**Weierstraß-Institut**
**für Angewandte Analysis und Stochastik**
**Leibniz-Institut im Forschungsverbund Berlin e. V.**

Preprint ISSN 2198-5855

# Curvature effects in pattern formation: Well-posedness and optimal control of a sixth-order Cahn–Hilliard equation

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submitted: January 15, 2024

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No. 3085
Berlin 2024

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2020 *Mathematics Subject Classification.* 35K55, 35K51, 49J20, 49K20, 49J50.

*Key words and phrases.* Sixth-order Cahn–Hilliard equation, functionalized Cahn–Hilliard equation, Willmore regularization, curvature effects, well-posedness, optimal control, first-order necessary optimality conditions.

This research was supported by the Italian Ministry of Education, University and Research (MIUR): Dipartimenti di Eccellenza Program (2018–2022) – Dept. of Mathematics "F. Casorati", University of Pavia. In addition, PC and AS gratefully acknowledge some other support from the MIUR-PRIN Grant 2020F3NCPX "Mathematics for industry 4.0 (Math4I4)" and their affiliation to the GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica). AS has been supported by "MUR GRANT Dipartimento di Eccellenza" 2023-2027.

Edited by
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# Curvature effects in pattern formation: Well-posedness and optimal control of a sixth-order Cahn–Hilliard equation

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**Abstract**

This work investigates the well-posedness and optimal control of a sixth-order Cahn–Hilliard equation, a higher-order variant of the celebrated and well-established Cahn–Hilliard equation. The equation is endowed with a source term, where the control variable enters as a distributed mass regulator. The inclusion of additional spatial derivatives in the sixth-order formulation enables the model to capture curvature effects, leading to a more accurate depiction of isothermal phase separation dynamics in complex materials systems. We provide a well-posedness result for the aforementioned system when the corresponding nonlinearity of double-well shape is regular and then analyze a corresponding optimal control problem. For the latter, existence of optimal controls is established, and the first-order necessary optimality conditions are characterized via a suitable variational inequality. These results aim at contributing to improve the understanding of the mathematical properties and control aspects of the sixth-order Cahn–Hilliard equation, offering potential applications in the design and optimization of materials with tailored microstructures and properties.

## 1 Introduction

The sixth-order Cahn–Hilliard equation represents an extension of the classical Cahn-Hilliard equation that accounts for curvature effects and higher-order variations of the order parameter. Besides, it provides a more accurate description of complex materials systems with intricate interface structures. By employing a rigorous analysis, our aim is to contribute to the comprehension of the evolution of complex materials, thereby paving the way for advanced applications in materials science.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded and smooth domain and let $T > 0$. The initial-boundary value problem under investigation reads as follows:

$$\partial_t\varphi - \operatorname{div}(m(\varphi)\nabla\mu) = S(\varphi,u) := u - \sigma\varphi \qquad \text{in } Q, \tag{1.1}$$

$$\mu = -\varepsilon\Delta w + \tfrac{1}{\varepsilon} f'(\varphi)w + \nu w \qquad \text{in } Q, \tag{1.2}$$

$$w = -\Delta\varphi + f(\varphi) \qquad \text{in } Q, \tag{1.3}$$

$$\partial_{\boldsymbol{n}}\varphi = m(\varphi)\nabla\mu\cdot\boldsymbol{n} = \partial_{\boldsymbol{n}} w = 0 \qquad \text{on } \Sigma, \tag{1.4}$$

$$\varphi(0) = \varphi_0 \qquad \text{in } \Omega. \tag{1.5}$$

Here, the parabolic cylinder $Q$ and its boundary $\Sigma$ are defined by

$$Q := \Omega \times (0,T) \quad \text{and} \quad \Sigma := \Gamma \times (0,T), \tag{1.6}$$

where $\Gamma := \partial\Omega$ denotes the boundary of $\Omega$ that is associated with a unit normal vector field $\boldsymbol{n}$ and the outward normal derivative $\partial_{\boldsymbol{n}}$. The above system (1.1)–(1.5) can be viewed as a variant of the

DOI 10.20347/WIAS.PREPRINT.3085 

classical fourth-order Cahn–Hilliard equation [2] in the unknowns $\varphi$, $\mu$, and $w$. The original Cahn–Hilliard equation is widely used to describe isothermal separation processes in binary mixtures. The variable $\varphi$ represents the local proportion of one of the two components in the binary material and serves as an order parameter. To simplify the analysis, it is usually normalized in such a way that the pure states correspond to $\varphi = \pm 1$, while $\{-1 < \varphi < 1\}$ represents the diffuse interface that occurs in a $\varepsilon$-tubular neighborhood of the interface, with thickness parameter $\varepsilon > 0$. The variable $\mu$ in equation (1.2) is referred to as the *chemical potential* and corresponds to the first variation of the free energy $\mathcal{E}$. Similarly, $w$ is the first variational derivative of the Ginzburg–Landau free energy. Namely, it holds that $\mu = \frac{\delta \mathcal{E}}{\delta \varphi}$ and $w = \frac{\delta \mathcal{G}}{\delta \varphi}$, where

$$\mathcal{E}(\varphi) := \mathcal{F}(\varphi) + \nu \mathcal{G}(\varphi) = \frac{1}{2} \int_\Omega (-\Delta \varphi + f(\varphi))^2 + \nu \int_\Omega \Big( \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} F(\varphi) \Big), \tag{1.7}$$

with natural definition of $\mathcal{F}$ and $\mathcal{G}$. It is worth pointing out that $\mathcal{E}$ is a higher-order extension of the Ginzburg–Landau free energy $\mathcal{G}$ which is associated with the classical Cahn–Hilliard equation. In (1.7), $F$ indicates a double-well shaped nonlinear potential, and $f$ indicates its derivative. A prototype of $F$ is the *classical regular potential* given by

$$F(s) := \frac{1}{4}(s^2 - 1)^2 \quad \text{for every } s \in \mathbb{R}. \tag{1.8}$$

Next, the function $u$ appearing in (1.1) is a prescribed distributed function that will play the role of the control in the second part of our investigation. Finally, $\sigma$ and $\nu$ are real constants, with $\sigma$ positive, and $\varphi_0$ is a prescribed initial datum.

From (1.1), we realize that the mass flux is assumed to be proportional to the gradient of the chemical potential $\mu$ through the mobility function $m$. This relationship leads to the variational structure:

$$\partial_t \varphi = \operatorname{div}(m(\varphi) \nabla \mu) + S(\varphi, u), \quad \text{with} \quad \mu = \frac{\delta \mathcal{E}}{\delta \varphi}.$$

As a consequence of the no-flux boundary condition in (1.4), integration of the equation over $\Omega$ leads to the ODE relation

$$\frac{d}{dt} \Big( \frac{1}{|\Omega|} \int_\Omega \varphi(t) \Big) = \frac{1}{|\Omega|} \int_\Omega S(\varphi(t), u(t)) \quad \text{for all } t \in [0, T],$$

which shows that the mass dynamics of the order parameter $\varphi$ is ruled by the source term $S(\varphi, u)$. This represents a novelty in comparison with previous works on the system, where $S \equiv 0$ which results in mass conservation (see, e.g., [16, 17, 25]).

In the Cahn–Hilliard context, the free energy reduces to $\mathcal{G}$ with $\nu = 1$. On the other hand, considering higher-order terms like the ones in $\mathcal{F}$, it also can make sense to allow $\nu$ to be negative. It turns out that the value and sign of $\nu$ in (1.7) are essential considerations in both modeling and practical applications involving deformations of elastic vesicles under volume and surface constraints. Specifically, when $\nu = 0$, the energy function $\mathcal{E}$ simplifies to the well-established *Willmore functional* within the phase-field formulation [4, 5]. This simplification effectively captures the Canham–Helfrich bending energy of surfaces, as demonstrated in, e.g., [8, 9]. Besides, when $\nu > 0$, the energy function $\mathcal{E}$ is associated with the Willmore regularization of the Ginzburg–Landau energy $\mathcal{G}$. This regularization was employed, for instance, in [1, 27], to investigate anisotropy effects arising during the growth and coarsening of thin films. Finally, the energy known as the *functionalized Cahn–Hilliard* (FCH) free energy arises when the parameter $\nu$ takes negative values. This formulation of the energy was primarily developed for mixtures characterized by an amphiphilic structure and nanoscale variations in functionalized polymer chains: see, e.g., [11,21]. Extensive research has been conducted on the FCH energy, covering topics such as minimization problems, bilayer structures, pearled patterns, and network bifurcations. Without the claim of being exhaustive, we refer to [6, 7, 22, 23] and the references therein. The study of sixth-order Cahn–Hilliard equations has sparked significant interest, and the applications are numerous: in relation to the dynamics of oil-water-surfactant mixtures, we refer to [19,20,24]; the faceting of growing surfaces is explored in works such as [14, 15], while the phase-field-crystal equation is investigated in references like [12, 13, 16, 17, 28]. The novelty in our contribution is the introduction of a source term $S$ into the system that accounts for mass transfer and exchange and leads to a deviation from the standard assumption of mass conservation. In addition, we introduce a control variable in the definition of the source whose inclusion allows us to manipulate and regulate the behavior of the source term.

Since we are not interested in the asymptotic behavior as the interface thickness parameter approaches zero, we set for convenience $\varepsilon = 1$. Moreover, due to our interest in the optimal control application, we consider from the very beginning a constant mobility function $m$, that is, we set $m$ to unity for simplicity. Thus, the overall system we are going to analyze is the following:

$$\partial_t\varphi - \Delta\mu = S(\varphi, u) := u - \sigma\varphi \qquad \text{in } Q, \tag{1.9}$$
$$\mu = -\Delta w + f'(\varphi)w + \nu w \qquad \text{in } Q, \tag{1.10}$$
$$w = -\Delta\varphi + f(\varphi) \qquad \text{in } Q, \tag{1.11}$$
$$\partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\mu = \partial_{\boldsymbol{n}} w = 0 \qquad \text{on } \Sigma, \tag{1.12}$$
$$\varphi(0) = \varphi_0 \qquad \text{in } \Omega. \tag{1.13}$$

As anticipated above, we address in the subsequent stage of our study an associated distributed control problem, where $u$ acts as the control variable. The tracking-type *cost functional* under consideration is given by

$$\mathcal{J}(u,\varphi) := \frac{\alpha_1}{2}\int_Q |\varphi - \phi_Q|^2 + \frac{\alpha_2}{2}\int_\Omega |\varphi(T) - \phi_\Omega|^2 + \frac{\alpha_3}{2}\int_Q |u|^2, \tag{1.14}$$

where the coefficients $\alpha_i$, $i = 1, 2, 3$, are nonnegative real numbers (not all zero, to avoid a trivial situation), and $\phi_Q$ and $\phi_\Omega$ are given target functions defined in $Q$ and $\Omega$, respectively. The set of admissible controls is given by

$$\mathcal{U}_{\rm ad} := \{u \in L^\infty(Q) : \ u_{\min} \le u \le u_{\max} \text{ a.e. in } Q\}, \tag{1.15}$$

where $u_{\min}, u_{\max} \in L^\infty(Q)$ are prescribed functions satisfying $u_{\min} \le u_{\max}$ a.e. in $Q$. The control problem then consists in minimizing $\mathcal{J}$ on $\mathcal{U}_{\rm ad}$ under the constraint that $\varphi$ is the first component of the solution $(\varphi, \mu, w)$ to the state system associated with $u$. To summarize, we study the following optimal control problem:

**(P)** $\min_{u\in\mathcal{U}_{\rm ad}} \mathcal{J}(u,\varphi)$ subject to the constraint that $(\varphi, \mu, w)$ solves (1.9)–(1.13).

The plan of the paper is as follows. In the upcoming section, we provide a comprehensive list of the specific assumptions we make and present our results. The well-posedness of the system (1.9)–(1.13) is then studied in Section 3, and the control problem is investigated in the last Section 4.

## 2 Notation, assumptions and results

To begin with, let us introduce some notation and conventions. Let $\Omega$ be an open set in $\mathbb{R}^d$, with either $d = 2$ or $d = 3$, which is assumed to be bounded, connected and smooth, and whose Lebesgue measure is denoted by $|\Omega|$. We denote the outward unit normal field on the boundary $\Gamma := \partial\Omega$ by $\boldsymbol{n}$ and the corresponding outward normal derivative by $\partial_{\boldsymbol{n}}$. We fix a final time $T > 0$, recall the definitions (1.6) of $Q$ and $\Sigma$, and set

$$Q_t := \Omega \times (0,t) \quad \text{for } t \in (0,T]\,. \tag{2.1}$$

In this paper, we employ for any Banach space $X$ the notation $\|\cdot\|_X$, $X^*$, and $\langle\,\cdot\,,\,\cdot\,\rangle_X$, to indicate the corresponding norm, its dual space, and the related duality pairing between $X^*$ and $X$. The only exceptions from this notation for norms are given by the space $H$ introduced below and the standard Lebesgue spaces $L^p(\Omega)$ with $p \in [1,+\infty]$, whose norms are denoted by $\|\cdot\|$ and $\|\cdot\|_p$, respectively. Moreover, the symbol $\|\cdot\|_\infty$ might denote the norm in $L^\infty(Q)$ as well, if no confusion may arise. For simplicity, we use the same symbol for the norm in some space and that in any power thereof. Accordingly, when dealing with vector-valued functions like the gradient of some scalar function, we adopt shorthands like $L^p(0,T;X)$ or $H^1(0,T;X)$ to denote Bochner spaces involving powers of $X$. Next, besides the space $H$ announced before, we introduce two further spaces $V$ and $W$. Indeed, we set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad \text{and} \quad W := \{v \in H^2(\Omega) :\ \partial_{\boldsymbol{n}} v = 0 \text{ on } \Gamma\}. \tag{2.2}$$

Norm and inner product are in the special case $H$ indicated by $\|\cdot\|$ and $(\,\cdot\,,\,\cdot\,)$, respectively. In connection with the above spaces, we adopt the usual framework of Hilbert triplets by identifying $H$ and $V^*$ with subsets of $V^*$ and $W^*$ in the usual way. Namely, we have that

$$\langle z,v\rangle_V = \int_\Omega zv \quad \text{and} \quad \langle z,v\rangle_W = \langle z,v\rangle_V,$$
$$\text{for every } z \in H, \text{ and } v \in V \text{ and every } z \in V^* \text{ and } v \in W, \text{ respectively.}$$

Then, we have that $W \hookrightarrow V \hookrightarrow H \hookrightarrow V^* \hookrightarrow W^*$, with dense and compact embeddings.

Now, we list our assumptions on the structure of the system to be analyzed. We generally assume:

$$\sigma \in (0,+\infty), \quad \lambda \in [0,+\infty) \quad \text{and} \quad \nu \in \mathbb{R}. \tag{2.3}$$

$$F \in C^4(\mathbb{R}) \text{ can be written as } \quad F(s) = \widehat{\beta}(s) - \frac{\lambda}{2}\,s^2, \quad s \in \mathbb{R}, \quad \text{with } \widehat{\beta} \text{ convex.} \tag{2.4}$$

We set

$$f := F', \quad \beta := \widehat{\beta}' \quad \text{and} \quad \gamma := \beta\beta', \tag{2.5}$$

so that

$$f(s) = F'(s) = \beta(s) - \lambda s \quad \text{and} \quad f'(s) = \beta'(s) - \lambda, \quad \text{for every } s \in \mathbb{R},$$

and require that

$$\beta(0) = \beta''(0) = 0, \quad \text{and} \quad \beta'''(s) \ge 0 \quad \text{for every } s \in \mathbb{R}, \tag{2.6}$$

$$\lim_{|s|\to+\infty} \frac{\beta'(s)}{|s|} = +\infty, \tag{2.7}$$

$$|\beta''(s)| \le C_\beta(|\beta'(s)| + 1) \quad \text{for some } C_\beta > 0 \text{ and every } s \in \mathbb{R}. \tag{2.8}$$

**Remark 2.1.** Let us remark that our structural assumptions imply that

$$\beta,\ \beta'' \text{ and } \gamma \text{ are monotone.} \tag{2.9}$$

This is clear for $\beta$ and $\beta''$, since $\widehat{\beta}$ is convex and $\beta''' = \widehat{\beta}^{(4)}$ is nonnegative. As for $\gamma$, notice that $\gamma'$ is nonnegative, since $\beta$ and $\beta''$ possess the same sign. Moreover, we notice that $|\beta(s)|$ must tend to infinity as $|s|$ approaches infinity as a consequence of (2.7), that the same is true for $\widehat{\beta}$ since it is convex, and that $\beta(0) = 0$ implies that $\widehat{\beta}$ has a minimum point at zero. Hence, by l'Hôpital's rule, we deduce that

$$\lim_{|s|\to\infty} \frac{s}{\beta(s)} = \lim_{|s|\to\infty} \frac{1}{\beta'(s)} = 0 \quad \text{and} \quad \lim_{|s|\to\infty} \frac{s^3}{\widehat{\beta}(s)} = \lim_{|s|\to\infty} \frac{3s^2}{\beta(s)} = \lim_{|s|\to\infty} \frac{6s}{\beta'(s)} = 0.$$

Thus, two consequences follow: first, it holds that $\widehat{\beta}$ grows faster than $|s|^3$ as $|s|$ tends to infinity, so that

$$F \text{ is bounded from below.} \tag{2.10}$$

The second consequence regards the function $g$ defined by

$$g(s) := -\lambda s\beta'(s) + (\nu - \lambda)\beta(s) + (\lambda^2 - \lambda\nu)s \quad \text{for every } s \in \mathbb{R}, \tag{2.11}$$

which we have to consider later on. We have that

$$\lim_{|s|\to\infty} \frac{g(s)}{\gamma(s)} = 0 \quad \text{and} \quad \lim_{|s|\to\infty} \frac{sg(s)}{\gamma(s)} = 0\,. \tag{2.12}$$

Finally, we observe that the above assumptions are not too restrictive and satisfied by a wide class of smooth potentials with either polynomial or exponential growth; in particular, they are met by the classical regular potential given by the expression (1.8).

At this point, we are ready to give a precise formulation of the problem (1.9)–(1.13) presented in the Introduction. As anticipated, the control variable $u$ plays the role of a prescribed and bounded forcing term in the well-posedness part. Given a constant $M > 0$, as well as data $\varphi_0$ and $u$ satisfying

$$\varphi_0 \in W \quad \text{and} \quad u \in L^\infty(Q) \quad \text{with} \quad \|u\|_\infty \le M\,, \tag{2.13}$$

we look for a triplet $(\varphi, \mu, w)$ with the regularity

$$\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;W), \tag{2.14}$$

$$\mu \in L^2(0,T;V), \tag{2.15}$$

$$w \in L^2(0,T;W), \tag{2.16}$$

that solves the following problem:

$$\langle \partial_t\varphi, v\rangle_V + \int_\Omega \nabla\mu\cdot\nabla v + \sigma\int_\Omega \varphi v = \int_\Omega uv \quad \text{a.e. in } (0,T), \text{ for every } v \in V, \tag{2.17}$$

$$-\Delta w + \beta'(\varphi)w + (\nu - \lambda)w = \mu \quad \text{a.e. in } Q, \tag{2.18}$$

$$-\Delta\varphi + \beta(\varphi) - \lambda\varphi = w \quad \text{a.e. in } Q, \tag{2.19}$$

$$\varphi(0) = \varphi_0\,. \tag{2.20}$$

Notice that the boundary conditions $\partial_{\boldsymbol{n}}\varphi = 0$ and $\partial_{\boldsymbol{n}} w = 0$ are contained in (2.14) and (2.16), while the analogue for $\mu$ just holds in a generalized sense as a consequence of (2.17).

As for the assumption on $u$, we observe that $M$ does not play any role if just well-posedness is considered. However, when dealing with the control problem, we let $u$ vary and need uniform bounds for the corresponding solutions, so that the introduction of $M$ is useful: it is worth noticing that the bounds we are going to find are independent of $u$ and depend just on $M$.

The first result of ours deals with well-posedness, regularity, and stability of the system (2.17)–(2.20).

**Theorem 2.2.** *Let the assumptions* (2.3)–(2.8) *and* (2.13) *on the structure of the system and the data be satisfied. Then, problem* (2.17)–(2.20) *has a unique solution* $(\varphi, \mu, w)$ *satisfying* (2.14)–(2.16)*. Moreover, the estimate*

$$\|\varphi\|_{H^1(0,T;V^*)\cap L^\infty(0,T;W)} + \|\mu\|_{L^2(0,T;V)} + \|w\|_{L^2(0,T;W)} \le K_1 \tag{2.21}$$

*holds true with a positive constant $K_1$ that depends only on $\Omega$, $T$, the structure of the system, the initial datum $\varphi_0$ and $M$.*

**Remark 2.3.** The unique solution $(\varphi, \mu, w)$ to problem (2.17)–(2.20) satisfying (2.14)–(2.16) actually enjoys the further regularity properties

$$\varphi \in L^2(0,T;H^4(\Omega)) \quad \text{and} \quad w \in L^\infty(0,T;H) \cap L^2(0,T;H^3(\Omega)), \tag{2.22}$$

as we immediately prove by repeated use of elliptic regularity theory. Since $\varphi$ is uniformly bounded as a consequence of (2.14) and the continuity of the embedding $W \hookrightarrow L^\infty(\Omega)$, we see that (2.14) and (2.19) imply that $w \in L^\infty(0,T;H)$. Now, we observe that (2.14) also implies that $|\nabla\varphi|^2 \in L^\infty(0,T;H)$, so that $\Delta\beta(\varphi) \in L^\infty(0,T;H)$ and $\beta(\varphi) \in L^\infty(0,T;H^2(\Omega))$. Thus, (2.19) and (2.16) yield that $\varphi \in L^2(0,T;H^4(\Omega))$. Next, the regularity just obtained, and that of $w$ in (2.16), imply that $\nabla\varphi \in L^2(0,T;L^\infty(\Omega))$ and $\nabla w \in L^2(0,T;H)$. Hence, as $w \in L^\infty(0,T;H)$, it turns out that $\nabla(\beta'(\varphi)w) \in L^2(0,T;H)$, i.e., $\beta'(\varphi)w \in L^2(0,T;V)$. Therefore, by also accounting for (2.15), we conclude from (2.18) that $w \in L^2(0,T;H^3(\Omega))$. We could continue in improving the regularity of $\varphi$. However, we just observe one more property of $\nabla\varphi$ that will be needed in the sequel. Namely, it holds that

$$|\nabla\varphi| \in L^4(0,T;L^\infty(\Omega)), \tag{2.23}$$

by virtue of the embedding $L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)) \hookrightarrow L^4(0,T;L^\infty(\Omega))$. We conclude this remark by observing that if (2.21) holds true for $(\varphi, \mu, w)$, then analogous estimates associated to the regularity (2.22) and (2.23) are satisfied as well by the solution, since each of the above steps also provides a corresponding estimate.

The second one regards continuous dependence of the solution with respect to the control variable $u$.

**Theorem 2.4.** *Let the assumptions* (2.3)–(2.8) *on the structure of the system be fulfilled, as well as* (2.13) *for $\varphi_0$, and let $u_i \in L^\infty(Q)$, $i = 1, 2$, satisfy $\|u_i\|_\infty \le M$. If $(\varphi_i, \mu_i, w_i)$ are the corresponding solutions to system* (2.17)–(2.20)*, then the estimate*

$$\begin{aligned} &\|\varphi_1 - \varphi_2\|_{C^0([0,T];V)\cap L^2(0,T;H^4(\Omega))} + \|\mu_1 - \mu_2\|_{L^2(0,T;H)} \\ &\quad + \|w_1 - w_2\|_{L^2(0,T;W)} \le K_2 \, \|u_1 - u_2\|_{L^2(0,T;V^*)} \end{aligned} \tag{2.24}$$

*holds true with a positive constant $K_2$ that depends only on $\Omega$, $T$, the structure of the system, the initial datum $\varphi_0$ and $M$.*

As announced in the Introduction, we address an associated distributed control problem for system (1.9)–(1.13). Here, we make our precise assumptions:

$$\alpha_i \text{ are nonnegative real numbers for } i = 1, 2, 3. \tag{2.25}$$

$$\phi_Q \in L^2(Q) \quad \text{and} \quad \phi_\Omega \in V. \tag{2.26}$$

$$u_{\min},\, u_{\max} \in L^\infty(Q) \quad \text{with} \quad u_{\min} \le u_{\max} \quad \text{a.e. in } Q. \tag{2.27}$$

Moreover, we define the cost functional, the control space, and the set of admissible controls, by setting

$$\mathcal{J}(u,\varphi) := \frac{\alpha_1}{2}\int_Q |\varphi - \phi_Q|^2 + \frac{\alpha_2}{2}\int_\Omega |\varphi(T) - \phi_\Omega|^2 + \frac{\alpha_3}{2}\int_Q |u|^2\,, \tag{2.28}$$

$$\mathcal{U} := L^\infty(Q), \quad \text{and} \quad \mathcal{U}_{\rm ad} := \{u \in \mathcal{U} : u_{\min} \le u \le u_{\max} \text{ a.e. in } Q\}. \tag{2.29}$$

Of course, for (2.28) to be meaningful, it suffices to require $\phi_\Omega \in H$, but that would not be enough to handle the control problem for technical reasons that will be clarified later on (cf. Theorem 4.8). Thus, the control problem under investigation is the following:

Minimize $\mathcal{J}(u,\varphi)$ under the conditions:
$u \in \mathcal{U}_{\rm ad}$, and $\varphi$ is the first component of the solution $(\varphi, \mu, w)$
to the state system (2.17)–(2.20) associated with $u$. (2.30)

We have considered a basic form for the cost functional for simplicity. However, let us claim that we can actually afford to include in the cost functional (2.28) an additional term involving the component $w$ and some target function $w_Q$ (see the forthcoming Remark 4.7).

In Section 4, we prove that there exist optimal controls and provide first-order necessary conditions for $u^*$ to be an optimal control (or, more generally, a locally optimal control in the sense of $L^p$ for $1 \le p \le +\infty$, see the concluding Remark 4.10), which reads as follows: if $u^* \in \mathcal{U}_{\rm ad}$ is the optimal control and $(\varphi^*, \mu^*, w^*)$ is the corresponding state, then the variational inequality

$$\int_Q (\alpha_3\, u^* + p)(u - u^*) \ge 0$$

is satisfied for every $u \in \mathcal{U}_{\rm ad}$, where $p$ is the first component of the solution $(p, q, r)$ to a proper adjoint system associated to (2.17)–(2.20). Here, for simplicity, we just confine ourselves to mention that it is a suitable weak formulation of the following formal backward-in-time system

$$\begin{aligned}
&-\partial_t p - \Delta r + \sigma p + f''(\varphi^*) w^* q - f'(\varphi^*) r = \alpha_1(\varphi^* - \phi_Q) && \text{in } Q,\\
&q = -\Delta p && \text{in } Q,\\
&r + \Delta q - \nu q - f'(\varphi^*) q = 0 && \text{in } Q,
\end{aligned}$$

complemented with the homogeneous Neumann boundary conditions for all the variables and the final condition

$$p(T) = \alpha_2(\varphi^*(T) - \phi_\Omega) \quad \text{in } \Omega.$$

We conclude this section by collecting some useful tools that will be employed throughout the paper. Besides the Hölder inequality, we often account for the Young, Sobolev and Poincaré inequalities as well as for some inequalities associated to elliptic regularity theory and to the compact embeddings $V \hookrightarrow L^p(\Omega)$ and $W \hookrightarrow V$. In fact, we have

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \in \mathbb{R} \text{ and } \delta > 0, \tag{2.31}$$

$$\|v\|_p \leq C_\Omega \|v\|_V \quad \text{for every } v \in V \text{ and } p \in [1,6], \tag{2.32}$$

$$\|v\|_V \leq C_\Omega \left(\|\nabla v\| + |\overline{v}|\right) \quad \text{for every } v \in V, \tag{2.33}$$

$$\|v\|_W \leq C_\Omega \big(\|\Delta v\| + \|v\|_*\big) \quad \text{for every } v \in W, \tag{2.34}$$

$$\|v\|_p \leq \delta \, \|\nabla v\| + C_{\Omega,p,\delta} \, \|v\|_* \quad \text{for every } v \in V,\, p \in [1,6) \text{ and } \delta > 0, \tag{2.35}$$

$$\|v\|_V \leq \delta \, \|\Delta v\| + C_{\Omega,\delta} \, \|v\|_* \quad \text{for every } v \in W \text{ and } \delta > 0, \tag{2.36}$$

where $\overline{v}$ denotes the mean value of $v$ and $\|\cdot\|_*$ a norm in $V^*$ which will be introduced below in (2.40). In the above inequalities, the constant $C_\Omega$ depends only on $\Omega$, while $C_{\Omega,\delta}$ and $C_{\Omega,p,\delta}$ also depend on $\delta$ and $(p,\delta)$, respectively.

More generally, we define the generalized mean value $\overline{v}$ of a generic element $v \in W^*$ by setting

$$\overline{v} := \frac{1}{|\Omega|} \, \langle v, 1 \rangle_W \,, \tag{2.37}$$

where $1$ stands for the constant function that takes the value $1$ everywhere in $\Omega$. For any $s \in \mathbb{R}$, we still denote by $s$ the corresponding constant functions in $\Omega$ and $Q$. Notice that the above definition (2.37) is meaningful, since $1$ actually belongs to $W$, and that $\overline{v}$ reduces to the usual mean value when $v \in H$. The same notation $\overline{v}$ is also employed if $v$ is a time-dependent function.

Next, we recall an important tool which is commonly used when working with problems connected to the Cahn–Hilliard equation. To this end, consider the weak formulation of the Poisson equation $-\Delta z = \zeta$ with homogeneous Neumann boundary conditions. Namely, for a given $\zeta \in V^*$ (which does not necessarily belong to $H$), we consider the problem of finding

$$z \in V \quad \text{such that} \quad \int_\Omega \nabla z \cdot \nabla v = \langle \zeta, v \rangle_V \quad \text{for every } v \in V. \tag{2.38}$$

Since $\Omega$ is connected and smooth, it is well known that the above problem admits solutions $z$ if and only if $\zeta$ has zero mean value. Hence, we can introduce the following solution operator $\mathcal{N}$ by setting

$$\mathcal{N} : \mathrm{dom}(\mathcal{N}) := \{\zeta \in V^* : \ \overline{\zeta} = 0\} \to \{z \in V : \ \overline{z} = 0\}, \quad \mathcal{N} : \zeta \mapsto z, \tag{2.39}$$

where $z$ is the unique solution to (2.38) coupled with $\overline{z} = 0$. It turns out that $\mathcal{N}$ is an isomorphism between the above spaces, and it follows that the formula

$$\|\zeta\|_*^2 := \|\nabla \mathcal{N}(\zeta - \overline{\zeta})\|^2 + |\overline{\zeta}|^2 \quad \text{for every } \zeta \in V^* \tag{2.40}$$

defines a Hilbert norm in $V^*$ that is equivalent to the standard dual norm of $V^*$. From the above properties, one can obtain the following identities:

$$\int_\Omega \nabla \mathcal{N}\zeta \cdot \nabla v = \langle \zeta, v \rangle_V \quad \text{for every } \zeta \in \mathrm{dom}(\mathcal{N}) \text{ and } v \in V, \tag{2.41}$$

$$\langle \zeta, \mathcal{N}\xi \rangle_V = \langle \xi, \mathcal{N}\zeta \rangle_V \quad \text{for every } \zeta, \xi \in \mathrm{dom}(\mathcal{N}), \tag{2.42}$$

$$\langle \zeta, \mathcal{N}\zeta \rangle_V = \int_\Omega |\nabla \mathcal{N}\zeta|^2 = \|\zeta\|_*^2 \quad \text{for every } \zeta \in \mathrm{dom}(\mathcal{N}). \tag{2.43}$$

Moreover, we point out that

$$\langle \partial_t \zeta(t), \mathcal{N}\zeta(t)\rangle_V = \frac{1}{2}\,\frac{d}{dt}\,\|\zeta(t)\|_*^2 \quad \text{for a.a. } t \in (0,T)\,, \tag{2.44}$$

which holds true for every $\zeta \in H^1(0,T;V^*)$ satisfying $\overline{\zeta} = 0$ a.e. in $(0,T)$.

Finally, without further reference later on, we are going to employ the following convention: the small-case symbol $c$ denotes a generic constant that depends only on the structure of the system, $\Omega$, $T$, the initial datum $\varphi_0$, and the constant $M$ that appears in (2.13). In particular, the values of $c$ are independent of $u$ and the approximation parameter $n$ we introduce in Section 3.1. Notice that the meaning of $c$ may vary from line to line and even within the same line. In addition, whenever a positive constant $\delta$ enters the computation, the related symbol $c_\delta$, in place of a general $c$, denotes constants that depend on $\delta$, in addition. We use different notations for precise constants we could refer to, like, e.g., in (2.32).

# 3 The state system

In this section, we prove Theorems 2.2 and 2.4. To this end, it is convenient to observe that our problem can be formulated in an alternative equivalent way.

**Proposition 3.1.** *If $(\varphi,\mu,w)$ satisfies* (2.14)–(2.16) *and solves* (2.17)–(2.20)*, then the pair $(\varphi,\mu)$ also solves the variational equation*

$$\begin{aligned} &\int_\Omega \Delta\varphi\,\Delta v - \int_\Omega \Delta\beta(\varphi)\,v - \int_\Omega \beta'(\varphi)\Delta\varphi\,v \\ &\quad + \int_\Omega \gamma(\varphi)\,v + (2\lambda-\nu)\int_\Omega \Delta\varphi\,v + \int_\Omega g(\varphi)\,v \\ &= \int_\Omega \mu v \quad \text{a.e. in } (0,T), \text{ for every } v \in W, \end{aligned} \tag{3.1}$$

*where the function $g:\mathbb{R}\to\mathbb{R}$ is defined by* (2.11)*. Conversely, if $\varphi$ and $\mu$ satisfy the regularities in* (2.14)–(2.15) *and the pair $(\varphi,\mu)$ solves* (3.1)*, then we have the following:*
(i) *The function $w$ given by*

$$w := -\Delta\varphi + \beta(\varphi) - \lambda\varphi \tag{3.2}$$

*satisfies* (2.16) *and the estimate*

$$\|w\|_{L^2(0,T;W)} \le C\big(\|\varphi\|_{L^2(0,T;W)} + \|\mu\|_{L^2(0,T;H)} + 1\big)\,, \tag{3.3}$$

*with a positive constant $C$ that depends only on $\Omega$, $T$ and the structure of the system.*
(ii) *The triplet $(\varphi,\mu,w)$ solves* (2.17)–(2.20)*.*

*Proof.* We observe once and for all that (2.14), the continuous embedding $W \hookrightarrow L^\infty(\Omega)$, and the smoothness of $\beta$, imply that $\varphi$, $\beta(\varphi)$, and $\beta'(\varphi)$, are bounded in $L^\infty(Q)$. Besides, throughout the proof, we repeatedly owe to elliptic regularity theory.

Assume first that $(\varphi,\mu,w)$ satisfies (2.14)–(2.16) and solves the problem (2.17)–(2.19). Then, we eliminate $w$ in (2.18) by means of (2.19) to obtain that

$$-\Delta\big(-\Delta\varphi + \beta(\varphi) - \lambda\varphi\big) + \big(\beta'(\varphi) - \lambda + \nu\big)\big(-\Delta\varphi + \beta(\varphi) - \lambda\varphi\big) = \mu \quad \text{a.e. in } Q. \tag{3.4}$$

Now, we recall that $w \in L^2(0,T;W)$ and the identity

$$\Delta\beta(\varphi) = \beta''(\varphi)|\nabla\varphi|^2 + \beta'(\varphi)\Delta\varphi\,,$$

where both $|\nabla\varphi|^2$ and $\Delta\varphi$ belong to $L^\infty(0,T;H)$, the former since $W \hookrightarrow W^{1,4}(\Omega)$. Hence, $\Delta\beta(\varphi)$ belongs to $L^\infty(0,T;H)$. Since $\partial_{\boldsymbol{n}}\beta(\varphi) = \beta'(\varphi)\partial_{\boldsymbol{n}}\varphi = 0$ on $\Sigma$, we conclude that $\beta(\varphi) \in L^2(0,T;W)$, whence, by comparison in (2.19), we also have that $\Delta\varphi \in L^2(0,T;W)$. Thus, we can distribute the first Laplacian in (3.4) to the single summands. By recalling (2.11), we obtain that

$$\Delta^2\varphi - \Delta\beta(\varphi) - \beta'(\varphi)\Delta\varphi + \gamma(\varphi) + (2\lambda-\nu)\Delta\varphi + g(\varphi) = \mu \quad \text{a.e. in } Q\,. \tag{3.5}$$

To derive (3.1), it suffices to multiply (3.5) by an arbitrary $v \in W$, integrate over $\Omega$ and use integration by parts in the first integral along with the property that both $\Delta\varphi$ and $v$ have a zero normal derivative on $\Sigma$.

Conversely, assume that $\varphi$ and $\mu$ fulfill (2.14)–(2.15) and the pair $(\varphi,\mu)$ solves (3.1). Then, since $\partial_{\boldsymbol{n}}\beta(\varphi) = 0$ on $\Sigma$, it holds that

$$\int_\Omega \Delta\beta(\varphi)\, v = \int_\Omega \beta(\varphi)\Delta v \quad \text{a.e. in } (0,T), \text{ for every } v \in W.$$

Hence, by recalling the definitions (2.5) and (2.11) of $\gamma$ and $g$, we can rewrite (3.1) in the form

$$\int_\Omega \big(-\Delta\varphi - \beta(\varphi) - \lambda\varphi\big)(-\Delta v) + \int_\Omega \big(\beta'(\varphi) - \lambda + \nu\big)\big(-\Delta\varphi + \beta(\varphi) - \lambda\varphi\big)v$$
$$= \int_\Omega \mu v \quad \text{a.e. in } (0,T), \text{ for every } v \in W.$$

Therefore, if $w$ is given by (3.2), then $w \in L^\infty(0,T;H)$, and the equation (2.19) is satisfied. Moreover, since (3.1) has the form

$$\int_\Omega \Delta\varphi\,\Delta v = \int_\Omega hv \quad \text{for a.a. } t \in (0,T), \text{ for every } v \in W,$$

with $h \in L^2(0,T;H)$, we deduce that $\Delta^2\varphi \in L^2(0,T;H)$ and $\partial_{\boldsymbol{n}}\Delta\varphi = 0$ on $\Sigma$. Thus, we also have that $w \in L^2(0,T;H^2(\Omega))$ and $\partial_{\boldsymbol{n}} w = 0$ on $\Sigma$, i.e., (2.16). Furthermore, it is clear that

$$\int_\Omega w(-\Delta v) + \int_\Omega \big(\beta'(\varphi) - \lambda + \nu\big)\, w\, v = \int_\Omega \mu v \quad \text{a.e. in } (0,T), \text{ for every } v \in W,$$

from which we obtain (2.18) since $w \in L^2(0,T;W)$. Finally, we have that

$$\|w\|_{L^2(0,T;W)} \le c\left(\|\mu\|_{L^2(0,T;H)} + \|w\|_{L^2(0,T;H)}\right) \le c\left(\|\mu\|_{L^2(0,T;H)} + \|\varphi\|_{L^2(0,T;W)} + 1\right),$$

that is, (3.3) with some computable constant $C$ as in the statement. □

**Remark 3.2.** Since both $\varphi$ and $w$ belong to $L^2(0,T;W)$, we conclude from (2.19) that

$$\Delta\varphi \in L^2(0,T;W)\,. \tag{3.6}$$

Hence, the first integral of (3.1) can be rewritten as $\int_\Omega \Delta^2\varphi\, v$, and (3.1) itself is equivalent to

$$\Delta^2\varphi - \Delta\beta(\varphi) - \beta'(\varphi)\Delta\varphi + \gamma(\varphi) + (2\lambda-\nu)\Delta\varphi + g(\varphi) = \mu \quad \text{a.e. in } Q\,,$$

that is, equation (3.5).

In summary, the problem (2.17)–(2.20) may be reformulated as:

*Find* $(\varphi,\mu)$ *satisfying* (2.14)–(2.15)*,* (2.17)*,* (3.1) *and* (2.20)*.* (3.7)

In this new framework, due to (3.3), the stability estimate (2.21) is equivalent to the estimate obtained by (2.21) itself by ignoring the part regarding $w$.

## 3.1 Existence

In this section, we prove the existence and stability part of Theorem 2.2 by constructing a solution that satisfies the estimate (2.21). By Proposition 3.1, we can consider the problem in the form expressed in (3.7). To this end, we introduce a corresponding discretization by means of a Faedo–Galerkin scheme.

In this direction, let $\{\lambda_j\}_{j\geq 1}$ and $\{e_j\}_{j\geq 1}$ be the sequence of the eigenvalues and an orthonormal system of corresponding eigenfunctions of the Neumann problem for the Laplace equation, i.e., we have that

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \quad \text{and} \quad \lim_{j\to\infty} \lambda_j = +\infty\,, \tag{3.8}$$

$$e_j \in V \quad \text{and} \quad \int_\Omega \nabla e_j \cdot \nabla v = \lambda_j \int_\Omega e_j v \quad \text{for every } v \in V \text{ and } j = 1, 2, \dots, \tag{3.9}$$

$$\int_\Omega e_i e_j = \delta_{ij} \quad \text{for } i, j = 1, 2, \dots, \quad \text{and} \quad \{e_j\}_{j\geq 1} \text{ is a complete system in } H\,, \tag{3.10}$$

where $\delta_{ij}$ is the Kronecker symbol. Notice that actually $e_j \in W$, since $\Omega$ is assumed to be smooth, and $e_1$ is a constant eigenfunction. We set

$$V_n := \operatorname{span}\{e_1, \dots, e_n\}, \quad \text{for } n = 1, 2, \dots, \tag{3.11}$$

and observe that the union of these spaces is dense in both $V$ and $H$. Besides, we point out once and for all that $V_1 = \operatorname{span}\{e_1\}$ consists of the space of constant functions, which are therefore admissible test functions in the Galerkin scheme.

**Remark 3.3.** We also notice that $V_n \subset W$ for every $n$ and that

$$\mathcal{N}v \in V_n \quad \text{for every } v \in V_n \text{ with zero mean value.} \tag{3.12}$$

Indeed, if $v \in V_n$ has zero mean value, then it can be represented as $v = \sum_{j=2}^n c_j e_j$ for suitable coefficients $c_j$, $j = 2, \dots, n$. On the other hand, the condition $z = \mathcal{N}v$ means that

$$z \in W, \quad \overline{z} = 0 \quad \text{and} \quad -\Delta z = v \text{ in } \Omega, \tag{3.13}$$

whence from (3.9) it easily follows that $z := \sum_{j=2}^n \lambda_j^{-1} c_j e_j \in V_n$ solves (3.13).

Next, let $\mathbb{P}_n : H \to V_n$ be the orthogonal projection operator onto $V_n$, and let $Y$ be any of the spaces $H$, $V$ and $W$. Then, we have that

$$\|\mathbb{P}_n v\|_Y \leq C_\Omega \|v\|_Y \quad \text{for every } v \in Y, \tag{3.14}$$

where the constant $C_\Omega$ (with $C_\Omega = 1$ if $Y = H$) depends only on $\Omega$. Moreover, if $v \in L^2(0,T;Y)$ and $v_n$ is defined by $v_n(t) := \mathbb{P}_n(v(t))$ for a.a. $t \in (0,T)$, then

$$\|v_n\|_{L^2(0,T;Y)} \leq C_\Omega \|v\|_{L^2(0,T;Y)}, \quad \text{and} \quad v_n \to v \quad \text{strongly in } L^2(0,T;Y). \tag{3.15}$$

These properties are straightforward if $Y = H$, whereas for the other cases, we refer, e.g., to [3, Rem 3.3].

We are now ready to state the discrete problem: find a pair $(\varphi_n, \mu_n)$ with

$$\varphi_n \in H^1(0,T;V_n) \quad \text{and} \quad \mu_n \in L^2(0,T;V_n), \tag{3.16}$$

$$\int_\Omega \partial_t\varphi_n\, v + \int_\Omega \nabla\mu_n\cdot\nabla v + \sigma\int_\Omega \varphi_n v = \int_\Omega uv \quad \text{a.e. in } (0,T), \text{ for every } v\in V_n, \tag{3.17}$$

$$\int_\Omega \Delta\varphi_n\,\Delta v - \int_\Omega \Delta\beta(\varphi_n)\,v - \int_\Omega \beta'(\varphi_n)\Delta\varphi_n\,v$$
$$+\int_\Omega \gamma(\varphi_n)\,v + (2\lambda-\nu)\int_\Omega \Delta\varphi_n\,v + \int_\Omega g(\varphi_n)\,v$$
$$= \int_\Omega \mu_n\,v \quad \text{a.e. in } (0,T), \text{ for every } v\in V_n, \tag{3.18}$$

$$\int_\Omega \varphi_n(0)\,v = \int_\Omega \varphi_0\,v \quad \text{for every } v\in V_n, \tag{3.19}$$

where the last condition implies that $\varphi_n(0)=\mathbb{P}_n(\varphi_0)$.

**Theorem 3.4.** *For every $n\in\mathbb{N}$, there exists a unique pair $(\varphi_n,\mu_n)$ satisfying* (3.16)–(3.19).

*Proof.* First of all, let us fix $n\in\mathbb{N}$. We expand the sought solution in the form

$$\varphi_n(t)=\sum_{j=1}^n \varphi_{nj}(t)e_j\,, \quad \mu_n(t)=\sum_{j=1}^n \mu_{nj}(t)e_j\,,$$

and introduce the real unknowns of the problem, which are the vector-valued functions $\boldsymbol{\varphi}_n$, $\boldsymbol{\mu}_n$ and $\boldsymbol{u}_n$ (to be intended as columns), by setting, for a.a. $t\in(0,T)$,

$$\boldsymbol{\varphi}_n(t) := (\varphi_{nj}(t))_{j=1}^n\,, \quad \boldsymbol{\mu}_n(t) := (\mu_{nj}(t))_{j=1}^n \quad\text{and}\quad \boldsymbol{u}_n(t) := (u_{nj}(t))_{j=1}^n\,,$$
$$\text{where}\quad u_{nj}(t)=\int_\Omega u(t)e_j \quad \text{for } j=1,\dots,n.$$

We aim at presenting the above problem as a system of ordinary differential equations in the unknown $(\boldsymbol{\varphi}_n,\boldsymbol{\mu}_n)$. To this end, we recall the identity

$$-\Delta\beta(\varphi_n) = -\beta''(\varphi_n)|\nabla\varphi_n|^2 - \beta'(\varphi_n)\Delta\varphi_n.$$

From this, for $i=1,\dots,n$, we have that

$$\int_\Omega \bigl(-\Delta\beta(\varphi_n) - \beta'(\varphi_n)\Delta\varphi_n\bigr)\,e_i$$
$$= -\int_\Omega \beta''(\varphi_n)\Bigl|\sum_{j=1}^n \varphi_{nj}\nabla e_j\Bigr|^2 e_i - 2\int_\Omega \beta'(\varphi_n)\sum_{j=1}^n \varphi_{nj}\lambda_j e_j e_i\,.$$

Then, the system (3.16)–(3.18) is equivalent to the vector-valued system

$$\boldsymbol{\varphi}_n \in H^1(0,T;\mathbb{R}^n) \quad\text{and}\quad \boldsymbol{\mu}_n\in L^2(0,T;\mathbb{R}^n)\,, \tag{3.20}$$
$$\boldsymbol{\varphi}_n' + A\boldsymbol{\mu}_n + \sigma\boldsymbol{\varphi}_n = \boldsymbol{u}_n\,, \tag{3.21}$$
$$B^2\boldsymbol{\varphi}_n + \mathcal{B}(\boldsymbol{\varphi}_n) - 2\mathcal{F}(\boldsymbol{\varphi}_n) + \mathcal{G}(\boldsymbol{\varphi}_n) - (2\lambda-\nu)B\boldsymbol{\varphi}_n = \boldsymbol{\mu}_n\,, \tag{3.22}$$

where the matrices $A=(a_{ij})_{i,j=1}^n$ and $B=(b_{ij})_{i,j=1}^n$ are defined by

$$a_{ij} := \int_\Omega \nabla e_j\cdot\nabla e_i \quad\text{and}\quad b_{ij}:=\lambda_i\delta_{ij}, \quad \text{for } i,j=1,\dots,n,$$

and $\mathcal{B}:\mathbb{R}^n\to\mathbb{R}^n$, $\mathcal{F}:\mathbb{R}^n\to\mathbb{R}^n$ and $\mathcal{G}:\mathbb{R}^n\to\mathbb{R}^n$ are the functions whose $i$th components, for $i=1,\dots,n$, are defined by

$$\mathcal{B}_i(s_1,\dots,s_n) := -\int_\Omega \Big\{\beta''\Big(\textstyle\sum_{j=1}^n s_j e_j\Big)\Big|\displaystyle\sum_{j=1}^n s_j \nabla e_j\Big|^2 e_i\Big\},$$

$$\mathcal{F}_i(s_1,\dots,s_n) := \int_\Omega \Big\{\beta'\Big(\textstyle\sum_{j=1}^n s_j e_j\Big)\displaystyle\sum_{j=1}^n s_j\lambda_j e_j e_i\Big\},$$

$$\mathcal{G}_i(s_1,\dots,s_n) := \int_\Omega \Big\{(\gamma+g)\Big(\textstyle\sum_{j=1}^n s_j e_j\Big)e_i\Big\}.$$

Notice that these functions are well-defined, since the eigenfunctions are bounded in view of the embedding $W\hookrightarrow L^\infty(\Omega)$. For the same reason, one can differentiate under the integral sign, so that they are (at least) of class $C^1$. At this point, we eliminate $\boldsymbol{\mu}_n$ in (3.21) by using (3.22) as definition of $\boldsymbol{\mu}_n$, thus obtaining a system of ordinary differential equations in the only unknown $\boldsymbol{\varphi}_n$. The local Lipschitz continuity just observed ensures that every Cauchy problem for this system has a unique maximal solution (which is even more regular than required). This is the case if the initial condition for $\boldsymbol{\varphi}_n$ is that derived from (3.19), so that the discrete problem (3.16)–(3.19) has a unique maximal solution $\boldsymbol{\varphi}_n$ defined in some interval $[0,T_n]$, with $T_n\in(0,T]$. To complete the proof, we have to show that $T_n=T$. We do this by proving some a priori estimates that are uniform with respect to $n$. Since the established bounds prove to be useful in the following, we start to enumerate them.

**First a priori estimate.** To begin with, we recall the previous comment concerning the space $V_1$. We then test (3.17) by $1/|\Omega|\in V_1$ to obtain an ordinary differential equation for $\overline{\varphi_n}$. Namely, we have that

$$\frac{d}{dt}\,\overline{\varphi_n}+\sigma\overline{\varphi_n}=\overline{u}\quad\text{a.e. in }(0,T_n)\,. \tag{3.23}$$

By also observing that $\overline{\varphi_n}(0)=\overline{\varphi_0}$, we immediately deduce that

$$\|\overline{\varphi_n}\|_{L^\infty(0,T_n)}\le C^*\,, \tag{3.24}$$

where, for further reference, we have used the special symbol $C^*$ instead of $c$.

**Second a priori estimate.** We recall the definition (2.39) of $\mathcal{N}$ and (3.12), and we interpret the functions $\overline{\varphi_n}$ and $\overline{u}$ as space-independent functions defined in $\Omega\times(0,T_n)$, and thus (3.23) as a partial differential equation. We multiply it by $v$, integrate over $\Omega$, subtract the result from (3.17), and test the resulting equation by $\mathcal{N}(\varphi_n-\overline{\varphi_n})$. At the same time, we test (3.18) by $\varphi_n-\overline{\varphi_n}$. Then, we add the resulting equalities to each other. By recalling the properties (2.39)–(2.44) of $\mathcal{N}$, which also produce a cancellation, we rearrange the terms and infer that a.e. in $(0,T_n)$ it holds that

$$\begin{aligned}
&\frac{d}{dt}\,\|\varphi_n-\overline{\varphi_n}\|_*^2+\sigma\|\varphi_n-\overline{\varphi_n}\|_*^2\\
&\quad+\int_\Omega|\Delta\varphi_n|^2+\int_\Omega\beta'(\varphi_n)|\nabla\varphi_n|^2+\int_\Omega\gamma(\varphi_n)(\varphi_n-\overline{\varphi_n})\\
&=\int_\Omega\beta'(\varphi_n)\Delta\varphi_n\,(\varphi_n-\overline{\varphi_n})-(2\lambda-\nu)\int_\Omega\Delta\varphi_n\,(\varphi_n-\overline{\varphi_n})\\
&\quad-\int_\Omega g(\varphi_n)(\varphi_n-\overline{\varphi_n})+\int_\Omega u\,\mathcal{N}(\varphi_n-\overline{\varphi_n})\,,
\end{aligned} \tag{3.25}$$

where in the last term we have written $u$ in place of $u-\overline{u}$ since $\mathcal{N}(\varphi_n-\overline{\varphi_n})$ is orthogonal to the space of constant functions $V_1$ to which $\overline{u}$ belongs. As for the left-hand side, we notice that $\beta'$ is nonnegative, so that we just have to deal with the last term, but this can be readily handled by using that $\gamma$ is monotone, as remarked in (2.9). Namely, there exist some $\delta_0>0$ and $C_0>0$, such that

$$\gamma(s)(s-s_0)\geq\delta_0|\gamma(s)|-C_0\quad\text{for every } s\in\mathbb{R} \text{ and } s_0\in[-C^*,C^*]\,, \tag{3.26}$$

where $C^*$ is the constant introduced in (3.24). This inequality is similar to the one proved in [18, Appendix, Prop. A.1] (see also the detailed argument given in [10, p. 908] with a fixed $s_0$, which works in the present case with only minor modifications). By applying it, we find that

$$\int_\Omega\gamma(\varphi_n)(\varphi_n-\overline{\varphi_n})\geq\delta_0\int_\Omega|\gamma(\varphi_n)|-c\quad\text{a.e. in } (0,T_n)\,. \tag{3.27}$$

Now, we consider the terms on the right-hand side of (3.25). We have that

$$\begin{aligned}
&\int_\Omega\beta'(\varphi_n)\Delta\varphi_n\,(\varphi_n-\overline{\varphi_n})=-\int_\Omega\nabla\varphi_n\cdot\big(\beta''(\varphi_n)(\varphi_n-\overline{\varphi_n})\nabla\varphi_n+\beta'(\varphi_n)\nabla\varphi_n\big)\\
&=-\int_\Omega\big(\beta''(\varphi_n)-\beta''(\overline{\varphi_n})\big)(\varphi_n-\overline{\varphi_n})|\nabla\varphi_n|^2-\int_\Omega\big(\beta''(\overline{\varphi_n})(\varphi_n-\overline{\varphi_n})+\beta'(\varphi_n)\big)|\nabla\varphi_n|^2\\
&\leq-\int_\Omega\big(\beta''(\overline{\varphi_n})(\varphi_n-\overline{\varphi_n})+\beta'(\varphi_n)\big)|\nabla\varphi_n|^2\quad\text{a.e. in } (0,T_n),
\end{aligned}\tag{3.28}$$

where we have used the monotonicity of $\beta''$ implied by (2.6). To estimate the last integral, we perform almost everywhere in $\Omega\times(0,T_n)$ a Taylor expansion of $\beta'$ about $\overline{\varphi_n}$ using the integral remainder, and then first account for the sign of $\beta'''$ and then for (3.24). Then, we have almost everywhere in $\Omega\times(0,T_n)$ that

$$\begin{aligned}
&-\big(\beta''(\overline{\varphi_n})(\varphi_n-\overline{\varphi_n})+\beta'(\varphi_n)\big)\\
&=-\beta'(\overline{\varphi_n})-2\beta''(\overline{\varphi_n})(\varphi_n-\overline{\varphi_n})-\Big[\int_0^1\beta'''\big(\overline{\varphi_n}+s(\varphi_n-\overline{\varphi_n})\big)(1-s)\,ds\Big](\varphi_n-\overline{\varphi_n})^2\\
&\leq-\beta'(\overline{\varphi_n})-2\beta''(\overline{\varphi_n})(\varphi_n-\overline{\varphi_n})\leq\widehat{C}\,(|\varphi_n|+1)\,,
\end{aligned}\tag{3.29}$$

where we have employed the special symbol $\widehat{C}$ instead of $c$, since now we use its precise value. Now observe that by (2.7) we can find some $s_1>0$ such that

$$\widehat{C}(|s|+1)\;\leq\;\frac{1}{2}\beta'(s)\quad\text{for } |s|>s_1.$$

At this point, we fix an arbitrary $t\in(0,T_n)$ for which (3.25) and (3.27)–(3.29), written at the argument $t$, hold true (we know that this is actually the case for almost every $t\in(0,T_n)$). We then put $\Omega_1:=\{x\in\Omega:\,|\varphi_n(x,t)|>s_1\}$ and conclude that

$$\begin{aligned}
&\widehat{C}\int_\Omega(|\varphi_n(t)|+1)|\nabla\varphi_n(t)|^2=\widehat{C}\int_{\Omega_1}(|\varphi_n(t)|+1)|\nabla\varphi_n(t)|^2+\widehat{C}\int_{\Omega\setminus\Omega_1}(|\varphi_n(t)|+1)|\nabla\varphi_n(t)|^2\\
&\leq\frac{1}{2}\int_\Omega\beta'(\varphi_n(t))|\nabla\varphi_n(t)|^2+\widehat{C}\,(s_1+1)\int_\Omega|\nabla\varphi_n(t)|^2\,,
\end{aligned}$$

where we also used that $\beta'$ is nonnegative. For the last integral, we make use of the compactness inequality (2.36), which yields that

$$\widehat{C}\,(s_1+1)\int_\Omega |\nabla\varphi_n(t)|^2 \le \frac{1}{4}\int_\Omega |\Delta\varphi_n(t)|^2 + c\,\|\varphi_n(t)\|_*^2$$
$$\le \frac{1}{4}\int_\Omega |\Delta\varphi_n(t)|^2 + c\,\|\varphi_n(t)-\overline{\varphi_n}(t)\|_*^2 + c\,,$$

where $c$ is independent of both $t$ and $n$. Thus, we deduce that

$$-\int_\Omega \big(\beta''(\overline{\varphi_n})(\varphi_n-\overline{\varphi_n}) + \beta'(\varphi_n)\big)|\nabla\varphi_n|^2 \le c\int_\Omega (|\varphi_n|+1)|\nabla\varphi_n|^2$$
$$\le \frac{1}{2}\int_\Omega \beta'(\varphi_n)|\nabla\varphi_n|^2 + \frac{1}{4}\int_\Omega |\Delta\varphi_n|^2 + c\,\|\varphi_n-\overline{\varphi_n}\|_*^2 + c \quad \text{a.e. in } (0,T_n). \tag{3.30}$$

With this, the estimation of the first term on the right-hand side of (3.25) is completed. Next, by Young's inequality and the compactness inequality (2.36), we have that

$$-(2\lambda-\nu)\int_\Omega \Delta\varphi_n\,(\varphi_n-\overline{\varphi_n}) \le \frac{1}{8}\int_\Omega |\Delta\varphi_n|^2 + c\int_\Omega |\varphi_n-\overline{\varphi_n}|^2$$
$$\le \frac{1}{4}\int_\Omega |\Delta\varphi_n|^2 + c\,\|\varphi_n-\overline{\varphi_n}\|_*^2 \quad \text{a.e. in } (0,T_n). \tag{3.31}$$

To treat the next term of (3.25), we observe that (2.12) ensures, similar to above, the existence of some $s_2>0$ such that

$$|g(s)|\big(|s|+C^*\big) \le \frac{\delta_0}{2}\,|\gamma(s)| \quad \text{whenever } |s|>s_2\,,$$

where $C^*$ is the constant appearing in (3.24) and $\delta_0$ is the same as in (3.26).

We now consider an arbitrary $t\in(0,T_n)$ for which (3.25) and (3.27)–(3.31), evaluated at $t$, hold true (this is the case for a.e. $t\in(0,T_n)$), and put $\Omega_2 := \{x\in\Omega:\ |\varphi_n(x,t)|>s_2\}$. Then, it turns out that

$$-\int_\Omega g(\varphi_n(t))(\varphi_n(t)-\overline{\varphi_n}(t))$$
$$\le \int_{\Omega_2} |g(\varphi_n(t))|\,(|\varphi_n(t)|+C^*) + \int_{\Omega\setminus\Omega_2} |g(\varphi_n(t))|\,(|\varphi_n(t)|+C^*)$$
$$\le \frac{\delta_0}{2}\int_\Omega |\gamma(\varphi_n(t))| + c\,,$$

where, again, $c$ is independent of both $t$ and $n$. Finally, we easily see that

$$\int_\Omega u\,\mathcal{N}(\varphi_n-\overline{\varphi_n}) \le c\,\|u\|_*\,\|\mathcal{N}(\varphi_n-\overline{\varphi_n})\|_V \le c\,\|\varphi_n-\overline{\varphi_n}\|_*^2 + c \quad \text{a.e. in } (0,T_n),$$

where we have used the inequalities $\|v\|_* \le c\,\|v\|_\infty$ for $v\in L^\infty(\Omega)$ with $v=u$, and $\|u\|_\infty \le M$, which we assumed in (2.13). At this point, we go back to (3.25), collect the estimates just proved, and rearrange the terms. Ignoring a nonnegative term on the left-hand side, we deduce that

$$\frac{d}{dt}\,\|\varphi_n-\overline{\varphi_n}\|_*^2 + \frac{1}{2}\int_\Omega |\Delta\varphi_n|^2 + \frac{1}{2}\int_\Omega \beta'(\varphi_n)|\nabla\varphi_n|^2 + \frac{\delta_0}{2}\int_\Omega |\gamma(\varphi_n)|$$
$$\le c\,\|\varphi_n-\overline{\varphi_n}\|_*^2 + c \quad \text{a.e. in } (0,T_n).$$

Now, we integrate over $(0,t)$ with an arbitrary $t \in (0,T_n]$. This integration produces the initial value $\varphi_n(0)$, and since $\varphi_n(0)$ is the $H$-orthogonal projection of $\varphi_0$ onto $V_n$ by (3.19), we have that

$$\|\varphi_n(0)\|_* \le c\,\|\varphi_n(0)\| \le c\,\|\varphi_0\| \,.$$

Thus, by applying the Gronwall lemma, we can conclude that

$$\|\varphi_n\|_{L^\infty(0,T_n;V^*)} \le c \,. \tag{3.32}$$

Now, since $\mathbb{R}^n$ is finite dimensional, there holds with some $c_n > 0$ the inequality

$$|(s_1,\dots,s_n)| \le c_n \left\| \textstyle\sum_{j=1}^n s_j e_j \right\|_{V^*} \quad \text{for every } (s_1,\dots,s_n) \in \mathbb{R}^n.$$

Hence, (3.32) implies that

$$\|\boldsymbol{\varphi}_n\|_{L^\infty(0,T_n;\mathbb{R}^n)} \le c_n \,,$$

that is, that $\boldsymbol{\varphi}_n$ is uniformly bounded. Since $\boldsymbol{\varphi}_n$ is also maximal, it has to be global. Thus, $T_n = T$, and the proof is complete. □

The rest of the present subsection is devoted to the existence and stability part of Theorem 2.2. To this end, we first upgrade and improve the estimates already obtained, and then perform further ones. The first estimate (3.24) yields an $L^\infty$ bound for $\overline{\varphi_n}$. However, we immediately find from (3.23) that the derivative of $\overline{\varphi_n}$ is bounded as well. Hence, we have that

$$\|\overline{\varphi_n}\|_{W^{1,\infty}(0,T)} \le c \,. \tag{3.33}$$

Next, the above argument involving Gronwall's lemma, which now can be applied for arbitrary $t \in (0,T]$, leads to the estimate

$$\|\varphi_n - \overline{\varphi_n}\|_{L^\infty(0,T;V^*)} + \|\Delta\varphi_n\|_{L^2(0,T;H)} + \|\beta'(\varphi_n)|\nabla\varphi_n|^2\|_{L^1(Q)} + \|\gamma(\varphi_n)\|_{L^1(Q)} \le c \,. \tag{3.34}$$

Thus, on account of (3.33), we realize that $\varphi_n$ is bounded in $L^\infty(0,T;V^*)$, and by also invoking the elliptic regularity theory, we conclude that

$$\|\varphi_n\|_{L^\infty(0,T;V^*)\cap L^2(0,T;W)} \le c \,. \tag{3.35}$$

**Third a priori estimate.** We now test (3.18) by $1 \in V_1$, obtaining that

$$|\Omega|\,\overline{\mu_n} = -\int_\Omega \beta'(\varphi_n)\Delta\varphi_n + \int_\Omega \bigl(\gamma(\varphi_n) + g(\varphi_n)\bigr) \quad \text{a.e. in } (0,T). \tag{3.36}$$

Now, by also accounting for the growth condition in (2.8), we infer that

$$\begin{aligned} &\Bigl|-\int_\Omega \beta'(\varphi_n)\Delta\varphi_n\Bigr| = \Bigl|\int_\Omega \nabla\varphi_n \cdot \nabla\beta'(\varphi_n)\Bigr| \\ &\le \int_\Omega |\beta''(\varphi_n)|\,|\nabla\varphi_n|^2 \le c\int_\Omega (\beta'(\varphi_n)+1)|\nabla\varphi_n|^2 \quad \text{a.e. in } (0,T). \end{aligned} \tag{3.37}$$

On the other hand, the definition (2.11) of $g$, (2.12), and the continuity of the involved functions, imply that

$$|g(s)| \le c\,(|\gamma(s)|+1) \quad \text{for every } s \in \mathbb{R}. \tag{3.38}$$

Hence, recalling (3.34)–(3.35), we conclude that

$$\|\overline{\mu_n}\|_{L^1(0,T)} \le c\,. \tag{3.39}$$

**Fourth a priori estimate.** We test (3.17) by $\mu_n$ and have a.e. in $(0,T)$ that

$$\int_\Omega \partial_t\varphi_n\,\mu_n + \int_\Omega |\nabla\mu_n|^2 = \int_\Omega (u-\sigma\varphi_n)\mu_n\,. \tag{3.40}$$

Next, we rewrite (3.18) in terms of the original potential, as

$$\begin{aligned}&\int_\Omega \Delta\varphi_n\,\Delta v - \int_\Omega \Delta f(\varphi_n)\,v - \int_\Omega f'(\varphi_n)\Delta\varphi_n\,v\\ &\quad - \nu\int_\Omega \Delta\varphi_n\,v + \int_\Omega f(\varphi_n)\,f'(\varphi_n)\,v + \nu\int_\Omega f(\varphi_n)\,v\\ &= \int_\Omega \mu_n\,v \quad \text{a.e. in } (0,T), \text{ for every } v\in V_n\,,\end{aligned}$$

and test it by $\partial_t\varphi_n$. Moreover, we perform an integration by parts on the left-hand side, where we recall that both $\varphi_n$ and $\partial_t\varphi_n$ are $W$-valued. We obtain that

$$\begin{aligned}&\int_\Omega (-\Delta\varphi_n)\,(-\Delta\partial_t\varphi_n) - \int_\Omega f(\varphi_n)\Delta\partial_t\varphi_n - \int_\Omega f'(\varphi_n)\Delta\varphi_n\,\partial_t\varphi_n\\ &\quad - \nu\int_\Omega \Delta\varphi_n\,\partial_t\varphi_n + \int_\Omega f(\varphi_n)\,f'(\varphi_n)\,\partial_t\varphi_n + \nu\int_\Omega f(\varphi_n)\,\partial_t\varphi_n\\ &= \int_\Omega \mu_n\,\partial_t\varphi_n \quad \text{a.e. in } (0,T),\end{aligned} \tag{3.41}$$

and recognize the time derivative of $\mathcal{E}(\varphi_n)$ (see (1.7)) on the left-hand side. Hence, by adding the equalities (3.40) and (3.41) to each other, rearranging, and noting an obvious cancellation, we deduce that

$$\frac{d}{dt}\,\mathcal{E}(\varphi_n) + \int_\Omega |\nabla\mu_n|^2 = \int_\Omega (u-\sigma\varphi_n)\mu_n \quad \text{a.e. in } (0,T).$$

Integration over $(0,t)$, for an arbitrary $t\in(0,T]$, then yields that, recalling the definition (2.1) of $Q_t$,

$$\begin{aligned}&\frac12\int_\Omega |-\Delta\varphi_n(t)+f(\varphi_n(t))|^2 + \frac{\nu}{2}\int_\Omega |\nabla\varphi_n(t)|^2 + \nu\int_\Omega F(\varphi_n(t)) + \int_{Q_t} |\nabla\mu_n|^2\\ &= \frac12\int_\Omega |-\Delta\varphi_n(0)+f(\varphi_n(0))|^2 + \frac{\nu}{2}\int_\Omega |\nabla\varphi_n(0)|^2 + \nu\int_\Omega F(\varphi_n(0))\\ &\quad + \int_{Q_t} (u-\sigma\varphi_n)\mu_n \quad \text{for every } t\in[0,T].\end{aligned} \tag{3.42}$$

The left-hand side is bounded from below (see (2.10)). As for the terms involving the initial value, we recall that $\varphi_0\in W$ and that $\|\varphi_n(0)\|_W$ is bounded (see (2.13) and Remark 3.3). Thus, also $\|\varphi_n(0)\|_\infty$ is bounded due to the continuous embedding $W\hookrightarrow L^\infty(\Omega)$, so that all of the terms involving the initial value are bounded. Let us estimate the last term on the right-hand side of (3.42). It holds that

$$\int_{Q_t} (u-\sigma\varphi_n)\mu_n = \int_{Q_t} (u-\sigma\varphi_n)(\mu_n-\overline{\mu_n}) + \int_{Q_t} (u-\sigma\varphi_n)\overline{\mu_n}\,,$$

where the Young and Poincaré inequalities imply that

$$\int_{Q_t} (u - \sigma\varphi_n)(\mu_n - \overline{\mu_n}) \le \frac{1}{2}\int_{Q_t} |\nabla\mu_n|^2 + c\int_{Q_t} (|u|^2 + |\varphi_n|^2) \le \frac{1}{2}\int_{Q_t} |\nabla\mu_n|^2 + c\,.$$

In the last inequality, we made use of (2.13) and (3.39), which also justify the computations to follow. Indeed, owing to (3.35) as well, we can estimate the remaining term this way:

$$\begin{aligned}&\int_{Q_t} (u - \sigma\varphi_n)\overline{\mu_n} \le c\int_0^T \|u(s) - \sigma\varphi_n(s)\|_* \, \|\overline{\mu_n}(s)\|_V \, ds \\ &= c\int_0^T \|u(s) - \sigma\varphi_n(s)\|_* \, |\overline{\mu_n}(s)| \, ds \le c\,\|u - \sigma\varphi_n\|_{L^\infty(0,T;V^*)} \, \|\overline{\mu_n}\|_{L^1(0,T)} \le c\,.\end{aligned}$$

Collecting the above computations, we conclude that

$$\begin{aligned}&\|-\Delta\varphi_n + f(\varphi_n)\|_{L^\infty(0,T;H)} + \|\nabla\varphi_n\|_{L^\infty(0,T;H)} \\ &\quad + \|F(\varphi_n)\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla\mu_n\|_{L^2(0,T;H)} \le c\,.\end{aligned}$$

By a standard argument regarding the elliptic operator $-\Delta + f(\cdot)$, and accounting for (3.34) once more, we deduce that

$$\|\varphi_n\|_{L^\infty(0,T;W)} + \|f(\varphi_n)\|_{L^\infty(0,T;H)} + \|F(\varphi_n)\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla\mu_n\|_{L^2(0,T;H)} \le c\,. \tag{3.43}$$

**Fifth a priori estimate.** Thanks to the continuous embedding $W \hookrightarrow L^\infty(\Omega)$, we have that $L^\infty(0,T;W) \hookrightarrow L^\infty(Q)$. Therefore, also using the regularity of the nonlinearities, we deduce from (3.43) that

$$\|\varphi_n\|_\infty \le c\,, \quad \|\beta'(\varphi_n)\|_\infty \le c\,, \quad \text{and} \quad \|\gamma(\varphi_n)\|_\infty \le c\,, \quad \text{a.e. in } (0,T)\,.$$

Hence, coming back to (3.36) and (3.38), we conclude that

$$\|\overline{\mu_n}\|_{L^\infty(0,T)} \le c\,,$$

which, together with (3.43) and Poincaré's inequality, produces

$$\|\mu_n\|_{L^2(0,T;V)} \le c\,. \tag{3.44}$$

**Sixth a priori estimate.** We take any $v \in L^2(0,T;V)$ and define $v_n \in L^2(0,T;V_n)$ by setting $v_n(t) := \mathbb{P}_n v(t)$ for a.a. $t \in (0,T)$. Then, we test (3.17) by $v_n$. Noting that $\partial_t\varphi_n(t) \in V_n$ for a.a. $t \in (0,T)$, and owing to the above estimates along with Remark 3.3, we obtain that

$$\begin{aligned}&\int_Q \partial_t\varphi_n \, v = \int_Q \partial_t\varphi_n \, v_n = \int_Q (u - \sigma\varphi_n)v_n - \int_Q \nabla\mu_n \cdot \nabla v_n \\ &\le c\,\|v_n\|_{L^2(0,T;V)} \le c\,\|v\|_{L^2(0,T;V)}\,,\end{aligned}$$

from which it is readily seen that

$$\|\partial_t\varphi_n\|_{L^2(0,T;V^*)} \le c\,. \tag{3.45}$$

**Conclusion of the existence proof.** Eventually, by letting $n$ tend to infinity, we find a solution $(\varphi, \mu)$ to the equations (2.17) and (3.1) that satifies the initial condition (2.20). Recalling all the above estimates, and invoking well-known compactness results (for the strong compactness see, e.g., [26, Sect. 8, Cor. 4]), we have that, as $n \to \infty$,

$$\varphi_n \to \varphi \quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;W), \\ \text{strongly in } C^0([0,T];V) \text{ and uniformly in } Q\,, \tag{3.46}$$

$$\mu_n \to \mu \quad \text{weakly in } L^2(0,T;V)\,, \tag{3.47}$$

for some pair $(\varphi, \mu)$, and at least for a (nonrelabeled) subsequence. Notice that also the inequality obtained from (2.21) by ignoring the part regarding $w$ is satisfied, due to the semicontinuity of norms.

By (3.46), $\{\varphi_n(0)\}$ converges to $\varphi(0)$ strongly in $V$. On the other hand, since $\varphi_n(0)$ is the $H$-projection of $\varphi_0$ onto $V_n$, it converges to $\varphi_0$ strongly in $H$. Hence, the initial condition (2.20) is satisfied. We show that the equations (2.17) and (3.1) are satisfied as well. Since $\{\varphi_n\}$ is uniformly bounded in $L^\infty(Q)$, and since the involved nonlinearities in the original system are smooth and thus Lipschitz continuous on bounded sets, we can deduce that

$$\beta^{(i)}(\varphi_n) \to \beta^{(i)}(\varphi) \quad \text{uniformly in } Q, \quad \text{for } i = 0, 1, 2. \tag{3.48}$$

At this point, we write the time-integrated version of (3.17) and (3.18), namely

$$\int_Q \partial_t \varphi_n \, v_n + \int_Q \nabla \mu_n \cdot \nabla v_n + \sigma \int_Q \varphi_n v_n = \int_Q u v_n \quad \text{for every } v_n \in L^2(0,T;V_n)\,, \tag{3.49}$$

$$\int_Q \Delta\varphi_n \, \Delta v_n - \int_Q \Delta\beta(\varphi_n)\, v_n - \int_Q \beta'(\varphi_n)\Delta\varphi_n \, v_n \\ + \int_Q \gamma(\varphi_n)\, v_n + (2\lambda - \nu) \int_Q \Delta\varphi_n \, v_n + \int_Q g(\varphi_n)\, v_n \\ = \int_Q \mu_n \, v_n \quad \text{for every } v_n \in L^2(0,T;V_n)\,. \tag{3.50}$$

Our aim is to take the limit as $n \to \infty$ in these equations with $V$- or $W$-valued test functions. We fix $v \in L^2(0,T;V)$ and take as $v_n$ in (3.49) the function defined by setting $v_n(t) := \mathbb{P}_n v(t)$ for a.a. $t \in (0,T)$. Then, $\{v_n\}$ converges to $v$ strongly in $L^2(0,T;V)$ by Remark 3.3, and we have that

$$\int_Q \partial_t \varphi \, v + \int_Q \nabla\mu \cdot \nabla v + \sigma \int_Q \varphi \, v = \int_Q u v\,.$$

Since $v$ is arbitrary in $L^2(0,T;V)$, this is equivalent to (2.17). Similarly, given an arbitrary $v \in L^2(0,T;W)$, we test (3.50) by the function $v_n$ defined as before. Now, $\{v_n\}$ converges to $v$ strongly in $L^2(0,T;W)$. To take the limit in (3.50), it is convenient to transform one of the terms as follows:

$$-\int_Q \Delta\beta(\varphi_n)\, v_n = \int_Q \beta'(\varphi_n)\nabla\varphi_n \cdot \nabla v_n\,.$$

Then we obtain that

$$\int_Q \Delta\varphi \, \Delta v + \int_Q \beta'(\varphi)\nabla\varphi \cdot \nabla v - \int_Q \beta'(\varphi)\Delta\varphi \, v \\ + \int_Q \gamma(\varphi)\, v + (2\lambda - \nu) \int_\Omega \Delta\varphi \, v + \int_Q g(\varphi)\, v = \int_Q \mu \, v\,. \tag{3.51}$$

Now, we have that $\{\varphi_n\}$ converges to $\varphi$ strongly in $C^0([0,T];W^{1,4}(\Omega))$ thanks to the just quoted strong compactness result and the compactness of the embedding $W \hookrightarrow W^{1,4}(\Omega)$. It follows that $\{\nabla\varphi_n\}$ converges to $\nabla\varphi$ strongly in $C^0([0,T];L^4(\Omega))$, and $\{|\nabla\varphi_n|^2\}$ converges to $|\nabla\varphi|^2$ strongly in $C^0([0,T];H)$. Since

$$\Delta\beta(\varphi_n) = \beta''(\varphi_n)|\nabla\varphi_n|^2 + \beta'(\varphi_n)\Delta\varphi_n\,,$$

we deduce that $\{\Delta\beta(\varphi_n)\}$ strongly converges (at least) in $L^2(0,T;H)$, as $n\to\infty$, and that its limit must be $\Delta\beta(\varphi)$, due to (3.48). Hence we can write

$$\int_Q \beta'(\varphi)\nabla\varphi\cdot\nabla v = -\int_Q \Delta\beta(\varphi)\,v$$

in (3.51). Since $v$ is arbitrary in $L^2(0,T;W)$, we infer that (3.51) is equivalent to (3.1), thus concluding the proof.

## 3.2 Uniqueness and continuous dependence

This subsection is devoted to the proof of the uniqueness of the solution to the problem (2.17)–(2.20) and the continuous dependence estimate (2.24). Thanks to Proposition 3.1, we can deal with the alternative formulation in (3.7). The outline of our strategy is as follows: $(i)$ prove an inequality in the direction of (2.24) for arbitrary pairs of solutions; $(ii)$ derive uniqueness; $(iii)$ complete the proof of (2.24). First, let us fix two functions $u_i \in L^\infty(Q)$ with $\|u_i\|_\infty \le M$, $i=1,2$, consider arbitrary corresponding solutions $(\varphi_i,\mu_i)$ as given by Theorem 2.2, and set for convenience

$$u := u_1 - u_2\,, \quad \varphi := \varphi_1 - \varphi_2\,, \quad \text{and} \quad \mu := \mu_1 - \mu_2\,.$$

We first write (2.17) for both solutions and take the difference. We thus find that

$$\langle \partial_t\varphi, v\rangle_V + \int_\Omega \nabla\mu\cdot\nabla v + \sigma\int_\Omega \varphi v = \int_\Omega uv \quad \text{a.e. in } (0,T), \text{ for every } v\in V\,. \tag{3.52}$$

We test this equation by $1/|\Omega|$, obtaining an identity for $\overline{\varphi}$, which we then multiply by any $v\in V$ to obtain the equation

$$\langle \partial_t\overline{\varphi}, v\rangle_V + \sigma\int_\Omega \overline{\varphi}v = \int_\Omega \overline{u}v \quad \text{a.e. in } (0,T), \text{ for every } v\in V. \tag{3.53}$$

Next, we take the difference between (3.52) and (3.53) and test it by $\mathcal{N}(\varphi-\overline{\varphi})$. Recalling the properties (2.39)–(2.44), we have a.e. in $(0,T)$ that

$$\frac12\,\frac{d}{dt}\,\|\varphi-\overline{\varphi}\|_*^2 + \int_\Omega \mu(\varphi-\overline{\varphi}) + \sigma\,\|\varphi-\overline{\varphi}\|_*^2 = \int_\Omega (u-\overline{u})\mathcal{N}(\varphi-\overline{\varphi})\,. \tag{3.54}$$

At the same time, we write (3.1) for both solutions and test the difference by $\varphi-\overline{\varphi}$. Upon rearranging, we obtain that

$$\begin{aligned}\int_\Omega |\Delta\varphi|^2 &= -\int_\Omega \nabla\big(\beta(\varphi_1)-\beta(\varphi_2)\big)\cdot\nabla\varphi \\ &\quad + \int_\Omega \big(\beta'(\varphi_1)\Delta\varphi_1 - \beta'(\varphi_2)\Delta\varphi_2\big)(\varphi-\overline{\varphi}) - \int_\Omega \big(\gamma(\varphi_1)-\gamma(\varphi_2)\big)(\varphi-\overline{\varphi}) \\ &\quad + (2\lambda-\nu)\int_\Omega |\nabla\varphi|^2 - \int_\Omega \big(g(\varphi_1)-g(\varphi_2)\big)(\varphi-\overline{\varphi}) + \int_\Omega \mu(\varphi-\overline{\varphi})\,. \end{aligned} \tag{3.55}$$

Now, we estimate the integrals on the right-hand side. For a while, we allow the values of $c$ to depend on the solutions we have fixed. In particular, since $\varphi_1$ and $\varphi_2$ are bounded, we can make use of the Lipschitz continuity of the nonlinearities on their range, the Lipschitz constant being dependent on these solutions. To begin with, we have for the right-hand side of (3.54) that

$$\begin{aligned}&\int_\Omega (u-\overline{u})\mathcal{N}(\varphi-\overline{\varphi}) \;=\; \int_\Omega u\mathcal{N}(\varphi-\overline{\varphi}) \;\le\; \|\mathcal{N}(\varphi-\overline{\varphi})\|_V \, \|u\|_{V^*} \\ &\le\; c\,\|\varphi-\overline{\varphi}\|_* \, \|u\|_* \;\le\; \frac{\sigma}{2}\,\|\varphi-\overline{\varphi}\|_*^2 + c\,\|u\|_*^2 \,.\end{aligned}$$

Next, we deal with the right-hand side of (3.55). To estimate the first term, we recall that $\beta'$ is nonnegative, the continuity of the embedding $W \hookrightarrow W^{1,4}(\Omega)$, and the regularity (2.14) for $\varphi_1$. Moreover, we owe to the Lipschitz continuity of $\beta'$ and the Hölder inequality, and apply the compactness inequality (2.36). For every $\delta > 0$, it holds that

$$\begin{aligned}&-\int_\Omega \nabla\big(\beta(\varphi_1)-\beta(\varphi_2)\big)\cdot\nabla\varphi = -\int_\Omega \big(\beta'(\varphi_1)\nabla\varphi_1 - \beta'(\varphi_2)\nabla\varphi_2\big)\cdot\nabla\varphi \\ &= -\int_\Omega \big(\beta'(\varphi_1)-\beta'(\varphi_2)\big)\nabla\varphi_1\cdot\nabla\varphi - \int_\Omega \beta'(\varphi_2)|\nabla\varphi|^2 \le c\int_\Omega |\varphi|\,|\nabla\varphi_1|\,|\nabla\varphi| \\ &\le c\,\|\varphi\|_4\,\|\nabla\varphi_1\|_4\,\|\nabla\varphi\| \le c\,\|\varphi\|_V^2 \le \delta\int_\Omega |\Delta\varphi|^2 + c_\delta\,\|\varphi\|_*^2 \,.\end{aligned}$$

Now observe that the procedure used to derive (3.23) can be applied to $\varphi$ and $u$, so that the pointwise values of $\overline{\varphi}$ can be estimated by the $L^2$ norm of $\overline{u}$, and thus also by the norm of $u$ in $L^2(0,T;V^*)$. We therefore have

$$|\overline{\varphi}| \;\le\; c\,\|u\|_{L^2(0,T;V^*)} \quad \text{a.e. in } (0,T),$$

whence we conclude that

$$\|\varphi\|_*^2 \le 2\,\|\varphi-\overline{\varphi}\|_*^2 + 2\,\|\overline{\varphi}\|_*^2 \le 2\,\|\varphi-\overline{\varphi}\|_*^2 + c\,|\overline{\varphi}|^2 \le 2\,\|\varphi-\overline{\varphi}\|_*^2 + c\,\|u\|_{L^2(0,T;V^*)}^2 \,.$$

Therefore, we find that

$$-\int_\Omega \nabla\big(\beta(\varphi_1)-\beta(\varphi_2)\big)\cdot\nabla\varphi \le \delta\int_\Omega |\Delta\varphi|^2 + c_\delta\,\|\varphi-\overline{\varphi}\|_*^2 + c_\delta\,\|u\|_{L^2(0,T;V^*)}^2 \,.$$

Mimicking this argument, we find for the next term that

$$\begin{aligned}&\int_\Omega \big(\beta'(\varphi_1)\Delta\varphi_1 - \beta'(\varphi_2)\Delta\varphi_2\big)(\varphi-\overline{\varphi}) \\ &= \int_\Omega \big(\beta'(\varphi_1)-\beta'(\varphi_2)\big)\Delta\varphi_1\,(\varphi-\overline{\varphi}) + \int_\Omega \beta'(\varphi_2)\Delta\varphi\,(\varphi-\overline{\varphi}) \\ &\le c\,\|\varphi\|_4\,\|\Delta\varphi_1\|\,\|\varphi-\overline{\varphi}\|_4 + c\,\|\Delta\varphi\|\,\|\varphi-\overline{\varphi}\| \\ &\le c\,\|\varphi\|_V\,\|\varphi-\overline{\varphi}\|_V + c\,\|\Delta\varphi\|\,\|\varphi-\overline{\varphi}\| \\ &\le c\,\|\varphi-\overline{\varphi}\|_V^2 + c\,\|\overline{u}\|_{L^2(0,T)}^2 + \delta\int_\Omega |\Delta\varphi|^2 + c_\delta\,\|\varphi-\overline{\varphi}\|^2 \\ &\le 3\delta\int_\Omega |\Delta\varphi|^2 + c_\delta\,\|\varphi-\overline{\varphi}\|_*^2 + c\,\|u\|_{L^2(0,T;V^*)}^2 \,.\end{aligned}$$

Since $\gamma$ and $g$ are Lipschitz continuous on the range of the solutions under consideration, the sum $S$ of the next three terms can be treated as before using the compactness inequality, namely

$$S \le \|\varphi\|^2 \|\varphi - \overline{\varphi}\|^2 + \|\varphi - \overline{\varphi}\|_V^2 \le 2\delta \int_\Omega |\Delta\varphi|^2 + c_\delta \,\|\varphi - \overline{\varphi}\|_*^2 + c\,\|u\|_{L^2(0,T;V^*)}^2 \,.$$

At this point, we add (3.54) and (3.55) to each other and notice the cancellation of the terms involving the chemical potentials. Then, we account for all of the above estimates, choose $\delta$ small enough, integrate with respect to time, and apply the Gronwall lemma to conclude that

$$\|\varphi\|_{L^\infty(0,T;V^*)} + \|\Delta\varphi\|_{L^2(0,T;H)} \le c\,\|u\|_{L^2(0,T;V^*)} \,. \tag{3.56}$$

In this estimate, the constant $c$ depends on the solutions we have considered. Nevertheless, by applying it in the case $u_1 = u_2$, we derive that $\varphi_1 = \varphi_2$. Then, by also considering the third components $w_i$ of the solutions to the original problem, and recalling first (2.19) and then (2.18), we deduce that $w_1 = w_2$ and $\mu_1 = \mu_2$. This proves the uniqueness part of Theorem 2.2.

Now, we come back to (3.56) and observe that the uniqueness just established implies that the solutions $\varphi_i$, for $i = 1, 2$, at hand must coincide with those constructed in our existence proof. Therefore, their norms we have considered in this proof are uniformly bounded as specified in the stability inequality. We conclude that the constant $c$ that appears in (3.56) can be estimated in terms of $\Omega$, $T$, the structure of the system, the initial datum $\varphi_0$ and $M$.

So, from now on, the symbol $c$ denotes constants that just depend on $\Omega, T$, the structure of the system, the initial datum $\varphi_0$, and $M$, as before, since the norms of the solutions involved in our argument can be controlled by the stability estimate (2.21). In particular, this holds for the estimate

$$\|\varphi\|_{L^\infty(0,T;V^*)\cap L^2(0,T;W)} \le c\,\|u\|_{L^2(0,T;V^*)} \,, \tag{3.57}$$

which is obtained from (3.56) on account of elliptic regularity.

We are now going to continue the stability analysis of (2.17)–(2.20) with respect to the control variable. This will be a crucial result in the derivation of first-order optimality conditions for the associated optimal control problem. Thus, we write (2.17) for both solutions and test the difference by $\varphi$. At the same time, recalling (3.6), we write (3.1) for both solutions and test the difference by $-\Delta\varphi$. Then we add the resulting equalities to each other and rearrange, noticing that a cancellation occurs. By this procedure, we obtain a.e. in $(0,T)$ that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi|^2 + \sigma \int_\Omega |\varphi|^2 + \int_\Omega |\nabla\Delta\varphi|^2 \\
&= \int_\Omega u\varphi - \int_\Omega \big(\Delta\beta(\varphi_1) - \Delta\beta(\varphi_2)\big)\,\Delta\varphi \\
&\quad - \int_\Omega \big(\beta'(\varphi_1)\Delta\varphi_1 - \beta'(\varphi_2)\Delta\varphi_2\big)\Delta\varphi + \int_\Omega \big(\gamma(\varphi_1) - \gamma(\varphi_2)\big)\,\Delta\varphi \\
&\quad + (2\lambda - \nu)\int_\Omega |\Delta\varphi|^2 + \int_\Omega \big(g(\varphi_1) - g(\varphi_2)\big)\,\Delta\varphi \,,
\end{aligned} \tag{3.58}$$

and we have to estimate the terms on the right-hand side. The first one is trivial, and the second integral is estimated as follows:

$$\begin{aligned}
&- \int_\Omega \big(\Delta\beta(\varphi_1) - \Delta\beta(\varphi_2)\big)\,\Delta\varphi = \int_\Omega \big(\beta'(\varphi_1)\nabla\varphi_1 - \beta'(\varphi_2)\nabla\varphi_2\big)\cdot\nabla\Delta\varphi \\
&\le \delta \int_\Omega |\nabla\Delta\varphi|^2 + c_\delta \int_\Omega |\beta'(\varphi_1)\nabla\varphi_1 - \beta'(\varphi_2)\nabla\varphi_2|^2 \,,
\end{aligned}$$

where $\delta$ is an arbitrary positive constant to be chosen later on. Now, by accounting for the Lipschitz continuity of $\beta'$, Hölder's inequality, the regularity of $\varphi_1$, and the continuity of the embedding $V \hookrightarrow L^4(\Omega)$, we can bound the last integral in the following form:

$$\int_\Omega |\beta'(\varphi_1)\nabla\varphi_1 - \beta'(\varphi_2)\nabla\varphi_2|^2 \le 2\int_\Omega |\beta'(\varphi_1) - \beta'(\varphi_2)|^2 \, |\nabla\varphi_1|^2 + 2\int_\Omega |\beta'(\varphi_2)|^2 \, |\nabla\varphi|^2$$
$$\le c\,\|\varphi\|_4^2 \, \|\nabla\varphi_1\|_4^2 + c\,\|\varphi\|_V^2 \le c\,\|\varphi\|_V^2 .$$

For the third term on the right-hand side of (3.58), we proceed with a similar argument, obtaining that

$$-\int_\Omega \big(\beta'(\varphi_1)\Delta\varphi_1 - \beta'(\varphi_2)\Delta\varphi_2\big)\Delta\varphi = -\int_\Omega \big(\big(\beta'(\varphi_1) - \beta'(\varphi_2)\big)\Delta\varphi_1 + \beta'(\varphi_2)\Delta\varphi\big)\Delta\varphi$$
$$\le c\,\|\varphi\|_4 \, \|\Delta\varphi_1\| \, \|\Delta\varphi\|_4 + c\int_\Omega |\Delta\varphi|^2 \le c\,\|\varphi\|_V^2 + \delta\int_\Omega |\nabla\Delta\varphi|^2 + c_\delta\,\|\varphi\|_W^2 .$$

Finally, the last three terms of (3.58) can be treated in a straightforward way. At this point, we integrate (3.58) with respect to time, account for all of the estimates we have established, choose $\delta$ small enough, and apply (3.57) to conclude that

$$\|\varphi\|_{C^0([0,T];H)\cap L^2(0,T;H^3(\Omega))} \le c\,\|u\|_{L^2(0,T;V^*)} . \tag{3.59}$$

Now, we recall Remark 3.2, write (2.17) for both solutions, and test the difference by $-\Delta\varphi$. At the same time, we write (3.5) for both solutions, multiply the difference by $\Delta^2\varphi$, and integrate over $\Omega$. Then, we add the resulting equalities to each other. Since a cancellation occurs in the terms involving $\mu$, we have a.e. in $(0,T)$ that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi|^2 + \sigma\int_\Omega |\nabla\varphi|^2 + \int_\Omega |\Delta^2\varphi|^2$$
$$= \int_\Omega u(-\Delta\varphi) + \int_\Omega \big(\Delta\beta(\varphi_1) - \Delta\beta(\varphi_2)\big)\,\Delta^2\varphi$$
$$+ \int_\Omega \big(\beta'(\varphi_1)\Delta\varphi_1 - \beta'(\varphi_2)\Delta\varphi_2\big)\,\Delta^2\varphi - \int_\Omega \big(\gamma(\varphi_1) - \gamma(\varphi_2)\big)\,\Delta^2\varphi$$
$$- (2\lambda - \nu)\int_\Omega \Delta\varphi\,\Delta^2\varphi - \int_\Omega \big(g(\varphi_1) - g(\varphi_2)\big)\Delta^2\varphi . \tag{3.60}$$

As for the right-hand side, we immediately have that

$$\int_\Omega u(-\Delta\varphi) \le c\,\|u\|_* \, \|\Delta\varphi\|_V \le \|u\|_*^2 + c\,\|\varphi\|_{H^3(\Omega)}^2 ,$$

while the next terms need some treatment. Since $\Delta\beta(\varphi_i) = \beta'(\varphi_i)\Delta\varphi_i + \beta''(\varphi_i)|\nabla\varphi_i|^2$, for $i = 1, 2$, we can handle the second and third term on the right-hand side as

$$\int_\Omega \big(\Delta\beta(\varphi_1) - \Delta\beta(\varphi_2)\big)\,\Delta^2\varphi + \int_\Omega \big(\beta'(\varphi_1)\Delta\varphi_1 - \beta'(\varphi_2)\Delta\varphi_2\big)\,\Delta^2\varphi$$
$$= 2\int_\Omega \big(\beta'(\varphi_1)\Delta\varphi_1 - \beta'(\varphi_2)\Delta\varphi_2\big)\,\Delta^2\varphi + \int_\Omega \big(\beta''(\varphi_1)|\nabla\varphi_1|^2 - \beta''(\varphi_2)|\nabla\varphi_2|^2\big)\,\Delta^2\varphi$$
$$\le \delta\int_\Omega |\Delta^2\varphi|^2 + c_\delta\int_\Omega |\beta'(\varphi_1)\Delta\varphi_1 - \beta'(\varphi_2)\Delta\varphi_2|^2$$
$$+ c_\delta\int_\Omega \big|\beta''(\varphi_1)|\nabla\varphi_1|^2 - \beta''(\varphi_2)|\nabla\varphi_2|^2\big|^2 .$$

Besides, we have that

$$\int_\Omega |\beta'(\varphi_1)\Delta\varphi_1 - \beta'(\varphi_2)\Delta\varphi_2|^2 \le c\int_\Omega |\varphi|^2\,|\Delta\varphi_1|^2 + c\int_\Omega |\Delta\varphi|^2$$
$$\le c\,\|\varphi\|_4^2\,\|\Delta\varphi_1\|_4^2 + c\int_\Omega |\Delta\varphi|^2 \le c\,\|\varphi\|_V^2\,\|\varphi_1\|_{H^3(\Omega)}^2 + c\int_\Omega |\Delta\varphi|^2$$
$$\le c\,\|\varphi_1\|_{H^3(\Omega)}^2 \int_\Omega |\nabla\varphi|^2 + c\,\|\varphi_1\|_{H^3(\Omega)}^2\,\|\varphi\|_{L^\infty(0,T;H)}^2 + c\int_\Omega |\Delta\varphi|^2\,,$$

and we observe at once that the function $t \mapsto \|\varphi_1(t)\|_{H^3(\Omega)}$ is estimated in $L^2(0,T)$ (see Remark 2.3). This will be sufficient when we apply the Gronwall lemma to the first term of the last line after time integration, and useful to estimate the time integral of the second one in terms of $u$ on account of (3.59). In order to control the term containing $\beta''$, we observe that

$$\beta''(\varphi_1)|\nabla\varphi_1|^2 - \beta''(\varphi_2)|\nabla\varphi_2|^2$$
$$= \bigl(\beta''(\varphi_1) - \beta''(\varphi_2)\bigr)|\nabla\varphi_1|^2 + \beta''(\varphi_2)\nabla\varphi\cdot\nabla\varphi_1 + \beta''(\varphi_2)\nabla\varphi_2\cdot\nabla\varphi\,,$$

so that

$$\bigl|\beta''(\varphi_1)|\nabla\varphi_1|^2 - \beta''(\varphi_2)|\nabla\varphi_2|^2\bigr|^2 \le c\,|\varphi|^2\,|\nabla\varphi_1|^4 + c\,|\nabla\varphi|^2\,|\nabla\varphi_1|^2 + c\,|\nabla\varphi_2|^2\,|\nabla\varphi|^2$$
$$\le c\,\|\nabla\varphi_1\|_\infty^4\,|\varphi|^2 + c\,\bigl(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2\bigr)|\nabla\varphi|^2\,.$$

Therefore, we deduce that

$$\int_\Omega \bigl|\beta''(\varphi_1)|\nabla\varphi_1|^2 - \beta''(\varphi_2)|\nabla\varphi_2|^2\bigr|^2$$
$$\le c\,\|\nabla\varphi_1\|_\infty^4\|\varphi\|_{L^\infty(0,T;H)}^2 + c\int_\Omega \bigl(\|\nabla\varphi_1\|_\infty^2 + \|\nabla\varphi_2\|_\infty^2\bigr)|\nabla\varphi|^2\,,$$

and we observe that (2.23) applied to $\varphi_1$ ensures that, after integration over time, the last two terms can be controlled, the former by a direct estimate, and the latter via Gronwall's lemma. The last three terms of (3.60) can be treated without any difficulty. Then, by integrating (3.60) with respect to time, combining all the inequalities we have established, choosing $\delta$ small enough, accounting for (3.57) and (3.59), and applying the Gronwall lemma, we arrive at

$$\|\varphi\|_{C^0([0,T];V)\cap L^2(0,T;H^4(\Omega))} \le c\,\|u\|_{L^2(0,T;V^*)}\,. \tag{3.61}$$

This concludes the proof of the part of (2.24) concerning the difference $\varphi = \varphi_1 - \varphi_2$. It remains to prove the estimates regarding the other components of the solutions. To this end, it suffices to write (2.19) and (2.18) for both solutions, take the differences, and compare with the estimates already obtained.

# 4 The control problem

In this section, we deal with the control problem presented in Section 2. It is understood that in the entire section all of the assumptions made on the state system and the initial datum $\varphi_0$, which ensure the validity of Theorems 2.2 and 2.4, are in force, as well as those related to the control problem, i.e., conditions (2.25)–(2.27). Moreover, the generic constants denoted by $c$ can also depend on the latter. Here, in the control problem, the variable $u$ no longer plays just the role of a fixed and bounded source term, but it enters (2.17)–(2.20) as a control variable, which justifies the introduction of the upper bound given by $M$. Furthermore, we recall that the cost functional, the space $\mathcal{U}$, and the set $\mathcal{U}_{\rm ad}$ of admissible controls, are defined in (2.28)–(2.29). We also set

$$\mathcal{Y} := H^1(0,T;W^*) \cap L^2(0,T;W) \tag{4.1}$$

and recall that, for every $u \in \mathcal{U}$, there exists a unique solution $(\varphi, \mu, w)$ to problem (2.17)–(2.20). Since $\varphi$ belongs to $\mathcal{Y}$, we could define the *control-to-state* mapping as a mapping from $\mathcal{U}$ into $\mathcal{Y}$ as $u \mapsto \varphi$. However, it is convenient to consider just a neighborhood of the set $\mathcal{U}_{\rm ad}$ of admissible controls as the domain of $\mathcal{S}$. So, we introduce the set $\mathcal{U}_R$ and the solution map $\mathcal{S} : \mathcal{U}_R \to \mathcal{Y}$ by setting

$$\mathcal{U}_R := \{u \in \mathcal{U} : \ \|u\|_\infty < R\}, \quad \text{where} \quad R := \max\{\|u_{\min}\|_\infty, \|u_{\max}\|_\infty\} + 1, \tag{4.2}$$

$$\mathcal{U}_R \ni u \mapsto \mathcal{S}(u) := \varphi, \quad \text{where } (\varphi, \mu, w) \text{ is the unique solution to the problem (2.17)–(2.20) associated with } u. \tag{4.3}$$

We notice at once that we can apply Theorem 2.2 with $M = R$ and Remark 2.3. By also accounting for the regularity of $F$ in (2.4), we obtain, in particular, that

$$\|\varphi\|_{H^1(0,T;V^*)\cap L^\infty(0,T;W)\cap L^2(0,T;H^4(\Omega))} + \|\mu\|_{L^2(0,T;V)} + \|w\|_{L^\infty(0,T;H)\cap L^2(0,T;W)} + \sum_{i=0}^{4} \|F^{(i)}(\varphi)\|_\infty + \|\Delta f'(\varphi)\|_{L^\infty(0,T;H)} + \|\nabla f'(\varphi)\|_{L^2(0,T;L^\infty(\Omega))} \le c \tag{4.4}$$

for every $u \in \mathcal{U}_R$. This estimate will be used throughout the whole section. In the next three subsections, we prove the existence of an optimal control, the Fréchet differentiability of the solution mapping $\mathcal{S}$, and the first-order necessary condition for optimality presented in Section 2.

## 4.1 Existence of an optimal control

First, let us prove that the minimization problem introduced in (2.30) admits at least one solution.

**Theorem 4.1.** *There exists at least one optimal control for the control problem* (2.30)*, that is, there exists some* $u^* \in \mathcal{U}_{\rm ad}$ *such that*

$$\mathcal{J}(u^*, \mathcal{S}(u^*)) \le \mathcal{J}(u, \mathcal{S}(u)) \quad \text{for every } u \in \mathcal{U}_{\rm ad}. \tag{4.5}$$

*Proof.* We use the direct method of the calculus of variations. In this direction, we observe that $\mathcal{J}$ is bounded from below as it is nonnegative. Then, observing that $\mathcal{U}_{\rm ad}$ is nonempty, we take a minimizing sequence $\{u_n\}$ and, for every $n$, denote by $(\varphi_n, \mu_n, w_n)$ the corresponding state. Since $\mathcal{U}_{\rm ad}$ is bounded in $\mathcal{U}$, we can assume without loss of generality that

$$u_n \to u^* \quad \text{weakly star in } L^\infty(Q)$$

for some $u^* \in \mathcal{U}_{\rm ad}$, as $\mathcal{U}_{\rm ad}$ is convex and closed. Now, we notice that the stability estimate (4.4) holds true for $(\varphi_n, \mu_n, w_n)$ and some constant independent of $n$. Thus, we can assume without loss of generality that, as $n \to \infty$,

$$\begin{aligned} &\varphi_n \to \varphi^* \quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;W), \\ &\mu_n \to \mu^* \quad \text{weakly in } L^2(0,T;V), \\ &w_n \to w^* \quad \text{weakly in } L^2(0,T;W), \end{aligned}$$

for some limit triplet $(\varphi^*, \mu^*, w^*)$. Since the stability estimate implies that $\{\varphi_n\}$ is bounded in $L^\infty(Q)$, we can suppose that the nonlinearities involved in the state system are Lipschitz continuous. On the other hand, the above convergence property of $\{\varphi_n\}$, and well-known strong convergence results (see, e.g., [26, Sect. 8, Cor. 4]), imply the strong convergence of $\{\varphi_n\}$ to $\varphi^*$ as $n \to \infty$, e.g., in $L^\infty(Q)$. Thus, it is immediately seen that $(\varphi^*, \mu^*, w^*)$ is the solution to the state system corresponding to $u^*$, that is, $\varphi^* = \mathcal{S}(u^*)$. Therefore, we also conclude that

$$\lim_{n\to\infty} \mathcal{J}(u_n, \varphi_n) = \mathcal{J}(u^*, \varphi^*).$$

Now, since $\{u_n\}$ is supposed to be a minimizing sequence, the right-hand side of the equality is a minimum of $\mathcal{J}$, meaning that $u^*$ is an optimal control. $\square$

## 4.2 The control-to-state operator

This subsection is devoted to analyze some differentiability properties of the control-to-state mapping $\mathcal{S}$ defined in (4.3). Namely, we are going to show its Fréchet differentiability between suitable Banach spaces. Prior to exploring this further, it is crucial to introduce two preliminary lemmas.

**Lemma 4.2.** *The space $H^1(0,T;W)$ is dense in $H^1(0,T;W^*) \cap L^2(0,T;H)$.*

*Proof.* Let $z \in H^1(0,T;W^*) \cap L^2(0,T;H)$. We define $c_j$ and $z_n$ by means of the eigenfunctions of the eigenvalue problem (3.8)–(3.10) as follows:

$$c_j(t) := (z(t), e_j) \quad \text{for } j \ge 1 \quad \text{and} \quad z_n(t) := \sum_{j=1}^n c_j(t) e_j \quad \text{for } n \ge 1 \text{ and a.e. } t \in (0,T).$$

Then $c_j \in H^1(0,T)$ for every $j$, and $z_n \in H^1(0,T;W)$ for every $n$. We claim that $\{z_n\}$ converges strongly to $z$ in $H^1(0,T;W^*) \cap L^2(0,T;H)$. The strong convergence in $L^2(0,T;H)$ is rather straightforward, because

$$\lim_{n\to\infty} \|z - z_n\|^2_{L^2(0,T;H)} = \lim_{n\to\infty} \sum_{j>n} \int_0^T |c_j(t)|^2 \, dt = 0 \, .$$

Note that the function $v \mapsto |\overline{v}|^2 + \|\Delta v\|^2$ is the square of a norm in $W$ that is equivalent to the standard one, as one can see arguing by contradiction and using the compact embedding $W \hookrightarrow V$ and the elliptic regularity theory. Hence, the map $L : v \mapsto \{(v, e_j)\}_{j=1}^\infty$ is a topological isomorphism from $W$ onto the weighted space $\ell^2_\lambda$ defined by

$$\ell^2_\lambda := \Big\{ (a_j)_{j=1}^\infty : \sum_{j=1}^\infty |\lambda_j a_j|^2 < +\infty \Big\}, \quad \text{with the norm defined by}$$

$$\|(a_j)\|^2_\lambda := |a_1|^2 + \sum_{j=2}^\infty |\lambda_j a_j|^2 \, .$$

It follows that the adjoint operator $L^* : (\ell^2_\lambda)^* \to W^*$ is a topological isomorphism. Therefore, every $\zeta \in W^*$ admits a representation of the form

$$\zeta = \sum_{j=1}^\infty b_j e_j \quad \text{with} \quad \sum_{j=2}^\infty |\lambda_j^{-1} b_j|^2 < +\infty \, .$$

Moreover, this representation is unique, and $b_j = \langle \zeta, e_j \rangle_W$ for every $j$. Furthermore, the function

$$W^* \ni \zeta \mapsto |\langle \zeta, e_1 \rangle_W|^2 + \sum_{j=2}^{\infty} |\lambda_j^{-1} \langle \zeta, e_j \rangle_W|^2$$

is the square of a norm in $W^*$ that is equivalent to the standard one. It thus remains to prove that

$$\lim_{n\to\infty} \sum_{j>n} \lambda_j^{-2} \int_0^T |c_j'(t)|^2 \, dt = 0 \,,$$

but this is readily seen, since

$$\sum_{j=2}^{\infty} \lambda_j^{-2} \int_0^T |c_j'(t)|^2 \, dt \le c \, \|\partial_t z\|^2_{L^2(0,T;W^*)} \le c \,.$$

□

**Lemma 4.3.** *The identity*

$$\langle \partial_t \zeta(t), \mathcal{N}\zeta(t) \rangle_W = \frac{1}{2} \frac{d}{dt} \|\zeta(t)\|_*^2 \quad \text{for a.a. } t \in (0,T) \tag{4.6}$$

*holds true for every* $\zeta \in H^1(0,T;W^*) \cap L^2(0,T;H)$ *satisfying* $\overline{\zeta} = 0$ *a.e. in* $(0,T)$.

*Proof.* Let $\zeta \in H^1(0,T;W^*) \cap L^2(0,T;H)$ satisfy $\overline{\zeta} = 0$ a.e. in $(0,T)$. By Lemma 4.2, we can find a sequence $\{\xi_n\}$ in $H^1(0,T;W)$ that converges to $\zeta$ strongly in $H^1(0,T;W^*) \cap L^2(0,T;H)$. We next set $\zeta_n := \xi_n - \overline{\xi}_n$, so that $\overline{\zeta}_n = 0$. Thus, the sequence $\{\zeta_n\}$ converges to $\zeta - \overline{\zeta} = \zeta$ strongly in $H^1(0,T;W^*) \cap L^2(0,T;H)$, and the identity (4.6) holds for $\zeta_n$. Thus, for every fixed $s \le t$ in $[0,T]$, we have that

$$\int_s^t \langle \partial_t \zeta_n(\tau), \mathcal{N}\zeta_n(\tau) \rangle_W \, d\tau = \frac{1}{2} \big( \|\zeta_n(t)\|_*^2 - \|\zeta_n(s)\|_*^2 \big).$$

On the other hand, since (the restriction of) $\mathcal{N}$ is a continuous linear operator from the subspace of $H$ of the zero mean value functions into $W$, by the elliptic regularity theory, the convergence of $\{\zeta_n\}$ to $\zeta$ in $L^2(0,T;H)$ implies that $\{\mathcal{N}\zeta_n\}$ converges to $\mathcal{N}\zeta$ strongly in $L^2(0,T;W)$. Hence, we have that

$$\lim_{n\to\infty} \int_s^t \langle \partial_t \zeta_n(\tau), \mathcal{N}\zeta_n(\tau) \rangle_W \, d\tau = \int_s^t \langle \partial_t \zeta(\tau), \mathcal{N}\zeta(\tau) \rangle_W \, d\tau \,.$$

Now, we observe that $V$ coincides with the real interpolation space $(W,H)_{1/2}$, the interpolation being understood in the sense of the trace method in Hilbert spaces. We deduce that $V^* = (H,W^*)_{1/2}$, so that we have the continuous embedding

$$H^1(0,T;W^*) \cap L^2(0,T;H) \hookrightarrow C^0([0,T];V^*) \,.$$

It follows that the convergence of $\{\zeta_n\}$ to $\zeta$ in $H^1(0,T;W^*) \cap L^2(0,T;H)$ as $n \to \infty$ implies the strong convergence in $V^*$ of the pointwise values. As a result, it follows that

$$\lim_{n\to\infty} \|\zeta_n(s)\|_* = \|\zeta(s)\|_* \quad \text{and} \quad \lim_{n\to\infty} \|\zeta_n(t)\|_* = \|\zeta(t)\|_* \,,$$

and we conclude that

$$\int_s^t \langle \partial_t \zeta(\tau), \mathcal{N}\zeta(\tau) \rangle_W \, d\tau = \frac{1}{2} \big( \|\zeta(t)\|_*^2 - \|\zeta(s)\|_*^2 \big).$$

Since $s$ and $t$ are arbitrary in $[0,T]$, this is equivalent to (4.6), and the proof is complete. □

**Remark 4.4.** Let us remark that, arguing along the same lines as above, the analogous formula

$$\langle \partial_t z, \Delta z \rangle_W = -\frac{1}{2} \frac{d}{dt} \|\nabla z\|^2 \quad \text{a.e. in } (0,T), \tag{4.7}$$

can be shown for every $z \in H^1(0,T;W^*) \cap L^2(0,T;W)$ such that $\Delta z \in L^2(0,T;W)$. Let us sketch the corresponding proof. First, we set

$$W_1 := \{v \in W : \ \Delta v \in W\},$$

and consider the formal sum

$$\sum_{j=1}^{\infty} c_j e_j \,. \tag{4.8}$$

Then, (4.8) represents an element of $W$ if and only if $\sum_{j=1}^{\infty} |\lambda_j c_j|^2 < +\infty$. Besides, it follows that (4.8) characterizes an element of either $W_1$, or $W^*$, or $V$, if and only if

$$\sum_{j=1}^{\infty} |\lambda_j^2 c_j|^2 < +\infty \,, \quad \text{or} \quad \sum_{j=2}^{\infty} |\lambda_j^{-1} c_j|^2 < +\infty \,, \quad \text{or} \quad \sum_{j=1}^{\infty} |\lambda_j^{1/2} c_j|^2 < +\infty \,,$$

respectively. It follows that the interpolation space $(W_1, W^*)_{1/2}$ coincides with $V$, implying the continuity of the embedding

$$H^1(0,T;W^*) \cap L^2(0,T;W_1) \hookrightarrow C^0([0,T];V) \,.$$

Finally, the same argument as that given in the proof of the above lemma entails (4.7).

Our result on the Fréchet differentiability of $\mathcal{S}$ involves the linearized system we introduce in the subsequent lines. For fixed $u \in \mathcal{U}_R$ and $h \in L^2(0,T;H)$, the linearized system corresponding to $u$ and the variation $h$ consists in looking for a triplet $(\psi, \eta, \omega)$ satisfying

$$(\psi, \eta, \omega) \in \bigl(H^1(0,T;W^*) \cap L^2(0,T;W)\bigr) \times L^2(0,T;H) \times L^2(0,T;W) \,, \tag{4.9}$$

$$\langle \partial_t \psi, v \rangle_W - \int_\Omega \eta \, \Delta v + \sigma \int_\Omega \psi v = \int_\Omega h v \quad \text{a.e. in } (0,T), \text{ for every } v \in W \,, \tag{4.10}$$

$$\int_\Omega \nabla \omega \cdot \nabla v + \int_\Omega f''(\varphi) \, w \psi v + \int_\Omega f'(\varphi) \, \omega v + \nu \int_\Omega \omega v = \int_\Omega \eta v \quad \text{a.e. in } (0,T), \text{ for every } v \in V \,, \tag{4.11}$$

$$\int_\Omega \nabla \psi \cdot \nabla v + \int_\Omega f'(\varphi) \, \psi v = \int_\Omega \omega v \quad \text{a.e. in } (0,T), \text{ for every } v \in V \,, \tag{4.12}$$

$$\psi(0) = 0 \,, \tag{4.13}$$

where $\varphi$ and $w$ are the components of the solution $(\varphi, \mu, w)$ to the state system associated with $u$. We observe that the continuous embedding $H^1(0,T;W^*) \cap L^2(0,T;W) \hookrightarrow C^0([0,T];H)$ gives a precise meaning to the initial condition (4.13). We also notice that (4.11) and (4.12) could have been written in the form

$$-\Delta \omega + f''(\varphi) \, w \, \psi + f'(\varphi) \, \omega + \nu \, \omega = \eta \quad \text{and} \quad -\Delta \psi + f'(\varphi) \, \psi = \omega \quad \text{a.e. in } Q \,,$$

due to the regularity of $\omega$ and $\psi$ given by (4.9), because the space $W$ inherently encodes the boundary conditions. Nevertheless, in the following we prefer to keep on working with the aforementioned variational formulation.

**Theorem 4.5.** *For every $u \in \mathcal{U}_R$ and $h \in L^2(0,T;H)$, problem* (4.10)–(4.13) *has a unique solution $(\psi, \eta, \omega)$ with the regularity specified in* (4.9)*. Moreover, the estimate*

$$\|\psi\|_{H^1(0,T;W^*)\cap L^2(0,T;W)} + \|\eta\|_{L^2(0,T;H)} + \|\omega\|_{L^2(0,T;W)} \leq K_3 \, \|h\|_{L^2(0,T;H)} \tag{4.14}$$

*holds true with a positive constant $K_3$ that depends only on $\Omega$, $T$, the structure of the system, and the upper bound $R$.*

*Proof.* We recall (4.4), and we start by proving uniqueness. It is worth noting that a portion of the insights we are about to unveil also contributes significantly to the proof of existence.

**First a priori estimate.** Testing (4.10) by $1/|\Omega|$ yields an ordinary differential equation for the mean value $\overline{\psi}$, namely

$$\frac{d}{dt}\,\overline{\psi} + \sigma\,\overline{\psi} = \overline{h} \quad \text{a.e. in } (0,T)\,. \tag{4.15}$$

From (4.13), we infer that $\overline{\psi}(0) = 0$. Thus, we readily deduce that

$$\|\overline{\psi}\|_{H^1(0,T)} \leq c\, \|h\|_{L^2(0,T;H)}\,. \tag{4.16}$$

**Second a priori estimate.** We first multiply (4.15) by a generic $v \in W$ and integrate over $\Omega$. Then, we take the difference between (4.10) and the equality just obtained and test it by $\mathcal{N}(\psi - \overline{\psi})$ (recall (2.39)). Thanks to Lemma 4.3, along with the definition of $\mathcal{N}$, we find that

$$\frac{1}{2}\,\frac{d}{dt}\,\|\psi - \overline{\psi}\|_*^2 + \int_\Omega \eta\,(\psi - \overline{\psi}) + \sigma\,\|\psi - \overline{\psi}\|_*^2 = \int_\Omega h\,\mathcal{N}(\psi - \overline{\psi}) \quad \text{a.e. in } (0,T)\,.$$

At the same time, we test (4.11) and (4.12) by $\psi - \overline{\psi}$ and $\psi - \omega$, respectively, and rearrange the terms to obtain, in the order, that

$$\int_\Omega \nabla\omega\cdot\nabla\psi = \int_\Omega \eta(\psi - \overline{\psi}) - \int_\Omega \bigl(f''(\varphi)\,\psi\,w + f'(\varphi)\,\omega + \nu\omega\bigr)(\psi - \overline{\psi})\,,$$
$$\int_\Omega |\nabla\psi|^2 - \int_\Omega \nabla\psi\cdot\nabla\omega + \int_\Omega |\omega|^2 = \int_\Omega \omega\,\psi - \int_\Omega f'(\varphi)\,\psi\,(\psi - \omega)\,.$$

At this point, we add the above equalities to each other, noticing that some cancellations occur. We infer that

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\,\|\psi - \overline{\psi}\|_*^2 + \sigma\,\|\psi - \overline{\psi}\|_*^2 + \int_\Omega |\nabla\psi|^2 + \int_\Omega |\omega|^2 \\
&= \int_\Omega h\,\mathcal{N}(\psi - \overline{\psi}) - \int_\Omega f''(\varphi)\,\psi\,w\,(\psi - \overline{\psi}) + \int_\Omega \omega\,\psi \\
&\quad + \int_\Omega f'(\varphi)\,\omega\,\overline{\psi} - \int_\Omega f'(\varphi)\,|\psi|^2 + \nu\int_\Omega \omega(\psi - \overline{\psi}) \quad \text{a.e. in } (0,T)\,.
\end{aligned} \tag{4.17}$$

The right-hand side is easily dealt with, in particular, by using the compacness inequality (2.35) to handle the $H$ norm of $\psi - \overline{\psi}$, since one can owe to the Gronwall lemma after time integration. We just detail how to handle the most delicate term, i.e., the one involving $w$. On account of (3.33), it suffices to estimate the analogue obtained by replacing $\psi$ by $\psi - \overline{\psi}$. We have that

$$-\int_\Omega f''(\varphi)\,w\,|\psi - \overline{\psi}|^2 \leq \|f''(\varphi)\|_\infty\,\|w\|\,\|\psi - \overline{\psi}\|_4^2\,.$$

Observing that the product $\|f''(\varphi)\|_\infty\,\|w\|$ is estimated in $L^\infty(0,T)$ by virtue of (4.4), and owing to the compactness inequality (2.35), we deduce that

$$-\int_\Omega f''(\varphi)\,w\,|\psi-\overline{\psi}|^2 \le \frac12\int_\Omega|\nabla\psi|^2 + c\,\|\psi-\overline{\psi}\|_*^2 \quad \text{a.e. in } (0,T)\,.$$

Therefore, after integrating (4.17) with respect to time, we apply Gronwall's lemma as announced and obtain, recalling also (4.16), that

$$\|\psi\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)} + \|\omega\|_{L^2(0,T;H)} \le c\,\|h\|_{L^2(0,T;H)}\,. \tag{4.18}$$

**Consequence.** By applying (4.18) in the case $h=0$, we obtain that $\psi=\omega=0$. Then (4.11) yields that $\eta=0$. Since the system (4.10)–(4.13) is linear, this proves the uniqueness part of the statement.

It remains to prove the existence of a solution and the estimate (4.14). This can be done by a discretization procedure like the Faedo–Galerkin scheme, with the same set of eigenfunctions as that used for Theorem 2.2: the unknowns of the discrete problem are $V_n$-valued, and the analogues of (4.10)–(4.12) are satisfied for every $v\in V_n$. One obtains a linear system that can be solved on account of the zero initial condition derived from (4.13). Then, one performes a priori estimates and let $n$ tend to infinity. However, for the sake of brevity, we limit ourselves to demonstrating formal estimates, acknowledging that the test functions we employ are admissible at the discrete level, especially when only the variational form of the problem is considered since $\Delta^k v\in V_n\subset W$ for every $k$, $n$ and $v\in V_n$. Since also estimates (4.16) and (4.18) can be performed at the discrete level, we can account for them in the sequel.

**Third a priori estimate.** We test (4.12) by $-\Delta\psi$. By virtue of (4.18), we derive an estimate for $\Delta\psi$ in $L^2(0,T;H)$; from elliptic regularity theory we then infer that

$$\|\psi\|_{L^2(0,T;W)} \le c\,\|h\|_{L^2(0,T;H)}\,. \tag{4.19}$$

**Fourth a priori estimate.** We test (4.10), (4.11), and (4.12), by $\psi$, $-\Delta\psi$, and $\Delta\omega$, respectively. Then, we sum up and rearrange. Thanks to the occurring cancellations and an integration by parts in the terms involving $\Delta\omega$, we obtain that

$$\begin{aligned}&\frac12\,\frac{d}{dt}\int_\Omega|\psi|^2 + \sigma\int_\Omega|\psi|^2 + \int_\Omega|\nabla\omega|^2\\&= \int_\Omega h\psi - \int_\Omega\big(f''(\varphi)\,w\psi + f'(\varphi)\,\omega + \nu\omega\big)(-\Delta\psi) - \int_\Omega\big(f''(\varphi)\,\psi\nabla\varphi + f'(\varphi)\nabla\psi\big)\cdot\nabla\omega\,,\end{aligned}$$

and we are left with estimating the right-hand side. We just consider the most delicate terms, since the others are easier and can be left as an exercise to the interested reader. We recall (4.4) and (4.19) and point out that

$$\begin{aligned}&-\int_\Omega f''(\varphi)\,w\psi(-\Delta\psi) - \int_\Omega f''(\varphi)\,\psi\nabla\varphi\cdot\nabla\omega\\&\le c\,\|w\|^2\,\|\psi\|_\infty^2 + c\,\|\Delta\psi\|^2 + \frac12\int_\Omega|\nabla\omega|^2 + c\,\|\nabla\varphi\|^2\,\|\psi\|_\infty^2\\&\le c\,\|\psi\|_W^2 + \frac12\int_\Omega|\nabla\omega|^2\,.\end{aligned}$$

Hence, after integrating over time, and also accounting for (4.18) and (4.19), we conclude that

$$\|\psi\|_{L^\infty(0,T;H)} + \|\omega\|_{L^2(0,T;V)} \le c\,\|h\|_{L^2(0,T;H)}\,. \tag{4.20}$$

**Fifth a priori estimate.** We test (4.10), (4.11), and (4.12), by $-\Delta\psi$, $\Delta^2\psi$, and $-\Delta^3\psi$, respectively, sum up, perform some integrations by parts, and account for some cancellations. This leads us to the identity

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\psi|^2 + \sigma\int_\Omega |\nabla\psi|^2 + \int_\Omega |\Delta^2\psi|^2 \\
&= -\int_\Omega h\Delta\psi - \nu\int_\Omega \omega\Delta^2\psi - \int_\Omega f''(\varphi)\,w\psi\Delta^2\psi \\
&\quad - \int_\Omega f'(\varphi)\,\omega\Delta^2\psi + \int_\Omega f'(\varphi)\,\psi\Delta^3\psi\,.
\end{aligned} \tag{4.21}$$

As before, we just handle the terms on the right-hand side that need some treatment, while leaving the simpler ones for the reader to follow. Integrating with respect to time, and owing to (4.4) and (4.20), we obtain that

$$\begin{aligned}
&-\int_{Q_t} f''(\varphi)\,w\psi\Delta^2\psi \le c\int_0^t \|w(s)\|_\infty\,\|\psi(s)\|\,\|\Delta^2\psi(s)\|\,ds \\
&\le \frac{1}{4}\int_{Q_t}|\Delta^2\psi|^2 + c\,\|w\|^2_{L^2(0,T;W)}\,\|\psi\|^2_{L^\infty(0,T;H)} \le \frac{1}{4}\int_{Q_t}|\Delta^2\psi|^2 + c\,\|h\|^2_{L^2(0,T;H)}\,.
\end{aligned}$$

Next, integrating with respect to time, invoking (4.4) once more and the above estimates, we have that

$$\begin{aligned}
&\int_{Q_t} f'(\varphi)\,\psi\Delta^3\psi = \int_{Q_t}\big((\Delta f'(\varphi))\psi + 2(\nabla f'(\varphi))\cdot\nabla\psi + f'(\varphi)\Delta\psi\big)\Delta^2\psi \\
&\le \frac{1}{4}\int_{Q_t}|\Delta^2\psi|^2 + c\,\|\Delta f'(\varphi)\|^2_{L^\infty(0,T;H)}\,\|\psi\|^2_{L^2(0,T;W)} \\
&\quad + c\int_0^t \|\nabla f'(\varphi(s))\|^2_\infty\,\|\nabla\psi(s)\|^2\,ds + c\int_{Q_t}|\Delta\psi|^2 \\
&\le \frac{1}{4}\int_{Q_t}|\Delta^2\psi|^2 + c\,\|h\|^2_{L^2(0,T;H)} + c\int_0^t \|\nabla f'(\varphi(s))\|^2_\infty\,\|\nabla\psi(s)\|^2\,ds\,,
\end{aligned}$$

and we observe that the function $t \mapsto \|\nabla f'(\varphi(s))\|^2_\infty$ is estimated in $L^1(0,T)$. Therefore, after integration of (4.21) and application of Gronwall's lemma, we conclude that

$$\|\nabla\psi\|_{L^\infty(0,T;H)} + \|\Delta^2\psi\|_{L^2(0,T;H)} \le c\,\|h\|_{L^2(0,T;H)}\,,$$

whence, by elliptic regularity theory,

$$\|\psi\|_{L^\infty(0,T;V)\cap L^2(0,T;H^4(\Omega))} \le c\,\|h\|_{L^2(0,T;H)}\,. \tag{4.22}$$

**Consequences.** By testing (4.12) by $\Delta^2\omega$ and integrating by parts we get the identity

$$\int_\Omega |\Delta\omega|^2 \;=\; -\int_\Omega \Delta^2\psi\,\Delta\omega \;+\; \int_\Omega \Delta(f'(\varphi)\psi)\,\Delta\omega\,,$$

whence, integrating over time and invoking Young's inequality,

$$\int_{Q_t} |\Delta\omega|^2 \leq c \int_{Q_t} |\Delta^2\psi|^2 + c \int_{Q_t} |\Delta(f'(\varphi)\psi)|^2 .$$

Using (4.4) and the estimates proved above, it is not difficult to see that the right-hand side is bounded by $c \|h\|^2_{L^2(0,T;H)}$, whence, by elliptic regularity,

$$\|\omega\|_{L^2(0,T;W)} \leq c \|h\|_{L^2(0,T;H)} . \tag{4.23}$$

Testing (4.11) by $\eta$ then easily yields that

$$\|\eta\|_{L^2(0,T;H)} \leq c \|h\|_{L^2(0,T;H)} . \tag{4.24}$$

Finally, by comparison in (4.10) (at the discrete level one should argue as we did to obtain (3.45)), we readily find that

$$\|\partial_t\psi\|_{L^2(0,T;W^*)} \leq c \|h\|_{L^2(0,T;H)} . \tag{4.25}$$

This concludes the proof. □

We are now ready to prove the Fréchet differentiability of the map $\mathcal{S}$ defined in (4.3).

**Theorem 4.6.** *The control-to-state mapping $\mathcal{S}$ is Fréchet differentiable in $\mathcal{U}_R$ as a mapping from $\mathcal{U}_R \subset \mathcal{U}$ into $\mathcal{Y}$. Moreover, given $u \in \mathcal{U}_R$, the Fréchet derivative $D\mathcal{S}(u) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is the linear operator that maps any $h \in \mathcal{U}$ into the component $\psi$ of the solution $(\psi, \eta, \omega)$ to the linearized problem* (4.10)–(4.13) *associated with $u$ and the variation $h$.*

*Proof.* The map described in the statement actually belongs to $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ as a consequence of (4.14) in Theorem 4.5. To prove the claim, we fix some $u \in \mathcal{U}_R$ and show that

$$\lim_{\|h\|_\infty \to 0} \frac{\|\mathcal{S}(u+h) - \mathcal{S}(u) - \psi\|_{\mathcal{Y}}}{\|h\|_\infty} = 0 , \tag{4.26}$$

where $\psi$ is the first component of the solution to the linearized associated to $u$ and $h$. Without loss of generality, we assume that $\|h\|_\infty < R - \|u\|_\infty$, so that $u + h$ also belongs to the open set $\mathcal{U}_R$. We denote by $(\varphi, \mu, w)$ and $(\varphi^h, \mu^h, w^h)$ the solutions to the state system corresponding to $u$ and $u+h$, respectively, and notice that the uniform bound (2.21) given by Theorem 2.2 holds true for both of them. Moreover, $(\psi, \eta, \omega)$ denotes the solution to the linearized system as in the statement, and we set for convenience

$$\psi^h := \varphi^h - \varphi - \psi , \quad \eta^h := \mu^h - \mu - \eta , \quad \text{and} \quad \omega^h := w^h - w - \omega .$$

We aim at proving an inequality that implies (4.26). Namely, we are going to prove that

$$\|\psi^h\|_{H^1(0,T;W^*) \cap L^2(0,T;W)} + \|\omega^h\|_{L^2(0,T;V)} \leq c \|h\|^2_{L^2(0,T;V^*)} , \tag{4.27}$$

where the first norm, due to the above definitions, is nothing but the numerator of (4.26). To this end, we observe that $(\psi^h, \eta^h, \omega^h)$ solves the following problem:

$$\langle \partial_t \psi^h, v\rangle_W - \int_\Omega \eta^h \, \Delta v + \sigma \int_\Omega \psi^h v = 0 \quad \text{a.e. in } (0,T), \text{ for every } v \in W\,, \tag{4.28}$$

$$\int_\Omega \nabla\omega^h \cdot \nabla v + \nu \int_\Omega \omega^h v = \int_\Omega \eta^h v - \int_\Omega \Lambda_1 \, v$$
$$\text{a.e. in } (0,T), \text{ for every } v \in V\,, \tag{4.29}$$

$$\int_\Omega \nabla\psi^h \cdot \nabla v = \int_\Omega \omega^h v - \int_\Omega \Lambda_2 \, v \quad \text{a.e. in } (0,T), \text{ for every } v \in V\,, \tag{4.30}$$

$$\psi^h(0) = 0\,, \tag{4.31}$$

where we have set for brevity

$$\begin{aligned}\Lambda_1 &:= f'(\varphi^h)w^h - f'(\varphi)w - f''(\varphi)\psi w - f'(\varphi)\omega \\ &= [f'(\varphi^h) - f'(\varphi) - f''(\varphi)\psi]w + [f'(\varphi^h) - f'(\varphi)](w^h - w) + f'(\varphi)\omega^h \\ \Lambda_2 &:= f(\varphi^h) - f(\varphi) - f'(\varphi)\psi\,.\end{aligned}$$

We notice at once that expanding $f'$ and $f$ around $\varphi$ (namely, around $\varphi(x,t)$ for almost all fixed $(x,t)$ in $Q$) by means of Taylor's formula with integral remainder, we find that

$$\begin{aligned}f'(\varphi^h) - f'(\varphi) - f''(\varphi)\psi &= f''(\varphi)\psi^h + R_1\,, \\ f(\varphi^h) - f(\varphi) - f'(\varphi)\psi &= f'(\varphi)\psi^h + R_2\,,\end{aligned}$$

where the remainders $R_1$ and $R_2$ have the form

$$\begin{aligned}R_1 &= \Big[\int_0^1 (1-s) f'''(s\varphi^h + (1-s)\varphi))\, ds\Big](\varphi^h - \varphi)^2\,, \\ R_2 &= \Big[\int_0^1 (1-s) f''(s\varphi^h + (1-s)\varphi))\, ds\Big](\varphi^h - \varphi)^2\,.\end{aligned}$$

Hence, we have that

$$\begin{aligned}&\Lambda_1 = [f''(\varphi)\psi^h + R_1]w + [f'(\varphi^h) - f'(\varphi)](w^h - w) + f'(\varphi)\omega^h\,, \\ &\text{and} \quad \Lambda_2 = f'(\varphi)\psi^h + R_2\,,\end{aligned} \tag{4.32}$$

where the remainders satisfy

$$|R_1| \le c\,(\varphi^h - \varphi)^2 \quad \text{and} \quad |R_2| \le c\,(\varphi^h - \varphi)^2 \quad \text{a.e. in } Q\,, \tag{4.33}$$

since both $\varphi$ and $\varphi^h$ are uniformly bounded in $L^\infty$ as a consequence of (2.21) and the integration variable $s$ attains values in $[0,1]$. In the estimates to follow, it is understood that $\delta$ is a positive parameter whose value will be chosen at the end of the computations.

**First a priori estimate.** First observe that $\psi^h$ has zero mean value, which directly follows from testing (4.28) by $1/|\Omega|$ and using (4.31). Next, we test (4.28), (4.29), and (4.30), by $\mathcal{N}\psi^h$, $\psi^h$, and

$\psi^h - \omega^h$, respectively, and add the resulting equalities to each other. By noting some cancellations, and recalling Lemma 4.3, we have that, a.e. in $(0,T)$,

$$\frac{1}{2}\frac{d}{dt}\,\|\psi^h\|_*^2 + \sigma\,\|\psi^h\|_*^2 + \int_\Omega |\nabla\psi^h|^2 + \int_\Omega |\omega^h|^2$$
$$= - \int_\Omega \Lambda_1\,\psi^h + \int_\Omega \Lambda_2\,(\omega^h - \psi^h) + (1-\nu)\int_\Omega \omega^h\psi^h\,. \tag{4.34}$$

As the last three terms on the left-hand side are nonnegative, we just have to estimate the integrals on the right-hand side. By the expression of $\Lambda_1$ given in (4.32), the term $\int_\Omega \Lambda_1\,\psi^h$ may be written as sum of three contributions, namely,

$$I_1 := \int_\Omega [f''(\varphi)\psi^h + R_1]\,w\psi^h\,, \quad I_2 := \int_\Omega [f'(\varphi^h) - f'(\varphi)](w^h - w)\psi^h\,,$$
$$\text{and}\quad I_3 := \int_\Omega f'(\varphi)\omega^h\psi^h\,.$$

Now, observe that $I_3$, as well as the last term of (4.34), can easily be treated using the Young and compactness inequalities. The latter inequality is also used to deal with $I_1$. Indeed, on account of (4.33), we have that

$$|I_1| \le c\,I_1' + c\,I_1''\,, \quad \text{where}\quad I_1' := \int_\Omega |w|\,|\psi^h|^2 \quad\text{and}\quad I_1'' := \int_\Omega (\varphi^h - \varphi)^2\,|w|\,|\psi^h|\,.$$

The term $I_1'$ can immediately be estimated with the help of (2.21) and (2.22) (cf. Remark 2.3) as

$$I_1' \le \|w\|_{L^\infty(0,T;H)}\,\|\psi^h\|_4^2 \le \delta \int_\Omega |\nabla\psi^h|^2 + c_\delta\,\|\psi^h\|_*^2\,,$$

while the estimation of $I_1''$ requires more work. We estimate its time integral by first using the Hölder, Sobolev, and Poincaré inequalities, where we recall that $\psi^h$ has zero mean value. Then, we apply the stability estimate for $w$ related to (2.22), Young's inequality, and the continuous dependence inequality (2.24) with $u_1 = u + h$ and $u_2 = u$. Thus, we find out that

$$\int_{Q_t} |\varphi^h - \varphi|^2\,|w|\,|\psi^h| \le \int_0^t \|\varphi^h(s) - \varphi(s)\|_6^2\,\|w(s)\|_2\,\|\psi^h(s)\|_6\,ds$$
$$\le c \int_0^t \|\varphi^h(s) - \varphi(s)\|_V^2\,\|w(s)\|\,\|\psi^h(s)\|_V\,ds$$
$$\le c \int_0^t \|\varphi^h(s) - \varphi(s)\|_V^2\,\|w(s)\|\,\|\nabla\psi^h(s)\|\,ds$$
$$\le \frac{1}{2}\int_{Q_t} |\nabla\psi^h|^2 + c\,\|w\|_{L^\infty(0,T;H)}^2\,\|\varphi^h - \varphi\|_{L^\infty(0,T;V)}^4$$
$$\le \frac{1}{2}\int_{Q_t} |\nabla\psi^h|^2 + c\,\|h\|_{L^2(0,T;V^*)}^4\,, \tag{4.35}$$

completing the estimate of $|I_1|$. Next, we observe that

$$|I_2| \le \|\psi^h\|^2 + c\,\|\varphi^h - \varphi\|_4^2\,\|w^h - w\|_4^2$$

and we can apply the compactness inequality once more to the first term, while we observe that (2.24) ensures that

$$\int_0^t \|\varphi^h(s) - \varphi(s)\|_4^2 \, \|w^h(s) - w(s)\|_4^2 \, ds \\ \le c \, \|\varphi^h - \varphi\|_{L^\infty(0,T;V)}^2 \, \|w^h - w\|_{L^2(0,T;V)}^2 \le c \, \|h\|_{L^2(0,T;V^*)}^4 \, .$$

Now, we consider the term involving $\Lambda_2$. By recalling (4.32) and (4.33), we have that

$$\int_\Omega \Lambda_2 \, (\omega^h - \psi^h) = \int_\Omega \big(f'(\varphi)\psi^h + R_2\big)(\omega^h - \psi^h) \le c \int_\Omega \big(|\psi^h| + (\varphi^h - \varphi)^2\big)\big(|\omega^h| + |\psi^h|\big).$$

Hence, it can be estimated with the same ideas as we used to prove (4.35) and is even easier. At this point, we return to (4.34), integrating it with respect to time. By choosing $\delta$ small enough, and applying the Gronwall lemma, we conclude that

$$\|\psi^h\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)} + \|\omega^h\|_{L^2(0,T;H)} \le c \, \|h\|_{L^2(0,T;V^*)}^2 \, . \tag{4.36}$$

Then, using (4.36) to estimate the right-hand side of (4.30), we deduce that

$$\|\psi^h\|_{L^2(0,T;W)} \le c \, \|h\|_{L^2(0,T;V^*)}^2 \, . \tag{4.37}$$

**Second a priori estimate.** For the sake of brevity, we argue only formally. However, we notice that $(\psi^h, \eta^h, \omega^h)$ can be approximated by $V_n$-valued functions (recall (3.11)), since this is true for $(\varphi^h, \mu^h, w^h)$, $(\varphi, \mu, w)$, and $(\psi, \eta, \omega)$. Hence, the estimate we are going to perform can be justified rigorously. We test (4.28), (4.29), and (4.30), by $\psi^h$, $-\Delta\psi^h$, and $\Delta\omega^h$, add the resulting identities, and account for some cancellations. We then obtain that a.e. in $(0, T)$ it holds

$$\frac{1}{2}\frac{d}{dt} \|\psi^h\|^2 + \sigma \, \|\psi^h\|^2 + \|\nabla\omega^h\|^2 = \nu \int_\Omega \omega^h \Delta\psi^h + \int_\Omega \Lambda_1 \, \Delta\psi^h - \int_\Omega \Lambda_2 \, \Delta\omega^h \, . \tag{4.38}$$

The first term on the right-hand side can easily be treated by the Hölder inequality and (4.36)–(4.37). As for the next term, we apply the Young inequality and (4.37). We then see that we are led to treat the integral of $|\Lambda_1|^2$. By recalling (4.32) and (4.33), we have that

$$\int_\Omega |\Lambda_1|^2 \le c \, \|w\|_\infty^2 \, \|\psi^h\|^2 + c \, \|\varphi^h - \varphi\|_4^4 \, \|w\|_\infty^2 + c \, \|\varphi^h - \varphi\|_4^2 \, \|w^h - w\|_4^2 + c \, \|\omega^h\|^2 \, .$$

By noting that the function $s \mapsto \|w(s)\|_\infty^2$ is bounded in $L^1(0, T)$ by (2.21), we can treat the first addend by the Gronwall lemma after time integration. On the other hand, the time integrals of the remaining terms on the right-hand side can be treated by owing to (2.21), (2.24) and (4.36). Namely, we conclude that

$$\int_{Q_t} |\Lambda_1|^2 \le c \, \|h\|_{L^2(0,T;V^*)}^4 \, .$$

Finally, by (4.32) and (4.33), we have that

$$- \int_\Omega \Lambda_2 \, \Delta\omega^h \le \delta \, \|\nabla\omega^h\|^2 + c_\delta \, \|\nabla\Lambda_2\|^2 \le \delta \, \|\nabla\omega^h\|^2 + c_\delta \, \|\psi^h\|_V^2 + c_\delta \, \|\nabla R_2\|^2 \, .$$

Since we can owe to (4.37), it remains to estimate the last term. To this aim, we compute $\nabla R_2$ by differentiating, under the integral sign as well, and subsequently obtain

$$\nabla R_2 = 2(\varphi^h - \varphi)\nabla(\varphi^h - \varphi) \int_0^1 (1-s) f''(s\varphi^h + (1-s)\varphi)\, ds$$
$$+ (\varphi^h - \varphi)^2 \int_0^1 (1-s) f'''(s\varphi^h + (1-s)\varphi)\big(s\nabla\varphi^h + (1-s)\nabla\varphi\big)\, ds\,.$$

Then, we deduce that

$$|\nabla R_2| \le c\,|\varphi^h - \varphi|\,|\nabla(\varphi^h - \varphi)| + c\,\big(|\nabla\varphi^h| + |\nabla\varphi|\big)(\varphi^h - \varphi)^2 \quad \text{a.e. in } Q.$$

Therefore, it turns out that

$$\|\nabla R_2\|^2 \le c\,\|\varphi^h - \varphi\|_4^2\,\|\nabla(\varphi^h - \varphi)\|_4^2 + c\,\big(\|\nabla\varphi^h\|_6^2 + \|\nabla\varphi\|_6^2\big)\,\|\varphi^h - \varphi\|_6^4\,,$$

from which we infer

$$\int_0^t \|\nabla R_2(s)\|^2\, ds \le c\,\|\varphi^h - \varphi\|^2_{C^0([0,T];V)}\,\|\varphi^h - \varphi\|^2_{L^2(0,T;W)}$$
$$+ c\,\big(\|\varphi^h\|^2_{L^\infty(0,T;W)} + \|\varphi\|^2_{L^\infty(0,T;W)}\big)\,\|\varphi^h - \varphi\|^4_{L^\infty(0,T;V)}$$
$$\le c\,\|h\|^4_{L^2(0,T;V^*)}\,, \tag{4.39}$$

where the last inequality follows from (2.21) and (2.24). At this point, we come back to (4.38), integrate it with respect to time, choose $\delta$ small enough, and apply Gronwall's lemma. By also accounting for (4.36), we conclude that

$$\|\psi^h\|_{L^\infty(0,T;H)} + \|\omega^h\|_{L^2(0,T;V)} \le c\,\|h\|^2_{L^2(0,T;V^*)}\,. \tag{4.40}$$

**Conclusion.** By revisiting the expressions of $\Lambda_1$ and $\Lambda_2$ and their treatement in the above calculations, and accounting for the estimates on $\psi^h$ and $\omega^h$ we have obtained, we deduce that

$$\|\Lambda_1\|_{L^2(0,T;H)} \le c\,\|h\|^2_{L^2(0,T;V^*)} \quad \text{and} \quad \|\Lambda_2\|_{L^2(0,T;H)} \le c\,\|h\|^2_{L^2(0,T;V^*)}\,.$$

Hence, by comparison, first in (4.29) and then in (4.28), we deduce that

$$\|\eta^h\|_{L^2(0,T;H)} \le c\,\|h\|^2_{L^2(0,T;V^*)} \quad \text{and} \quad \|\partial_t\psi^h\|_{L^2(0,T;W^*)} \le c\,\|h\|^2_{L^2(0,T;V^*)}\,.$$

The last inequality, when combined with (4.37) and (4.40), yields (4.27), and the proof is complete. □

**Remark 4.7.** The inequality (4.27) also shows the Fréchet differentiability of the map $u \mapsto w$ as a mapping from $\mathcal{U}_R$ into $L^2(0,T;V)$, where $w$ is the third component of the solution to the state system corresponding to $u$. Thus, we could have considered in the cost functional (2.28) an integral depending on $w$ of the type $(\alpha_4/2)\int_Q |w - w_Q|^2$ with some target function $w_Q \in L^2(Q)$ and a nonnegative constant $\alpha_4$. The whole theory could have been developed with minor changes, indeed. However, such a target term is not that relevant from the modeling viewpoint. Therefore, its inclusion would be rather a mathematical extension, wherefore we decided to not include it, being aware that it may be handled from a mathematical standpoint.

### 4.3 Necessary conditions for optimality

By virtue of the Fréchet differentiability result given by Theorem 4.6, along with the continuous embedding $H^1(0,T;W^*) \cap L^2(0,T;W) \hookrightarrow C^0([0,T];H)$ and the quadratic structure of the cost functional (2.28), we can apply the chain rule to the composite mapping

$$\mathcal{U}_R \to (\mathcal{U}_R \times \mathcal{Y}) \to \mathbb{R} \quad \text{given by} \quad u \mapsto (u,\varphi) := (u, \mathcal{S}(u)) \;\mapsto\; \mathcal{J}(u,\varphi)\,.$$

Since the control problem consists in minimizing this mapping on $\mathcal{U}_{\rm ad}$ and $\mathcal{U}_{\rm ad}$ is convex, a necessary condition for some $u^*$ to be an optimal control is that the variational inequality

$$\alpha_1 \int_Q (\varphi^* - \phi_Q)\psi + \alpha_2 \int_\Omega (\varphi^*(T) - \phi_\Omega)\psi(T) + \alpha_3 \int_Q u^*(u - u^*) \geq 0 \tag{4.41}$$

holds true for every $u \in \mathcal{U}_{\rm ad}$, where $\psi$ is the first component of the solution $(\psi,\eta,\omega)$ to the linearized problem associated with $h := u - u^*$ and the components $\varphi^*$ and $w^*$ of the state $(\varphi^*,\mu^*,w^*)$ corresponding to $u^*$. However, this condition is not satisfactory, since it requires to solve the linearized system infinitely many times as $u$ varies in $\mathcal{U}_{\rm ad}$. Let us briefly mention that proceeding as suggested in Remark 4.7 would introduce an additional term of the form $\alpha_4 \int_Q (w^* - w_Q)\omega$ in (4.41), with $\omega$ being the third component of the solution $(\psi,\eta,\omega)$ to the linearized problem associated with $h := u - u^*$.

As usual, this unpleasant situation is overcome by the introduction of a proper adjoint problem associated with the above optimal control and the corresponding state. This problem consists in looking for a triplet $(p,q,r)$ satisfying

$$p \in H^1(0,T;W^*) \cap L^\infty(0,T;V) \cap L^2(0,T;H^4(\Omega)\cap W), \quad q \in L^2(0,T;W), \quad \text{and} \quad r \in L^2(0,T;H), \tag{4.42}$$

$$- \langle \partial_t p, v \rangle_W - \int_\Omega r\,\Delta v + \sigma \int_\Omega pv + \int_\Omega \xi^* qv + \int_\Omega \zeta^* rv = \int_\Omega \rho_1 v \quad \text{a.e. in } (0,T), \text{ for every } v \in W, \tag{4.43}$$

$$q + \Delta p = 0 \quad \text{a.e. in } Q, \tag{4.44}$$

$$\int_\Omega rv - \int_\Omega \nabla q \cdot \nabla v - \nu \int_\Omega qv - \int_\Omega \zeta^* qv = 0 \quad \text{a.e. in } (0,T), \text{ for every } v \in V, \tag{4.45}$$

$$p(T) = \rho_2, \tag{4.46}$$

where, for brevity, we have set

$$\xi^* := f''(\varphi^*)\,w^*, \quad \zeta^* := f'(\varphi^*), \quad \rho_1 := \alpha_1(\varphi^* - \phi_Q) \quad \text{and} \quad \rho_2 := \alpha_2(\varphi^*(T) - \phi_\Omega)\,. \tag{4.47}$$

**Theorem 4.8.** *In addition to the current assumptions, suppose $\phi_\Omega \in V$. Then the adjoint problem has a unique solution $(p,q,r)$ satisfying* (4.42)*.*

*Proof.* We first perform a basic estimate and prove uniqueness. Then we sketch how the same estimate can be used to prove existence.

**The basic estimate.** We test (4.43) first by $p$, and then by $q$, and respectively obtain that

$$
\begin{aligned}
& -\langle \partial_t p, p\rangle_W - \int_\Omega r\Delta p + \sigma \int_\Omega |p|^2 = -\int_\Omega \xi^* qp - \int_\Omega \zeta^* rp + \int_\Omega \rho_1 p\,, \\
& -\langle \partial_t p, q\rangle_W - \int_\Omega r\,\Delta q = -\sigma \int_\Omega pq - \int_\Omega \xi^* |q|^2 - \int_\Omega \zeta^* rq + \int_\Omega \rho_1 q\,.
\end{aligned}
$$

Now, we notice that (4.42) and (4.44) yield that $\Delta p \in L^2(0,T;W)$. Thus, we can take the duality beween $\partial_t p$ and equation (4.44) leading to

$$\langle \partial_t p, q\rangle_W + \langle \partial_t p, \Delta p\rangle_W = 0\,.$$

Next, for a positive constant $K$ whose value is chosen later on, we test (4.44) by $Kq$ and integrate by parts to infer that

$$K\int_\Omega |q|^2 = K\int_\Omega \nabla q\cdot\nabla p\,.$$

Now, we test (4.44) by $r$. Moreover, we perform an integration by parts in (4.45), write the resulting equality as a differential equation, multiply by $r$ and integrate over $\Omega$. Finally, we test (4.45) as it is by $-q$. We obtain the following identities:

$$
\begin{aligned}
& \int_\Omega qr + \int_\Omega r\Delta p = 0 \\
& \int_\Omega |r|^2 + \int_\Omega \Delta q\, r = \nu\int_\Omega qr + \int_\Omega \zeta^* qr \\
& -\int_\Omega rq + \int_\Omega |\nabla q|^2 = -\nu\int_\Omega |q|^2 - \int_\Omega \zeta^* |q|^2\,.
\end{aligned}
$$

At this point, we add all of the equalities we have listed above to each other and notice several cancellations leading to the identity

$$
\begin{aligned}
& -\langle \partial_t p, p\rangle_W + \langle \partial_t p, \Delta p\rangle_W + \sigma\int_\Omega |p|^2 + K\int_\Omega |q|^2 + \int_\Omega |r|^2 + \int_\Omega |\nabla q|^2 \\
& = -\int_\Omega \xi^* pq + \int_\Omega \zeta^* pr + \int_\Omega \rho_1 p - \sigma\int_\Omega pq - \int_\Omega \xi^*|q|^2 + \int_\Omega \zeta^* qr + \int_\Omega \rho_1 q \\
& \quad + K\int_\Omega \nabla q\cdot\nabla p + \nu\int_\Omega qr + \int_\Omega \zeta^* qr - \nu\int_\Omega |q|^2 - \int_\Omega \zeta^*|q|^2\,.
\end{aligned}
\tag{4.48}
$$

In order to treat the left-hand side, we first note that

$$-\langle \partial_t p, p\rangle_W = -\frac{1}{2}\,\frac{d}{dt}\,\|p\|^2\,.$$

On the other hand, as a direct consequence of Remark 4.4, we have that the second term on the left-hand side can be interpreted as

$$\langle \partial_t p, \Delta p\rangle_W = -\frac{1}{2}\,\frac{d}{dt}\,\|\nabla p\|^2\,.$$

Therefore, by integrating (4.48) over $(t,T)$ and observing that the assumption $\phi_\Omega \in V$ and the regularity of $\varphi^*$ ensure that $\rho_2 \in V$, the sum of the first two terms is given by

$$\frac{1}{2}\,\|p(t)\|_V^2 - \frac{1}{2}\,\|\rho_2\|_V^2\,.$$

As for the time integral of the right-hand side, we recall (2.21) but do not detail how to estimate it. However, it is easy to see that one can play with the Young inequality and choose $K$ large enough in order to apply the Gronwall lemma. In this connection, the terms involving $\zeta^*$ are easy to handle, since $\zeta^*$ is bounded. The terms involving $\xi^*$ require some attention. However, we know that $\xi^* \in L^\infty(0,T;H)$, and thus we can estimate the only critical term by means of the compactness inequality (2.35) as follows:

$$
\begin{aligned}
&- \int_t^T \int_\Omega \xi^* |q|^2 \leq \int_t^T \|\xi^*(s)\| \, \|q(s)\|_4^2 \, ds \\
&\leq c \int_t^T \|q(s)\|_4^2 \, ds \leq \frac{1}{4} \int_t^T \int_\Omega |\nabla q|^2 + c \int_t^T \int_\Omega |q|^2,
\end{aligned}
$$

where the last term can be absorbed by choosing a sufficiently large $K$. Thus, we can conclude that

$$
\|p\|_{L^\infty(0,T;V)} + \|q\|_{L^2(0,T;V)} + \|r\|_{L^2(0,T;H)} \leq c \left( \|\rho_1\|_{L^2(0,T;H)} + \|\rho_2\|_V \right). \tag{4.49}
$$

**Uniqueness.** Recalling that (4.49) has been rigorously obtained under the regularity assumptions (4.42), and applying it with $\rho_1$ and $\rho_2$ replaced by zero, we conclude that the solution is $(0,0,0)$. Since the problem is linear, this proves uniqueness.

**Existence.** As in the case of the original system (2.17)–(2.20) and of the linearized system (4.10)–(4.13), one can construct a solution to (4.43)–(4.46) by starting from a Faedo–Galerkin scheme with the same set of eigenfunctions as considered before. Then, estimate (4.49) can be rigorously performed at the discrete level. In fact, the argument we have used can even be simplified, since the discrete solution is smooth. Once the analogue of (4.49) is obtained, one can derive further estimates by comparison in the discrete equations (see, e.g., the argument used to prove (3.45)). The formal analogues on our system are

$$
\begin{aligned}
&\|\Delta q\|_{L^2(0,T;H)} \leq c, \quad \text{whence} \quad \|q\|_{L^2(0,T;W)} \leq c, \\
&\|\Delta p\|_{L^2(0,T;W)} \leq c, \quad \text{whence} \quad \|p\|_{L^2(0,T;H^4(\Omega))} \leq c, \\
&\|\partial_t p\|_{L^2(0,T;W^*)} \leq c,
\end{aligned}
$$

which are obtained by comparison in (4.45), (4.44), and (4.43), respectively. Once estimates like these are proved at the discrete setting, one can easily let the discretization parameter tend to infinity and obtain a solution to problem (4.43)–(4.46). □

We conclude the paper by proving a first-order necessary condition for optimality.

**Theorem 4.9.** *Let $u^* \in \mathcal{U}_{\mathrm{ad}}$ be an optimal control, and let $(\varphi^*, \mu^*, w^*)$ be the corresponding state. Then*

$$
\int_Q (\alpha_3 \, u^* + p)(u - u^*) \geq 0 \quad \textit{for every } u \in \mathcal{U}_{\mathrm{ad}}, \tag{4.50}
$$

*where $p$ is the first component of the solution $(p,q,r)$ to the corresponding adjoint problem. In particular, if $\alpha_3 > 0$, then the optimal control $u^*$ is the $H$-projection of $-p/\alpha_3$ onto $\mathcal{U}_{\mathrm{ad}}$.*

*Proof.* We fix $u$ in $\mathcal{U}_{\mathrm{ad}}$ and write the linearized system (4.10)–(4.13) associated with the optimal control $u^*$, the corresponding state $(\varphi^*, \mu^*, w^*)$, and the variation $h := u - u^*$, and test the three equations by $p$, $q$, and $r$, respectively. More precisely, regarding (4.12), on account of the regularity of

$\psi$ given in (4.9), we replace the first integral by $-\int_\Omega \Delta\psi\, v$ and then test by $r$. At the same time, we test (4.43) by $-\psi$, multiply (4.44) by $\eta$, and integrate over $\Omega$, and test (4.45) by $\omega$. Hence, by recalling the definitions of $\xi^*$ and $\zeta^*$ given in (4.47), we obtain a.e. in $(0,T)$ the identities

$$\begin{aligned}
&\langle\partial_t\psi, p\rangle_W - \int_\Omega \eta\,\Delta p + \sigma\int_\Omega \psi p = \int_\Omega (u-u^*)p\,,\\
&\int_\Omega \nabla\omega\cdot\nabla q + \int_\Omega f''(\varphi^*)\,w^*\psi q + \int_\Omega f'(\varphi^*)\,\omega q + \nu\int_\Omega \omega q = \int_\Omega \eta q\,,\\
&-\int_\Omega \Delta\psi\, r + \int_\Omega f'(\varphi^*)\,\psi r = \int_\Omega \omega r\,,\\
&\langle\partial_t p, \psi\rangle_W + \int_\Omega r\,\Delta\psi - \sigma\int_\Omega p\psi - \int_\Omega f''(\varphi^*)\,w^* q\psi - \int_\Omega f'(\varphi^*)\,r\psi\\
&= -\int_\Omega \rho_1\psi\,,\\
&\int_\Omega q\eta + \int_\Omega \Delta p\,\eta = 0\,,\\
&\int_\Omega r\omega - \int_\Omega \nabla q\cdot\nabla\omega - \nu\int_\Omega q\omega - \int_\Omega f'(\varphi^*)\,q\omega = 0\,.
\end{aligned}$$

At this point, we add all of them to each other and integrate over $(0,T)$. Due to obvious cancellations, what remains is the identity

$$\int_0^T \bigl(\langle\partial_t\psi(t), p(t)\rangle_W + \langle\partial_t p(t), \psi(t)\rangle_W\bigr)\,dt = \int_Q (u-u^*)p - \int_Q \rho_1\psi\,,$$

and an application of the well-known integration-by-parts formula in the framework of the Hilbert triplet $(W,H,W^*)$, combined with (4.13) and (4.46), yields that

$$\int_Q \rho_1\psi + \int_\Omega \rho_2\psi(T) = \int_Q (u-u^*)p\,.$$

Therefore, by recalling (4.47) and using the above equality in (4.41), we obtain (4.50). □

**Remark 4.10.** Since the optimal control problem **(P)** is nonconvex, it will usually have many local minima. In this connection, recall that a control $u^*\in\mathcal{U}_{\rm ad}$ is called *locally optimal for* **(P)** *in the sense of $L^p$ for $p\in[1,+\infty]$* if and only if there is some $\tau>0$ such that $\mathcal{J}(u^*,\mathcal{S}(u^*)) \le \mathcal{J}(u,\mathcal{S}(u))$ for all $u\in\mathcal{U}_{\rm ad}$ with $\|u-u^*\|_p\le\tau$. It is easily seen that any locally optimal control in the sense of $L^p$ for some $p\in[1,+\infty)$ is also locally optimal in the sense of $L^\infty$. We now claim that the variational inequality (4.50) is valid also for every control which is locally optimal in the sense of any $p\in[1,+\infty]$. Indeed, it is easily observed that for every locally optimal control $u^*$ in the sense of $L^\infty$ the variational inequality (4.41) must be satisfied. By the same argument as above, then also (4.50) must be valid.

# References


[1] M. Burger, S. Y. Chu, P. A. Markowich and C. B. Schonlieb, The Willmore functional and instabilities in the Cahn–Hilliard equation, *Commun. Math. Sci.* **6** (2008), 309-329.

[2] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, *J. Chem. Phys.* **28** (1958), 258–267.

[3] P. Colli, G. Gilardi, A. Signori and J. Sprekels, Cahn–Hilliard–Brinkman model for tumor growth with possibly singular potentials, *Nonlinearity* **36** (2023), 4470-4500.

[4] P. Colli and P. Laurençot, *A phase-field approximation of the Willmore flow with volume constraint*, Interfaces Free Bound. **13** (2011), 341-351.

[5] P. Colli and P. Laurençot, *A phase-field approximation of the Willmore flow with volume and area constraints*, SIAM J. Math. Anal. **44** (2012), 3734-3754.

[6] S. Dai and K. Promislow, Competitive geometric evolution of amphiphilic interfaces, *SIAM J. Math. Anal.* **47** (2015), 347–380.

[7] A. Doelman, G. Hayrapetyan, K. Promislow and B. Wetton, Meander and pearling of single-curvature bilayer interfaces in the functionalized Cahn–Hilliard equation, *SIAM J. Math. Anal.* **46** (2014), 3640-3677.

[8] Q. Du, C. Liu, R. Ryham and X. Wang, A phase field formulation of the Willmore problem, *Nonlinearity* **18** (2005), 1249–1267.

[9] Q. Du, C. Liu and X. Wang, A phase field approach in the numerical study of the elastic bending energy for vesicle membranes, *J. Comput. Phys.* **198** (2004), 450–468.

[10] G. Gilardi, A. Miranville and G. Schimperna, On the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions, *Commun. Pure Appl. Anal.* **8** (2009) 881-912.

[11] G. Gompper and M. Schick, Correlation between structural and interfacial properties of amphiphilic systems, *Phys. Rev. Lett.* **65** (1990), 1116–1119.

[12] M. Grasselli and H. Wu, Well-posedness and long-time behavior for the modified phase-field crystal equation, *Math. Models Methods Appl. Sci.* **24** (2014), 2743–2783.

[13] M. Grasselli and H. Wu, Robust exponential attractors for the modified phase-field crystal equation, *Discrete Contin. Dyn. Syst.* **35** (2015), 2539–2564.

[14] M. Korzec, P. Nayar and P. Rybka, Global weak solutions to a sixth order Cahn–Hilliard type equation, *SIAM J. Math. Anal.* **44** (2012), 3369–3387.

[15] M. Korzec and P. Rybka, On a higher order convective Cahn–Hilliard-type equation, *SIAM J. Appl. Math.* **72** (2012), 1343–1360.

[16] A. Miranville, Sixth-order Cahn–Hilliard equations with singular nonlinear terms, *Appl. Anal.* **94** (2015), 2133–2146.

[17] A. Miranville, On the phase-field-crystal model with logarithmic nonlinear terms, *RACSAM* **110** (2016), 145–157.

[18] A. Miranville and S. Zelik, Robust exponential attractors for Cahn–Hilliard type equations with singular potentials, *Math. Methods Appl. Sci.* **27** (2004) 545-582.

[19] I. Pawłow and W. Zajączkowski, A sixth order Cahn–Hilliard type equation arising in oil-water- surfactant mixtures, *Comm. Pure Appl. Anal.* **10** (2011), 1823–1847.

[20] I. Pawłow and W. Zajączkowski, On a class of sixth order viscous Cahn–Hilliard type equations, *Discrete Contin. Dyn. Syst. Ser. S* **6** (2013), 517–546.

[21] K. Promislow and B. Wetton, PEM fuel cells: a mathematical overview, *SIAM J. Appl. Math.* **70** (2009), 369–409.

[22] K. Promislow and Q. Wu, Existence of pearled patterns in the planar functionalized Cahn–Hilliard equation, *J. Differential Equations* **259** (2015), 3298-3343.

[23] K. Promislow and Q. Wu, Existence, bifurcation, and geometric evolution of quasi-bilayers in the multicomponent functionalized Cahn–Hilliard equation, *J. Math. Biol.* **75** (2017), 443–489.

[24] G. Schimperna and I. Pawłow, A Cahn–Hilliard equation with singular diffusion, *J. Differential Equations* **254** (2013), 779–803.

[25] G. Schimperna and H. Wu, On a class of sixth-order Cahn–Hilliard-type equations with logarithmic potential, *SIAM J. Math. Anal.* **52** (2020) 5155-5195.

[26] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl. (4)* **146** (1987), 65-96.

[27] S. Torabi, J. Lowengrub, A. Voigt and S. Wise, A new phase-field model for strongly anisotropic systems, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **46** (2009), 1337–1359.

[28] C. Wang and S. Wise, Global smooth solutions of the three-dimensional modified phase field crystal equation, *Methods Appl. Anal.* **17** (2010), 191–211.