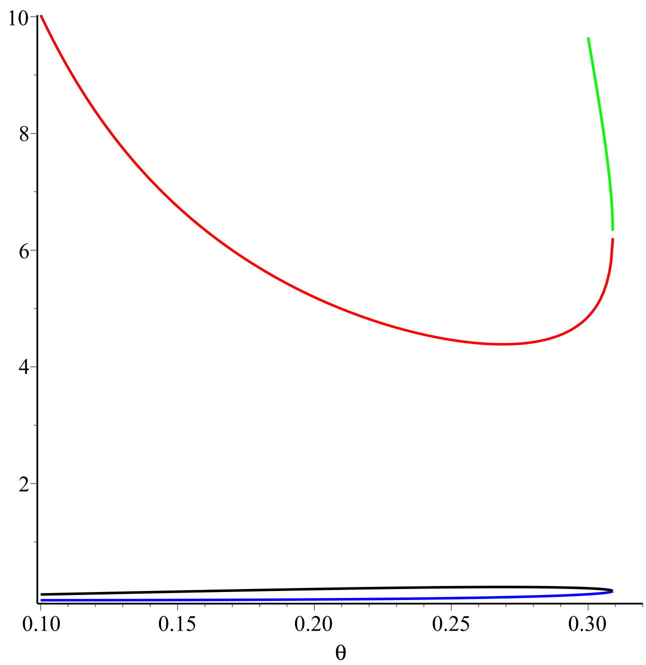

Figure 3: The graphs of the functions $x_i = x_i(\theta, 10)$, $i = 4, 5, 6, 7$. The coloring represents $x_4 = blue$, $x_5 = black$, $x_6 = red$ and $x_7 = green$.

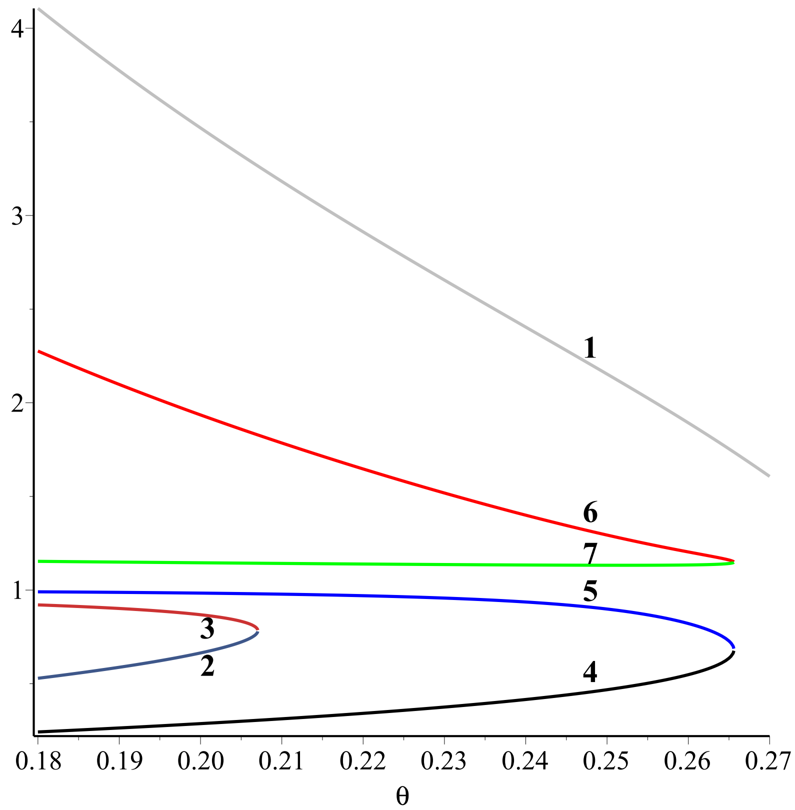

Figure 4: The graphs of the functions $y_i = y_i(\theta, 0.1)$, $i = 1, 2, ..., 7$. Here, $y_1 = grey$, $y_2 = azure$, $y_3 = orange$, $y_4 = black$, $y_5 = blue$, $y_6 = red$, and $y_7 = green$.

$$S(\theta,p)=\begin{cases}\{v_1=(1,y_1)\}, & \textit{if } \theta\geq 1,\ p>0 \ \textit{ or } \theta\in\left(\frac{2\sqrt{2}-1}{7},1\right), p<\min\{m(\theta),M(\theta)\}\\ \{v_1, v_4=(x_4,y_4), v_6=(x_6,y_6)\}, & \textit{if } p=m(\theta) \ \textit{ or } p=M(\theta)\\ \{v_1, v_i=(x_i,y_i), i=4,5,6,7\}, & \textit{if } (\theta,p)\in\mathcal{Q}_-\cap\mathcal{P}\\ \{v_1, v_i=(x_i,y_i), i=3,4,5,6,7\}, & \textit{if } (\theta,p)\in\mathcal{Q}_0\\ \{v_i=(x_i,y_i), i=1,2,3,4,5,6,7\}, & \textit{if } (\theta,p)\in\mathcal{Q}_+,\end{cases} \tag{24}$$

where y_i, $i = 1, 2, 3$ *are solutions of* (11), *which can be given explicitly by Cardano's formula,* $x_1 = x_2 = x_3 = 1$, x_i *and* y_i *for* $i = 4, 5, 6, 7$ *are given by* (21) *and* (23).

We present in Fig. 2 and 4 the graphs of the functions mentioned in Proposition 2 for the case $p = 0.1$ and Fig. 3 for the case $p = 10$.

Denote by μ_i the TISGM corresponding to v_i, $i = 1, \ldots, 7$. As an immediate corollary to Propositions 1 and 2 we get the following statement.

Theorem 1. *Assume that, for the parameters of the* p*-SOS model, the Conditions* (20) *and* (22) *are satisfied. Then, the number of TISGMs* $\mathcal{M}(\theta, p)$ *changes under variations of the parameters* θ *and* p

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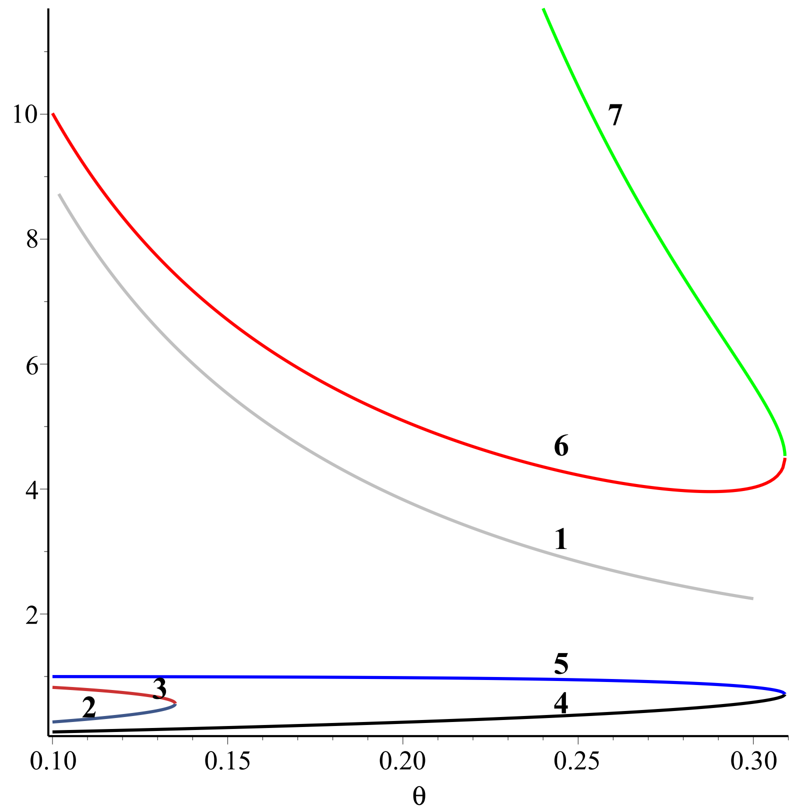

Figure 5: The graphs of the functions $y_i = y_i(\theta, 10)$, $i = 1, 2, ..., 7$. Here, $y_1 = grey$,$y_2 = azure$, $y_3 = orange$, $y_4 = black$, $y_5 = blue$, $y_6 = red$, $y_7 = green$.

as follows:

$$\mathcal{M}(\theta, p) = \begin{cases} 1, & \textit{if } \theta \geq 1,\ p > 0 \ \textit{ or } \ \theta \in \big((2\sqrt{2}-1)/7), 1\big), p < \min\{m(\theta), M(\theta)\} \\ 3, & \textit{if } p = m(\theta) \ \textit{ or } \ p = M(\theta) \\ 5, & \textit{if } (\theta, p) \in \mathcal{Q}_- \cap \mathcal{P} \\ 6, & \textit{if } (\theta, p) \in \mathcal{Q}_0 \\ 7, & \textit{if } (\theta, p) \in \mathcal{Q}_+ \end{cases} \tag{25}$$

Remark 3. *Note that the first part of the first line in* (25) *refers to the antiferromagnetic p-SOS model, which features only one TISGM for all $p > 0$.*

4 Tree-indexed Markov chains of TISGMs.

First note that a TISGM corresponding to a vector $v = (x, y) \in \mathbb{R}^2$ (which is a solution to the system (8), (9)) is a tree-indexed Markov chain with states $\{0, 1, 2\}$, see [4, Definition 12.2], and for the transition matrix

$$\mathbb{P} := \begin{pmatrix} \frac{x^2}{x^2+\theta y^2+\theta^{2^p}} & \frac{\theta y^2}{x^2+\theta y^2+\theta^{2^p}} & \frac{\theta^{2^p}}{x^2+\theta y^2+\theta^{2^p}} \\ \frac{\theta x^2}{\theta x^2+y^2+\theta} & \frac{y^2}{\theta x^2+y^2+\theta} & \frac{\theta}{\theta x^2+y^2+\theta} \\ \frac{\theta^{2^p} x^2}{\theta^{2^p} x^2+\theta y^2+1} & \frac{\theta y^2}{\theta^{2^p} x^2+\theta y^2+1} & \frac{1}{\theta^{2^p} x^2+\theta y^2+1} \end{pmatrix}. \tag{26}$$

For each given solution (x_i, y_i), $i = 1, \dots, 7$ of the system (8), (9), we need to calculate the eigenvalues of $\mathbb{P}$. The first eigenvalue is one since we deal with a stochastic matrix, the other two eigenvalues

$$\lambda_j(x_i, y_i, \theta, p), \qquad j = 1, 2, \tag{27}$$

can be found via computer, but they have bulky formulas. For example, in the case $x = 1$ for y we have up to three possible values, as mentioned in Lemma 1, and the matrix (26) has three eigenvalues, 1 and

$$\lambda_1(1, y, \theta, p) = \frac{(\theta^{2^p} - 2\theta^2 + 1)y^2}{\theta y^4 + (\theta^{2^p} + 2\theta^2 + 1)y^2 + 2\theta(\theta^{2^p} + 1)} \quad \text{and} \quad \lambda_2(1, y, \theta, p) = \frac{1 - \theta^{2^p}}{\theta^{2^p} + \theta y^2 + 1}.$$

4.1 Conditions for non-extremality

A sufficient condition for non-extremality of a Gibbs measure μ corresponding to the matrix $\mathbb{P}$ on a Cayley tree of order $k \geq 1$ is given by the Kesten–Stigum Condition $k\lambda_2^2 > 1$, where λ_2 is the second largest (in absolute value) eigenvalue of $\mathbb{P}$, see [5]. Using this criterion, in this section, we find the regions of the parameters θ and p where the TISGMs μ_i, $i = 1, 2, 3, 4, 5, 6, 7$ are not extreme in the set of all Gibbs measures. Let us denote

$$\lambda_{\max,i}(\theta, p) := \max\{|\lambda_1(x_i, y_i, \theta, p)|, |\lambda_2(x_i, y_i, \theta, p)|\}, \ \ i = 1, \dots, 7,$$

$$\eta_i(\theta, p) := 2\lambda^2_{\max,i}(\theta, p) - 1, \ \ i = 1, \dots, 7,$$

and

$$\mathbb{K}_i := \{(\theta, p) \in (0, 1) \times (0, +\infty) : \eta_i(\theta, p) > 0\}, \ \ i = 1, \dots, 7.$$

Then, the Kesten–Stigum Condition provides us with the following criterion.

Proposition 3. *If $(\theta, p) \in \mathbb{K}_i$ is such that μ_i exists then, μ_i is non-extremal.*

Let us illustrate this proposition for the measures μ_i, $i = 1, 2, 3$ for two values of p, namely $p = 0.1$ and $p = 10$. The precise choice of the values is personal taste.

4.1.1 Case: $p = 0.1$

We note that y_1 exists for any $\theta > 0$ (see Lemma 1). For the case y_1, via computer analysis, one can check that there is $\theta_1 \approx 0.32$ such that

$$\lambda_{\max,1}(\theta, 0.1) = \begin{cases} |\lambda_1(1, y_1, \theta, 0.1)|, & \text{if } \theta \in (0, \theta_1) \\ |\lambda_2(1, y_1, \theta, 0.1)|, & \text{if } \theta \geq \theta_1. \end{cases}$$

To see that the function $\eta_1(\theta, 0.1)$ is monotone increasing for $\theta > \tilde{\theta}_1 \approx 1523.4$ we draw the graph of the function $\eta_1(1/\theta, 0.1)$ for $\theta \in (0, 0.001)$ (see Fig. 6). This leads to the following conclusion.

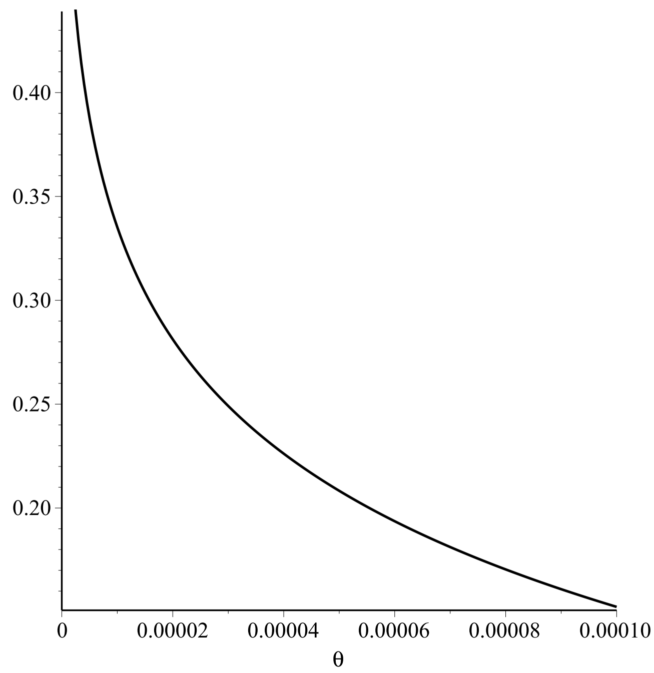

Figure 6: Graph of the function $\eta_1(1/\theta, 0.1)$, for $\theta \in (0, 0.001)$.

Result 1. *Considering Fig. 7 and 6 we conclude that the Kesten–Stigum Condition does not hold for* $(1, y_1(\theta, 0.1))$*, with* $\theta \in (0, \tilde{\theta}_1]$*, where* $\tilde{\theta}_1 \approx 1523.4$*. However, the condition holds for* $\theta > \tilde{\theta}_1$*.*

Next, from Fig. 4, for $p = 0.1$ we know that y_2 and y_3 exist only for $\theta < \theta_2 \approx 0.206$ and thus, using again computer algebra, one can see that

$$\lambda_{\max,i}(\theta, 0.1) = |\lambda_2(1, y_i, \theta, 0.1)|, \ \ \forall \theta \in (0, \theta_2), i = 2, 3.$$

Result 2. *Considering Fig. 8, we see that there are* $\hat{\theta}_2, \hat{\theta}_3 < \theta_2$ *(*$\hat{\theta}_2 \approx 0.175$*,* $\hat{\theta}_3 \approx 0.139$*) such that the Kesten–Stigum Condition holds for solutions* $(1, y_i(\theta, 0.1))$*, with* $\theta \in (0, \hat{\theta}_i)$*,* $i = 2, 3$*.*

4.1.2 Case: $p = 10$

For $p = 10$, in the case y_1, by computer analysis, one can check that

$$\lambda_{\max,1}(\theta, 10) = |\lambda_2(1, y_1, \theta, 10)|.$$

Result 3. *Fig. 9 shows that the Kesten–Stigum Condition never holds for the solution* $(1, y_1(\theta, 10))$*, with* $\theta \in (0, 1]$*. But it holds for* $\theta > 1$*.*

The fact that the threshold for $p = 10$ is precisely given by $\theta = 1$ is remarkable. We also note that in the antiferromagnetic case $\theta > 1$, there is a unique TISGM which is however not extremal in case $p = 10$. From Fig. 4 we know that y_2 and y_3 exist only for $\theta < \theta_2' \approx 0.136$ and hence, using computer algebra, one can see that

$$\lambda_{\max,i}(\theta, 10) = |\lambda_2(1, y_i, \theta, 10)|, \ \ \forall \theta \in (0, \theta_2'), i = 2, 3.$$

Result 4. *Considering Fig. 10, we see that the Kesten–Stigum Condition always holds for the solution* $(1, y_i(\theta, 10))$*, with* $\theta \in (0, \theta_2')$*,* $i = 2, 3$*.*

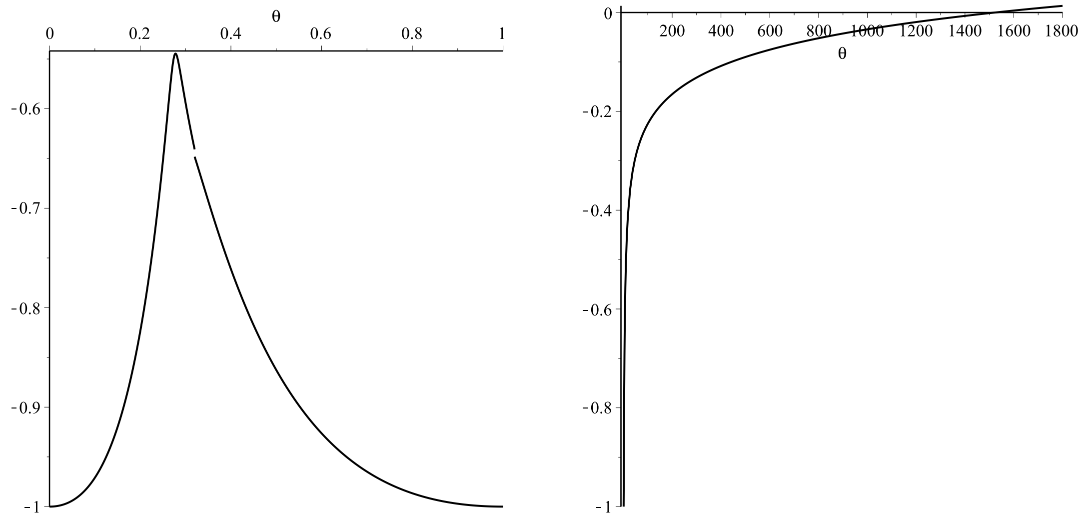

Figure 7: Graph of the function $\eta_1(\theta, 0.1)$, for $\theta \in (0,1)$ (left) and for $\theta > 1$ (right).

4.2 Conditions for extremality

In this section we find sufficient conditions for extremality (or non-reconstructability in information-theoretic language [8, 10, 11, 17]) of TISGMs for the 3-state p-SOS model, depending on parameters θ, p and the boundary law. We shall consider the TISGMs: $\mu_i, i = 1, \dots, 7$. In order to check extremality, we will use a result of [8] to establish a bound for reconstruction impossibility that corresponds to the matrix (26)) of a solution (x_i, y_i), $i = 1, \dots, 7$.

Let us start by recalling some definitions from [8]. Considering finite complete subtrees $\mathcal{T}$ that are *initial* with respect to the Cayley tree Γ^k, i.e., share the same root. If $\mathcal{T}$ has depth d, i.e., the vertices of $\mathcal{T}$ are within distance $\leq d$ from the root, then it has $(k^{d+1}-1)/(k-1)$ vertices, and its boundary $\partial\mathcal{T}$ consists the neighbors (in $\Gamma^k \setminus \mathcal{T}$) of its vertices, i.e., $|\partial\mathcal{T}| = k^{d+1}$. We identify subgraphs of $\mathcal{T}$ with their vertex sets and write $E(A)$ for the edges within a subset A and ∂A for the boundary of A, i.e., the neighbors of A in $(\mathcal{T} \cup \partial\mathcal{T}) \setminus A)$.

Consider Gibbs measures $\{\mu^{\tau}_{\mathcal{T}}\}$, where the boundary condition τ is fixed and $\mathcal{T}$ ranges over all initial finite complete subtrees of Γ^k. For a given subtree $\mathcal{T}$ of Γ^k and a vertex $x \in \mathcal{T}$, we write $\mathcal{T}_x$ for the (maximal) subtree of $\mathcal{T}$ rooted at x. When x is not the root of $\mathcal{T}$, let $\mu^s_{\mathcal{T}_x}$ denote the (finite-volume) Gibbs measure in which the parent of x has its spin fixed to s and the configuration on the bottom boundary of $\mathcal{T}_x$ (i.e., on $\partial\mathcal{T}_x \setminus \{\text{parent of } x\}$) is specified by τ.

For two measures μ_1 and μ_2 on Ω, $\|\mu_1 - \mu_2\|_x$ denotes the variational distance between the projections of μ_1 and μ_2 onto the spin at x, i.e.,

$$\|\mu_1 - \mu_2\|_x := \frac{1}{2}\sum_{i=0}^{2} |\mu_1(\sigma(x) = i) - \mu_2(\sigma(x) = i)|.$$

Let $\eta^{x,s}$ be the configuration η with the spin at x set to s. Following [8], we define

$$\kappa := \kappa(\mu) = \sup_{x\in\Gamma^k} \max_{x,s,s'} \|\mu^s_{\mathcal{T}_x} - \mu^{s'}_{\mathcal{T}_x}\|_x \quad \text{and} \quad \gamma := \gamma(\mu) = \sup_{A\subset\Gamma^k} \max \|\mu^{\eta^{y,s}}_A - \mu^{\eta^{y,s'}}_A\|_x,$$

where the maximum on the right-hand side is taken over all boundary conditions η, all sites $y \in \partial A$, all neighbors $x \in A$ of y, and all spins $s, s' \in \{0, 1, 2\}$. We apply [8, Theorem 9.3] which goes as follows.

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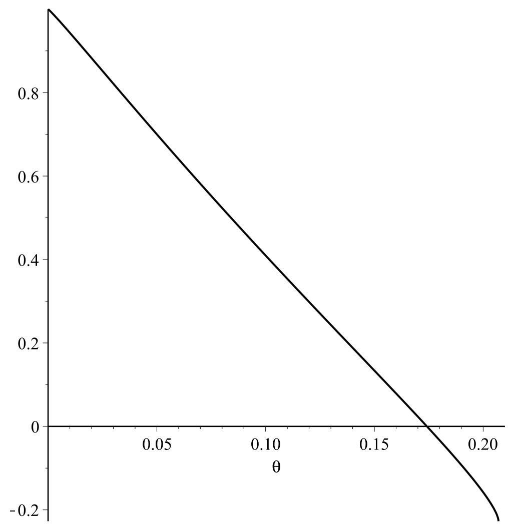

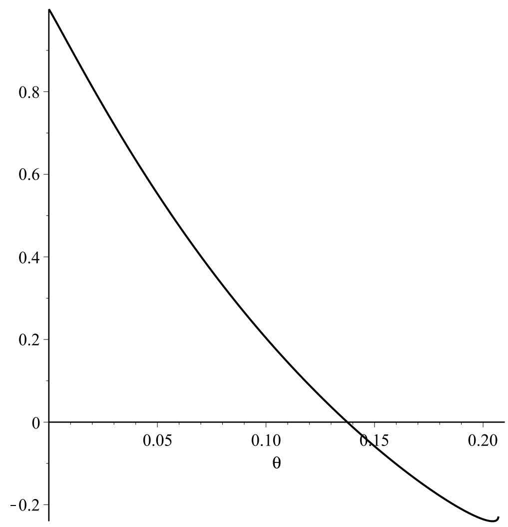

Figure 8: Graphs of the functions $\eta_2(\theta, 0.1)$ (left) and $\eta_3(\theta, 0.1)$ (right), $\theta \in (0, \theta_2)$.

Theorem 2. *For an arbitrary (ergodic[1] and permissive[2]) channel $\mathbb{P} = (P_{ij})_{i,j=1}^{q}$ on a tree, the reconstruction of the corresponding tree-indexed Markov chain is impossible if $k\kappa\gamma < 1$.*

Since each TISGM μ corresponds to a solution (x, y) of the system of equations (8) and (9), we can write $\gamma(\mu) = \gamma(x, y)$ and $\kappa(\mu) = \kappa(x, y)$.

It is easy to see that the channel $\mathbb{P}$ corresponding to a TISGM of the p-SOS model is ergodic and permissive. Thus the criterion of *extremality* of a TISGM is $k\kappa\gamma < 1$. Note that κ has the particularly simple form (see [8])

$$\kappa = \frac{1}{2} \max_{i,j} \sum_{l} |P_{il} - P_{jl}| \tag{28}$$

and γ is a constant that does not have a clean general formula, but it can be estimated.

4.2.1 Estimation of γ.

To estimate the constant $\gamma(x_i, y_i)$ depending on the boundary law labeled by i, for our model, we prove several lemmas.

Lemma 2. *Recall the matrix $\mathbb{P}$ given by (26) and denote by $\mu = \mu(\theta, p)$ the corresponding Gibbs measure. Then, for any subset $A \subset \mathcal{T}$, (where $\mathcal{T}$ is a complete subtree of Γ^k) any boundary configuration η, any pair of spins (s_1, s_2), any site $y \in \partial A$, and any neighbor $x \in A$ of y, we have*

$$\|\mu_A^{\eta^{y,s_1}} - \mu_A^{\eta^{y,s_2}}\|_x \leq \max\{p^0(0) - p^1(0), p^0(0) - p^2(0), |p^i(1) - p^j(1)|, p^2(2) - p^0(2), p^2(2) - p^1(2)\},$$

where $p^t(s) := \mu_A^{\eta^{y,t}}(\sigma(x) = s)$.

[1]Ergodicity here means irreduciblity and aperiodicity. In this case, we have a unique stationary distribution $\pi = (\pi_1, \ldots, \pi_q)$ with $\pi_i > 0$ for all i.

[2]Permissive here means that, for arbitrary finite A and boundary condition outside A being η, the conditioned Gibbs measure on A, corresponding to the channel, is positive for at least one configuration.

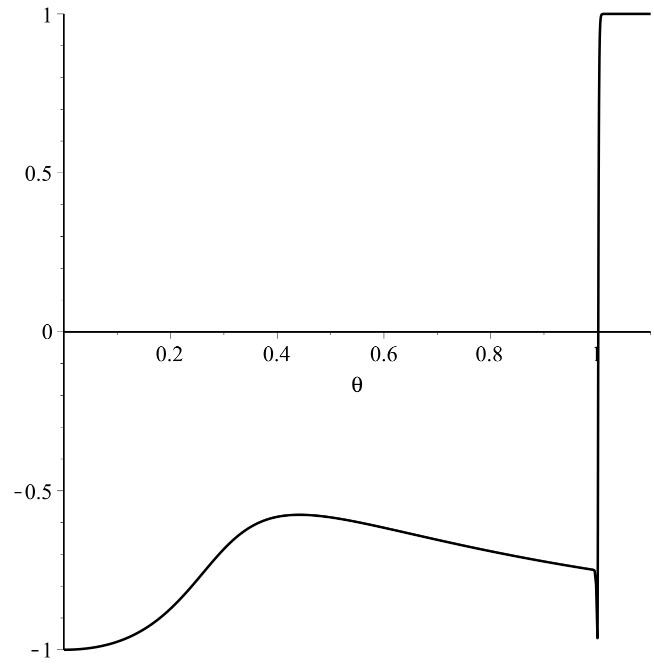

Figure 9: Graph of the function $\eta_1(\theta, 10)$, $\theta \in (0, 1)$.

Proof. Denote $p_s = \mu_A^{\eta^{y,\text{free}}}(\sigma(x) = s)$, $s = 0, 1, 2$. By definition of the matrix $\mathbb{P}$, we have

$$p^0(0) = \frac{x^2 p_0}{x^2 p_0 + \theta y^2 p_1 + \theta^{2^p} p_2}, \ p^0(1) = \frac{\theta y^2 p_1}{x^2 p_0 + \theta y^2 p_1 + \theta^{2^p} p_2}, \ p^0(2) = \frac{\theta^{2^p} p_2}{x^2 p_0 + \theta y^2 p_1 + \theta^{2^p} p_2};$$

$$p^1(0) = \frac{\theta x^2 p_0}{\theta x^2 p_0 + y^2 p_1 + \theta p_2}, \ p^1(1) = \frac{y^2 p_1}{\theta x^2 p_0 + y^2 p_1 + \theta p_2}, \ p^1(2) = \frac{\theta p_2}{\theta x^2 p_0 + y^2 p_1 + \theta p_2}; \tag{29}$$

$$p^2(0) = \frac{\theta^{2^p} x^2 p_0}{\theta^{2^p} x^2 p_0 + \theta y^2 p_1 + p_2}, \ p^2(1) = \frac{\theta y^2 p_1}{\theta^{2^p} x^2 p_0 + \theta y^2 p_1 + p_2}, \ p^2(2) = \frac{p_2}{\theta^{2^p} x^2 p_0 + \theta y^2 p_1 + p_2},$$

and hence, the proposition follows from the following Lemma 3 and Lemma 4. □

Lemma 3. *If $\theta < 1$ then*

a) $p^0(0) \geq \max\{p^1(0), p^2(0)\}$*;*

b) If $p \leq 1$, then $p^1(1) \geq \max\{p^0(1), p^2(1)\}$. If $p > 1$, then, the values of $p^0(1)$, $p^1(1)$ and $p^2(1)$ may have any order depending on (p_0, p_2).

c) $p^2(2) \geq \max\{p^0(2), p^1(2)\}$*.*

Proof. We shall prove some of the inequalities (all others are proved similarly):

a) By the Formula (29), we get

$$p^0(0) - p^1(0) = x^2 p_0 \frac{(1-\theta^2) y^2 p_1 + (1-\theta^{2^p}) \theta p_2}{(x^2 p_0 + \theta y^2 p_1 + \theta^{2^p} p_2)(\theta x^2 p_0 + y^2 p_1 + \theta p_2)},$$

$$p^0(0) - p^2(0) = x^2 p_0 \frac{\theta(1-\theta^{2^p}) y^2 p_1 + (1-\theta^{2^{p+1}} p_2)}{(x^2 p_0 + \theta y^2 p_1 + \theta^{2^p} p_2)\,(\theta^{2^p} x^2 p_0 + \theta y^2 p_1 + p_2)},$$

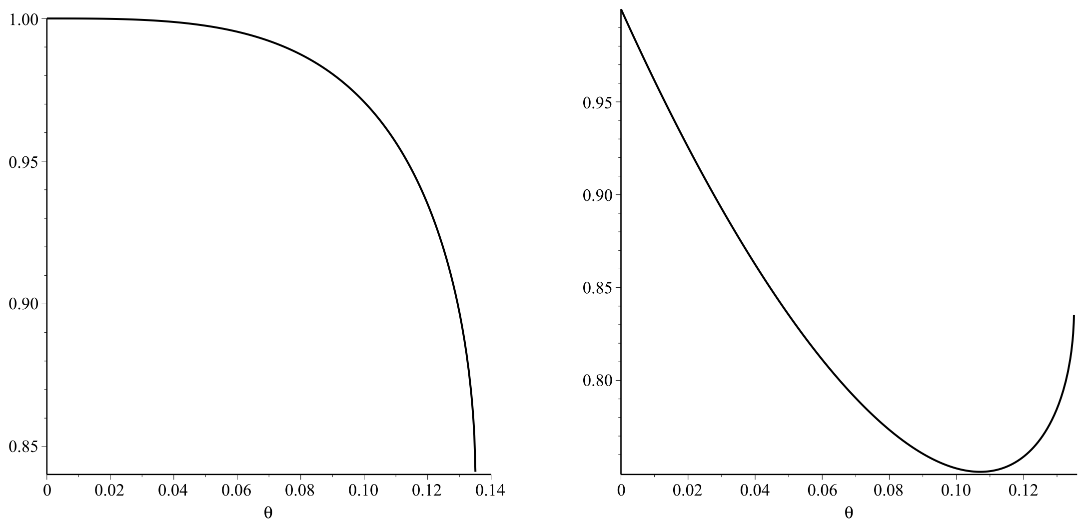

Figure 10: The graphs of the functions $\eta_2(\theta, 10)$ (left) and $\eta_3(\theta, 10)$ (right), $\theta \in (0, \theta_2')$.

and both are positive iff $\theta < 1$.

b) Consider

$$p^1(1) - p^0(1) = y^2 p_1 \frac{(1-\theta^2)p_0x^2 + (\theta^{2^p} - \theta^2)p_2}{(\theta x^2 p_0 + y^2 p_1 + \theta p_2)\,(x^2 p_0 + \theta y^2 p_1 + \theta^{2^p} p_2)},$$
$$p^1(1) - p^2(1) = y^2 p_1 \frac{(\theta^{2^p} - \theta^2)p_0x^2 + (1-\theta^2)p_2}{(\theta x^2 p_0 + y^2 p_1 + \theta p_2)\,(\theta^{2^p} x^2 p_0 + \theta y^2 p_1 + p_2)},$$

which are non-negative if $\theta < 1$ and $p \le 1$. Moreover, if $p > 1$, then, for $\theta < 1$, we have $\theta^{2^p} - \theta^2 < 0$, then assuming $p^1(1) - p^2(1) < 0$, that is $(\theta^{2^p} - \theta^2)p_0x^2 + (1-\theta^2)p_2 < 0$, we get the condition $p_0 > (1-\theta^2)p_2/((\theta^2 - \theta^{2^p})x^2)$. It is easy to see that there are values $p > 1$ and $\theta < 1$ such that all possible inequalities may hold.

c) Similar to the case a). □

Next, if $\theta < 1$ then we have that

$$\max_{i,j,k} \left\{|p^i(k) - p^j(k)|\right\}$$
$$= \max_{i,j}\{p^0(0) - p^1(0), p^0(0) - p^2(0), |p^i(1) - p^j(1)|, p^2(2) - p^0(2), p^2(2) - p^1(2)\}.$$

Indeed, the case $k = 0$ and $k = 2$ follow from Lemma 3. For $k = 1$ some differences can be reduced to the case $k = 0$ or $k = 2$, by the following equality

$$p^i(1) - p^j(1) = p^j(0) - p^i(0) + p^j(2) - p^i(2).$$

Let us give an upper bound of $|p^i(k) - p^j(k)|$, for the maximal ones mentioned above. For (p_0, p_1, p_2) (i.e., a probability distribution on $\{0, 1, 2\}$) denote $t = p_0$, $u = p_2$, $0 \le t + u \le 1$ and define the following functions

$$\begin{aligned}
f(t,u,\theta,p) &= p^0(0) - p^2(0)\\
&= \frac{x^2 t}{(x^2-\theta y^2)t + \theta(\theta^{2^p-1} - y^2)u + \theta y^2} - \frac{x^2\theta^{2^p} t}{\theta(\theta^{2^p-1}x^2 - y^2)t + (1-\theta y^2)u + \theta y^2}\\
\varphi(t,u,\theta,p) &= p^0(0) - p^1(0)\\
&= \frac{x^2 t}{(x^2-\theta y^2)t + (\theta^{2^p} - \theta y^2)u + \theta y^2} - \frac{\theta x^2 t}{(\theta x^2 - y^2)t + (\theta - y^2)u + y^2},\\
\psi(t,u,\theta,p) &= p^1(1) - p^0(1)\\
&= \frac{y^2 u}{\theta(x^2-1)t + (y^2-\theta)u + \theta} - \frac{\theta y^2 u}{(x^2-\theta^{2^p})t + (\theta y^2 - \theta^{2^p})u + \theta^{2^p}},\\
g(t,u,\theta,p) &= p^2(2) - p^0(2)\\
&= \frac{u}{\theta(\theta^{2^p-1}x^2 - y^2)t + (1-\theta y^2)u + \theta y^2} - \frac{\theta^{2^p} u}{(x^2-\theta y^2)t + \theta(\theta^{2^p-1} - y^2)u + \theta y^2}.
\end{aligned}$$

Lemma 4. *If $\theta < 1$ then*

$$\max\{|f(t,u,\theta,p)|, |\varphi(t,u,\theta,p)|, |\psi(t,u,\theta,p)|, |g(t,u,\theta,p)|\} \le \frac{1-\theta^{2^p}}{1+\theta^{2^p}}.$$

Proof. We present our calculation only for the function f, the other functions are checked similarly. To find the maximal value of the function f we have to solve the following system

$$\begin{aligned}
f'_u(t,u,\theta,p) = &\frac{\theta x^2 t(y^2 - \theta^{2^p-1})}{((x^2-\theta y^2)t + \theta(\theta^{2^p-1} - y^2)u + \theta y^2)^2}\\
&+ \frac{\theta^{2^p} x^2 t(1-\theta y^2)}{(\theta(\theta^{2^p-1}x^2 - y^2)t + (1-\theta y^2)u + \theta y^2)^2} = 0,
\end{aligned} \tag{30}$$

$$\begin{aligned}
f'_t(t,u,\theta,p) = &\frac{\theta x^2 u(\theta^{2^p-1} - y^2) + \theta x^2 y^2}{((x^2-\theta y^2)t + \theta(\theta^{2^p-1} - y^2)u + \theta y^2)^2}\\
&- \frac{\theta^{2^p} x^2 u(1-\theta y^2) + \theta^{2^p+1} x^2 y^2}{(\theta(\theta^{2^p-1}x^2 - y^2)t + (1-\theta y^2)u + \theta y^2)^2} = 0.
\end{aligned} \tag{31}$$

From (30) one has either $t = 0$, or if $t \neq 0$ we note that if $y^2 = 1/\theta$ then $y^2 = \theta^{2^p-1}$, i.e., $\theta = 1$. So we can assume $y^2 \neq 1/\theta$. Then from (30) (for $t \neq 0$) and (31) we get

$$\frac{\theta^{2^p-1} - y^2}{1-\theta y^2} = \frac{(\theta^{2^p-1} - y^2)u + y^2}{(1-\theta y^2)u + \theta y^2},$$

which is possible only iff $\theta = 1$. So it remains to check only the case $t = 0$, which gives a minimum $(= 0)$ of the function f. Hence the maximal value of f is reached on the boundary of the set $\{(t,u) \in [0,1]^2 : t + u \le 1\}$. We note that similar results hold for the function φ too. We discuss the three line segments of the boundary separately:

Case: $t = 0$. In this case it was already mentioned above that the function has a minimum which is equal to zero.

Case: $u = 0$. In this case simple calculations show that

$$\max f(t,0,\theta,p) = f\left(\frac{y^2}{\theta^{2^{p-1}-1}x^2+y^2},0,\theta,p\right) = \frac{1-\theta^{2^{p-1}}}{1+\theta^{2^{p-1}}} \text{ and}$$
$$\max \varphi(t,0,\theta,p) = \varphi\left(\frac{y^2}{x^2+y^2},0,\theta,p\right) = \frac{1-\theta}{1+\theta}.$$

Case: $t + u = 1$. In this case we have

$$\max f(t,1-t,\theta,p) = f\left(\frac{1}{1+x^2},\frac{x^2}{1+x^2},\theta,p\right) = \frac{1-\theta^{2^p}}{1+\theta^{2^p}} \text{ and}$$
$$\max \varphi(t,1-t,\theta,p) = \varphi\left(\frac{\theta^{2^{p-1}}}{\theta^{2^{p-1}}+x^2},\frac{x^2}{\theta^{2^{p-1}}+x^2},\theta,p\right) = \frac{1-\theta^{2^{p-1}}}{1+\theta^{2^{p-1}}}.$$

Similarly for ψ one can show that

$$|\psi(t,u,\theta,p)| \leq \max\left\{\frac{|1-\theta|}{1+\theta},\frac{\left|1-\theta^{2^{p-1}-1}\right|}{1+\theta^{2^{p-1}-1}}\right\}.$$

Next, for $\theta < 1$ and $t > -1$ consider the following function

$$\Theta(t) = \frac{1-\theta^t}{1+\theta^t}.$$

It is easy to check that this function is monotone increasing and therefore we have

$$\max\left\{\frac{1-\theta}{1+\theta},\frac{\left|1-\theta^{2^{p-1}-1}\right|}{1+\theta^{2^{p-1}-1}},\frac{1-\theta^{2^{p-1}}}{1+\theta^{2^{p-1}}}\right\} \leq \frac{1-\theta^{2^p}}{1+\theta^{2^p}}.$$

This completes the proof for f. For g the proof is very similar. □

In the following proposition we now present our bound on γ.

Proposition 4. *Independent of the possible values of (x, y) (i.e., the solutions to the system* (8) *and* (9)*) for $\theta < 1, p > 0$ we have*

$$\gamma(x,y) \leq \frac{1-\theta^{2^p}}{1+\theta^{2^p}}. \tag{32}$$

Proof. This is a corollary of the above-mentioned lemmas. □

4.2.2 Computation of κ.

Now we shall compute the constant κ. Since (x, y) is a solution to the system (8), (9), the matrix (26) can be written in the following form

$$\mathbb{P} = \frac{1}{Z}\begin{pmatrix} x & \theta y^2/x & \theta^{2^p}/x \\ \theta x^2/y & y & \theta/y \\ \theta^{2^p}x^2 & \theta y^2 & 1 \end{pmatrix}, \tag{33}$$

where $Z=\theta^{2^p}x^2+\theta y^2+1$. Using (28) and (33), we get

$$\kappa(x,y)=\frac{1}{2}\max_{i,j}\sum_{l=0}^{2}|P_{il}-P_{jl}|=\frac{1}{2Z}$$
$$\max\left\{\tfrac{x^2|y-\theta x|+y^2|x-\theta y|+|\theta^{2^p}y-\theta x|}{xy},\ \tfrac{x^2|1-\theta^{2^p}x|+\theta y^2|1-x|+|\theta^{2^p}-x|}{x},\ \tfrac{x^2|\theta-\theta^{2^p}y|+y^2|1-\theta y|+|\theta-y|}{y}\right\}, \quad (34)$$

where $Z=\theta^{2^p}x^2+\theta y^2+1$. We are interested in computing $\kappa(x,y)$ for

$$(x,y)\in\{(1,y_1),(1,y_2),(1,y_3),(x_4,y_4),(x_5,y_5),(x_6,y_6),(x_7,y_7)\}.$$

Since we have an explicit formula for the solutions (x,y) (mentioned in Section 3), the value of $\kappa(x,y)$ will be a function of the parameters θ,p. Unfortunately, the explicit formulas for the solutions are very bulky, so we start with $x=1$.

Case $x=1$. In this case we have

$$\kappa(1,y)=\frac{1}{2Z_1}\max\left\{\frac{|y-\theta|+y^2|1-\theta y|+|\theta^{2^p}y-\theta|}{y},2|1-\theta^{2^p}|\right\},$$

where $Z_1=\theta^{2^p}+\theta y^2+1$ and y is a solution to (11).

Subcase: $p=0.1$ and for solution y_1: For the solution y_1 of (11) from (34), by computer analysis, one can see that there exists $\hat{\theta}_1\approx 0.335$ such that

$$\kappa(1,y_1)=\frac{1}{2Z_1}\begin{cases}\frac{|y_1-\theta|+y_1^2|1-\theta y_1|+|\theta^{2^p}y_1-\theta|}{y_1}, & \text{if } \theta\in(0,\hat{\theta}_1)\\ 2|1-\theta^{2^p}|, & \text{if } \theta\geq\hat{\theta}_1.\end{cases} \quad (35)$$

Denote

$$U_1(\theta,p)=2\frac{1-\theta^{2^p}}{1+\theta^{2^p}}\kappa(1,y_1)-1.$$

Result 5. *Fig. 11 shows that the extremality condition holds for the solution $(1,y_1(\theta,0.1))$, with $\theta\in(0,\theta_1^*)$, where $\theta_1^*\approx 19.08$.*

Subcase: $p=0.1$ and for solutions y_i, $i=2,3$: For solutions y_2 and y_3 of (11) from (34), by computer analysis, we get

$$\kappa(1,y_i)=\frac{|1-\theta^{2^{0.1}}|}{Z_1},\quad i=2,3. \quad (36)$$

Denote

$$U_i(\theta,p)=2\frac{1-\theta^{2^p}}{1+\theta^{2^p}}\kappa(1,y_i)-1.$$

Result 6. *Fig. 12 shows that the extremality condition for solutions $(1,y_i(\theta,0.1))$, $i=2,3$, where they exist (i.e., when $\theta\in(0,\theta_2^*)$), holds for:*

- y_2 if $\theta\in(\bar{\theta}_2,\theta_2^*)$, where $\bar{\theta}_2\approx 0.1817$ and

- y_3 if $\theta\in(\bar{\theta}_3,\theta_2^*)$, where $\bar{\theta}_3\approx 0.1625$.

Subcase: $p=10$. Consider the case $y_1(\theta,10)$, then our computer analysis shows the following:

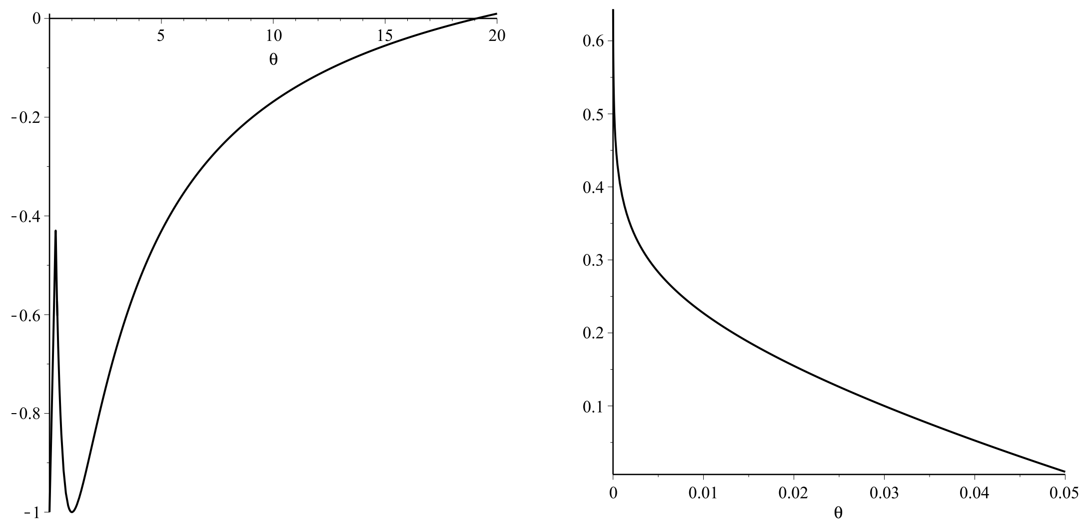

Figure 11: Graphs of the functions $U_1(\theta, 0.1)$ for $\theta \in (0, 20)$ (left) and $U_1(1/\theta, 0.1)$ for $\theta \in (0, 0.05)$ (right).

Result 7. *For the solution $y_1(\theta, 10)$ the extremality condition is satisfied (see Fig. 13) if $\theta \in (0, 1)$ and does not hold if $\theta > 1$.*

Remark 4. *In [6], it was demonstrated that for $p = 1$, the measure $\mu_1(\theta, 1)$ associated with $y_1(\theta, 1)$ is extreme when $\theta <\approx 2.655$ and non-extreme if $\theta >\approx 2.87$. For $p = 0.1$, we have established that $\mu_1(\theta, 0.1)$ is extreme when $\theta <\approx 19$ and non-extreme when $\theta >\approx 1523$. However, the unexpected finding emerged in the case of $p = 10$, where the critical value distinguishing between extremality and non-extremality is exactly 1. This critical value aligns with the boundary between the ferromagnetic and anti-ferromagnetic cases.*

Result 8. *For the solutions $y_i(\theta, 10)$, $i = 2, 3$ Fig. 14 shows that the extremality condition is never satisfied.*

Recall $\mu_i = \mu_i(\theta, p)$ is the SGM corresponding to solution v_i. Let us summarize Results 1–8 in the following theorem.

Theorem 3. *The following is a fact.*

1. *If $p = 0.1$ then*
   - 1.a) *There are values $\theta_1^* \ (\approx 19.08)$ and $\tilde{\theta}_1 \ (\approx 1523.4)$ such that the measure μ_1 is extreme if $0 < \theta < \theta_1^*$ and is non-extreme if $\theta > \tilde{\theta}_1$.*
   - 1.b) *There are values $\hat{\theta}_2 \ (\approx 0.175)$ and $\bar{\theta}_2 \ (\approx 0.1817)$ such that the measure[3] μ_2 is non-extreme if $0 < \theta < \hat{\theta}_2$ and is extreme if $\theta \in (\bar{\theta}_2, \theta_2^*)$.*
   - 1.c) *There are values $\hat{\theta}_3 \ (\approx 0.139)$ and $\bar{\theta}_3 \ (\approx 0.1625)$ such that the measure μ_3 is non-extreme if $0 < \theta < \hat{\theta}_3$ and is extreme if $\theta \in (\bar{\theta}_3, \theta_2^*)$.*
2. *If $p = 10$ then*

[3]Note that μ_2 and μ_3 exist for $\theta \in (0, \theta_2^*)$, where $\theta_2^* \approx 0.206$.

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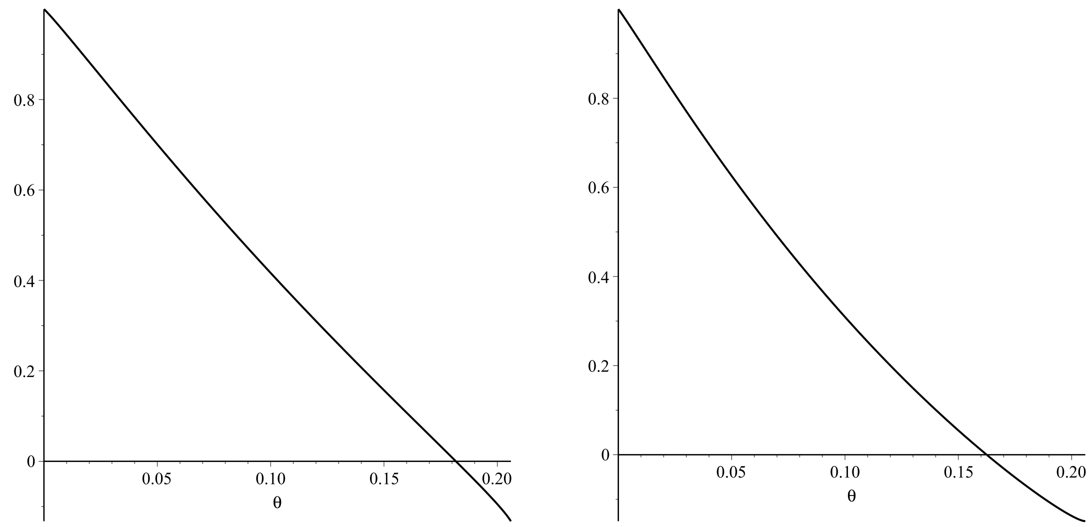

Figure 12: Graphs of the functions $U_2(\theta, 0.1)$ (left) and $U_3(\theta, 0.1)$ (right) for $\theta \in (0, \theta_2^*)$.

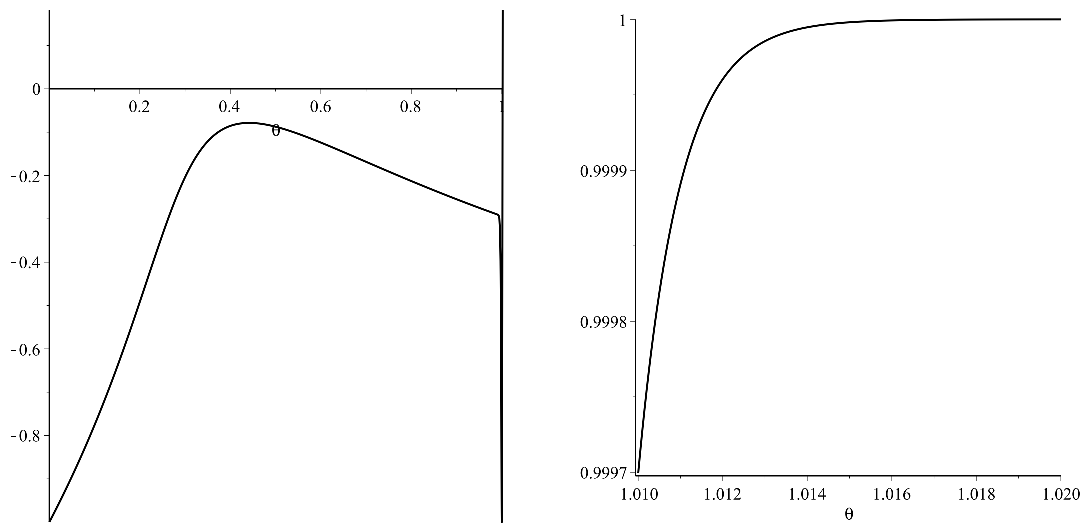

Figure 13: Graph of the function $U_1(\theta, 10)$ for $\theta \in (0, 1)$ (left) and $\theta > 1$ (right).

2.a) The measure μ_1 is extreme if $0 < \theta \leq 1$ and is non-extreme if $\theta > 1$.

2.b) The measures μ_2 and μ_3 are non-extreme (where they exist).

Let us finally also present a criterion for extremality for the remaining solutions.

Case $x \neq 1$. Now we compute κ for (x_i, y_i), $i = 4, 5, 6, 7$. Recall that all of them exist only for $\theta < 1$,

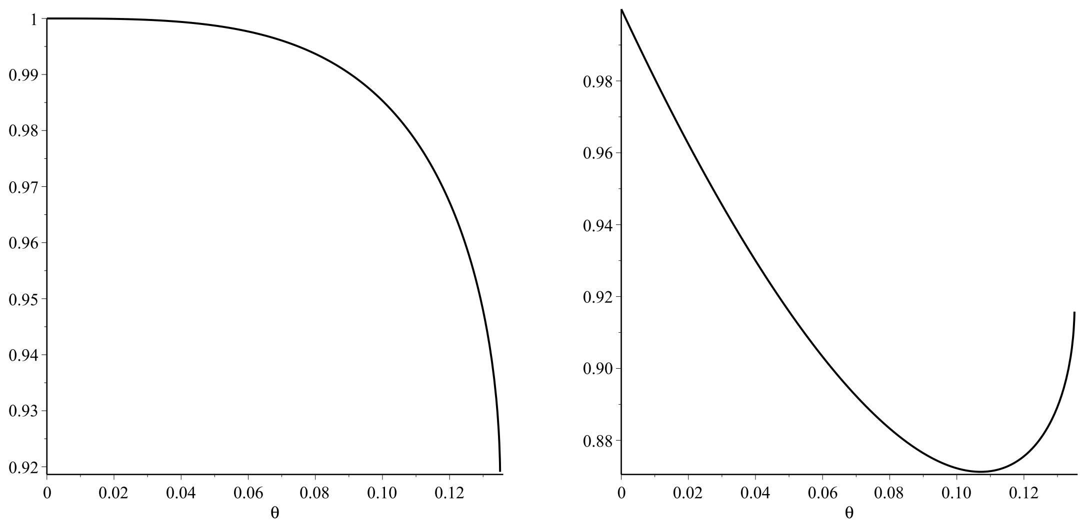

Figure 14: Graphs of the functions $U_2(\theta, 10)$ (left) and $U_3(\theta, 10)$ (right) for $\theta \in (0, \theta_2')$.

therefore, from the system (8), (9) we get the following inequalities

$$
\begin{aligned}
y - \theta &= \frac{(1-\theta^2)y^2 + \theta(1-\theta^{2^p})}{Z} > 0,\\
1 - \theta^{2^p} x &= \frac{(1-\theta^{2^p})\theta y^2 + (1-\theta^{2^{p+1}})}{Z} > 0,\\
x - \theta^{2^p} &= \frac{(1-\theta^{2^p})((\theta^{2^p}+1)x^2 + \theta y^2)}{Z} > 0,\\
y - \theta &= \frac{(1-\theta^{2^p})\theta x^2 + (1-\theta^2)y^2)}{Z} > 0.
\end{aligned}
$$

These inequalities are useful in order to omit the absolute value of the corresponding difference, but still the form of $\kappa(x, y)$ remains bulky. Recall that, in order to check the extremality of a given TISGM μ_i we need to verify that $2\kappa\gamma < 1$. Using the above mentioned bound of γ and Formula (34), it suffices to check

$$
2\kappa(x_i, y_i)\gamma(x_i, y_i) \leq 2\frac{1-\theta^{2^p}}{1+\theta^{2^p}}\kappa(x_i, y_i) < 1.
$$

Denote

$$
U_i(\theta, p) = 2\frac{1-\theta^{2^p}}{1+\theta^{2^p}}\kappa(x_i, y_i) - 1 \qquad \text{and} \qquad \mathbb{E}_i = \{(\theta, p) : U_i(\theta, p) < 0\}.
$$

Thus we obtained the following proposition.

Proposition 5. *If $(\theta, p) \in \mathbb{E}_i$ is such that μ_i exists then μ_i is extreme.*

Remark 5. *In Propositions 3 and 5, we were unable to explicitly provide the regions of (θ, p) for (non-)extremality due to the complex nature of the solutions (x_i, y_i). However, our results may still provide the groundwork for future numerical studies of these regions. In this manuscript we prototyped this analysis for the cases where3 $p = 0.1$, $p = 10$ and $x = 1$.*

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