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Augmenting the grad-div stabilization for Taylor–Hood finite elements with a vorticity stabilization

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Abstract

The least squares vorticity stabilization (LSVS), proposed in [2] for the Scott–Vogelius finite element discretization of the Oseen equations, is studied as an augmentation of the popular grad-div stabilized Taylor–Hood pair of spaces. An error analysis is presented which exploits the situation that the velocity spaces of Scott–Vogelius and Taylor–Hood are identical. Convection-robust error bounds are derived under the assumption that the Scott–Vogelius discretization is well posed on the considered grid. Numerical studies support the analytic results and they show that the LSVS-grad-div method might lead to notable error reductions compared with the standard grad-div method.

1 Introduction

The Navier–Stokes equations are the fundamental equations of fluid dynamics. For incompressible fluids in a domain $\Omega \subset \mathbb{R}^d$, $d = \{2, 3\}$, the steady-state Navier–Stokes equations are given by

$$-\nu \nabla \cdot (\mathbb{D}\boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} + \nabla p = \boldsymbol{f} \ \text{ in } \ \Omega, \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0 \ \text{ in } \ \Omega, \tag{1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \ \text{ on } \ \partial\Omega. \tag{1c}$$

Here $\boldsymbol{u} := (\boldsymbol{u}_j)_{j=1,\ldots,d}$ is the (vector-valued) velocity field, p is the (scalar-valued) pressure field and ν is the viscosity coefficient. The differential operators involve the symmetric part of the stress tensor $\mathbb{D}(\boldsymbol{u}) := \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right)/2$, the nonlinear convection term $(\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} = (\nabla \boldsymbol{u})\,\boldsymbol{u}$, where $\nabla \boldsymbol{u}$ is the Jacobian of $\boldsymbol{u}$, and the divergence operator $\nabla \cdot \boldsymbol{u}$.

Equation (1a) models the conservation of linear momentum under external forces $\boldsymbol{f}$, equation (1b) models the conservation of mass and equation (1c) poses a (for simplicity homogeneous) Dirichlet boundary condition along the boundary of the domain. The gradient of the pressure ∇p can be understood, from the mathematical point of view, as a Lagrange multiplier for the divergence constraint $\nabla \cdot \boldsymbol{u} = 0$ and, from the physical point of view, as a counter force that balances conservative forces (= gradient fields) in the momentum balance to avoid compression of the fluid.

There are two main difficulties in discretizing the Navier–Stokes equations for small viscosity coefficients. First, the nonlinear convection term dominates the viscous term, a phenomenon called dominant convection. That leads to large errors if (1a) – (1c) is discretized using a standard Galerkin method. Second, classical finite element discretizations, like the popular Taylor–Hood method [18], often relax the divergence constraint $\nabla \cdot \boldsymbol{u} = 0$ and only compute discretely divergence-free discrete velocity solutions $\boldsymbol{u}_h$, which in general do not satisfy $\nabla \cdot \boldsymbol{u}_h = 0$ weakly, i.e., in the sense of $L^2(\Omega)$. On the one hand, this affects the mass conservation and on the other hand, this also perturbs the balancing of gradient forces in the momentum equation which can lead to additional discretization errors due to a coupling of discrete velocity and the (unknown) exact pressure [13, 14, 16]. If a method admits velocity estimates with bounds that do not depend on the pressure, it is called pressure-robust.

Historically, convection stabilization ideas from scalar convection-diffusion equations were also applied to the vector-valued momentum equation with some success, e.g., see the review article [9] and references therein. To tackle errors from a relaxed divergence constraint, the so-called grad-div stabilization is very popular, which adds a zero term, i.e., $-\gamma \nabla (\nabla \cdot \boldsymbol{u})$, $\gamma > 0$, to the momentum equation. In this way, a penalization of the divergence of the discrete velocity is introduced in the finite element equation, [7, 15]. The grad-div stabilization is studied in [11] for the Stokes equations and in [1] for the Oseen equations with investigations how to choose the stabilization parameter optimally. Finite element error analysis shows that grad-div stabilization does not remove the presence of the pressure in velocity error estimates, however it reduces the impact for small viscosity coefficients. In addition, it mitigates dominating convection. To be precise, for steady-state problems, the impact of small viscosity is reduced, compared with the Galerkin discretization, e.g., see [11]. For time-dependent Navier–Stokes problems, it is even possible to derive error estimates (in appropriate norms) with constants that do not depend on the viscosity, e.g., see [6, 9]. This property is certainly another reason for the popularity of grad-div stabilization.

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In [2], the so-called least squares vorticity stabilization (LSVS) convection stabilization was proposed that is motivated by the underlying vorticity equation (obtained by applying the curl operator $\nabla\times$ to the momentum equation), which has the advantage that it adds no (additional) coupling between discrete velocity and exact pressure. This is opposite to the classical SUPG convection stabilization. Applied to a pressure-robust finite element method, optimal convergence results are obtained in [2] for the Oseen problem and the exactly divergence-free Scott–Vogelius finite element method proposed in [17], on appropriate meshes. The approach was extended to divergence-free virtual element methods [3] and time-dependent Navier–Stokes equations in [4], but still with the restriction to divergence-free finite element methods.

In this paper we explore, for the steady-state Oseen equations, the LSVS convection stabilization in combination with the grad-div stabilization for Taylor–Hood pairs of spaces, called the LSVS-grad-div stabilized method. For the grad-div stabilized method, [5] shows that for larger grad-div parameter γ the Taylor–Hood solutions converge to the solution of the Scott–Vogelius finite element method, at least on barycentric-refined meshes. It will be shown that the convergence property for $\gamma \to \infty$ holds also for the LSVS-grad-div stabilized Taylor–Hood method. In addition, an error analysis of this method is presented, which is based on the observation that it is possible to introduce the Scott–Vogelius velocity solution in the error equation, instead as some interpolant or projection as it is usually done. Numerical studies support the convergence result for $\gamma \to \infty$ and the proved orders of convergence. Likewise important, it will turn out that the results obtained with the LSVS-grad-div stabilized method are, for appropriately chosen stabilization parameters, notably more accurate than those computed with the traditional grad-div stabilized method.

The paper is organized as follows. Section 2 presents the steady-state Oseen equations and inf-sup stable pairs of finite element spaces. Some results from the literature are recalled. The LSVS discretization for weakly divergence-free pairs of spaces and the LSVS-grad-div stabilization for Taylor–Hood pairs are introduced in Section 3 and their connection is explained in Section 4. Section 5 presents the error analysis and the numerical studies are provided in Section 6. The paper concludes with a summary and outlook.

2 Preliminaries

This section introduces the model Oseen problem and its standard Galerkin finite element discretization, as well as necessary notation.

2.1 The Oseen equations

Let $\Omega \subset \mathbb{R}^d$, $d = \{2,3\}$, be a bounded and connected domain with Lipschitz boundary $\partial\Omega$. The Oseen equations are linear and stationary equations that possess a convective term and, in their more general form, also a reactive term in the momentum equation. They are given by

$$\mathcal{L}\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{2a}$$
$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2b}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{2c}$$

where

$$\mathcal{L}\boldsymbol{u} := -\nu\Delta\boldsymbol{u} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + \sigma\boldsymbol{u}.$$

Here, $\nu > 0$ denotes the viscosity coefficient, $\sigma > 0$ is assumed to be a constant and the convection field $\boldsymbol{b}$ is assumed to belong to $\left(W^{1,\infty}(\Omega)\right)^d$ and to satisfy $\nabla\cdot\boldsymbol{b} = 0$. Let $\boldsymbol{V} := \left(H_0^1(\Omega)\right)^d$ and $Q := L_0^2(\Omega)$ be equipped with the norms $\|\boldsymbol{v}\|_{\boldsymbol{V}} := \|\nabla\boldsymbol{v}\|_{L^2(\Omega)}$ and $\|q\|_Q := \|q\|_{L^2(\Omega)}$. The subscripts mean that the functions from $\boldsymbol{V}$ possess a vanishing trace on $\partial\Omega$ and the functions from Q have integral mean value zero. The weak form of (2) reads as follows: Given $\boldsymbol{f} \in \left(H^{-1}(\Omega)\right)^d := \boldsymbol{V}^*$, find $(\boldsymbol{u}, p) \in \boldsymbol{V}\times Q$ such that

$$\begin{array}{rcll} \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + \sigma\boldsymbol{u},\boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, p) & = & \langle \boldsymbol{f},\boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} & \forall\, \boldsymbol{v}\in\boldsymbol{V}, \\ (\nabla\cdot\boldsymbol{u}, q) & = & 0 & \forall\, q\in Q. \end{array} \tag{3}$$

Here, $(\cdot,\cdot)_{L^2(D)}$ denotes the L^2-inner product on a domain D, where D is omitted for $D = \Omega$. This is a linear saddle point problem with the bilinear forms

$$a(\boldsymbol{u},\boldsymbol{v}) := \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + \sigma\boldsymbol{u},\boldsymbol{v}), \quad b(\boldsymbol{v}, q) := -\left(\nabla\cdot\boldsymbol{v}, q\right). \tag{4}$$

A key property of the convection term is the skew-symmetry, that is

$$((\boldsymbol{b}\cdot\nabla)\boldsymbol{v},\boldsymbol{v}) = 0 \;\; \forall\, \boldsymbol{v}\in\boldsymbol{V}, \tag{5}$$

which follows from the assumption that $\boldsymbol{b}$ is divergence-free. Let

$$\boldsymbol{V}_{\text{div}} := \Big\{\boldsymbol{v}\in\boldsymbol{V} : (\nabla\cdot\boldsymbol{v},q)=0,\;\; \forall\, q\in Q\Big\}$$

be the space of weakly divergence-free functions. The inf-sup stability of the spaces $\boldsymbol{V}$ and Q plus the $\boldsymbol{V}$-coercivity of the bilinear form $a(\cdot,\cdot)$ guarantee existence and uniqueness of the solution to the weak Oseen problem (3). The following stability estimate can also be proved, e.g., see [12, Lemma 5.8],

$$\frac{\nu}{2}\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\sigma^{\frac{1}{2}}\boldsymbol{u}\|^2_{L^2(\Omega)} \le \frac{1}{2\nu}\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)}. \tag{6}$$

The analysis will use Poincaré's inequality: there is a constant $C>0$, depending only on the diameter of Ω, so that

$$\|\boldsymbol{v}\|_{L^2(\Omega)} \le C\|\nabla\boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v}\in\boldsymbol{V}. \tag{7}$$

2.2 Inf-sup stable mixed finite elements

Let $\mathcal{T}_h$ be a triangulation of Ω into simplices that is admissible in the usual sense. For every mesh cell $K\in\mathcal{T}_h$ its diameter is denoted by h_K and we define $h:=\max\{h_K :\; K\in\mathcal{T}_h\}$. The set of all facets of $\mathcal{T}_h$ is denoted by $\mathcal{F}$, where $\mathcal{F}^i\subset\mathcal{F}$ is the subset of all interior facets. In addition, let $\mathcal{F}_K\subset\mathcal{F}$ be the set of all facets of $K\in\mathcal{T}_h$ and let h_F be the diameter of $F\in\mathcal{F}$.

For a vector-valued function $\boldsymbol{v}$ we define the tangential jumps across $F=K_1\cap K_2$ with $K_1,K_2\in\mathcal{T}_h$ as

$$[\![\boldsymbol{v}\times\boldsymbol{n}]\!] := \boldsymbol{v}_1\times\boldsymbol{n}_1 + \boldsymbol{v}_2\times\boldsymbol{n}_2,$$

where $\boldsymbol{v}_i = \boldsymbol{v}|_{K_i}$ and $\boldsymbol{n}_i$ is the unit normal pointing out of K_i. If $F\in\partial\Omega$, then we define

$$[\![\boldsymbol{v}\times\boldsymbol{n}]\!] := \boldsymbol{v}\times\boldsymbol{n}.$$

With respect to $\mathcal{T}_h$, $\mathcal{F}$ and $\mathcal{F}^i$, piecewise L^2 products are defined by

$$\begin{aligned}
(\boldsymbol{v},\boldsymbol{w})_h &:= \sum_{K\in\mathcal{T}_h}(\boldsymbol{v},\boldsymbol{w})_{L^2(K)}, \quad (\boldsymbol{v},\boldsymbol{w})_{\mathcal{F}} := \sum_{F\in\mathcal{F}}(\boldsymbol{v},\boldsymbol{w})_{L^2(F)},\\
(\boldsymbol{v},\boldsymbol{w})_{\mathcal{F}^i} &:= \sum_{F\in\mathcal{F}^i}(\boldsymbol{v},\boldsymbol{w})_{L^2(F)},
\end{aligned}$$

with associated norms $\|\cdot\|_h$, $\|\cdot\|_{\mathcal{F}}$ and $\|\cdot\|_{\mathcal{F}^i}$, respectively.

Let $\boldsymbol{V}_h/Q_h$ be a pair of conforming finite elements spaces on $\mathcal{T}_h$ and assume that they satisfy the discrete inf-sup compatibility condition

$$\inf_{q_h\in Q_h\setminus\{0\}}\;\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h\setminus\{\boldsymbol{0}\}}\frac{b(\boldsymbol{v}_h,q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}\;\|q_h\|_Q} \ge \beta_h > 0. \tag{8}$$

Then, the associated Galerkin finite element method for the Oseen equations admits a unique solution $(\boldsymbol{u}_h,p_h)\in\boldsymbol{V}_h\times Q_h$ such that, for all $\boldsymbol{v}_h\in\boldsymbol{V}_h$ and $q_h\in Q_h$,

$$\begin{aligned}
\nu(\nabla\boldsymbol{u}_h,\nabla\boldsymbol{v}_h) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{u}_h + \sigma\boldsymbol{u}_h,\boldsymbol{v}_h) - (\nabla\cdot\boldsymbol{v}_h,p_h) &= \langle\boldsymbol{f},\boldsymbol{v}_h\rangle_{\boldsymbol{V}^\star,\boldsymbol{V}},\\
(\nabla\cdot\boldsymbol{u}_h,q_h) &= 0.
\end{aligned}$$

In this paper we study the Taylor–Hood (TH) finite element family with vector-valued globally continuous piecewise polynomials of up to order $k\ge 2$ for the velocity, denoted by $\boldsymbol{V}_h := \boldsymbol{P}_k$, and continuous piecewise polynomials of up to order $k-1$ for the pressures, denoted by $Q_h := P_{k-1}$. For our studies, we need to consider also Scott–Vogelius (SV) finite element pairs, which use the same velocity space but discontinuous pressure functions of the same polynomial degree as the Taylor–Hood pair, denoted by $Q_h = P_{k-1}^{\text{disc}}$.

The following velocity error estimate in the norm of $\boldsymbol{V}$ holds, e.g., see [12, Theorem 5.14].

Theorem 2.1. *Let $\boldsymbol{u} \in (H^{k+1}(\Omega))^d$ and $p \in H^k(\Omega)$. For $\boldsymbol{V}_h = \boldsymbol{P}_k$ and $Q_h = P_{k-1}$, there holds*

$$\begin{aligned} &\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\Omega)} \\ &\qquad \le Ch^k\left(1+\frac{\sigma^{\frac12}}{\nu^{\frac12}}+\frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{\nu^{\frac12}}\min\left\{\frac{1}{\nu^{\frac12}},\frac{1}{\sigma^{\frac12}}\right\}\right)\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}+\frac{C}{\nu}h^k\|p\|_{H^k(\Omega)}. \end{aligned} \tag{9}$$

In the case of dominating convection, i.e., when $\nu \ll \|\boldsymbol{b}\|_{L^\infty(\Omega)}h$, the estimate (9) indicates that large velocity errors are to be expected. Therefore, convection-stabilization methods are required. Additionally, the error estimate shows a lack of pressure-robustness: for small ν and complicated pressure p, the error induced by the pressure term can be very large. For divergence-free methods like Scott–Vogelius there is no lack of pressure-robustness and an estimate of type (9) can be shown without the pressure term. The next section introduces stabilizations that mitigate both problems.

This section closes with recalling a couple of tools that are used in the finite element analysis. The first one is a local trace inequality: there exists a constant $C > 0$ such that for every $K \in \mathcal{T}_h$, $F \in \mathcal{F}_K$ and all $\boldsymbol{v} \in H^1(K)$

$$\|\boldsymbol{v}\|_{L^2(F)} \le C\left(h_K^{-\frac12}\|\boldsymbol{v}\|_{L^2(K)}+h_K^{\frac12}\|\nabla\boldsymbol{v}\|_{L^2(K)}\right). \tag{10}$$

Next, an inverse estimate is needed, which states that for all $K \in \mathcal{T}_h$, all $0 \le \ell \le s \le m$ and all $q \in P_m(K)$, there is a constant $C > 0$ with

$$\|D^s q\|_{L^2(K)} \le Ch^{-(s-\ell)}\|D^\ell q\|_{L^2(K)}. \tag{11}$$

Finally, let $I_K : C^s(K) \to P(K)$ be the Lagrangian interpolation operator, where $P(K)$ is a polynomial space defined on K and let $2(m+1-s) > 1$. Then there is a constant C, which is independent of $\boldsymbol{v} \in H^{m+1}(K)$, such that for all $\boldsymbol{v} \in H^{m+1}(K)$ the following local interpolation estimate is satisfied:

$$\|D^k(\boldsymbol{v}-I_k\boldsymbol{v})\|_{L^2(K)} \le Ch^{m+1-k}\|D^{m+1}\boldsymbol{v}\|_{L^2(K)},\ 0 \le k \le m+1.$$

3 Stabilized discretizations

This section introduces two stabilizations, namely the popular grad-div stabilization and the rather novel LSVS convection stabilization.

3.1 Grad-div stabilization

The grad-div stabilization introduces a penalty term in the Galerkin finite element formulation which penalizes the violation of mass conservation. So we add $\mathbf{0} = -\gamma\nabla(\nabla\cdot\boldsymbol{u})$ to the continuous momentum equation (2a), apply integration by parts to derive the weak formulation, and then replace the infinite-dimensional spaces with finite element spaces. Assuming from now on for simplicity that $\boldsymbol{f} \in (L^2(\Omega))^d$, then the grad-div stabilization scheme for the Oseen equations seeks $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that, for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $q_h \in Q_h$,

$$\begin{aligned} a(\boldsymbol{u}_h,\boldsymbol{v}_h)+\gamma\left(\nabla\cdot\boldsymbol{u}_h,\nabla\cdot\boldsymbol{v}_h\right)-\left(\nabla\cdot\boldsymbol{v}_h,p_h\right) &= \left(\boldsymbol{f},\boldsymbol{v}_h\right), \\ \left(\nabla\cdot\boldsymbol{u}_h,q_h\right) &= 0, \end{aligned}$$

where $\gamma \ge 0$ is the grad-div stabilization parameter and $a(\cdot,\cdot)$ is the standard bilinear form from (4) associated with the Oseen equations. Since generally $\boldsymbol{V}_{h,\text{div}} \not\subset \boldsymbol{V}_{\text{div}}$, which means that $\nabla\cdot\boldsymbol{u}_h \ne 0$, in most classical finite element spaces, such as the Taylor–Hood pair, the grad-div term is non-zero and has an effect on the discrete solution. For finite element pairs that give a weakly divergence-free numerical solution, like the Scott–Vogelius pair, the stabilization has no effect.

3.2 The LSVS stabilization method for divergence-free methods

The LSVS stabilization method introduced in [2] adds a stabilization term to the Oseen equations in the form of a residual-based stabilization of the vorticity equation $\operatorname{curl}\mathcal{L}\boldsymbol{u} = \operatorname{curl}\boldsymbol{f}$, supplemented by a penalty term on the jump of the convection term over the facets of the mesh cells. Note that the vorticity equation, which is obtained from the continuous momentum balance (2a) by applying the curl operator, is independent of the pressure, in contrast to the SUPG stabilization for the full residual equation. Originally the LSVS stabilization was designed for divergence-free pairs, like the Scott–Vogelius finite element pair, to preserve the pressure-robustness of the stabilized method.

The LSVS method reads as follows: find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that, for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $q_h \in Q_h$,

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + S(\boldsymbol{u}_h, \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{v}_h, p_h) &= L(\boldsymbol{v}_h), \\ (\nabla \cdot \boldsymbol{u}_h, q_h) &= 0. \end{aligned} \tag{12}$$

Here, $a(\cdot, \cdot)$ is the standard bilinear form from (4) associated with the Oseen equations. The stabilizing bilinear form and the right-hand side are given by:

$$\begin{aligned} S(\boldsymbol{u}_h, \boldsymbol{v}_h) &= \delta_0 \Big[(\tau \operatorname{curl} \mathcal{L}\boldsymbol{u}_h, \operatorname{curl} \mathcal{L}\boldsymbol{v}_h)_h + \big(h^2 [\![(\boldsymbol{b} \cdot \nabla)\boldsymbol{u}_h \times \boldsymbol{n}]\!], [\![(\boldsymbol{b} \cdot \nabla)\boldsymbol{v}_h \times \boldsymbol{n}]\!] \big)_{\mathcal{F}^i} \Big], \\ L(\boldsymbol{v}_h) &= (\boldsymbol{f}, \boldsymbol{v}_h) + \delta_0 (\tau \operatorname{curl} \boldsymbol{f}, \operatorname{curl} \mathcal{L}\boldsymbol{v}_h)_h. \end{aligned}$$

Following [2], the stabilization parameter τ is defined as

$$\tau|_K = \tau_K := \min \left\{ \frac{h_K^3}{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}, \frac{h_K^4}{\nu} \right\} \tag{13}$$

and $\delta_0 \geq 0$ is another scaling parameter.

We introduce the mesh-dependent norm

$$\|\boldsymbol{v}\|_{\mathrm{LSVS}}^2 := \|\sqrt{\sigma}\boldsymbol{v}\|_{L^2(\Omega)}^2 + \|\sqrt{\nu}\nabla\boldsymbol{v}\|_{L^2(\Omega)}^2 + S(\boldsymbol{v}, \boldsymbol{v}). \tag{14}$$

From the skew-symmetry property of the convection term (5), one obtains that

$$\begin{aligned} \|\boldsymbol{v}_h\|_{\mathrm{LSVS}}^2 &= \|\sqrt{\sigma}\boldsymbol{v}_h\|_{L^2(\Omega)}^2 + \|\sqrt{\nu}\nabla\boldsymbol{v}_h\|_{L^2(\Omega)}^2 + ((\boldsymbol{b} \cdot \nabla)\boldsymbol{v}_h, \boldsymbol{v}_h) + S(\boldsymbol{v}_h, \boldsymbol{v}_h) \\ &= (a + S)(\boldsymbol{v}_h, \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{aligned} \tag{15}$$

Next the LSVS method velocity error estimate from [2] is recalled. There are a number of technical assumptions stated in [2], most prominently that the reaction coefficient σ is positive and that $\boldsymbol{V}_{h,\mathrm{div}}$ is part of an exact sequence with $\boldsymbol{V}_{h,\mathrm{div}} = \operatorname{curl} Z_h$ for a suitable space Z_h with optimal approximation properties of $\operatorname{curl} \boldsymbol{u}$. Note that the Scott–Vogelius finite element pair satisfies the assumptions from [2]. Then, the following optimal error estimate can be proved [2, Corollary 7].

Theorem 3.1 (LSVS velocity error estimate for weakly divergence-free methods)**.** *In addition to the hypotheses of [2, Theorem 6] assume that $\boldsymbol{u} \in (H_0^1(\Omega))^d \cap (H^{k+1}(\Omega))^d$. Then there exists a constant $C > 0$ independent of h and ν such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{LSVS}} \leq C h^k \left(h^{\frac{1}{2}} + \nu^{\frac{1}{2}} \right) \|\boldsymbol{u}\|_{H^{k+1}(\Omega)}. \tag{16}$$

Denote by $\pi_h : L^2(\Omega) \to Q_h$ the $L^2(\Omega)$ orthogonal projection onto Q_h. The following result from [2] presents an error estimate for the distance of the finite element pressure to the pressure projection, which depends only on the velocity error.

Theorem 3.2 (LSVS pressure error estimate for weakly divergence-free methods)**.** *Assume that the hypothesis of [2, Theorem 6] are satisfied. Then there exists a constant $C > 0$, which is independent of h and ν, such that*

$$\|\pi_h p - p_h\|_{L^2(\Omega)} \leq C \left(1 + \nu^{\frac{1}{2}} + h^{\frac{1}{2}} \right) \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{LSVS}}.$$

Remark 3.3. *Theorem 3.1 and Theorem 3.2 state that the difference $\pi_h p - p_h$ satisfies basically the same error estimate as the velocity, independently of the value of ν, since*

$$\|\pi_h p - p_h\|_{L^2(\Omega)} \leq C h^k \left(1 + \nu^{\frac{1}{2}} + h^{\frac{1}{2}} \right) \left(h^{\frac{1}{2}} + \nu^{\frac{1}{2}} \right) \|\boldsymbol{u}\|_{H^{k+1}(\Omega)}.$$

Denote by $\boldsymbol{V}_h^{\mathrm{SV}} / Q_h^{\mathrm{SV}} := \boldsymbol{P}_k / P_{k-1}^{disc}$ the Scott–Vogelius space and let p_h^{SV} be the Scott–Vogelius pressure solution to the LSVS stabilization scheme (12)*. Utilizing that Q_h^{SV} contains piecewise polynomials of order $k-1$, using the triangle inequality and applying the approximation properties of π_h, see [10], we get the following error estimate for the Scott–Vogelius LSVS pressure:*

$$\begin{aligned} \|p - p_h^{\mathrm{SV}}\|_{L^2(\Omega)} &\leq \|p - \pi_{Q_h^{\mathrm{SV}}}(p)\|_{L^2(\Omega)} + \|p_h^{\mathrm{SV}} - \pi_{Q_h^{\mathrm{SV}}}(p)\|_{L^2(\Omega)} \\ &\leq C h^k \|p\|_{H^k(\Omega)} + C h^k \left(1 + \nu^{\frac{1}{2}} + h^{\frac{1}{2}} \right) \left(\nu^{\frac{1}{2}} + h^{\frac{1}{2}} \right) \|\boldsymbol{u}\|_{H^{k+1}(\Omega)}. \end{aligned} \tag{17}$$

3.3 The LSVS-grad-div stabilized method

In this section we apply the LSVS method to (non weakly divergence-free) Taylor–Hood pairs of spaces and also add the grad-div term (12). This is the method that is proposed and analyzed in this paper.

Let $\boldsymbol{V}_h/Q_h$ be either a Taylor–Hood or Scott–Vogelius pair of spaces. The LSVS-grad-div stabilized method seeks $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + S(\boldsymbol{u}_h, \boldsymbol{v}_h) + \gamma\,(\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v}_h) - (\nabla\cdot\boldsymbol{v}_h, p_h) &= L(\boldsymbol{v}_h), \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ (\nabla\cdot\boldsymbol{u}_h, q_h) &= 0,\ \ \forall\, q_h \in Q_h. \end{aligned} \tag{18}$$

The existence and uniqueness of a solution for Taylor–Hood pairs follows from the satisfaction of the discrete inf-sup condition (8) and the $\boldsymbol{V}_h$-ellipticity of the bilinear form whose arguments are both from $\boldsymbol{V}_h$. For Scott–Vogelius pairs, (18) is equivalent to (12). From the definition of the LSVS-grad-div method (18), it is natural to consider the following norm including the divergence term:

$$\|\boldsymbol{v}\|^2_{\mathrm{LSVS},\gamma} := \|\boldsymbol{v}\|^2_{\mathrm{LSVS}} + \gamma\|\nabla\cdot\boldsymbol{v}\|^2_{L^2(\Omega)}. \tag{19}$$

Denote by Q_h^{TH} and Q_h^{SV} the pressure spaces of Taylor–Hood and Scott–Vogelius pairs, respectively. Recall that both Taylor–Hood and Scott–Vogelius finite element spaces approximate the velocity by polynomials of degree k, so $\boldsymbol{V}_h^{\mathrm{TH}} = \boldsymbol{V}_h^{\mathrm{SV}} = \boldsymbol{P}_k$, while the pressure space P_{k-1} of Taylor–Hood is just a subspace of the Scott–Vogelius pressure space P_{k-1}^{disc}. We denote the discretely divergence-free subspaces for Taylor–Hood and Scott–Vogelius spaces by

$$\begin{aligned} \boldsymbol{V}_{h,\mathsf{div}}^{\mathrm{TH}} &:= \left\{\boldsymbol{v}_h \in \boldsymbol{P}_k : (\nabla\cdot\boldsymbol{v}_h, q_h) = 0\ \forall\, q_h \in Q_h^{\mathrm{TH}}\right\}, \\ \boldsymbol{V}_{h,\mathsf{div}}^{\mathrm{SV}} &:= \left\{\boldsymbol{v}_h \in \boldsymbol{P}_k : (\nabla\cdot\boldsymbol{v}_h, q_h) = 0\ \forall\, q_h \in Q_h^{\mathrm{SV}}\right\}. \end{aligned}$$

Note that the elements of $\boldsymbol{V}_{h,\mathsf{div}}^{\mathrm{SV}}$ satisfy more conditions since $Q_h^{\mathrm{TH}} \subset Q_h^{\mathrm{SV}}$, and therefore $\boldsymbol{V}_{h,\mathsf{div}}^{\mathrm{SV}} \subset \boldsymbol{V}_{h,\mathsf{div}}^{\mathrm{TH}}$. In addition, it is $\boldsymbol{V}_{h,\mathsf{div}}^{\mathrm{SV}} \subset \boldsymbol{V}_{\mathsf{div}}$ but $\boldsymbol{V}_{h,\mathsf{div}}^{\mathrm{TH}} \not\subset \boldsymbol{V}_{\mathsf{div}}$.

The Galerkin orthogonality in case of Taylor–Hood spaces reads: for arbitrary $q_h \in Q_h$ and $\boldsymbol{v}_h \in \boldsymbol{V}_{h,\mathsf{div}}^{\mathrm{TH}}$, it holds

$$a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) + S(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) + \gamma\,(\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h), \nabla\cdot\boldsymbol{v}_h) = (\nabla\cdot\boldsymbol{v}_h, p - q_h)\,, \tag{20}$$

where the pressure term on the right-hand side does not vanish. So we expect the appearance of some norm of the pressure in the velocity error bound if Taylor–Hood spaces are used in method (18). The grad-div stabilization is added to reduce the impact of the pressure.

For the numerical analysis, a stability estimate for the Taylor–Hood velocity solution $\boldsymbol{u}_h^{\mathrm{TH}}$ is needed. This estimate is derived in the usual way by using $\boldsymbol{u}_h^{\mathrm{TH}}$ as test function in (18). Assuming that $\operatorname{curl} \boldsymbol{f} \in L^2(\Omega)$, neglecting some non-negative terms on the left-hand side, applying the Cauchy–Schwarz inequality and Young's inequality yields with a straightforward calculation

$$\nu\|\nabla\boldsymbol{u}_h^{\mathrm{TH}}\|^2_{L^2(\Omega)} + \delta_0\tau\|\operatorname{curl}\mathcal{L}\boldsymbol{u}_h^{\mathrm{TH}}\|^2_h \le \frac{\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)}}{\nu} + \delta_0\tau\|\operatorname{curl}\boldsymbol{f}\|^2_{L^2(\Omega)}. \tag{21}$$

4 A connection between LSVS Scott–Vogelius and LSVS-grad-div stabilized Taylor–Hood solutions

A main motivation of applying the LSVS method in combination with Taylor–Hood spaces, in addition to the grad-div stabilization, is a theoretical result from [5]. It was proved that the limit of the grad-div stabilized Taylor–Hood solutions to the Navier–Stokes problem converges to the Galerkin solution computed with the corresponding Scott–Vogelius pair as the stabilization parameter γ tends to infinity, if the Scott–Vogelius pair is inf-sup stable on the considered grid. This result, plus the fact that the LSVS method is pressure-robust for Scott–Vogelius pairs, initiated the hope of improving the results obtained with the grad-div stabilized method for Taylor–Hood pairs by additionally applying the LSVS stabilization.

This section shows that a convergence result as presented in [5, Theorem 3.1] is also valid if both discrete problems, for the Taylor–Hood and the corresponding Scott–Vogelius pair, contain additionally the LSVS stabilization term.

Theorem 4.1 (Convergence of LSVS-grad-div stabilized Taylor–Hood solutions to the LSVS stabilized Scott–Vogelius solution)**.** *Let $\boldsymbol{b} \in W^{1,\infty}(\Omega)$ and $\operatorname{curl} \boldsymbol{f} \in L^2(\Omega)$. Under conditions where the Scott–Vogelius pair $\boldsymbol{P}_k/P_{k-1}^{disc}$ is inf-sup stable it holds for the corresponding Taylor–Hood pair $\boldsymbol{P}_k/P_{k-1}$: any sequence $\{\boldsymbol{u}_h^{\mathrm{TH}}\}_{\gamma_i}$ of Taylor–Hood velocity* *solutions of the LSVS-grad-div scheme* (18) *with* $\gamma_i \to \infty$ *has a subsequence that converges to a Scott–Vogelius velocity solution in the norm* (14)*. The corresponding sequence of Taylor–Hood 'modified pressure' solutions* $\{p_h^{\mathrm{TH}} - \gamma_i \nabla \cdot \boldsymbol{u}_h^{\mathrm{TH}}\}_{\gamma_i}$ *converges to the corresponding Scott–Vogelius pressure.*

Proof. The proof uses similar techniques as those from the proof of [5, Theorem 3.1]. However, estimating the LSVS term requires a number of additional technical steps.

i). We begin with an a priori estimate for the Taylor–Hood velocity solutions, which follows by choosing $\boldsymbol{v}_h = \boldsymbol{u}_h^{\mathrm{TH}}$ and $q_h = p_h^{\mathrm{TH}}$ in (18) and adding both equations. Using the skew-symmetry, the inverse estimate (11), the value of τ in (13), the stability estimate (21) and $h \leq \mathrm{diam}(\Omega)$ yields

$$\begin{aligned}
&\nu\|\nabla \boldsymbol{u}_h^{\mathrm{TH}}\|_{L^2(\Omega)}^2 + \sigma\|\boldsymbol{u}_h^{\mathrm{TH}}\|_{L^2(\Omega)}^2 + \big((\boldsymbol{b}\cdot\nabla)\boldsymbol{u}_h^{\mathrm{TH}}, \boldsymbol{u}_h^{\mathrm{TH}}\big) + S(\boldsymbol{u}_h^{\mathrm{TH}}, \boldsymbol{u}_h^{\mathrm{TH}}) + \gamma\|\nabla\cdot\boldsymbol{u}_h^{\mathrm{TH}}\|_{L^2(\Omega)}^2 \\
&\quad= \big(\boldsymbol{f}, \boldsymbol{u}_h^{\mathrm{TH}}\big) + \delta_0\tau\left(\mathrm{curl}\,\boldsymbol{f}, \mathrm{curl}\left(-\nu\Delta\boldsymbol{u}_h^{\mathrm{TH}} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}_h^{\mathrm{TH}} + \sigma\boldsymbol{u}_h^{\mathrm{TH}}\right)\right)_h \\
&\quad\leq \|\boldsymbol{f}\|_{H^{-1}(\Omega)}\|\nabla\boldsymbol{u}_h^{\mathrm{TH}}\|_{L^2(\Omega)} \\
&\qquad + \delta_0\tau\|\mathrm{curl}\,\boldsymbol{f}\|_h\left(\nu\|D^3\boldsymbol{u}_h^{\mathrm{TH}}\|_h + \|\boldsymbol{b}\|_{L^\infty(\Omega)}\|D^2\boldsymbol{u}_h^{\mathrm{TH}}\|_h + \sigma\|\nabla\boldsymbol{u}_h^{\mathrm{TH}}\|_h\right) \\
&\quad\leq \|\nabla\boldsymbol{u}_h^{\mathrm{TH}}\|_{L^2(\Omega)}\left(\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + \delta_0\tau\|\mathrm{curl}\,\boldsymbol{f}\|_h\left(\nu h^{-2} + \|\boldsymbol{b}\|_{L^\infty(\Omega)}h^{-1} + \|\nabla\boldsymbol{b}\|_{L^\infty(\Omega)} + \sigma\right)\right) \\
&\quad\leq \left(\frac{\|\boldsymbol{f}\|_{H^{-1}(\Omega)}^2}{\nu^2} + \frac{\delta_0}{\nu}\min\left\{\frac{h^3}{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}, \frac{h^4}{\nu}\right\}\|\mathrm{curl}\,\boldsymbol{f}\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\
&\qquad \times\left(\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + \delta_0\|\mathrm{curl}\,\boldsymbol{f}\|_h\left(2h^2 + \min\left\{\frac{h^3}{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}, \frac{h^4}{\nu}\right\}\left(\|\nabla\boldsymbol{b}\|_{L^\infty(\Omega)} + \sigma\right)\right)\right) \\
&\quad=: C_1(\mathrm{data}).
\end{aligned}$$

Hence, we obtain the following bound of the Taylor–Hood velocity solution:

$$\|\boldsymbol{u}_h^{\mathrm{TH}}\|_{\mathrm{LSVS}}^2 + \gamma\|\nabla\cdot\boldsymbol{u}_h^{\mathrm{TH}}\|_{L^2(\Omega)}^2 = \|\boldsymbol{u}_h^{\mathrm{TH}}\|_{\mathrm{LSVS},\gamma}^2 \leq C_1(\mathrm{data}). \tag{22}$$

For a given sequence of grad-div stabilization parameters $\{\gamma_i\}_{i=1}^\infty \to \infty$ and corresponding Taylor–Hood velocity solutions $\boldsymbol{u}_i^{\mathrm{TH}}$ on a fixed mesh, we therefore have a bounded sequence in a finite-dimensional space. Hence, there exists $\boldsymbol{w} \in \boldsymbol{V}_h$ such that a subsequence $\boldsymbol{u}_{i_k}^{\mathrm{TH}} \to \boldsymbol{w}$ in the $\|\cdot\|_{\mathrm{LSVS},\gamma}$ norm. For notational simplicity we identify the subsequence $\boldsymbol{u}_{i_k}^{\mathrm{TH}}$ with the entire sequence $\boldsymbol{u}_i^{\mathrm{TH}}$. From (22), we obtain in particular

$$\|\nabla\cdot\boldsymbol{w}\|_{L^2(\Omega)}^2 = \lim_{i\to\infty}\|\nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}}\|_{L^2(\Omega)}^2 \leq \lim_{i\to\infty}\frac{1}{\gamma_i}C_1(\mathrm{data}) = 0,$$

and we conclude that $\|\nabla\cdot\boldsymbol{w}\|_{L^2(\Omega)} = 0$, i.e, $\boldsymbol{w} \in \boldsymbol{V}_{h,\mathrm{div}}^{\mathrm{SV}}$.

ii). To show that $\boldsymbol{w}$ is the Scott–Vogelius velocity solution, consider the residual

$$\mathrm{res}(\boldsymbol{v}_h) = a(\boldsymbol{w}, \boldsymbol{v}_h) + S(\boldsymbol{w}, \boldsymbol{v}_h) - L(\boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_{h,\mathrm{div}}^{\mathrm{TH}}.$$

Using $\nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}} \to 0$ and the Lebesgue dominated convergence theorem, we find for all $\boldsymbol{v}_h \in \boldsymbol{V}_{h,\mathrm{div}}^{\mathrm{TH}}$

$$\begin{aligned}
\mathrm{res}(\boldsymbol{v}_h) &= a\left(\lim_{i\to\infty}\boldsymbol{u}_i^{\mathrm{TH}}, \boldsymbol{v}_h\right) + S\left(\lim_{i\to\infty}\boldsymbol{u}_i^{\mathrm{TH}}, \boldsymbol{v}_h\right) + \lim_{i\to\infty}\gamma_i\left(\nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}}, \nabla\cdot\boldsymbol{v}_h\right) - L(\boldsymbol{v}_h) \\
&= \lim_{i\to\infty}\left(a(\boldsymbol{u}_i^{\mathrm{TH}}, \boldsymbol{v}_h) + S(\boldsymbol{u}_i^{\mathrm{TH}}, \boldsymbol{v}_h) + \gamma_i\left(\nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}}, \nabla\cdot\boldsymbol{v}_h\right) - L(\boldsymbol{v}_h)\right) = 0,
\end{aligned}$$

since $\boldsymbol{u}_i^{\mathrm{TH}}$ is for every i a Taylor–Hood solution satisfying (18). Since $\boldsymbol{V}_{h,\mathrm{div}}^{\mathrm{SV}} \subset \boldsymbol{V}_{h,\mathrm{div}}^{\mathrm{TH}}$, it follows that

$$a(\boldsymbol{w}, \boldsymbol{v}_h) + S(\boldsymbol{w}, \boldsymbol{v}_h) = L(\boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_{h,\mathrm{div}}^{\mathrm{SV}},$$

i.e, $\boldsymbol{w}$ is the Scott–Vogelius velocity solution of the LSVS Oseen problem (18).

iii). It remains to show the convergence of the pressure. Using the inf-sup stability of the Scott–Vogelius pair and since $p_i^{\mathrm{TH}} - \gamma_i\nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}} \in Q_h^{\mathrm{SV}}$ and $(\boldsymbol{u}_i^{\mathrm{TH}}, p_i^{\mathrm{TH}})$ is the Taylor–Hood solution of (18), the first step of bounding the pressure consists in

$$\|p_i^{\mathrm{TH}} - \gamma_i\nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}}\|_{L^2(\Omega)}$$

$$
\begin{aligned}
&\leq \frac{1}{\beta_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{h,\mathrm{div}}^{\mathrm{SV}}} \frac{\left(\nabla\cdot\boldsymbol{v}_h, p_i^{\mathrm{TH}} - \gamma_i \nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}}\right)}{\|\nabla \boldsymbol{v}_h\|_{L^2(\Omega)}} \\
&\leq \frac{1}{\beta_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{h,\mathrm{div}}^{\mathrm{SV}}} \left(\frac{\nu\left(\nabla \boldsymbol{u}_i^{\mathrm{TH}}, \nabla \boldsymbol{v}_h\right) + \sigma\left(\boldsymbol{u}_i^{\mathrm{TH}}, \boldsymbol{v}_h\right) + \left((\boldsymbol{b}\cdot\nabla)\boldsymbol{u}_i^{\mathrm{TH}}, \boldsymbol{v}_h\right)}{\|\nabla \boldsymbol{v}_h\|_{L^2(\Omega)}} \right. \\
&\qquad + \frac{\delta_0 \tau \left(\operatorname{curl}\mathcal{L}\boldsymbol{u}_i^{\mathrm{TH}}, \operatorname{curl}\mathcal{L}\boldsymbol{v}_h\right)_h + \delta_0 h^2 \left(\llbracket(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}_i^{\mathrm{TH}}\times\boldsymbol{n}\rrbracket, \llbracket(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}_h\times\boldsymbol{n}\rrbracket\right)_{\mathcal{F}^i}}{\|\nabla \boldsymbol{v}_h\|_{L^2(\Omega)}} \\
&\qquad \left. - \frac{(\boldsymbol{f}, \boldsymbol{v}_h) + \delta_0 \tau \left(\operatorname{curl}\boldsymbol{f}, \operatorname{curl}\mathcal{L}\boldsymbol{v}_h\right)_h}{\|\nabla \boldsymbol{v}_h\|_{L^2(\Omega)}} \right).
\end{aligned} \tag{23}
$$

We estimate the right-hand side of (23) term by term. The first three terms are bounded easily using the Cauchy–Schwarz inequality and the Poincaré inequality (7). Estimating the fourth term starts also with the Cauchy–Schwarz inequality

$$
\delta_0 \tau \left(\operatorname{curl}\mathcal{L}\boldsymbol{u}_i^{\mathrm{TH}}, \operatorname{curl}\mathcal{L}\boldsymbol{v}_h\right)_h \leq \delta_0 \sum_{K\in\mathcal{T}_h} \tau_K \|\operatorname{curl}\mathcal{L}\boldsymbol{u}_i^{\mathrm{TH}}\|_{L^2(K)} \|\operatorname{curl}\mathcal{L}\boldsymbol{v}_h\|_{L^2(K)}.
$$

Applying the inverse inequality (11), the definition of the curl operator, the product rule, and the value of τ from (13) yields

$$
\begin{aligned}
&\tau_K^{1/2} \|\operatorname{curl}\mathcal{L}\boldsymbol{u}_i^{\mathrm{TH}}\|_{L^2(K)} \\
&\leq C\tau_K^{1/2}\left(\nu h_K^{-2} + \|\nabla\boldsymbol{b}\|_{L^\infty(K)} + h_K^{-1}\|\boldsymbol{b}\|_{L^\infty(\Omega)}K + \sigma\right)\|\nabla\boldsymbol{u}_i^{\mathrm{TH}}\|_{L^2(K)} \\
&\leq C\left(\sqrt{\nu} + \sqrt{h\|\boldsymbol{b}\|_{L^\infty(\Omega)}} + \min\left\{\frac{h^{3/2}}{\sqrt{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}}, \frac{h^2}{\sqrt{\nu}}\right\}\left(\|\nabla\boldsymbol{b}\|_{L^\infty(\Omega)} + \sigma\right)\right)\|\nabla\boldsymbol{u}_i^{\mathrm{TH}}\|_{L^2(K)}.
\end{aligned}
$$

Using the same estimate for the second factor gives the bound

$$
\begin{aligned}
&\delta_0\tau\left(\operatorname{curl}\mathcal{L}\boldsymbol{u}_i^{\mathrm{TH}}, \operatorname{curl}\mathcal{L}\boldsymbol{v}_h\right)_h \\
&\leq C\delta_0\left(\sqrt{\nu} + \sqrt{h\|\boldsymbol{b}\|_{L^\infty(\Omega)}} + \min\left\{\frac{h^{3/2}}{\sqrt{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}}, \frac{h^2}{\sqrt{\nu}}\right\}\left(\|\nabla\boldsymbol{b}\|_{L^\infty(\Omega)} + \sigma\right)\right)^2 \\
&\qquad \times\|\nabla\boldsymbol{u}_i^{\mathrm{TH}}\|_{L^2(\Omega)}\|\nabla\boldsymbol{v}_h\|_{L^2(\Omega)}.
\end{aligned}
$$

For the jump term we apply the Cauchy–Schwarz inequality, the local trace inequality (10), Hölder's inequality, $\nabla\cdot\boldsymbol{b} = 0$ and the inverse inequality to obtain

$$
\begin{aligned}
&\delta_0 h^2\left(\llbracket(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}_i^{\mathrm{TH}}\times\boldsymbol{n}\rrbracket, \llbracket(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}_h\times\boldsymbol{n}\rrbracket\right)_{\mathcal{F}^i} \\
&\qquad\qquad \leq C\delta_0 h\|\boldsymbol{b}\|_{L^\infty(\Omega)}^2\|\nabla\boldsymbol{u}_i^{\mathrm{TH}}\|_{L^2(\Omega)}\|\nabla\boldsymbol{v}_h\|_{L^2(\Omega)}.
\end{aligned}
$$

And finally the last term is estimated with the same techniques as the fourth term, giving

$$
\begin{aligned}
&\delta_0\tau\left(\operatorname{curl}\boldsymbol{f}, \operatorname{curl}\mathcal{L}\boldsymbol{v}_h\right)_h \\
&\leq C\delta_0\left(2h^2 + \min\left\{\frac{h^3}{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}, \frac{h^4}{\nu}\right\}\left(\|\nabla\boldsymbol{b}\|_{L^\infty(\Omega)} + \sigma\right)\right)\|\operatorname{curl}\boldsymbol{f}\|_{L^2(\Omega)}\|\nabla\boldsymbol{v}_h\|_{L^2(\Omega)}.
\end{aligned}
$$

We then substitute all the terms back in (23), all the factors $\|\nabla\boldsymbol{v}_h\|_{L^2(\Omega)}$ can be cancelled. Finally, the stability estimate (21) for $\|\nabla\boldsymbol{u}_i^{\mathrm{TH}}\|_{L^2(\Omega)}$ is applied to get a bound of the pressure sequence that only depends on the data

$$
\|p_i^{\mathrm{TH}} - \gamma_i\nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}}\|_{L^2(\Omega)} \leq C_2(\text{data}).
$$

Hence, there is a subsequence of $p_i^{\mathrm{TH}} - \gamma_i\nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}}$ that converges to some $\overline{p}$, for simplicity denote it also by $p_i^{\mathrm{TH}} - \gamma_i\nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}}$. Thus, we obtain with (18) and the result of the first part of the proof that for all fixed $\boldsymbol{v}_h \in \boldsymbol{V}_h$

$$
\begin{aligned}
(\nabla\cdot\boldsymbol{v}_h, \overline{p}) &= \lim_{i\to\infty}\left(\nabla\cdot\boldsymbol{v}_h, p_i^{\mathrm{TH}} - \gamma_i\nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}}\right) = \lim_{i\to\infty}\left((a+S)(\boldsymbol{u}_i^{\mathrm{TH}}, \boldsymbol{v}_h) - L(\boldsymbol{v}_h)\right) \\
&= (a+S)(\boldsymbol{w}, \boldsymbol{v}_h) - L(\boldsymbol{v}_h) = \left(\nabla\cdot\boldsymbol{v}_h, p^{\mathrm{SV}}\right).
\end{aligned}
$$

Since $\left(Q_h^{\mathrm{TH}} + \nabla\cdot\boldsymbol{V}_h\right) \subset Q_h^{\mathrm{SV}}$, we find from the inf-sup stability of the Scott–Vogelius pair that $\overline{p} = p^{\mathrm{SV}}$. Since the pressure is unique, we conclude that the entire sequence $p_i^{\mathrm{TH}} - \gamma_i\nabla\cdot\boldsymbol{u}_i^{\mathrm{TH}}$ converges to p^{SV}. ☐

5 Error analysis

This section presents an error analysis of the LSVS-grad-div stabilized method for Taylor–Hood pairs of spaces. Bounds for velocity and pressure errors are derived on grids where the corresponding Scott–Vogelius pair of spaces is inf-sup stable.

Theorem 5.1 (Velocity error estimate for the LSVS-grad-div stabilized method)**.** *Let the assumptions of [2, Theorem 6] be satisfied, i.e., assume in particular that there exists a unique Scott–Vogelius solution of* (18)*, and let, in addition,* $(\boldsymbol{u},p) \in ((H_0^1(\Omega))^d \cap (H^{k+1}(\Omega))^d) \times (L_0^2(\Omega) \cap H^k(\Omega))$ *be the solution of the weak Oseen equations* (3)*. Let* $\boldsymbol{u}_h^{\mathrm{TH}}$ *be the velocity solution of the LSVS-grad-div scheme* (18) *when discretized with a Taylor–Hood space. Then the velocity error can be bounded by*

$$\|\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{TH}}\|_{\mathrm{LSVS},\gamma} \le \frac{C}{\gamma^{\frac12}} h^k \left(\left(1+\gamma^{\frac12}+\nu^{\frac12}+h^{\frac12}\right)\left(\nu^{\frac12}+h^{\frac12}\right)\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)}\right), \tag{24}$$

where the constant C *is independent of* h *and* ν*.*

Proof. The principal approach of the error analysis is as usual: a discrete function is added and subtracted to the error, which is generally an interpolant or projection of the solution of the continuous problem. Then the main task consists in bounding the difference of the discrete function and the finite element solution. The crucial observation that is used in the following proof is that due to the special setup studied in this paper, we can apply an unusual discrete function, namely the velocity solution of the Scott–Vogelius discretization.

i). Let $\boldsymbol{u}_h^{\mathrm{SV}}$ be the velocity solution of the LSVS-grad-div scheme (18) when discretized with the Scott–Vogelius space. We decompose the Taylor–Hood velocity error by adding and subtracting the Scott–Vogelius velocity solution $\boldsymbol{u}_h^{\mathrm{SV}}$

$$\|\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{TH}}\|_{\mathrm{LSVS},\gamma} \le \|\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{SV}}\|_{\mathrm{LSVS},\gamma} + \|\boldsymbol{u}_h^{\mathrm{SV}}-\boldsymbol{u}_h^{\mathrm{TH}}\|_{\mathrm{LSVS},\gamma}. \tag{25}$$

For the first term observe that $\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{SV}}$ is divergence free, so we already have an estimate from Theorem 3.1:

$$\|\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{SV}}\|_{\mathrm{LSVS},\gamma} = \|\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{SV}}\|_{\mathrm{LSVS}} \le C h^k \left(\nu^{\frac12}+h^{\frac12}\right)\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}. \tag{26}$$

ii). Subtracting the Scott–Vogelius discretization of the LSVS-grad-div scheme (18) from the Taylor–Hood discretization, both tested with $\boldsymbol{v}_h^{\mathrm{TH}} \in \boldsymbol{V}_{h,\mathrm{div}}^{\mathrm{TH}} \subset \boldsymbol{V}_h$, gives the error equation

$$(a+S)(\boldsymbol{u}_h^{\mathrm{TH}}-\boldsymbol{u}_h^{\mathrm{SV}},\boldsymbol{v}_h^{\mathrm{TH}}) + \gamma\left(\nabla\cdot(\boldsymbol{u}_h^{\mathrm{TH}}-\boldsymbol{u}_h^{\mathrm{SV}}),\nabla\cdot\boldsymbol{v}_h^{\mathrm{TH}}\right) + \left(\nabla\cdot\boldsymbol{v}_h^{\mathrm{TH}},p_h^{\mathrm{SV}}\right) = 0.$$

Observe that the pressure term $\left(\nabla\cdot\boldsymbol{v}_h^{\mathrm{TH}},p_h^{\mathrm{SV}}\right)$ does not vanish since $\boldsymbol{v}_h^{\mathrm{TH}} \in \boldsymbol{V}_{h,\mathrm{div}}^{\mathrm{TH}} \supset \boldsymbol{V}_{h,\mathrm{div}}^{\mathrm{SV}}$ and $p_h^{\mathrm{SV}} \in Q_h^{\mathrm{SV}} \not\subset Q_h^{\mathrm{TH}}$. Let $\pi_{Q_h^{\mathrm{TH}}}(p)$ be the $L^2(\Omega)$-orthogonal projection of the pressure solution of (3) onto Q_h^{TH}. Then, the zero term $\left(\nabla\cdot\boldsymbol{v}_h^{\mathrm{TH}},\pi_{Q_h^{\mathrm{TH}}}(p)\right) = 0$ can be subtracted from the left-hand side of the error equation.

Setting $\boldsymbol{v}_h^{\mathrm{TH}} = \boldsymbol{u}_h^{\mathrm{TH}}-\boldsymbol{u}_h^{\mathrm{SV}} \in \boldsymbol{V}_h$, using the norm definition (15), and then utilizing the Cauchy–Schwarz inequality and Young's inequality yields

$$\begin{aligned}\|\boldsymbol{u}_h^{\mathrm{TH}}-\boldsymbol{u}_h^{\mathrm{SV}}\|_{\mathrm{LSVS}}^2 &+ \gamma\|\nabla\cdot\left(\boldsymbol{u}_h^{\mathrm{TH}}-\boldsymbol{u}_h^{\mathrm{SV}}\right)\|_{L^2(\Omega)}^2 \\ &= -\left(\nabla\cdot(\boldsymbol{u}_h^{\mathrm{TH}}-\boldsymbol{u}_h^{\mathrm{SV}}),p_h^{\mathrm{SV}}-\pi_{Q_h^{\mathrm{TH}}}(p)\right) \\ &\le \frac{\gamma}{2}\|\nabla\cdot\left(\boldsymbol{u}_h^{\mathrm{TH}}-\boldsymbol{u}_h^{\mathrm{SV}}\right)\|_{L^2(\Omega)}^2 + \frac{1}{2\gamma}\|p_h^{\mathrm{SV}}-\pi_{Q_h^{\mathrm{TH}}}(p)\|_{L^2(\Omega)}^2.\end{aligned}$$

Absorbing the divergence term in the left-hand and then multiplying by 2, we get an estimate in the $\|\cdot\|_{\mathrm{LSVS},\gamma}$ norm defined in (19). Then with the triangle inequality we obtain

$$\begin{aligned}\|\boldsymbol{u}_h^{\mathrm{TH}}-\boldsymbol{u}_h^{\mathrm{SV}}\|_{\mathrm{LSVS},\gamma}^2 &\le \frac{1}{\gamma}\|p_h^{\mathrm{SV}}-\pi_{Q_h^{\mathrm{TH}}}(p)\|_{L^2(\Omega)}^2 \\ &\le \frac{1}{\gamma}\left(\|p-p_h^{\mathrm{SV}}\|_{L^2(\Omega)}+\|p-\pi_{Q_h^{\mathrm{TH}}}(p)\|_{L^2(\Omega)}\right)^2.\end{aligned} \tag{27}$$

The first term on the right-hand side is the LSVS Scott–Vogelius pressure solution, which was estimated in (17). The second term is the $L^2(\Omega)$-orthogonal projection error of the pressure in the Taylor–Hood pressure space, which can be bounded by

$$\|p-\pi_{Q_h^{\mathrm{TH}}}(p)\|_{L^2(\Omega)} \le C h^k \|p\|_{H^k(\Omega)}. \tag{28}$$

Therefore, we arrive at

$$\|\boldsymbol{u}_h^{\mathrm{TH}} - \boldsymbol{u}_h^{\mathrm{SV}}\|_{\mathrm{LSVS},\gamma} \le \frac{C}{\gamma^{\frac{1}{2}}} h^k \left(\left(1 + \nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \left(\nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)} \right). \tag{29}$$

iii). Now, the bounds in (26) and (29) are substituted in (25) and after similar terms have been collected, estimate (24) is proved. □

Remark 5.2 (On the difference of the Taylor–Hood and Scott–Vogelius velocity solutions)**.** *Note that the estimate of the difference between the Taylor–Hood and Scott–Vogelius velocity solutions in* (29) *converges to zero as the grad-div parameter γ goes to infinity, which is the same finding as in the first part of Theorem 4.1 and thus an alternative proof is provided.*

Estimate (29) *also implies the following bounds in other norms:*

$$\begin{aligned} \|\boldsymbol{u}_h^{\mathrm{TH}} - \boldsymbol{u}_h^{\mathrm{SV}}\|_{L^2(\Omega)} &\le \frac{C}{\sigma^{\frac{1}{2}}\gamma^{\frac{1}{2}}} h^k \left(\left(1 + \nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \left(\nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)} \right), \\ \|\nabla(\boldsymbol{u}_h^{\mathrm{TH}} - \boldsymbol{u}_h^{\mathrm{SV}})\|_{L^2(\Omega)} &\le \frac{C}{\nu^{\frac{1}{2}}\gamma^{\frac{1}{2}}} h^k \left(\left(1 + \nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \left(\nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)} \right). \end{aligned} \tag{30}$$

Remark 5.3 (On the Taylor–Hood velocity estimate)**.** *Note that for $\gamma \to \infty$ the Taylor–Hood velocity estimate in* (24) *converges to the Scott–Vogelius velocity estimate* (16)*. In contrast to estimate* (16)*, which gives for $\nu \le h$ the order $k + 1/2$ of velocity error reduction for Scott–Vogelius pairs of spaces, one has in* (24) *only order k for Taylor–Hood pairs. The reason is that using the latter pairs does not lead to a pressure-robust method. It can be seen clearly in the proof of Theorem 5.1, e.g., in* (27) *and* (28)*, that the reduction to order k arises from the necessity of estimating pressure terms.*

The norm on the left-hand side of the error estimate (24) *is stronger than the norm for which an error bound for the grad-div stabilization, without LSVS term, can be derived.*

Again, one finds from (24) *that the velocity error of the LSVS-grad-div Taylor–Hood stabilized method in the $L^2(\Omega)$-norm and the $L^2(\Omega)$-norm of the gradient have the error bounds*

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h^{\mathrm{TH}}\|_{L^2(\Omega)} &\le \frac{C}{\sigma^{\frac{1}{2}}\gamma^{\frac{1}{2}}} h^k \left(\left(1 + \gamma^{\frac{1}{2}} + \nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \left(\nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)} \right), \\ \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h^{\mathrm{TH}})\|_{L^2(\Omega)} &\le \frac{C}{\nu^{\frac{1}{2}}\gamma^{\frac{1}{2}}} h^k \left(\left(1 + \gamma^{\frac{1}{2}} + \nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \left(\nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)} \right). \end{aligned}$$

Theorem 5.4 (Pressure error estimate for the LSVS-grad-div stabilized method)**.** *Let the assumptions of Theorem 5.1 be satisfied, then the pressure error of the LSVS-grad-div Taylor–Hood stabilized method can be bounded in the following way*

$$\begin{aligned} \|p - p_h^{\mathrm{TH}}\|_{L^2(\Omega)} \quad \le \quad & \frac{C}{\gamma^{\frac{1}{2}}} h^k \left\{ \left(1 + \gamma^{\frac{1}{2}} + \nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \left(\nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \left(1 + \nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} \right. \\ & \left. + \left(1 + \gamma^{\frac{1}{2}} + \nu^{\frac{1}{2}} + h^{\frac{1}{2}}\right) \|p\|_{H^k(\Omega)} \right\}, \end{aligned} \tag{31}$$

where the constant C is independent of h and ν.

Proof. i). Let $\boldsymbol{v}_h \in \boldsymbol{V}_h^{\mathrm{TH}}$ be arbitrary, then by Galerkin orthogonality (20) we see that

$$\begin{aligned} & a(\boldsymbol{u} - \boldsymbol{u}_h^{\mathrm{TH}}, \boldsymbol{v}_h) + S(\boldsymbol{u} - \boldsymbol{u}_h^{\mathrm{TH}}, \boldsymbol{v}_h) + \gamma\left(\nabla\cdot(\boldsymbol{u} - \boldsymbol{u}_h^{\mathrm{TH}}), \nabla\cdot\boldsymbol{v}_h\right) \\ = \quad & \left(\nabla\cdot\boldsymbol{v}_h, p - p_h^{\mathrm{TH}}\right) = \left(\nabla\cdot\boldsymbol{v}_h, p - \pi_{Q_h^{\mathrm{TH}}}(p)\right) + \left(\nabla\cdot\boldsymbol{v}_h, \pi_{Q_h^{\mathrm{TH}}}(p) - p_h^{\mathrm{TH}}\right). \end{aligned} \tag{32}$$

The pair $\boldsymbol{V}_h^{\mathrm{TH}}/Q_h^{\mathrm{TH}}$ is inf-sup stable, e.g., see [12, Lemma 3.58], so there exists $\boldsymbol{w}_h \in \boldsymbol{V}_h$ such that

$$\nabla\cdot\boldsymbol{w}_h = \pi_{Q_h^{\mathrm{TH}}}(p) - p_h^{\mathrm{TH}} \quad \text{and} \quad \|\nabla\boldsymbol{w}_h\|_{L^2(\Omega)} \le \frac{1}{\beta_h}\|\pi_{Q_h^{\mathrm{TH}}}(p) - p_h^{\mathrm{TH}}\|_{L^2(\Omega)}. \tag{33}$$

To decompose the pressure error, we substitute $\boldsymbol{v}_h = \boldsymbol{w}_h$ in (32) and use (33), the fact that the LSVS stabilization term is a symmetric positive semi-definite bilinear form and thus satisfies a Cauchy–Schwarz inequality, integration by parts, and $\nabla\cdot\boldsymbol{b} = 0$:

$$\|\pi_{Q_h^{\mathrm{TH}}}(p) - p_h^{\mathrm{TH}}\|_{L^2(\Omega)}^2$$

$$
\begin{aligned}
&= a(\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{TH}},\boldsymbol{w}_h)+S(\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{TH}},\boldsymbol{w}_h)+\gamma\left(\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{TH}}),\pi_{Q_h^{\mathrm{TH}}}(p)-p_h^{\mathrm{TH}}\right)\\
&\quad -\left(\pi_{Q_h^{\mathrm{TH}}}(p)-p_h^{\mathrm{TH}},p-\pi_{Q_h^{\mathrm{TH}}}(p)\right)\\
&\le \|\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{TH}}\|_{\mathrm{LSVS},\gamma}\|\boldsymbol{w}_h\|_{\mathrm{LSVS}}\\
&\quad +C\|\boldsymbol{b}\|_{L^\infty(\Omega)}\|\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{TH}}\|_{\mathrm{LSVS},\gamma}\|\pi_{Q_h^{\mathrm{TH}}}(p)-p_h^{\mathrm{TH}}\|_{L^2(\Omega)}\\
&\quad +\gamma\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{TH}})\|_{L^2(\Omega)}\|\pi_{Q_h^{\mathrm{TH}}}(p)-p_h^{\mathrm{TH}}\|_{L^2(\Omega)}\\
&\quad +\|p-\pi_{Q_h^{\mathrm{TH}}}(p)\|_{L^2(\Omega)}\|\pi_{Q_h^{\mathrm{TH}}}(p)-p_h^{\mathrm{TH}}\|_{L^2(\Omega)}.
\end{aligned}
\tag{34}
$$

From [2, Theorem 8] it is known that an estimate of the form

$$
\|\boldsymbol{w}_h\|_{\mathrm{LSVS}}\le C\left(1+\nu^{\frac12}+h^{\frac12}\right)\|\pi_{Q_h^{\mathrm{TH}}}(p)-p_h^{\mathrm{TH}}\|_{L^2(\Omega)} \tag{35}
$$

is valid, where the constant C depends on σ and different norms of $\boldsymbol{b}$ but not on ν. Inserting (35) into (34) we get the estimate

$$
\|\pi_{Q_h^{\mathrm{TH}}}(p)-p_h^{\mathrm{TH}}\|_{L^2(\Omega)}\le C\left(1+\nu^{\frac12}+h^{\frac12}\right)\|\boldsymbol{u}-\boldsymbol{u}_h^{\mathrm{TH}}\|_{\mathrm{LSVS},\gamma}+\|p-\pi_{Q_h^{\mathrm{TH}}}(p)\|_{L^2(\Omega)}.
$$

Using the bound proved in Theorem 5.1, the triangle inequality and the standard approximation properties of the projection $\pi_{Q_h^{\mathrm{TH}}}$ in (28) we get the pressure error bound (31). ☐

Remark 5.5 (On general meshes)**.** *The results derived in this section require that the Scott–Vogelius pair of finite element spaces is inf-sup stable on the considered meshes. It is well known that this situation is given only on special meshes, like barycentric-refined meshes for $k \ge d$. A finite element error analysis for Taylor–Hood pairs combined with the LSVS-grad-div stabilization on general meshes is still open. Numerical studies presented in Section 6 show on general simplicial meshes the same orders of convergence in this situation as those proved in the current section.*

6 Numerical studies

In the numerical studies below we consider the Oseen problem on $\Omega=(0,1)^2$ with the prescribed flow field

$$
\boldsymbol{u}=\begin{pmatrix}\sin(2\pi x)\sin(2\pi y)\\ \cos(2\pi x)\cos(2\pi y)\end{pmatrix}
$$

and pressure

$$
p=\frac14(\cos(4\pi x)-\cos(4\pi y)).
$$

The convection field is given by $\boldsymbol{b}=\boldsymbol{u}+(0,1)^T$ similar to a test problem suggested in [2] to have a convection term that contains both a divergence-free part and an irrotational part. Since $\nabla p=-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$, the fully divergence-free right-hand side is given by

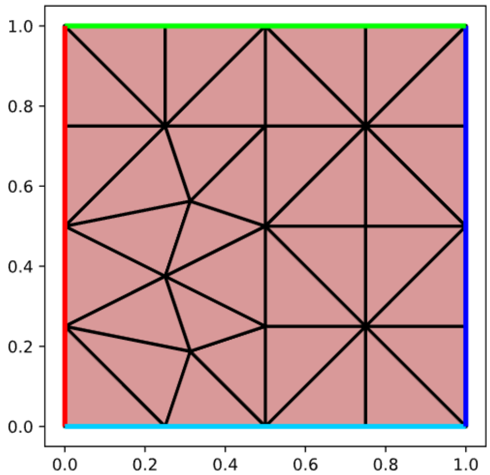
$$
\boldsymbol{f}=\sigma\boldsymbol{u}-\nu\Delta\boldsymbol{u}+((0,1)\cdot\nabla)\boldsymbol{u}.
$$

For small viscosity coefficients, the right-hand side is dominated for large σ by the first term and for small σ by the last one. It is well known that the counterpart of the stability bound (6) for $\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}$ under the assumption that $\boldsymbol{f}\in(L^2(\Omega))^d$ depends on the divergence-free part of the right-hand side and this part is scaled with ν^{-1}, e.g., see also [13, Lemma 4.2] for the Stokes equations. Hence, we think that this example is well suited for studying the convection-dominated regime.

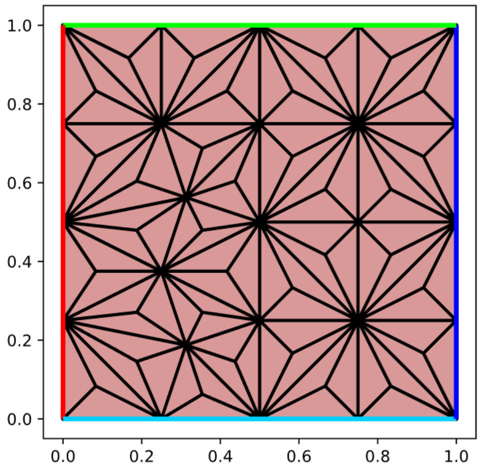
The values of ν and σ are specified below for the individual studies. We incorporated also the case $\sigma=0$, which is not covered by the analysis. All simulations were performed with the finite element Julia package ExtendableFEM.jl. We used the second order finite element spaces Taylor–Hood ($\boldsymbol{P}_2/P_1$) and Scott–Vogelius ($\boldsymbol{P}_2/P_1^{\mathrm{disc}}$). The simulations were carried out on a sequence of regularly refined unstructured grids and corresponding barycentric-refined grids. The coarsest grids (level 1) and the degrees of freedom on each grid for the first five levels are shown in Figure 1.

6.1 Numerical illustration of the result from Theorem 4.1

In Figure 2, differences of the LSVS-grad-div stabilized Taylor–Hood and LSVS stabilized Scott–Vogelius solutions are plotted with respect to $\|\boldsymbol{u}_h^{\mathrm{TH}}-\boldsymbol{u}_h^{\mathrm{SV}}\|_{L^2(\Omega)}$, $\|\nabla(\boldsymbol{u}_h^{\mathrm{TH}}-\boldsymbol{u}_h^{\mathrm{SV}})\|_{L^2(\Omega)}$, $\|p_h^{\mathrm{TH}}-\left(p_h^{\mathrm{SV}}+\gamma\nabla\cdot\boldsymbol{u}_h^{\mathrm{TH}}\right)\|_{L^2(\Omega)}$ and $\|\nabla\cdot$



level	ndof $\boldsymbol{u}_h$	ndof p_h	total ndof
1	170	26	196
2	846	115	916
3	3386	443	3829
4	13072	1672	14744
5	51316	6489	57805



level	ndof $\boldsymbol{u}_h$	ndof p_h	total ndof
1	442	60	502
2	2398	309	2707
3	9850	1251	11101
4	38616	4865	43481
5	152764	19170	171934

Figure 1: Initial mesh level 1 (left) and number of degrees of freedom of the Taylor–Hood method for five refinement levels (right). Top: standard unstructured triangular grid, bottom: corresponding barycentric-refined grid.

$\boldsymbol{u}_h^{\mathrm{TH}}\|_{L^2(\Omega)}$ versus the grad-div stabilization parameter γ. The coefficients of the Oseen problem were set to $\nu = 10^{-5}$, $\sigma = 1$ and the LSVS parameter to $\delta_0 = 10^{-2}$, which is the same order of magnitude as in [2], compare also Section 6.2 for a motivation of this choice. Results obtained on the barycentric-refined grid of level 4 are presented.

It can be observed that both velocity and pressure differences as well as the divergence are going to zero as we increase γ which confirms the finding of Theorem 4.1. Additionally, the L^2 velocity difference $\|\boldsymbol{u}_h^{\mathrm{TH}} - \boldsymbol{u}_h^{\mathrm{SV}}\|_{L^2(\Omega)}$ is smaller than the H^1 velocity difference $\|\nabla(\boldsymbol{u}_h^{\mathrm{TH}} - \boldsymbol{u}_h^{\mathrm{SV}})\|_{L^2(\Omega)}$ by a factor close to $({}^{\sigma}/_{\nu})^{1/2} = 10^{2.5}$, see estimate (30) in Remark 5.2. Furthermore, it is expected from (30) that the velocity difference depends on γ at least as $\mathcal{O}(\gamma^{-1/2})$. In practice we observe in Figure 2 even a linear order of convergence with respect to γ, which agrees with a result for the standard grad-div stabilization from [8]. This result states that the order $\mathcal{O}(\gamma^{-1/2})$ can be proved for a constant that is independent of the mesh, as in (30), whereas first order is obtained for a constant that might depend on the mesh width, see also [12, Lemma 4.118]. The differences are getting larger after $\gamma = 10^4$ due to round-off errors that are induced by the large condition number of the system matrix.

6.2 LSVS parameter studies

In Figure 3 we investigate the choice of the LSVS stabilization parameter δ_0. Our aims consist in exploring whether appropriate choices of δ_0 lead to error reductions compared with the standard grad-div stabilization ($\delta_0 = 0$) and if this is the case, to determine an appropriate order of magnitude for δ_0. The simulations were performed on barycentric-refined meshes and general meshes on a sequence of three levels. As recommended in the literature, e.g., see [13] and [1], the grad-div parameter was taken to be $\gamma = 1$. The viscosity coefficient was fixed at the small value $\nu = 10^{-5}$ and different

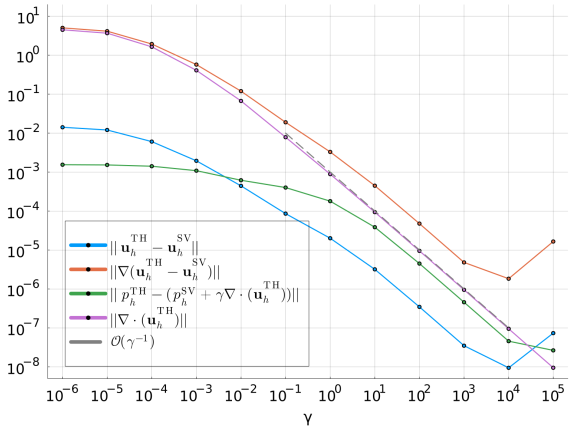


Figure 2: Convergence of LSVS-grad-div stabilized Taylor–Hood solutions to LSVS-stabilized Scott–Vogelius solution on a fixed barycentric-refined mesh.

values $\sigma \in \{0, 1, 100\}$ were studied for the reaction coefficient.

It can be observed first that including the LSVS stabilization term reduces the errors compared to the case when only grad-div stabilization is used. This effect is stronger for small values of σ. The reduction might be up to one order of magnitude. Second, the optimal parameter δ_0 does not depend strongly on the refinement level (observe the alignment of the minima for all levels). A good LSVS stabilization parameter can be chosen to be $\delta_0 = 10^{-2}$, which leads to comparatively small velocity errors for both norms and for all values of σ. This value is of a similar order of magnitude as proposed for the Scott–Vogelius pair of spaces in [2] ($\delta_0 = 0.006$).

6.3 Convergence histories

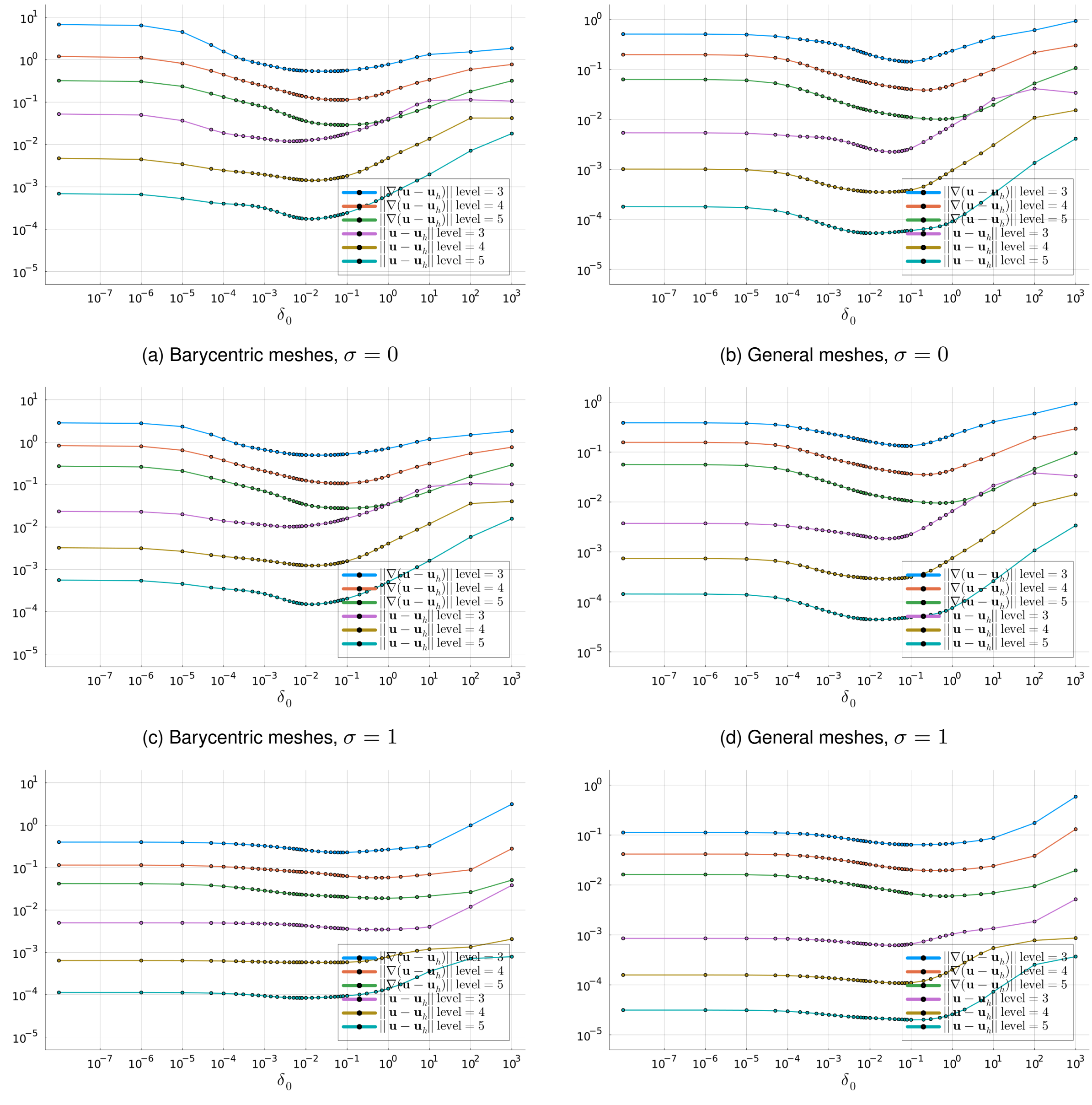
Numerical studies, not shown here for the sake of brevity, with the Taylor–Hood method without stabilization or only with LSVS stabilization showed very poor results for small values of the viscosity. Hence, here we concentrate on the LSVS-grad-div Taylor–Hood method. As in the previous study, the viscosity coefficient was chosen to be $\nu = 10^{-5}$ and the reaction coefficient was taken from the set $\{0, 1, 100\}$. The simulations were performed on both general and barycentric-refined meshes.

Figure 4 shows the convergence histories and convergence rates for LSVS and grad-div stabilization parameters $\gamma = 1$ and $\delta_0 = 10^{-2}$, respectively. The results on both types of meshes are of quite similar accuracy. The error $\|\boldsymbol{u} - \boldsymbol{u}_h^{\mathrm{TH}}\|_{\mathrm{LSVS},\gamma}$ reduces even faster than the expected second order which is proved in Theorem 5.1. We think that the reason is that the pressure term, which determines the order of convergence in the error bound, is small for the present example and the considered grids are yet too coarse for observing the full impact of this term. The $H^1(\Omega)$ velocity error and the $L^2(\Omega)$ pressure error converge with the optimal order $\mathcal{O}(h^2)$. The $L^2(\Omega)$ velocity error converges faster than $\mathcal{O}(h^{2.5})$, at least for the cases with $\sigma \in \{0, 1\}$. Overall the results confirm the theoretical findings and suggest that the chosen parameters for the grad-div and LSVS stabilizations are working well.

7 Summary and Outlook

This paper discussed the Taylor–Hood method with two stabilizations. The first one is the popular grad-div stabilization, which improves the pressure-robustness, and therefore reduces the impact of the pressure and of small viscosity coefficients on velocity error bounds already significantly. On barycentric meshes it is known that the method converges to the divergence-free Scott–Vogelius finite element method. However, even the Scott–Vogelius finite element method may suffer from dominant convection. The LSVS convection stabilization from [2] mitigates this situation and the accuracy of numerical solutions can be significantly improved. It is based on the residual of the vorticity equation and therefore does not compromise the pressure-robustness. The present paper investigated if and to which extent the grad-div-stabilized Taylor–Hood finite element method can profit from this additional LSVS stabilization.

An error analysis of this method showed for the grad-div parameter tending to infinity that the Scott–Vogelius velocity solution is recovered. Convection-robust error bounds for velocity, in a norm that contains contributions from both stabi-



(a) Barycentric meshes, $\sigma = 0$

(b) General meshes, $\sigma = 0$

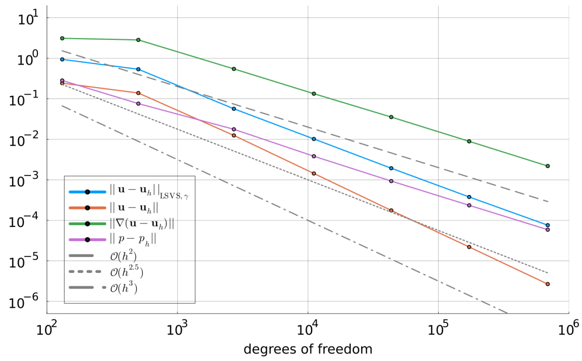
(c) Barycentric meshes, $\sigma = 1$

(d) General meshes, $\sigma = 1$

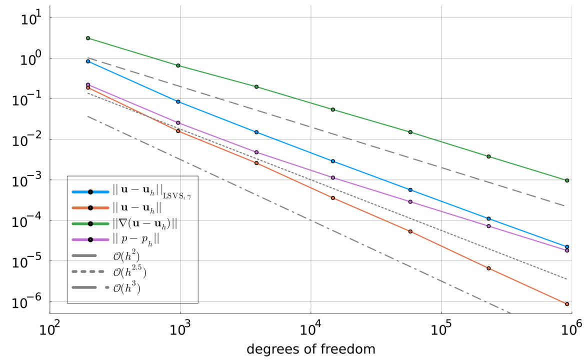
(e) Barycentric meshes, $\sigma = 100$

(f) General meshes, $\sigma = 100$

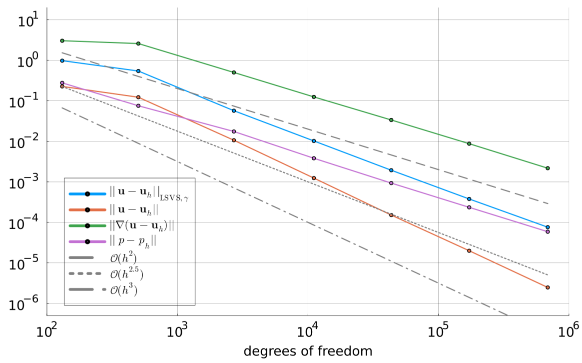
Figure 3: LSVS parameter studies for the Taylor–Hood LSVS-grad-div stabilization with $\gamma = 1$ and $\sigma \in \{0, 1, 100\}$.



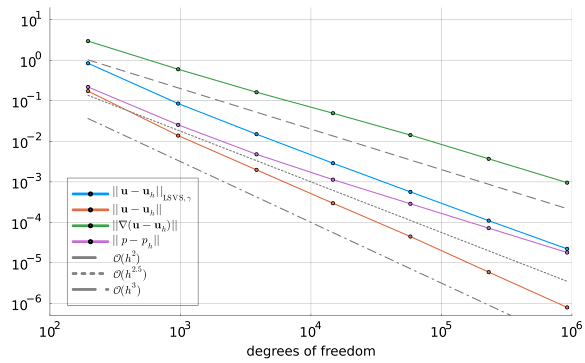
(a) Barycentric meshes, $\sigma = 0$



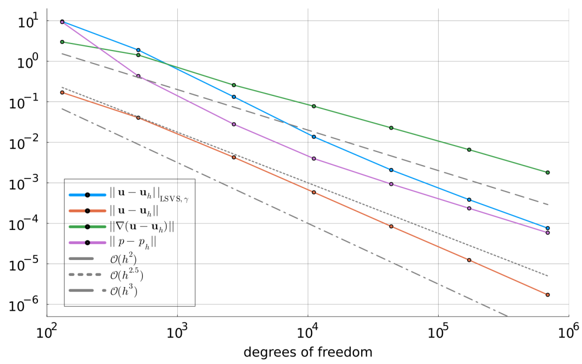
(b) General meshes, $\sigma = 0$



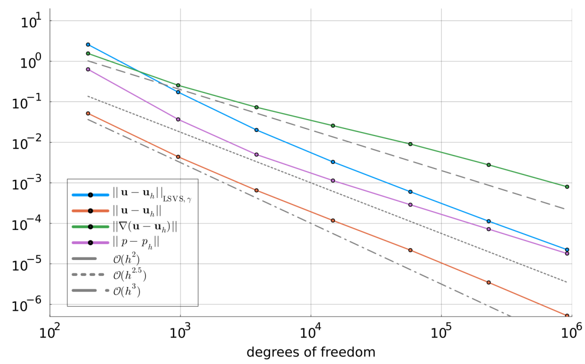
(c) Barycentric meshes, $\sigma = 1$



(d) General meshes, $\sigma = 1$



(e) Barycentric meshes, $\sigma = 100$



(f) General meshes, $\sigma = 100$

Figure 4: Convergence histories for simulations using the Taylor–Hood LSVS-grad-div stabilization with $\gamma = 1$ and $\delta_0 = 10^{-2}$.

lizations, and for the pressure were derived. The expected orders of convergence were proved, where the appearance of the pressure in the velocity error bound prevents a higher order of error reduction in the convection-dominated regime. Moreover, the numerical studies revealed that stabilization parameters close to available literature values for the separate stabilization, i.e., $\gamma \approx 1$ and $\delta_0 \approx 10^{-2}$, work well and and lead to significantly smaller errors than for a Taylor–Hood method without or with only one stabilization. While the theoretical results were based on barycentric meshes to ensure inf-sup stability of the Scott–Vogelius method, numerical experiments did not show any deterioration on general meshes.

Concerning general meshes, we were not able to perform an error analysis that utilizes in a similarly elegant way as in the proof of Theorem 5.1 already proved error bounds. We think that one has to perform a lengthy analysis of this case similarly to the one presented in [2]. The analysis of LSVS stabilized methods for more general problems than the Oseen equations and corresponding numerical studies is a further wide open field.

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