Reduced subcritical Galton-Watson processes in a random environment

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Abstract

We study the structure of genealogical trees of reduced subcritical Galton-Watson processes in a random environment assuming that all (in time randomly varying) offspring generating functions are fractional linear. We show that this structure may differ significantly from that for the "classical" reduced subcritical Galton-Watson processes. In particular, it may look like a complex "hybrid" of classical reduced super- and subcritical processes. Some relations with random walks in a random environment are discussed.

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1 Introduction and statement of results

This paper deals with the model of a *Galton-Watson process in a random environment* (*GWRE*) (see, for example, [AK72], [AN72, § VI.5] and [SW69]). A GWRE is specified by a family $\mathbf{f} = \{f_n : n \geq 0\}$ of *independent and identically distributed* offspring generating functions[1]. It is assumed that conditioned on the environment $\mathbf{f}$, the reproduction of particles follows the pattern of a "classical" inhomogeneous Galton-Watson branching process. That is, given $\mathbf{f}$ and the total number Z_n of particles in the n-th generation the reproduction law of the number Z_{n+1} of particles in the $(n+1)$-th generation is described via the generating function

$$\mathbf{E}_{\mathbf{f}} \left\{ s^{Z_{n+1}} \;\middle|\; Z_n \right\} = \left(f_n\left(s\right)\right)^{Z_n}, \qquad 0 \leq s \leq 1.$$

Here $\mathbf{E}_{\mathbf{f}}$ refers to expectation with respect to the *quenched* law $\mathbf{P}_{\mathbf{f}}$ of the model, that is the process law given the environment $\mathbf{f}$.

In this paper we always *assume* that $\log f_0'(1)$ is integrable with respect to the law $\mathbb{P}$ of the environment $\mathbf{f}$. (The prime refers to the derivative.) According to a standard classification, a GWRE is said to be *subcritical, critical* or *supercritical* if $\mathbb{E} \log f_0'(1) < 0$, $= 0$, or > 0, respectively ($\mathbb{E}$ refers to expectation with respect to $\mathbb{P}$).

[1] $f : [0,1] \to [0,1]$ is an *offspring generating function* if $f(s) = \sum_{i=0}^{\infty} \pi_i s^i$ with $\pi_i \geq 0$ and $\sum_i \pi_i = 1$.

In the sequel we *assume* that $Z_0 = 1$.

The quantity we want to study is the number $Z_{m,n}$, $0 \le m \le n$, of particles at time m having non-empty offspring at time n. Of course, this quantity requires a finer description of the model as used above. The most convenient way is to think in terms of *Galton-Watson trees*. For our purpose, we skip a formal description and only refer to [FP74] or [FSS77], for instance. Given $Z_n > 0$, the process $\{Z_{m,n} : 0 \le m \le n\}$ is called the *reduced GWRE.*

Reduced processes have been studied in the constant environment setting by several authors, see [FP74], [Zub75], [FSS77], [Dur78], [Vat79], [Sag95], providing a detailed description of the genealogical structure. *Applications* of some of those results to the problem of estimating the *age of the most recent common ancestor* (Eve) of all nowadays living people (given such an Eve exists) can be found in O'Connell [O'C95]; see also Jagers et al. [JNT91] and Vigilant et al. [VPH$^+$89] for related discussions.

In the present paper we are interested in reduced Galton-Watson processes in the *random environment* case. For this purpose, it is very convenient to *restrict completely to fractional linear offspring generating functions.* That is,

$$1 - f_n(s) \;=\; \frac{\alpha(n)}{1-\beta(n)} - \frac{\alpha(n)s}{1-\beta(n)s}\,, \qquad 0 \le s \le 1, \quad n \ge 0, \tag{1}$$

where

$$\alpha(n), \beta(n) \;>\; 0 \quad \text{and} \quad \alpha(n) + \beta(n) \;<\; 1. \tag{2}$$

(Note that the latter conditions exclude the cases $f_n(s) \equiv 1$ or $f_n(s) \equiv s$ for some n, hence, in particular, $f_n'(1) = 0$ and $f_n''(1) = 0$ are forbidden.)

A first step to deal with such reduced processes in a random environment was done by [BV96]. There it was shown that in the *critical* case the structure of the genealogical trees of such processes resembles a bit the classical *super*critical case. Our interest however concerns the *subcritical* case. It will turn out that here even more interesting effects may occur. In fact, under certain conditions, we get some kind of *"hybrid"* behavior from the point of view of classical processes: At the initial stage, the reduced tree might look as in the classical supercritical processes, whereas in the final stage it resembles again the classical subcritical case. Theorems 3 and 4 below will give a rigorous description of these phenomena.

Henceforth we will use the following *notation:*

$$X_i \;:=\; \log f_{i-1}'(1) \in \mathsf{R}, \qquad \eta_i \;:=\; \frac{f_{i-1}''(1)}{2\left(f_{i-1}'(1)\right)^2} \in (0,\infty), \qquad i \ge 1. \tag{3}$$

Throughout we assume that the random walk

$$S_n \;:=\; \sum_{1 \le i \le n} X_i\,, \qquad n \ge 0, \tag{4}$$

in R (starting from 0) *is non-lattice.*

Remark 1 (lattice case) The lattice case can also be studied by the methods of the present paper. However, for this one needs some additional assumptions such as, for instance, $\mathbb{P}\{X_1 = 0\} > 0$, and we simply want to avoid such technicalities. $\diamond$

Recall that for classical reduced subcritical Galton-Watson processes the finiteness of a so-called $Z \log Z$–moment is of some importance; see, for instance, [FP74], [Pre79]. In the present random environment case there is a similar quantity, the moment $\mathbb{E} f_0'(1) \log f_0'(1)$. But the situation is a bit delicate, since one has to distinguish between three different regions of finiteness, namely whether this moment is less than 0, equal to 0, and larger than 0. Accordingly, for the survival probability one has three different speeds (see Lemma 11 below). On the other hand, a new phenomenon concerning a *conditional limit theorem* for the reduced process is obtained only in the last case.

Now we are ready to formulate our *principal results.* They are expressed in terms of the *annealed* law $\mathcal{P} := \mathbb{E}\mathbf{P_f}$. Recall that we always assume (1) and (2), and that $\log f_0'(1)$ is $\mathbb{P}$–integrable.

Theorem 2 (conditional limit theorem of the classical type) *Let either condition* (a) *or* (b) *be fulfilled:*

(a) (strongly subcritical case)

$$\mathbb{E} f_0'(1) < 1, \qquad \mathbb{E} f_0'(1) \log f_0'(1) < 0, \tag{5}$$

and, in addition,

$$\min\left(\mathbb{E} f_0''(1), \mathbb{E}\,\eta_1\right) < \infty. \tag{6}$$

(b) (intermediate subcritical case)

$$\mathbb{E} f_0'(1) < 1, \qquad \mathbb{E} f_0'(1) \log f_0'(1) = 0, \tag{7}$$

and, additionally,

$$\mathbb{E}\,\eta_1 f_0'(1) < \infty, \;\; \mathbb{E}\,\eta_1 f_0'(1)\,|\log f_0'(1)| < \infty, \;\; \mathbb{E} f_0'(1)(\log f_0'(1))^2 < \infty. \tag{8}$$

Then, for $m \geq 0$ *and* $l \geq 1$, *the following limits*

$$\lim_{n\to\infty} \mathcal{P}\Big\{ Z_{n-m,\,n} = l \;\Big|\; Z_n > 0 \Big\} =: p_l(m) > 0 \tag{9}$$

of conditional probabilities exist and satisfy

$$\sum_{l=1}^{\infty} p_l(m) = 1 \tag{10}$$

and

$$\lim_{m\to\infty} p_1(m) = 1. \tag{11}$$

The quantity $\min\{m \geq 1 : Z_{n-m,\,n} = 1\}$ is called the *age of the most recent common ancestor* of the non-empty n–th generation. (In [FP74] and other papers this quantity was called the *source time* of the n–th generation.) Thus, according to Theorem 2, in the strongly and intermediate subcritical cases the most recent common ancestor is "located" *close* to the moment n. Thus, in this regime the situation is similar to that for classical subcritical Galton-Watson processes ([FP74]).

Theorem 3 (hybrid conditional limit theorem) *Assume*

(c) (weakly subcritical case)

$$\mathbb{E}\log f_0'(1) < 0, \qquad 0 < \mathbb{E} f_0'(1)\log f_0'(1) < \infty \tag{12}$$

and, in addition,

$$\mathbb{E}\,\eta_1 < \infty, \qquad \mathbb{E}\,\eta_1\, f_0'(1) < \infty. \tag{13}$$

Then for $m \geq 0$ *and* $l \geq 1$, *the limits* (9) *with* (10) *exist, and for* $m, l \geq 1$ *also the limits*

$$\lim_{n\to\infty} \mathcal{P}\Big\{ Z_{m,n} = l \;\Big|\; Z_n > 0 \Big\} =: \widehat{p}_l(m) \;>\; 0, \tag{14}$$

with

$$\sum_{l=1}^{\infty} \widehat{p}_l(m) \;=\; 1. \tag{15}$$

Consequently, in this weakly subcritical case, besides (9) and (10) as in the previous theorem, with a positive probability the most recent common ancestor is located exactly at the *beginning* of the genealogical tree just as for classical *super*critical Galton-Watson processes ([Zub75]). This phenomenon is now considered in more detail. For each $n \geq 1$, let u_n and v_n be integers such that

$$\lim_{n\to\infty} u_n \;=\; \lim_{n\to\infty} v_n \;=\; \infty, \qquad \lim_{n\to\infty} (n - u_n - v_n) \;=\; \infty. \tag{16}$$

Theorem 4 (branchless thick trunk) *Let conditions* (12), (13) *and* (16) *be fulfilled. Then*

$$\lim_{n\to\infty} \mathcal{P}\Big\{ Z_{u_n,n} = Z_{n-v_n,n} \;\Big|\; Z_n > 0 \Big\} \;=\; 1. \tag{17}$$

This means that for weakly subcritical GWRE at late times the reduced genealogical tree can be *interpreted* as follows. After the branching of the reduced process at the beginning as in (14), there are *very long branches without any branching* till the moment $n - v_n$, and after this the branching is allowed to continue. See the idealized picture in the figure below.

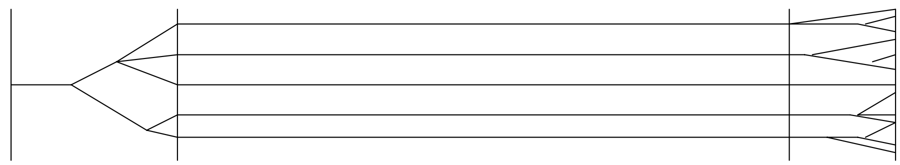

Figure 1: Reduced weakly subcritical GWRE at a late time

As will be seen later in the proof, this phenomenon is closely related to some well-known random walk effects: Random walks with negative drift but conditioned to stay positive, appropriately scaled converge to some Brownian *excursion* ([Kao78]). In our context, the initial period of excursion corresponds to a 'supercritical phase' whereas the final period reflects the 'subcritical phase' of branching. At least on a *heuristic* level this explains a bit the strange behavior of the reduced process as expressed in Theorem 4.

The rest of this paper is laid out as follows. After some preparations in Section 2, the proofs of the Theorems 2–4 will be provided in the Sections 3–5, respectively. For this we exploit an idea of Kozlov [Koz76] who studied the behavior of survival probabilities of critical GWRE, and which was modified by Afans'ev [Afa80] to investigate the analogous problem for subcritical GWRE.

Remark 5 (completeness of conditions) Conditions (5), (7) and (12) cover *all* the possibilities in the subcritical case provided that $f_0'(1)\log f_0'(1)$ is $\mathbb{P}$–integrable. Indeed, by Jensen's inequality, $\mathbb{E}\log f_0'(1) \le \log \mathbb{E} f_0'(1)$. Therefore, (5), (7) and (12) cover first of all the case $\mathbb{E} f_0'(1) < 1$. On the other hand, if $\mathbb{E} f_0'(1) \ge 1$, then in view of the elementary inequality $x\log x \ge x-1$ for $x \ge 0$, with equality if and only if $x=1$, we have $\mathbb{E} f_0'(1)\log f_0'(1) > \mathbb{E} f_0'(1) - 1 \ge 0$, where the strong inequality follows from the subcriticality $\mathbb{E}\log f_0'(1) < 0$, which forbids that $f_0'(1) = 1$, $\mathbb{P}$–a.s. $\diamond$

Remark 6 (technical conditions) Conditions (6), (8), and (13), however, are of purely technical nature. They had been used in [Afa80] to establish the asymptotics of the survival probability, on which we rely in this paper; see Lemma 11 below. $\diamond$

Remark 7 (terminal population in the quenched approach) Because of $Z_{n,n} = Z_n$, by Theorems 2 and 3 we get in particular, that for subcritical GWRE the conditional distribution of Z_n given $Z_n > 0$ tends to a (proper) law as $n \to \infty$ (*Yaglom type theorem*). For the case when there is no averaging over the environment (quenched approach) this fact has been proved in [AK72] without restricting to fractional linear offspring generating functions (1). $\diamond$

Remark 8 (relations to RWRE) The case of *geometric* offspring distributions is covered by the fractional linear offspring generating functions of (1). (Here we have $\alpha(n) \equiv (1-\beta(n))\,\beta(n)$ additionally.) In this particular case there is a natural correspondence between the models of GWRE and *random walk in a random environment* having negative drift and jumps ± 1, which starts at zero and is stopped after the first return to zero; see [Dwa69], [KKS75], [Koz73] and [VD96] for more details. Passing to reduced trees, this relation remains valid by dropping parts of the walk and stick together the remaining parts in the obvious way. Then all our results imply facts for such a random walk in a random environment. In particular, it follows from Theorem 3 that, given the maximum of the stopped random walk exceeds n, the walk may *oscillate* several times between the level n and the vicinity of 0 with positive probability before hitting zero. Such phenomenon differs significantly from the behavior of the classical random walk with negative drift. Note that a similar random medium effect was observed by Sinai [Sin82] and [BV96] concerning a random walk in a random environment without drift. $\diamond$

For standard facts on Galton-Watson processes we refer to [AN72], for random walks to [Fel71].

2 Preparation: Some basic facts on GWRE

Recalling notation (3) and (4), for $0 \le r \le n$, set

$$a_{r+1,n} := \exp\Big[-\sum_{r+1\le i\le n} X_i\Big], \qquad b_{r+1,n} := \sum_{r+1\le i\le n} \eta_i\, a_{r+1,i-1}\,, \tag{18}$$

and

$$a_n := a_{1,n}\,, \qquad b_n := b_{1,n}\,. \tag{19}$$

Note that $a_{r+1,n} > 0$ and $b_{r+1,n} > 0$, except $b_{n+1,n} \equiv 0$. Note also the multiplicativity

$$a_r\, a_{r+1,n} \equiv a_n\,, \quad \text{hence} \quad b_r + a_r\,(a_{r+1,n} + b_{r+1,n}) \equiv a_n + b_n > 0. \tag{20}$$

For $0 \le m \le n$, put

$$F_{m,n}(s) := f_m(f_{m+1}(\cdots f_{n-1}(s)\cdots)), \qquad F_n(s) := F_{0,n}(s),$$

reading $F_{n,n}(s)$ as s. Set

$$q_{m,n} := 1 - F_{m,n}(0), \qquad q_n := q_{0,n} = \mathbf{P}_{\mathbf{f}}\,(Z_n > 0), \tag{21}$$

and

$$Q(n) := \mathbb{E}\, q_n = \mathcal{P}\,\{Z_n > 0\} \tag{22}$$

(*survival probability* of Z_n). Recall we always consider fractional linear offspring generating functions as written in (1) and (2). For convenience, we expose the following statement due to Agresti [Agr75].

Lemma 9 (formulas for iterates) *For* $0 \le m \le n$ *and* $0 \le s \le 1$,

$$1 - F_{m,n}(s) \;=\; \frac{1}{\frac{a_{m+1,n}}{1-s} + b_{m+1,n}}\,. \tag{23}$$

In particular,

$$1 - F_n(s) \;=\; \frac{1}{\frac{a_n}{1-s} + b_n}\,. \tag{24}$$

Specializing (23) to $s = 0$ gives

$$\left(a_{m+1,n} + b_{m+1,n}\right)^{-1} \;\le\; 1. \tag{25}$$

Recalling (21), (23), and (20), for $0 \le m \le n$, put

$$0 \;<\; w_{m,n} \;:=\; \frac{a_m}{q_{m,n}} \;=\; a_n + a_m\, b_{m+1,n} \;=\; a_n + b_n - b_m \;\le\; a_n + b_n\,. \tag{26}$$

For $0 \le m \le n$ and $l \ge 1$, set

$$0 \;\le\; U_n(m,l) \;:=\; \frac{w_{m,n}\; b_m^{l-1}}{\left(a_n + b_n\right)^{l+1}} \;\le\; \frac{1}{a_n + b_n} \;=\; q_n \;\le\; \frac{1}{a_n} \;=\; \mathrm{e}^{S_n}, \tag{27}$$

and

$$0 \;\le\; V_n(m,l) \;:=\; \frac{b_m^{l-1}}{\left(a_n + b_n\right)^{l}}\,. \tag{28}$$

The next lemma is established in [BV96]. Recall notation (22).

Lemma 10 (representation of conditional probabilities) *For* $l \ge 1$ *and* $0 \le m \le n$,

$$\mathcal{P}\Big\{Z_{m,n} = l \;\Big|\; Z_n > 0\Big\} \;=\; Q^{-1}(n)\,\mathbb{E}\,U_n(m,l), \tag{29}$$

$$\mathcal{P}\Big\{Z_{m,n} \ge l \;\Big|\; Z_n > 0\Big\} \;=\; Q^{-1}(n)\,\mathbb{E}\,V_n(m,l). \tag{30}$$

It is known [AK72] that the extinction time for subcritical GWRE is *finite* $\mathcal{P}$–a.s. However, to prove the Theorems 2–4 we need more detailed information about the behavior of the survival probabilities $Q(n)$ as $n \to \infty$, taken from [Afa80].

Lemma 11 (asymptotics for the survival probability) *Let* Z *be a subcritical GWRE satisfying* (1) *and* (2). *Then*

$$Q(n) \;=\; \mathcal{P}\left\{Z_n > 0\right\} \;\sim\; h(n)\,G^{\,n} \quad as \quad n \to \infty$$

where

(i)	$h(n) \equiv c_1 > 0,$	$G = \mathbb{E} f_0'(1),$	*if* (5) *and* (6) *hold;*
(ii)	$h(n) = c_2\, n^{-1/2},$	$G = \mathbb{E} f_0'(1),$	*if* (7) *and* (8) *hold;*
(iii)	$h(n) = c_3\, n^{-3/2},$	$G = \min\limits_{0 \le t \le 1} \mathbb{E}\left(f_0'(1)\right)^{t},$	*if* (12) *and* (13) *hold.*

Here and in what follows the symbols c, $c_1, \ldots$ are used to denote positive constants not necessarily the same in different formulas.

Note that the previous three lemmas *reduce* the conditional probabilities in the center of our theorems to expressions in terms of the environment $\mathbf{f}$ only, in particular on assertions on the random walk S. This we will heavily use in the further procedure.

3 Classical type behavior

In this section we want to provide the *Proof of Theorem* 2. We will use ideas based on random walks a bit different from S, namely ones obtained by some standard transformations (see, for example, [VT76]). Based on Lemma 10, such transformations allow to reduce the problem under consideration to relatively easy arguments about a random walk with negative drift (strongly subcritical case) or to a known problem for a driftless random walk (intermediate subcritical case).

1° (*preparation*) Recall that $0 < \mathbb{E} f_0'(1) := g < 1$ by the assumptions in Theorem 2. Then, evidently,

$$g^{-n} \prod_{i=1}^{n} \mathrm{e}^{x_i}\, \mathbb{P}\Big(X_i \in \mathrm{d}x_i\,,\ \eta_i \in \mathrm{d}y_i\Big) \tag{31}$$

is a law of a random vector $\big\{\xi_i\,, \zeta_i :\ 1 \leq i \leq n\big\}$, say, with n independent and identically distributed pairs $(\xi_1\,, \zeta_1)\,, \ldots, (\xi_n\,, \zeta_n)$ in $\mathrm{R} \times (0, \infty)$. Clearly,

$$\mathbb{E}\, \xi_i \;=\; g^{-1} \int_{-\infty}^{\infty} \mathbb{P}(X_1 \in \mathrm{d}x)\ x\, \mathrm{e}^{x} \;=\; g^{-1}\, \mathbb{E} f_0'(1) \log f_0'(1) \;\leq\; 0 \tag{32}$$

(under (a) or (b)). Recalling notation (27), (26), (18), (19), and (31), one can see that

$$\mathbb{E}\, U_n(n-m, l) \;=\; g^n\, \mathbb{E} M_n(m, l), \qquad 0 \leq m \leq n, \quad l \geq 1, \tag{33}$$

where

$$M_n(m,l) \;:=\; \frac{\Big(1 + \sum_{n-m+1 \leq i \leq n} \zeta_i\, \mathrm{e}^{\xi_i + \cdots + \xi_n}\Big) \Big(\sum_{1 \leq i \leq n-m} \zeta_i\, \mathrm{e}^{\xi_i + \cdots + \xi_n}\Big)^{l-1}}{\Big(1 + \sum_{1 \leq i \leq n} \zeta_i\, \mathrm{e}^{\xi_i + \cdots + \xi_n}\Big)^{l+1}} \;\leq 1. \tag{34}$$

It will be more convenient to introduce the random variables

$$\mu_j \;:=\; -\xi_{n-j+1}\,, \quad \nu_j \;:=\; \zeta_{n-j+1}\,, \qquad 1 \leq j \leq n. \tag{35}$$

Substituting $i = n - j + 1$, formula (34) turns into

$$M_n(m,l) \;=\; \frac{\Big(1+\sum\limits_{1\le j\le m}\nu_j\,\mathrm{e}^{-\mu_1-\cdots-\mu_j}\Big)\Big(\sum\limits_{m+1\le j\le n}\nu_j\,\mathrm{e}^{-\mu_1-\cdots-\mu_j}\Big)^{l-1}}{\Big(1+\sum\limits_{1\le j\le n}\nu_j\,\mathrm{e}^{-\mu_1-\cdots-\mu_j}\Big)^{l+1}} \;\le\; 1. \quad (36)$$

Next we use the random walk

$$Y_n \;:=\; \sum_{1\le i\le n}\mu_i\,, \qquad n\ge 0, \qquad (37)$$

and, for $0\le m\le n$, the quantities

$$A_{m+1,n} \;:=\; \exp\Big[-\sum_{m+1\le i\le n}\mu_i\Big], \qquad B_{m+1,n} \;:=\; \sum_{m+1\le i\le n}\nu_i\,A_{m+1,i}\,, \quad (38)$$

and

$$A_n \;:=\; A_{1,n}\,, \qquad B_n \;:=\; B_{1,n}\,. \qquad (39)$$

Note that $A_{m+1,n}>0$ and $B_{m+1,n}>0$, except $B_{n+1,n}\equiv 0$. Then we can rewrite formula (36) as

$$M_n(m,l) \;=\; \frac{(1+B_m)\,(A_m B_{m+1,n})^{l-1}}{(1+B_n)^{l+1}} \;\le\; 1, \qquad 0\le m\le n, \quad l\ge 1. \quad (40)$$

2° (*convergence and positivity under positive drift*) First we consider the case

$$\mathbb{E}\,\mu_1 \;=\; -g^{-1}\,\mathbb{E}f_0'(1)\log f_0'(1) \;>\; 0 \qquad (41)$$

(that is the strongly subcritical case (5)). Applying arguments as in the standard proof of strong law of large numbers, by the additional moment condition (6),

$$0 \;<\; \lim_{n\to\infty} M_n(m,l) \;=:\; M_\infty(m,l) \;\le\; 1, \quad \mathbb{P}\text{–a.s.}, \qquad m\ge 0, \quad l\ge 1.$$

Hence, recalling notation (22), identity (33), and assumption (41), by the asymptotics Lemma 11 (i) we get

$$\lim_{n\to\infty} Q^{-1}(n)\,\mathbb{E}\,U_n(n-m,l) \;=\; c_1\,\mathbb{E}M_\infty(m,l) \;\in\; (0,\infty).$$

By the representation formula (29), this proves (9) in the strongly subcritical case (a).

3° (*driftless case*) To demonstrate (9) for the case $\mathbb{E}\,\mu_1 = 0$ (that is in the intermediate subcritical case (7)) we introduce the *weak lower ladder epochs*

$$\tau_0^* \;:=\; 0, \quad \tau_j^* \;:=\; \min\Big\{t>\tau_{j-1}^* :\; Y_t\le Y_{\tau_{j-1}^*}\Big\}, \qquad j\ge 1, \qquad (42)$$

and set

$$I_j^*(n) := I\Big\{\tau_{j-1}^* \le n < \tau_j^*\Big\}, \qquad \Theta_j^*(n) := I\left\{\tau_j^* \le n\right\}. \tag{43}$$

(Here and below the symbol $I\{E\}$ is used to denote the indicator of an event E.) By Lemma 11 (ii), for (9) it suffices to establish that the limit

$$\lim_{n\to\infty} \sqrt{n}\ \mathbb{E} M_n(m,l), \qquad m \ge 0, \quad l \ge 1, \tag{44}$$

exists and is positive.

4° (*convergence in the driftless case*) For any $J > 1$, we represent

$$\mathbb{E} M_n(m,l) = \sum_{j=1}^{J-1} \mathbb{E} M_n(m,l)\, I_j^*(n) + \mathbb{E} M_n(m,l)\, \Theta_J^*(n). \tag{45}$$

Because of $\mathbb{E}\,\mu_1 = 0$, it follows from [Koz76, (46)–(49)] and conditions (7) and (8) that

$$\lim_{J\to\infty} \limsup_{n\to\infty} \sqrt{n}\ \mathbb{E}\left(1+B_n\right)^{-1} \Theta_J^*(n) = 0.$$

But from (40),

$$M_n(m,l) \le \left(1+B_n\right)^{-1}.$$

Therefore it suffices to consider $\mathbb{E} M_n(m,l)\, I_j^*(n)$ for a fixed j. Following the pattern of [BV96, proof of Theorem 1] one can show that for each fixed j, m and l, the limit

$$\lim_{n\to\infty} \sqrt{n}\ \mathbb{E} M_n(m,l)\, I_j^*(n)$$

exists and is finite. In view of the representations (33) and (29), and the asymptotics Lemma 11 (ii), this gives the existence of the finite limit (44).

5° (*positivity in the driftless case*) To complete the proof of (9), it remains to show the positivity of (44), i.e.

$$\liminf_{n\to\infty} \sqrt{n}\ \mathbb{E} M_n(m,l) > 0, \qquad m \ge 0, \quad l \ge 1. \tag{46}$$

Clearly, from (40) and the multiplicativity of A,

$$M_n(m,l) \ge \frac{\left(\nu_{m+1}\, \mathrm{e}^{-Y_{m+1}}\right)^{l-1}}{\left(1+B_n\right)^{l+1}} = \frac{\left(\nu_{m+1}\, A_{m+1}\right)^{l-1}}{\Big(1 + B_{m+1} + A_{m+1} B_{m+2,n}\Big)^{l+1}},$$

hence,

$$M_n(m,l) \ge \frac{\left(\nu_{m+1}\, A_{m+1}\right)^{l-1}}{\Big(1 + B_{m+1} + A_{m+1}\Big)^{l+1} \Big(1 + B_{m+2,n}\Big)^{l+1}}. \tag{47}$$

Taking expectations in this inequality, we use the fact that the second factor in the denominator is *independent* of the remaining expressions. But from the i.i.d. property of the sequence $\{(\mu_i\,,\nu_i):i\geq 1\}$,

$$\mathbb{E}\,\frac{1}{\left(1+B_{m+2,n}\right)^{l+1}} \;=\; \mathbb{E}\,\mathbb{E}\left\{\frac{1}{\left(1+B_{m+2,n}\right)^{l+1}}\;\middle|\;Y_{m+1}\right\} \;=\; \mathbb{E}\,\frac{1}{\left(1+B_{n-m-1}\right)^{l+1}}\,,$$

thus

$$\mathbb{E}\left(1+B_{m+2,n}\right)^{-l-1} \;\geq\; \mathbb{E}\left(1+B_{n-m-1}\right)^{-l-1} I\left\{\tau_1^*>n-m-1\right\}. \tag{48}$$

By the arguments used in proving [Koz76, Lemma 2] one can demonstrate that the limit

$$\lim_{n\to\infty}\mathbb{E}\left\{\left(1+B_n\right)^{-1}\;\middle|\;\tau_1^*>n\right\}$$

exists and is positive. By Jensen's inequality we conclude that

$$\left.\begin{aligned}&\liminf_{n\to\infty}\mathbb{E}\left\{\left(1+B_n\right)^{-l-1}\;\middle|\;\tau_1^*>n\right\}\\&\geq\lim_{n\to\infty}\left(\mathbb{E}\left\{\left(1+B_n\right)^{-1}\;\middle|\;\tau_1^*>n\right\}\right)^{l+1}\;>\;0.\end{aligned}\right\} \tag{49}$$

Finally, see, for example, [Fel71, Ch. XII, Sec. 7],

$$\mathbb{P}\left(\tau_1^*>n\right)\;\sim\;c\,n^{-1/2}\quad\text{as}\quad n\to\infty. \tag{50}$$

Applying (48), (49), and (50) to the expectation of (47) shows that

$$\begin{aligned}&\liminf_{n\to\infty}\sqrt{n}\;\mathbb{E}M_n(m,l)\\&\geq\left(\mathbb{E}\,\frac{\left(\nu_{m+1}\,A_{m+1}\right)^{l-1}}{\left(1+B_{m+1}+A_{m+1}\right)^{l+1}}\right)\liminf_{n\to\infty}\sqrt{n}\;\mathbb{P}\left(\tau_1^*>n-m-1\right)\\&\qquad\qquad\times\,\mathbb{E}\left\{\left(1+B_{n-m-1}\right)^{-l-1}\;\middle|\;\tau_1^*>n-m-1\right\}\;>\;0.\end{aligned}$$

Summarizing, (46) is established and therefore (9) holds also in the intermediate subcritical case (b).

6° (*no loss of mass and degeneration under positive drift*) Now we turn to the proof of (10) and (11). From the representation (30), and similarly to (33),

$$\left.\begin{aligned}\mathcal{P}\left\{Z_{n-m,n}\geq l\;\middle|\;Z_n>0\right\}&\;=\;Q^{-1}(n)\,\mathbb{E}\,V_n(n-m,l)\\&\;=\;Q^{-1}(n)\,g^n\,\mathbb{E}N_n(m,l),\end{aligned}\right\} \tag{51}$$

where

$$N_n(m,l) \;=\; \frac{\left(A_m B_{m+1,n}\right)^{l-1}}{\left(1+B_n\right)^{l}} \;\le\; 1, \qquad 0 \le m \le n, \quad l \ge 1.$$

Let first again (41) be valid (positive drift). Then, as in step 2°,

$$\lim_{n\to\infty} N_n(m,l) \;=\; \frac{1}{1+\omega_1}\Big(\frac{\omega_{m+1}}{1+\omega_1}\Big)^{l-1} \underset{l\to\infty}{\searrow} 0 \tag{52}$$

with $\mathbb{P}$–probability 1, where

$$\omega_r \;:=\; \sum_{i=r}^{\infty} \nu_i\, \mathrm{e}^{-Y_i} < \infty, \quad \mathbb{P}\text{–a.s.}, \qquad r \ge 1.$$

Hence, by Lemma 11 (i),

$$\lim_{n\to\infty} \mathcal{P}\left\{ Z_{n-m,\,n} \ge l \;\middle|\; Z_n > 0 \right\} \underset{l\to\infty}{\searrow} 0, \tag{53}$$

and therefore

$$\lim_{l\to\infty} \Big(1 - \sum_{j=1}^{l} p_j(m)\Big) \;=\; 0$$

proving (10). Finally, since $\omega_m \downarrow 0$ as $m \uparrow \infty$, from (51) and (52),

$$\lim_{m\to\infty} \sum_{j=2}^{\infty} p_j(m) \;=\; 0, \quad \text{that is,} \quad \lim_{m\to\infty} p_1(m) \;=\; 1.$$

This yields (11) under (41), that is in the strongly subcritical case (a).

7° (*degeneration in the driftless case*) The proof of (11) in the case $\mathbb{E}\,\mu_1 = 0$ needs more delicate estimates. Analogously to (45), for any $J > 1$,

$$\mathbb{E}N_n(m,l) \;=\; \sum_{j=1}^{J-1} \mathbb{E}N_n(m,l)\, I_j^*(n) + \mathbb{E}N_n(m,l)\, \Theta_J^*(n). \tag{54}$$

Since $N_n(m,l) \le (1+B_n)^{-1}$, as in step 4° it suffices to consider the expectation $\mathbb{E}N_n(m,l)\, I_j^*(n)$, for a fixed j. Start with $j=1$. By (50), and according to [Koz76, Lemma 1] (whose correct proof can be found in [BV96]), for each $l \ge 2$,

$$\left.\begin{aligned}
&\limsup_{m\to\infty}\, \limsup_{n\to\infty}\, \sqrt{n}\; \mathbb{E}N_n(m,l)\, I_1^*(n) \\
&= c\, \limsup_{m\to\infty}\, \limsup_{n\to\infty}\, \mathbb{E}\left\{ N_n(m,l) \;\middle|\; \tau_1^* > n \right\} \\
&\le c\, \limsup_{m\to\infty}\, \limsup_{n\to\infty}\, \mathbb{E}\left\{ A_m B_{m+1,n} \;\middle|\; \tau_1^* > n \right\} \;=\; 0.
\end{aligned}\right\} \tag{55}$$

Hence, it remains to look at a fixed $j \geq 2$. For $T \geq 1$ (and $j, l \geq 2$) fixed,

$$\left.\begin{aligned} \mathbb{E} N_n(m,l)\, I_j^*(n) \;\leq\; & \mathbb{P}\left\{T \leq \tau_{j-1}^* \leq n < \tau_j^*\right\} \\ & + \mathbb{E} N_n(m,l)\, I\left\{\tau_{j-1}^* < T,\ \tau_j^* > n\right\}. \end{aligned}\right\} \tag{56}$$

Evidently,

$$\mathbb{P}\left(T \leq \tau_{j-1}^* \leq n < \tau_j^*\right) \;=\; \sum_{k=T}^{n} \mathbb{P}\left(\tau_{j-1}^* = k\right) \mathbb{P}\left(\tau_1^* > n-k\right).$$

Distinguishing additionally between $k < \frac{n}{2}$ and $k \geq \frac{n}{2}$, we may estimate above by

$$\leq\; \mathbb{P}\big(\tau_{j-1}^* \geq T\big)\, \mathbb{P}\big(\tau_1^* > \tfrac{n}{2}\big) + \sum_{\frac{n}{2} \leq k \leq n} \mathbb{P}(\tau_1^* > n-k) \max_{k \geq \frac{n}{2}} \mathbb{P}\big(\tau_{j-1}^* = k\big).$$

Therefore, in view of

$$\mathbb{P}\left(\tau_{j-1}^* = n\right) \;\sim\; c_j\, n^{-3/2} \quad \text{as} \quad n \to \infty$$

(see, for example, [Koz76, (17)]), we have

$$\sqrt{n}\, \mathbb{P}\left(T \leq \tau_{j-1}^* \leq n < \tau_j^*\right) \;\leq\; c_1\, \mathbb{P}\left(\tau_{j-1}^* \geq T\right) + c_2\, n^{-1/2}. \tag{57}$$

Letting first $n \to \infty$, and then $T \to \infty$, these terms will disappear. It remains to deal with the second term at the r.h.s. of (56), for $T \geq 1$ and $j, l \geq 2$ fixed. For $m > T$,

$$\mathbb{E} N_n(m,l)\, I\left\{\tau_{j-1}^* < T,\ \tau_j^* > n\right\} \;\leq\; \mathbb{E} A_m B_{m+1,n}\, I\left\{\tau_{j-1}^* < T,\ \tau_j^* > n\right\}.$$

Distinguishing between different values k of τ_{j-1}^*, using that $m > T$, and applying a renewal argument, the r.h.s. can be written as

$$\sum_{j-1 \leq k < T} \mathbb{E}\Big\{\mathrm{e}^{-Y_k};\, \tau_{j-1}^* = k\Big\}\, \mathbb{E}\Bigg(\sum_{m+1-k \leq i \leq n-k} \nu_{i+k}\, \mathrm{e}^{-Y_i'}\, \widehat{I}_1(n-k)\Bigg). \tag{58}$$

Here Y' denotes an independent copy of Y (but keeping the dependence structure with the ν_j), and $\widehat{I}_1(n)$ is the indicator of the event that the Y_i' are positive for $1 \leq i \leq n$. Multiplying the latter expectation expression with $\sqrt{n-k}$, by (55) and (57) (case $j = 2$ there), we see that (58) will vanish as first $n \to \infty$ and then $m \to \infty$. This finishes the proof of (11).

8° (*no loss of mass in the driftless case*) It remains to prove that the limiting measure has total mass 1 if $\mathbb{E}\,\mu_1 = 0$. From the results of [Koz76, Section 3] it follows that

$$\limsup_{J \to \infty}\, \limsup_{n \to \infty}\, Q^{-1}(n)\, g^{-n}\, \mathbb{E} N_n(m,l)\, \Theta_J^*(n) \;=\; 0,$$

and that for each fixed $j \geq 1$,

$$\lim_{n\to\infty} n^{-1/2}\, Q^{-1}(n)\, g^{-n}\, \mathbb{E}\left\{ N_n(m,l) \;\Big|\; \tau^*_{j-1} \leq n < \tau^*_j \right\}$$
$$= c_j\, \mathbb{E}\, \frac{1}{1+\theta_{1,j}} \left(\frac{\theta_{m+1,j}}{1+\theta_{1,j}}\right)^{l-1}$$

where the laws of the non-negative random variables $\theta_{r,j}$, $r \geq 1$, are specified by

$$\mathbb{P}(\theta_{r,j} \leq x) \;=\; \lim_{n\to\infty} \mathbb{P}\left\{ \sum_{i=r}^{n} \nu_i\, \mathrm{e}^{-Y_i} \leq x \;\Big|\; \tau^*_{j-1} \leq n < \tau^*_j \right\}.$$

Combining these facts with Lemma 11 (ii), it follows that (53) is valid for the case $\mathbb{E}\,\mu_1 = 0$ as well. Summarizing, we verified (10) also in the intermediate subcritical case, finishing the proof of Theorem 2. ∎

4 Hybrid behavior

In this section we will provide the *Proof of Theorem* 3. To this aim, analogously to step 3° (p.9) in the proof of Theorem 2, for the random walk S with negative drift we introduce weak lower ladder epochs

$$\tau_0 := 0, \quad \tau_j \;:=\; \min\left\{ t > \tau_{j-1} :\; S_t \leq S_{\tau_{j-1}} \right\}, \qquad j \geq 1. \tag{59}$$

According to the conditions of Theorem 3 the function $\varphi(t) := \mathbb{E}e^{tX_1}$, $0 \leq t \leq 1$, possesses the following properties:

$$\varphi(0) \;=\; 1, \qquad \varphi'(0) \;=\; \mathbb{E}\log f_0'(1) \;<\; 0, \tag{60}$$

$$\varphi(1) \;=\; \mathbb{E}f_0'(1) \;<\; \infty, \qquad \varphi'(1) \;=\; \mathbb{E}f_0'(1)\log f_0'(1) \;>\; 0. \tag{61}$$

Therefore, $\min_{0\leq t\leq 1} \varphi(t) < 1$ is attained at an interior point t of the interval $[0,1]$. Recall that in Lemma 11 (iii) this minimum is denoted by G. For the following facts, see for example, [Afa90] and [VT76].

Lemma 12 (random walk asymptotics) *Under the conditions* (60)–(61), *as* $n \to \infty$, *for each* $j \geq 1$, *there are positive constants* c_j, d_j, e_j *such that,*

$$\mathbb{P}(\tau_j > n) \;\sim\; c_j\, n^{-3/2}\, G^n, \tag{62}$$

$$\mathbb{P}(\tau_j = n) \;\sim\; d_j\, n^{-3/2}\, G^n, \tag{63}$$

$$\mathbb{P}(\tau_{j-1} \leq n < \tau_j) \;\sim\; e_j\, n^{-3/2}\, G^n, \tag{64}$$

and a constant K *satisfying*

$$\mathbb{P}\Big(\min_{1\leq p\leq n} S_p > -y \Big) \;\leq\; K e^{y}\, n^{-3/2}\, G^n, \qquad y > 0. \tag{65}$$

Introduce the events

$$D_j(n) := \{\tau_{j-1} \le n < \tau_j\}, \qquad j \ge 1, \quad n \ge 0, \tag{66}$$

and the indicators

$$I_j(n) := I\{D_j(n)\}, \qquad \Theta_j(n) := I\{\tau_j \le n\}. \tag{67}$$

Denote by $\mathbb{P}_{j,n}(\cdot)$ and $\mathbb{E}_{j,n}(\cdot)$, respectively, the conditional probability and conditional expectation given $D_j(n)$.

In the sequel we need the following result being a slight reformulation of Lemma 10 from [Afa90].

Lemma 13 (general error estimate) *Let $\{W_n = W_n(S_1, ..., S_n) : n \ge 1\}$ be a sequence of random variables such that for each n,*

$$W_n \le \mathrm{e}^{S_t}, \qquad 1 \le t \le n.$$

Then under the conditions (60)–(61),

$$\limsup_{J\to\infty} \limsup_{n\to\infty} Q^{-1}(n)\, \mathbb{E}\, W_n\, \Theta_J(n) = 0.$$

Now we are ready to describe the *scheme of proving Theorem* 3. For convenience, we introduce the *symbol* $h = h(n,m)$ to denote (depending on the situation) either m with $1 \le m \le n$, or $n-m$ with $0 \le m \le n$. (Later we will send $n \to \infty$ for fixed m.) In view of (29) in Lemma 10, in order to demonstrate Theorem 3 we first need to show that for each $l \ge 1$, the limit

$$\lim_{n\to\infty} \mathcal{P}\left\{ Z_{h,n} = l \;\middle|\; Z_n > 0 \right\} = \lim_{n\to\infty} Q^{-1}(n)\, \mathbb{E}\, U_n(h,l) \tag{68}$$

exists. For this purpose, we use for $J \ge 2$ the representation

$$\mathbb{E}\, U_n(h,l) = \sum_{j=1}^{J-1} \mathbb{P}(D_j(n))\, \mathbb{E}_{j,n}\, U_n(h,l) + \mathbb{E}\, U_n(h,l)\, \Theta_J(n) \tag{69}$$

(compare with (45)). First we will show that

$$\limsup_{J\to\infty} \limsup_{n\to\infty} Q^{-1}(n)\, \mathbb{E}\, U_n(h,l)\, \Theta_J(n) = 0, \tag{70}$$

which then allows to deal with $\mathbb{P}(D_j(n))\, \mathbb{E}_{j,n}\, U_n(h,l)$ for a fixed $j \ge 1$ to establish the existence of the limit (68). But in view of Lemma 12 and Lemma 11 (iii), the limit

$$\lim_{n\to\infty} Q^{-1}(n)\, \mathbb{P}(D_j(n)) \in (0,\infty) \tag{71}$$

exists. Thus it then only remains to show the existence and positivity of the limit

$$\lim_{n\to\infty} \mathbb{E}_{j,n}\, U_n(h,l). \tag{72}$$

To demonstrate this, we will proceed as follows. Recalling definitions (26) and (27), we see that

$$U_n(h,l) \;=\; \frac{(a_n + b_n - b_h)\; b_h^{l-1}}{(a_n + b_n)^{l+1}} \;\leq\; 1, \tag{73}$$

and therefore we will finish the proof if we show that the conditional distribution of the three-dimensional vector

$$\mathbf{g}(n,h) \;:=\; (a_n\,, b_h\,, b_n) \tag{74}$$

conditioned on the event $D_j(n)$, weakly converges as $n \to \infty$ to a law of a vector whose coordinates are positive a.s. However, b_n (and b_h if $h = n-m$) depend on a growing number of summands, which are difficult to handle with. To bypass this obstacle we fix a sufficiently large u and write for $n > u$,

$$b_n \;=\; b_u + a_u\, b_{u+1,n-u} + a_{n-u}\, b_{n-u+1,n} \tag{75}$$

and, if $h = n - m > u > m$,

$$b_h \;=\; b_u + a_u\, b_{u+1,n-u} + a_{n-u}\, b_{n-u+1,h}\,. \tag{76}$$

Then we show that the conditional distribution of the three-dimensional vector

$$\mathbf{g}_u(n,h) \;:=\; \begin{cases} (a_n\,, b_m\,, b_n - a_u\, b_{u+1,n-u}) & \text{if} \quad h = m, \\ (a_n\,, b_h\, - a_u\, b_{u+1,n-u}\,, b_n - a_u\, b_{u+1,n-u}) & \text{if} \quad h = n-m, \end{cases} \tag{77}$$

conditioned on the event $D_j(n)$ weakly converges as $n \to \infty$ to a law of a vector whose coordinates are positive a.s. To return to the vector $\mathbf{g}(n,h)$ we demonstrate that for sufficiently large u one can neglect the contribution given by $a_u\, b_{u+1,n-u}$ to (75) and (76) (conditioned on $D_j(n)$).

Let us proceed to fulfill this scheme. Recall the symbol $h = h(n,m)$ introduced after Lemma 13.

Lemma 14 (an error estimate) *Under the conditions* (60)–(61), *assertion* (70) *holds.*

Proof In view of (27), for $t \leq n$,

$$U_n(h,l) \;\leq\; q_n \;\leq q_t \;\leq\; \mathrm{e}^{S_t}. \tag{78}$$

To complete the proof it remains to recall Lemma 13. (Note that it is not disturbing that $U_n(h,l)$ depends on further random variables than those from the sequence $S_1\,, S_2\,, \ldots$) ■

For fixed $1 < u < n - v < n$, we introduce the $(2u + 2v + 1)$–dimensional vector

$$\mathbf{L}_{u,v}(n) := \Big(X_1, \eta_1, \dots, X_u, \eta_u, S_{n-v}, X_{n-v+1}, \eta_{n-v+1}, \dots, X_n, \eta_n\Big) \tag{79}$$

and denote by $\mathbf{L}_u$ the vector consisting of the first $2u$ coordinates of $\mathbf{L}_{u,v}(n)$. Let $\mathbf{L}^j_{u,v}(n)$ be a vector whose distribution coincides with that of $\mathbf{L}_{u,v}(n)$ *conditioned* on $D_j(n)$, and let $\widehat{\mathbf{L}}^j_{u,v}(n)$ denote a vector whose law coincides with the distribution of $\mathbf{L}_{u,v}(n)$ conditioned on $\tau_j = n$.

We use the symbol $\Longrightarrow$ for convergence in distribution.

Lemma 15 (convergence of conditioned auxiliary vectors) *Under the conditions of Theorem* 3*, for fixed* $j \geq 1$, *and* $u, v > 1$,

$$\mathbf{L}^j_{u,v}(n) \underset{n\to\infty}{\Longrightarrow} \textit{some } \mathbf{L}^j_{u,v}(\infty), \tag{80}$$

$$\widehat{\mathbf{L}}^j_{u,v}(n) \underset{n\to\infty}{\Longrightarrow} \textit{some } \widehat{\mathbf{L}}^j_{u,v}(\infty). \tag{81}$$

Proof As summarized in [Afa90], for fixed u and v, the distribution of the $(u+v+1)$–dimensional vector $(X_1, \dots, X_u, S_{n-v}, X_{n-v+1}, \dots, X_n)$, conditioned on the event $\{\tau_1 > n\}$, converges weakly as $n \to \infty$. We know that η_i depends on X_i only. With this in mind, following the same lines of arguments, one can demonstrate that the distribution of the $(2u + 2v + 1)$–dimensional vector $\mathbf{L}^1_{u,v}(n)$ also converges weakly as $n \to \infty$. This proves (80) for $j = 1$.

To verify (81) for $j = 1$, we fix a vector $\mathbf{z} := (z_1, \dots, z_{2u+2v+1})$ and write

$$\left.\begin{aligned}
&\mathbb{P}\Big\{\mathbf{L}_{u,v}(n) \leq \mathbf{z} \;\Big|\; \tau_1 = n\Big\} \\
&= \mathbb{P}\Big\{\mathbf{L}_{u,v}(n) \leq \mathbf{z} \;\Big|\; \tau_1 > n-1\Big\} \frac{\mathbb{P}(\tau_1 > n-1)}{\mathbb{P}(\tau_1 = n)} \\
&\quad - \mathbb{P}\Big\{\mathbf{L}_{u,v}(n) \leq \mathbf{z} \;\Big|\; \tau_1 > n\Big\} \frac{\mathbb{P}(\tau_1 > n)}{\mathbb{P}(\tau_1 = n)}.
\end{aligned}\right\} \tag{82}$$

Since on the event $\{\tau_1 > n-1\}$ the pair (X_n, η_n) is independent of the remaining coordinates of $\mathbf{L}_{u,v}(n)$, we have

$$\begin{aligned}
&\mathbb{P}\Big\{\mathbf{L}_{u,v}(n) \leq \mathbf{z} \;\Big|\; \tau_1 > n-1\Big\} \\
&= \mathbb{P}\Big\{\mathbf{L}_{u,v-1}(n-1) \leq \mathbf{z}^{(1)} \;\Big|\; \tau_1 > n-1\Big\} \mathbb{P}\Big(X_n \leq z_{2u+2v},\ \eta_n \leq z_{2u+2v+1}\Big)
\end{aligned}$$

where $\mathbf{z}^{(1)}$ is obtained from $\mathbf{z}$ by dropping the two last coordinates. From this identity, Lemma 12, and (80) with $j = 1$, it follows that the conditional distribution at the l.h.s. of (82) has a weak limit as $n \to \infty$, proving (81) for $j = 1$.

Now we proceed by induction on j and assume that (80) and (81) are true for some $j-1 \geq 1$. First we note that by Lemma 12, for $1 < 2T < n$ and $j \geq 2$,

$$\left.\begin{aligned} &\mathbb{P}\Big\{T < \tau_{j-1} \leq n-T;\ \tau_j > n\Big\} \;=\; \sum_{i=T+1}^{n-T} \mathbb{P}(\tau_{j-1}=i)\,\mathbb{P}(\tau_1 > n-i) \\ &\leq\; c\,G^n \sum_{i=T+1}^{n-T} (n-i)^{-3/2}\, i^{-3/2} \;\leq\; c_1\, n^{-3/2}\, G^n \sum_{i=T+1}^{\infty} i^{-3/2}. \end{aligned}\right\} \tag{83}$$

Hence, in view of (64), for large T, the event $\{T < \tau_{j-1} < n-T,\ \tau_j > n\}$ gives a negligible contribution to the probability $\mathbb{P}\{\tau_{j-1} \leq n < \tau_j\}$. This fact allows us in analyzing the asymptotic behavior of $\mathbb{P}\{\mathbf{L}_{u,v}(n) \leq \mathbf{z} \mid D_j(n)\}$ to deal only with the sum

$$\frac{1}{\mathbb{P}(D_j(n))} \sum_{i\leq T \text{ or } n-T\leq i\leq n} \mathbb{P}\Big\{\mathbf{L}_{u,v}(n) \leq \mathbf{z},\ \tau_{j-1}=i;\ \tau_j > n\Big\}. \tag{84}$$

Let us study the i–th summand. Given τ_{j-1} and $S_{\tau_{j-1}}$, the random walk $S'_r := S_{\tau_{j-1}+r} - S_{\tau_{j-1}}$, $r \geq 0$, is an independent copy of S. Therefore, if we first consider the case $n-v \leq i \leq n$, and write $i = n-k$ for k to be independent of n,

$$\left.\begin{aligned} &\mathbb{P}\Big\{\mathbf{L}_{u,v}(n) \leq \mathbf{z} \;\Big|\; \tau_{j-1}=i,\ \tau_j > n\Big\} \\ &=\; \mathbb{P}\Big\{\mathbf{L}_{u,v-(n-i)}(i) \leq \mathbf{z}^{(1)} \;\Big|\; \tau_{j-1}=i\Big\}\,\mathbb{P}\Big\{\mathbf{L}_{n-i} \leq \mathbf{z}^{(2)} \;\Big|\; \tau_1 > n-i\Big\} \\ &=\; \mathbb{P}\Big\{\mathbf{L}_{u,v-k}(n-k) \leq \mathbf{z}^{(1)} \;\Big|\; \tau_{j-1}=n-k\Big\}\,\mathbb{P}\Big\{\mathbf{L}_{k} \leq \mathbf{z}^{(2)} \;\Big|\; \tau_1 > k\Big\} \end{aligned}\right\} \tag{85}$$

where $\mathbf{z}^{(1)}$ consists of the first $2u+2(v-k)+1$ coordinates of $\mathbf{z}$, and $\mathbf{z}^{(2)}$ of the remaining $2k$ ones. By the induction hypothesis a weak limit (as $n \to \infty$) of the right most term in (85) exists.

In the case $n-T \leq i < n-v$ we again write $i = n-k$ for k fixed, and we first consider simply the vector $\mathbf{L}_{u,k}(n)$ which, as we know, has a weak limit (conditioned on the event $\{\tau_{j-1}=n-k,\ \tau_j > n\}$) as $n \to \infty$, and then return to the vector $\mathbf{L}_{u,v}(n)$. Thus, for all $i \in [n-T, n]$ a weak limit of the starting conditional probability in (85) exists.

Assume now that $1 \leq i \leq u$. Then, instead of $\mathbf{L}_{u,v}(n)$ we first consider the vector $\mathbf{L}'_{u,v}(n)$ which is obtained from $\mathbf{L}_{u,v}(n)$ by substituting $S_{n-v} - S_i$ for S_{n-v}. Clearly,

$$\left.\begin{aligned} &\mathbb{P}\Big\{\mathbf{L}'_{u,v}(n) \leq \mathbf{z} \;\Big|\; \tau_{j-1}=i,\ \tau_j > n\Big\} \\ &=\; \mathbb{P}\Big\{\mathbf{L}_{i} \leq \mathbf{z}^{(1)} \;\Big|\; \tau_{j-1}=i\Big\}\,\mathbb{P}\Big\{\mathbf{L}_{u-i,v}(n-i) \leq \mathbf{z}^{(2)} \;\Big|\; \tau_1 > n-i\Big\} \end{aligned}\right\} \tag{86}$$

where now $\mathbf{z}^{(1)}$ consists of the first $2i$ coordinates of $\mathbf{z}$, and $\mathbf{z}^{(2)}$ of the remaining ones. By the induction hypothesis, (86) has a weak limit as $n \to \infty$. From here it follows easily that (for such i) $\mathbb{P}\left(\mathbf{L}_{u,v}(n) \le \mathbf{z} \,\middle|\, \tau_{j-1} = i,\ \tau_j > n\right)$ has a weak limit as $n \to \infty$.

In order to consider the case $u < i \le T$, we first add additional coordinates $X_{u+1}, \eta_{u+1}, \ldots, X_i, \eta_i$ to $\mathbf{L}_{u,v}(n)$ and then proceed as in the previous case.

Thus we have established, the starting conditional probability in (85) has a weak limit for all i in question. From here, (84), and Lemma 12 we deduce the validity of (80).

To establish (81), one should use the same arguments replacing $\tau_j > n$ by $\tau_j = n$ (where needed). This then finishes the proof by induction. ■

Recalling definitions (74) and (77), denote by $\mathbf{g}_u^j(n,h)$ (and $\mathbf{g}^j(n,h)$) a vector whose distribution coincides with the law of $\mathbf{g}_u(n,h)$ (respectively $\mathbf{g}(n,h)$) *conditioned* on $D_j(n)$.

Lemma 16 (existence of a positive limit) *Under the conditions of Theorem* 3, *for fixed* m, j *and* $u > m$,

$$\mathbf{g}_u^j(n,h) \underset{n\to\infty}{\Longrightarrow} \textit{some } \mathbf{g}_u^j(\infty,m) \tag{87}$$

where $\mathbf{g}_u^j(\infty,m)$ *is a vector whose coordinates are positive a.s.*

(Note that the limit $\mathbf{g}_u^j(\infty,m)$ is different for the two choices of the symbol $h = h(n,m)$ introduced after Lemma 13.)

Proof The coordinates of the vector $\mathbf{g}_u(n,h)$ depend on the coordinates of $\mathbf{L}_{u,u}(n)$ in a simple way. Therefore the limit in (87) exists. The first component of $\mathbf{g}_u(n,h)$ is $a_n = e^{-S_n}$. Moreover, according to Lemma 15, $S_n = S_{n-v} + X_{n-v+1} + \cdots + X_n$ conditioned on $D_j(n)$ has a limit in law. Therefore, the first coordinate of $\mathbf{g}_u^j(\infty,m)$ is positive with probability one. The a.s.–positivity of the remaining coordinates follows easily from the preceding fact. ■

The next lemma is a crucial step in proving the existence of the limit in (72) for $j = 1$. Let $1 < u < n - v < n$.

Lemma 17 (intermediate part, case $j = 1$) *Under the conditions of Theorem* 3,

$$\limsup_{\min(u,v)\to\infty} \ \limsup_{n\to\infty} Q^{-1}(n)\, \mathbb{E}\Big\{a_u\, b_{u+1,\,n-v}\ ;\ \tau_1 > n\Big\} \;=\; 0. \tag{88}$$

Proof By definition,

$$a_u\, b_{u+1,\,n-v} \;=\; \sum_{i=u+1}^{n-v} \eta_i\, a_{i-1}\,. \tag{89}$$

We want to estimate the expectation of the i-th term restricted to $\tau_1 > n$, which implies that $\tau_1 > i-1$. We distinguish between the different values x of S_{i-1} and y of S_i. Write F for the distribution function of $X_1 = \log f_0'(1)$ (with respect to $\mathbb{P}$). Then

$$\mathbb{E}\{\eta_i\, a_{i-1}\,;\ \tau_1 > n\} \;=\; \int_0^\infty \mathbb{P}\Big\{S_{i-1} \in \mathrm{d}x\,;\ \tau_1 > i-1\Big\}\,\mathrm{e}^{-x}$$

$$\times\ \int_0^\infty \mathrm{F}(\mathrm{d}y - x)\ \mathbb{E}\Big\{\eta_i \;\Big|\; X_i = y-x\Big\}\,\mathbb{P}\Big\{\min_{i+1\le j\le n} S_j > 0 \;\Big|\; S_i = y\Big\}.$$

By (65),

$$\mathbb{P}\Big\{\min_{i+1\le j\le n} S_j > 0 \;\Big|\; S_i = y\Big\} \;=\; \mathbb{P}\Big(\min_{1\le t\le n-i} S_t > -y\Big)$$
$$\le\; c_1\,\mathrm{e}^{y}\,(n-i)^{-3/2}\,G^{\,n-i}$$

with $G<1$ from Lemma 11 (iii). Therefore,

$$\begin{aligned}\mathbb{E}\Big\{\eta_i\, a_{i-1}\,;\ \tau_1 > n\Big\} \;&\le\; c_1\,(n-i)^{-3/2}\,G^{\,n-i}\int_0^\infty \mathbb{P}\Big\{S_{i-1}\in\mathrm{d}x\,;\ \tau_1 > i-1\Big\}\\ &\qquad\times\ \int_{-x}^\infty \mathrm{F}(\mathrm{d}z)\,\mathrm{e}^{z}\ \mathbb{E}\big\{\eta_i \;\big|\; X_i = z\big\}\\ &\le\; c_1\,(n-i)^{-3/2}\,G^{\,n-i}\,\mathbb{P}\,(\tau_1 > i-1)\,\mathbb{E}\,\eta_1\mathrm{e}^{X_1} \;<\; \infty.\end{aligned}$$

Applying (62) we see that

$$\mathbb{E}\Big\{\eta_i\, a_{i-1}\,;\ \tau_1 > n\Big\} \;\le\; c_4\,(n-i)^{-3/2}\,i^{-3/2}\,G^{\,n}. \tag{90}$$

Hence, in view of Lemma 11 (iii),

$$\left.\begin{aligned}&Q^{-1}(n)\ \mathbb{E}\Big\{a_u\, b_{u+1,\,n-v}\,;\ \tau_1 > n\Big\}\\ &\le\; c_4\,n^{3/2}\sum_{u+1\le i\le n-v}(n-i)^{-3/2}\,i^{-3/2} \;\le\; c_5\sum_{i\,=\,\min(u,v)}^{\infty} i^{-3/2},\end{aligned}\right\} \tag{91}$$

that proves (88). ■

Lemma 18 (intermediate part, general j) *Under the conditions of Theorem 3, for any $\varepsilon > 0$, and $j \ge 1$,*

$$\limsup_{\min(u,v)\to\infty}\ \limsup_{n\to\infty}\mathbb{P}_{j,n}\Big(a_u\, b_{u+1,n-v} > \varepsilon\Big) \;=\; 0, \tag{92}$$

$$\limsup_{\min(u,v)\to\infty}\ \limsup_{n\to\infty}\mathbb{P}\Big\{a_u\, b_{u+1,n-v} > \varepsilon \;\Big|\; \tau_j = n\Big\} \;=\; 0. \tag{93}$$

Proof By Chebyshev's inequality,

$$\mathbb{P}_{1,n}\left(a_u\, b_{u+1,n-v} > \varepsilon\right) \le \varepsilon^{-1}\, \mathbb{E}_{1,n}\, a_u\, b_{u+1,n-v}$$

which, in view of Lemmas 17, 12, and 11 (iii), proves (92) for $j=1$. To demonstrate (93) it suffices to note that

$$\mathbb{P}\left\{a_u b_{u+1,n-v} > \varepsilon \;\middle|\; \tau_1 = n\right\} \le \frac{\mathbb{P}(\tau_1 > n-1)}{\mathbb{P}(\tau_1 = n)}\, \mathbb{P}_{1,n-1}\Big(a_u b_{u+1,n-1-(v-1)} > \varepsilon\Big).$$

Now we apply induction on j. Assume the statement is true for $j-1 \ge 1$, and consider $\mathbb{P}_{j,n}\left(a_u\, b_{u+1,n-v} > \varepsilon\right)$. By (83) it suffices to show that for a fixed T,

$$\frac{1}{\mathbb{P}\left(D_j(n)\right)} \sum_{i \le T \text{ or } n-T \le i \le n} \mathbb{P}\Big\{a_u\, b_{u+1,n-v} > \varepsilon,\ \tau_{j-1} = i;\ \tau_j > n\Big\} \tag{94}$$

vanishes as first $n \to \infty$ and then $\min(u,v) \to \infty$.

Let $n - T \le i \le n$. In this case the respective summand in (94) can be estimated from above:

$$\begin{aligned} &\le \frac{1}{\mathbb{P}\left(D_j(n)\right)}\, \mathbb{P}\Big\{a_u\, b_{u+1,n-v} > \varepsilon;\ \tau_{j-1} = i\Big\} \\ &= \frac{\mathbb{P}\left(\tau_{j-1} = i\right)}{\mathbb{P}\left(D_j(n)\right)}\, \mathbb{P}\Big\{a_u\, b_{u+1,n-v} > \varepsilon \;\Big|\; \tau_{j-1} = i\Big\} \end{aligned}$$

where the latter term tends to 0 under the required limit transition, by the induction hypothesis.

If $1 \le i \le T$ (and without loss of generality $T < u$), we write $a_u = a_i\, a_{i+1,u}$ and consider separately the events $\{S_i \le -L\}$ and $\{S_i > -L\}$, for $L > 0$. In view of $a_i = \mathrm{e}^{-S_i}$, we have

$$\begin{aligned} &\mathbb{P}\Big\{a_u\, b_{u+1,n-v} > \varepsilon,\ \tau_{j-1} = i;\ \tau_j > n\Big\} \\ &\le \mathbb{P}\Big\{S_i \le -L,\ \tau_{j-1} = i;\ \tau_j > n\Big\} \\ &\qquad + \mathbb{P}\Big\{a_{i+1,u}\, b_{u+1,n-v} > \varepsilon\, \mathrm{e}^{-L},\ \tau_{j-1} = i;\ \tau_j > n\Big\} \\ &\le \mathbb{P}\left(S_i \le -L\right)\, \mathbb{P}\left(\tau_1 > n-i\right) + \mathbb{P}\Big\{a_{i+1,u}\, b_{u+1,n-v} > \varepsilon\, \mathrm{e}^{-L};\ \tau_1 > n-i\Big\}. \end{aligned}$$

From this estimate, letting n, u and v tend to infinite in the needed order, one can show, using the induction hypothesis, that for the i in question

$$\limsup_{\min(u,v)\to\infty}\ \limsup_{n\to\infty} \frac{\mathbb{P}\Big\{a_u\, b_{u+1,n-v} > \varepsilon,\ \tau_{j-1} = i;\ \tau_j > n\Big\}}{\mathbb{P}\left(D_j(n)\right)} \le c\, \mathbb{P}(S_i \le -L)\,.$$

Of course, the latter probability vanishes as $L \to \infty$. Summarizing, the first statement of the lemma is proved.

To deduce the second statement from this fact, it remains to repeat the arguments used for $j = 1$. ■

Lemma 19 (convergence of conditioned vectors) *Under the conditions of Theorem* 3, *for fixed* m *and* j,

$$\mathbf{g}^j(n,h) \underset{n\to\infty}{\Longrightarrow} \text{ some } \mathbf{g}^j(\infty,m)$$

where $\mathbf{g}^j(\infty,m)$ *is a vector whose coordinates are positive a.s.*

Proof Combine Lemmas 18 and 16. ■

Remark 20 Using Lemmas 18 and 16 one can show also that for each $j \geq 1$ and admissible m, the vector $(a_n\,, a_n + b_n - b_h\,, b_h\,, b_n)$ conditioned on $D_j(n)$ weakly converges as $n \to \infty$ to a vector whose coordinates are positive a.s. ◇

Lemma 19 is the last preliminary result we need to prove (72), that is (68), and now the desired statement follows relatively easily:

Lemma 21 (general case) *Under the conditions of Theorem* 3, *for each fixed* $j \geq 1$, *admissible* m, *and* $l \geq 1$, *the limit in* (72) *exists and is positive.*

Proof Multiplying both sides of (69) by $Q^{-1}(n)$, applying Lemmas 14, 19, and appealing to (71), we establish the existence of the limit in (72). To prove that it is positive, we use Remark 20. This completes the proof of Lemma 21. ■

Lemma 22 (no loss of mass) *Under the conditions of Theorem* 3, *the total mass statement* (15) *holds.*

Proof First we note that $Z_{t,n} \leq Z_n$ for any $t \leq n$ (given $Z_n > 0$) and so it suffices to show that

$$\lim_{l\to\infty} \lim_{n\to\infty} \mathcal{P}\Big\{ Z_n \geq l \;\Big|\; Z_n > 0 \Big\} \;=\; 0. \tag{95}$$

To this end we recall that by Lemma 10,

$$\mathcal{P}\Big\{ Z_n \geq l \;\Big|\; Z_n > 0 \Big\} \;=\; Q^{-1}(n)\, \mathbb{E}\, V_n(n,l).$$

Now

$$\mathbb{E}\, V_n(n,l) \;=\; \sum_{j=1}^{J} \mathbb{P}\,(D_j(n))\, \mathbb{E}_{j,n} V_n(n,l) + \mathbb{E}\, V_n(n,l)\, \Theta_J(n). \tag{96}$$

Since

$$V_n(n,l) \leq (a_n + b_n)^{-1} \;\leq\; a_t^{-1} \;=\; \mathrm{e}^{S_t}$$

for $t \le n$,

$$\limsup_{J\to\infty} \limsup_{n\to\infty} Q^{-1}(n)\, \mathbb{E}\, V_n(n,l)\, \Theta_J(n) \;=\; 0 \tag{97}$$

by Lemma 13. On the other hand, by Lemma 19 for each fixed j,

$$\lim_{n\to\infty} \mathbb{E}_{j,n} V_n(n,l) \;=\; \mathbb{E}\, \frac{\lambda_j^{l-1}}{(\kappa_j+\lambda_j)^l}$$

where κ_j and λ_j are positive random variables. Therefore, by the dominated convergence theorem

$$\lim_{l\to\infty} \mathbb{E}\, \frac{\lambda_j^{l-1}}{(\kappa_j+\lambda_j)^l} \;=\; 0. \tag{98}$$

Relations (96)–(98) imply (95) and the proof of the lemma is finished. ■

Completion of proof of Theorem 3 Combine Lemmas 21 and 22. ■

5 Branchless intermediate period

In this section we will provide the *Proof of Theorem* 4. This is mainly based on ideas exploited in the previous section.

For $1 < u < n-v < n$ and $l \ge 1$, set

$$\Delta_{\mathbf{f}}(l; u, n-v) \;:=\; \mathbf{P}_{\mathbf{f}}\Big\{Z_{u,n} = l,\;\; Z_{u,n} \ne Z_{n-v,n}\,;\;\; Z_n > 0\Big\}. \tag{99}$$

Lemma 23 (sufficient condition) *Let u_n and v_n satisfy* (16). *If under the conditions of Theorem* 4,

$$\lim_{n\to\infty} Q^{-1}(n)\, \mathbb{E}\, \Delta_{\mathbf{f}}\,(l; u_n\,, n-v_n) \;=\; 0, \qquad l \ge 1, \tag{100}$$

holds, then claim (17) *is true.*

Proof By Theorem 3, the conditioned laws of Z_n given $Z_n > 0$ are relatively compact. Because of $Z_{u_n}, Z_{n-v_n} \le Z_n$ (under $Z_n > 0$), this implies the relative compactness of the conditioned laws of both $\{Z_{u_n} \mid Z_n > 0\}$ and $\{Z_{n-v_n} \mid Z_n > 0\}$. Then (100) gives

$$\lim_{n\to\infty} \mathcal{P}\Big\{Z_{u_n,n} \ne Z_{n-v_n,n} \;\Big|\; Z_n > 0\Big\} \;=\; 0,$$

that is the claim. ■

We continue with a calculation of some reduced process probabilities. For $1 < u < n-v < n$ and $l \ge 1$, set

$$\Gamma_n(u, n-v) \;:=\; \frac{a_{u+1,n-v}\, q_{u,n}}{a_{u+1,n-v} + b_{u+1,n-v}\, q_{n-v,n}} \tag{101}$$

and
$$\Phi_n(l,u) \;:=\; \frac{a_u\, b_u^{l-1}}{\left(a_u + b_u\, q_{u,n}\right)^{l+1}}\,. \tag{102}$$

Lemma 24 (some reduced process probabilities) *The following identities hold:*
$$\mathbf{P_f}\Big\{Z_{u,n} = Z_{n-v,n} = l\,;\; Z_n > 0\Big\} \;=\; \Phi_n(l,u)\Big(\Gamma_n(u,n-v)\Big)^l, \tag{103}$$
$$\mathbf{P_f}\Big\{Z_{u,n} = l\,;\; Z_n > 0\Big\} \;=\; \Phi_n(l,u)\, q_{u,n}^l\,. \tag{104}$$

Proof First we note that in view of (23)
$$\frac{\partial}{\partial s}F_{u,n-v}(s) \;=\; \frac{a_{u+1,n-v}}{\Big(a_{u+1,n-v} + b_{u+1,n-v}(1-s)\Big)^2}$$
and therefore
$$\begin{aligned}
&\mathbf{P_f}\Big\{Z_{n-v,n} = 1,\; Z_n > 0 \;\Big|\; Z_u = 1\Big\}\\
&= \sum_{j=1}^{\infty}\mathbf{P_f}\Big\{Z_{n-v} = j \;\Big|\; Z_u = 1\Big\}\; j\, F_{n-v,n}^{j-1}(0)\, q_{n-v,n}\\
&= q_{n-v,n}\frac{\partial}{\partial s}F_{u,n-v}(s)\bigg|_{s=F_{n-v,n}(0)} \;=\; \frac{a_{u+1,n-v}\, q_{n-v,n}}{\Big(a_{u+1,n-v} + b_{u+1,n-v}\, q_{n-v,n}\Big)^2}\,.
\end{aligned}$$
By (23), this chain of identities can be continued with
$$\begin{aligned}
&= \frac{a_{u+1,n-v}}{a_{u+1,n-v} + b_{u+1,n-v}\, q_{n-v,n}}\Big(1 - F_{u,n-v}(F_{n-v,n}(0))\Big)\\
&= \frac{a_{u+1,n-v}\, q_{u,n}}{a_{u+1,n-v} + b_{u+1,n-v}\, q_{n-v,n}} \;=\; \Gamma_n(u,n-v).
\end{aligned}$$
Hence,
$$\left.\begin{aligned}
&\mathbf{P_f}\Big\{Z_{u,n} = Z_{n-v,n} = l\,;\; Z_n > 0\Big\}\\
&= \sum_{j=l}^{\infty}\mathbf{P_f}\,(Z_u = j)\binom{j}{l}F_{u,n}^{j-l}(0)\,\Big(\Gamma_n(u,n-v)\Big)^l.
\end{aligned}\right\} \tag{105}$$
In view of (24),
$$\begin{aligned}
&\sum_{j=l}^{\infty}\mathbf{P_f}\,(Z_u = j)\binom{j}{l}F_{u,n}^{j-l}(0) \;=\; \frac{1}{l!}\,\frac{\partial^l}{\partial s^l}F_u(s)\bigg|_{s=F_{u,n}(0)}\\
&= \frac{1}{l!}\,\frac{l!\, a_u\, b_u^{l-1}}{\left(a_u + (1-s)b_u\right)^{l+1}}\bigg|_{s=F_{u,n}(0)} \;=\; \Phi_n(l,u),
\end{aligned}$$

that combined with (105) gives identity (103). Claim (104) is still simpler to verify. ■

Recall notations (99) and (67). For fixed $l \geq 1$ and $J \geq 2$ we write

$$\begin{aligned} & \mathbb{E}\, \Delta_{\mathbf{f}}\,(l; u_n\,, n - v_n) \\ & = \sum_{j=1}^{J} \mathbb{E}\, \Delta_{\mathbf{f}}\,(l; u_n\,, n - v_n)\, I_j(n) + \mathbb{E}\, \Delta_{\mathbf{f}}\,(l; u_n\,, n - v_n)\, \Theta_J(n). \end{aligned} \tag{106}$$

It follows from this representation and Lemma 23 above that in order to prove Theorem 4 it suffices to show that

$$\limsup_{J\to\infty}\, \limsup_{n\to\infty} Q^{-1}(n)\, \mathbb{E}\, \Delta_{\mathbf{f}}\,(l; u_n\,, n - v_n)\, \Theta_J(n) = 0, \tag{107}$$

and that for each fixed $j \geq 1$,

$$\lim_{n\to\infty} Q^{-1}(n)\, \mathbb{E}\, \Delta_{\mathbf{f}}\,(l; u_n\,, n - v_n)\, I_j(n) = 0. \tag{108}$$

To prove these two statements, we first estimate $\Delta_{\mathbf{f}}\,(l; u, n - v)$.

Lemma 25 (simplification) *For* $1 < u < n - v < n$,

$$\Delta_{\mathbf{f}}\,(l; u, n - v) \leq l\, a_u\, b_{u+1,n-v}\,, \qquad l \geq 1.$$

Proof From the definition (101) of $\Gamma_n(u, n - v)$ and Lemma 9 it follows that

$$q_{u,n} - \Gamma_n(u, n - v) = q_{u,n}\, \frac{b_{u+1,n-v}\, q_{n-v,n}}{a_{u+1,n-v} + b_{u+1,n-v}\, q_{n-v,n}} = q_{u,n}^2\, b_{u+1,n-v}\,.$$

In view of the elementary inequality

$$x^l - y^l \leq l\,(x - y)\, x^{l-1}, \qquad x > y > 0,$$

we have, by the formulas in Lemma 24,

$$\begin{aligned} & \Delta_{\mathbf{f}}\,(l; u, n - v) \\ & = \mathbf{P}_{\mathbf{f}}\Big\{Z_{u,n} = l;\ Z_n > 0\Big\} - \mathbf{P}_{\mathbf{f}}\Big\{Z_{u,n} = Z_{n-v\,,n} = l;\ Z_n > 0\Big\} \\ & \leq l\, \Phi_n(l, u)\, q_{u,n}^{l+1}\, b_{u+1,n-v}\,. \end{aligned}$$

By the definition (102) of Φ_n and identity (24), we may continue with

$$\begin{aligned} & = \frac{l\, a_u\, b_u^{l-1}\, q_{u,n}^{l+1}\, b_{u+1,n-v}}{\left(a_u + b_u\, q_{u,n}\right)^{l+1}} \leq \frac{l\, a_u\, q_{u,n}^2\, b_{u+1,n-v}}{\left(a_u + b_u\, q_{u,n}\right)^2} \\ & = l\, a_u\, q_n^2\, b_{u+1,n-v} \leq l\, a_u\, b_{u+1,n-v} \end{aligned}$$

finishing the proof. ■

Lemma 26 (estimation of the J–term) *Under the conditions of Theorem 4, statement* (107) *holds.*

Proof From (99) and (78),

$$\Delta_{\mathbf{f}}\,(l; u, n-v) \;\le\; \mathbf{P}_{\mathbf{f}}\,(Z_n > 0) \;=\; q_n \le \mathrm{e}^{S_t}, \qquad t \le n,$$

which according to Lemma 13 implies (107). ■

Lemma 27 (estimation of the j–terms) *For $j \ge 1$, and u_n and v_n satisfying* (16), *statement* (108) *holds.*

Proof From Lemma 25 and definition (99), it follows that

$$\begin{aligned}&\Delta_{\mathbf{f}}\,(l; u, n-v)\Big[I\,(a_u\, b_{u+1,n-v} \le \varepsilon) + I\,(a_u\, b_{u+1,n-v} > \varepsilon)\Big]\\ &\le\; l\,\varepsilon + I\,(a_u\, b_{u+1,n-v} > \varepsilon)\,.\end{aligned}$$

Hence,

$$\mathbb{E}\,\Delta_{\mathbf{f}}\,(l; u, n-v)\, I_j(n) \;\le\; l\,\varepsilon\, \mathbb{P}\,(D_j(n)) + \mathbb{P}\Big\{a_u\, b_{u+1,n-v} > \varepsilon;\; D_j(n)\Big\},$$

or, in view of Lemma 12,

$$Q^{-1}(n)\, \mathbb{E}\,\Delta_{\mathbf{f}}\,(l; u_n\,, n-v_n)\, I_j(n) \;\le\; c\,\varepsilon + c_1\, \mathbb{P}_{j,n}\,(a_{u_n}\, b_{u_n+1,n-v_n} > \varepsilon)\,.$$

Now, letting $n \to \infty$, and taking into account (92) and the fact that, by identity (89), $a_u\, b_{u+1,n-v}$ is monotonously non-increasing in u and v, we obtain

$$\limsup_{n\to\infty}\; Q^{-1}(n)\, \mathbb{E}\,\Delta_{\mathbf{f}}\,(l; u_n\,, n-v_n)\, I_j(n) \;\le\; c\,\varepsilon.$$

This proves (108) since $\varepsilon > 0$ is arbitrary. ■

Proof of Theorem 4 Lemmas 26–27 establish (108) and (107). This, in view of (106), implies (100), and therefore the validity of Theorem 4. ■

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