Weierstraß-Institut
für Angewandte Analysis und Stochastik

Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Exponential equivalence for misanthrope processes in contact with weak reservoirs and applications to totally asymmetric exclusion processes

Julian Kern

submitted: November 7, 2023

Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: julian.kern@wias-berlin.de

No. 3051
Berlin 2023

2020 *Mathematics Subject Classification.* 60K35, 60J27.

Key words and phrases. Exponential equivalence, misanthrope process, hydrodynamic limit, large deviations, boundary-driven particle systems.

This research has been funded by Deutsche Forschungsgemeinschaft (DFG) through grant CRC 1114 Scaling Cascades in Complex Systems, Project Number 235221301, Project C02 Interface dynamics: Bridging stochastic and hydrodynamic descriptions. I would like to thank Robert Patterson for valuable discussions.

Edited by

Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)

Leibniz-Institut im Forschungsverbund Berlin e. V.

Mohrenstraße 39

10117 Berlin

Germany

Fax: +49 30 20372-303

E-Mail: preprint@wias-berlin.de

World Wide Web: http://www.wias-berlin.de/

Exponential equivalence for misanthrope processes in contact with weak reservoirs and applications to totally asymmetric exclusion processes

Julian Kern

Abstract

We provide a short proof for the exponential equivalence between misanthrope processes in contact with weak reservoirs and those with impermeable boundaries. As a consequence, we can derive both the hydrodynamic limit and the large deviations of the totally asymmetric exclusion process (TASEP) in contact with weak reservoirs. This extends a recent result which proved the hydrodynamic behaviour of a vanishing viscocity approximation of the TASEP in contact with weak reservoirs. Furthermore, applications to a class of asymmetric exclusion processes with long jumps is discussed.

1 Introduction

The recent work [Xu22] makes use of the method of compensated compactness to investigate the hydrodynamic behaviour of the totally asymmetric exclusion process (TASEP) in contact with weak boundaries. An immediate drawback of this approach is that it can only be applied to a vanishing viscocity approximation of the model. In [GKX23+], a variant of the model with long jumps from [SS18] in contact with weak reservoirs has been proposed. In the regime of jumps with infinite mean, the hydrodynamic limit has been derived therein, but the case of jumps with finite mean remains open and is expected to be tightly linked to the TASEP studied in [Xu22],

In this paper, we provide a short proof of the exponential equivalence between processes in contact with weak reservoirs and with impermeable boundaries respectively for a large class of models. As an immediate consequence, we can extend the result from [Xu22] to the TASEP without vanishing viscocity and derive the hydrodynamic behaviour of the model studied in [GKX23+].

The remainder of the paper is divided into three sections: the presentation of the main result, its applications, and the proof. A generalization is discussed in Appendix A.

2 Notation and main result

In this section, we will restrict ourselves to the special case of exclusion processes on $\mathbb{Z}$. For the more general case, see Appendix A.

Write $\Lambda_N := \{1, \dots, N-1\}$ for the bulk and $\Omega_N := \{0,1\}^{\Lambda_N}$ for the space of configurations. We denote by $\eta^{x,y}$ the exchange of the sites x and y, and by η^x the flip of the site x in the sense that

$$\eta^{x,y}_z := \begin{cases} \eta_y & \text{if } z = x \\ \eta_x & \text{if } z = y \\ \eta_z & \text{otherwise} \end{cases} \qquad \text{and} \qquad \eta^x_z := \begin{cases} 1-\eta_x & \text{if } z = x \\ \eta_z & \text{otherwise} \end{cases}.$$

Next, define the generators

$$\begin{aligned}
\mathcal{L}^N_{bulk} f(\eta) &:= \sum_{x,y\in\Lambda_N} p(x,y)\cdot\eta_x(1-\eta_y)\cdot\Big(f(\eta^{x,y})-f(\eta)\Big),\\
\mathcal{L}^N_{influx} f(\eta) &:= \sum_{x\notin\Lambda_N}\sum_{y\in\Lambda_N} p(x,y)\cdot\alpha(x)(1-\eta_y)\cdot\Big(f(\eta^{x})-f(\eta)\Big),\\
\mathcal{L}^N_{outflux} f(\eta) &:= \sum_{x\in\Lambda_N}\sum_{y\notin\Lambda_N} p(x,y)\cdot\eta_x\beta(y)\cdot\Big(f(\eta^{x})-f(\eta)\Big),
\end{aligned}$$

where $p:\mathbb{Z}^2\to[0,+\infty)$ is a jump kernel and $\alpha,\beta:\mathbb{Z}\to[0,+\infty)$ are bounded. Here and in the following, we will use the shortcut $x\notin\Lambda_N$ to mean $x\in\mathbb{Z}\setminus\Lambda_N$.

Definition 2.1. 1 *An* exclusion process in contact with impermeable boundaries *is an Ω_N-valued Markov process with generator $\mathcal{L}^N_{bulk}$.*

2 *An* exclusion process in contact with weak reservoirs *is an Ω_N-valued Markov process with generator $\mathcal{L}^N := \mathcal{L}^N_{bulk} + \theta(N)\left(\mathcal{L}^N_{influx}+\mathcal{L}^N_{outflux}\right)$ for some $\theta(N)=o(1)$.*

In the following, we will identify Ω_N-valued processes with the corresponding measure-valued process via the map

$$\pi^N:\Omega_N\to\mathcal{M}_F([0,1]),\qquad \eta\mapsto\frac{1}{N-1}\sum_{x\in\Lambda_N}\eta_x\delta_{x/N},$$

where $\mathcal{M}_F([0,1])$ denotes the space of finite measures on $[0,1]$. To compare the different processes, we endow the space of measure-valued càdlàg processes $\mathbb{D}_{[0,T]}\big(\mathcal{M}_F([0,1])\big)$ with the Skorokhod topology induced by the weak topology on $\mathcal{M}_F([0,1])$. In the following, we will write d_{J_1} for any complete metric inducing this topology.

Definition 2.2 (Exponential equivalence). *We say that two sequences of probability measures $\mathbb{Q}^N_1$ and $\mathbb{Q}^N_2$ on $\mathbb{D}_{[0,T]}\big(\mathcal{M}_F([0,1])\big)$ are* exponentially equivalent *if there exists*

i) a sequence of (abstract) probability spaces $(\mathcal{X}^N,\mathcal{F}^N,\mathcal{Q}^N)$ and

ii) a sequence of random variables $\pi^N_1,\pi^N_2:\mathcal{X}^N\to\mathbb{D}_{[0,T]}\big(\mathcal{M}_F([0,1])\big)$ with respective laws $\mathbb{Q}^N_1$ and $\mathbb{Q}^N_2$

such that the sets $\{d_{J_1}(\pi^N_1,\pi^N_2)>\varepsilon\}$ are $\mathcal{F}^N$-measurable and

$$\limsup_N\frac{1}{N}\ln\mathcal{Q}^N\left\{\Delta(\pi^N_1,\pi^N_2)>\varepsilon\right\}=-\infty$$

for all $\varepsilon>0$.

In order to state the main result, we introduce the notion of a time change. We say that we *speed up* a process $(\pi_t)_{t\geq 0}$ by a factor ϑ if we consider the process $(\pi_{\vartheta t})_{t\geq 0}$ instead. In the case of a Markov process with generator $\mathcal{L}$, this is equivalent to considering the Markov process with generator $\vartheta\mathcal{L}$.

Theorem 2.3. *If both process are sped up by some factor $\vartheta(N)$, then the exclusion processes in contact with impermeable boundaries is exponentially equivalent to the exclusion process in contact with weak boundaries provided*

$$\vartheta(N)\theta(N)\sum_{x\in\Lambda_N, y\notin\Lambda_N}\Big(p(x,y)+p(y,x)\Big)=o(N).$$

Under the assumptions of the theorem, the two processes are indistinguishable at the level of large deviations. In particular, both have the same hydrodynamic behaviour.

3 Applications

In this section, we discuss applications of the exponential equivalence to prove the hydrodynamic behaviour and the large deviations for asymmetric exclusion processes.

3.1 The TASEP in contact with weak reservoirs

The recent paper [Xu22] considers the nearest-neighbour TASEP in contact with weak reservoirs given by the above through the choice $p(x,y)=\mathbb{1}_{y=x+1}$, and α,β some (possibly time-dependent, but locally) bounded rates. Due to the asymmetry, the process evolves on the timescale $\vartheta(N)=N$. This means in particular that

$$\theta(N)\vartheta(N)\sum_{x\in\Lambda_N, y\notin\Lambda_N}\Big(p(x,y)+p(y,x)\Big)=\mathcal{O}\big(N\theta(N)\big)=o(N),$$

verifying the condition of Theorem 2.3. To include time-dependent rates, we make use of the more general result discussed at the end of Appendix A.

Theorem 2.3, then, ensures that the process is exponentially equivalent to the (nearest-neighbour) TASEP with impermeable boundaries. We can extend the latter to the left by zeros and to the right by ones without changing the dynamics, transforming it into the TASEP on $\mathbb{Z}$. Using [Sep98] (see also [Var04, Theorem 2.1]), we obtain the hydrodynamic behaviour. The large deviations are considered in [Var04] and completed in [QT22, Main Theorem]. Through the exponential equivalence, this translate directly to the TASEP in contact with weak reservoirs.

In contrast to [Xu22], this result does not necessitate the addition of a vanishing viscocity to the model in order to prove the hydrodynamic behaviour.

3.2 The ALJEP in contact with weak reservoirs

Similarly, we may consider the jump kernel

$$p(x,y)=\frac{\mathbb{1}_{y>x}}{|y-x|^{1+\gamma}}$$

for some $\gamma > 0$. As a process on $\mathbb{Z}$, this is a special case of the model from [SS18]. This Asymmetric Long Jump Exclusion Process (ALJEP) in contact with reservoirs has first been introduced in [GKX23+].

The ALJEP on the whole of $\mathbb{Z}$ undergoes a phase transition at $\gamma = 1$: for $\gamma \in (0, 1)$, the mean jump size is infinite and the long range effects remain at the macroscopic level; for $\gamma > 1$, the mean becomes finite and the model behaves like the TASEP. The phase transition can also be read off of the correct time scales given by

$$\vartheta(N) = \begin{cases} N^\gamma & \text{if } \gamma < 1 \\ \frac{N}{\ln N} & \text{if } \gamma = 1 \\ N & \text{if } \gamma > 1 \end{cases},$$

see [SS18] for details. In the setting with reservoirs, more phase transitions (depending on the reservoir strength) are expected, see [GKX23+, Figure 1].

In the case of weak boundaries, we may check that for every $\gamma > 0$ and the above choice of $\vartheta(N)$, one has

$$\vartheta(N)\theta(N) \sum_{x \in \Lambda_N, y \notin \Lambda_N} \Big(p(x, y) + p(y, x)\Big) = o(N),$$

so that Theorem 2.3 is applicable. As in the case of the TASEP, we can extend the ALJEP with impermeable boundaries to the left with zeros and the right with ones to recover the ALJEP on $\mathbb{Z}$. However, this does not yet allow us to deduce the hydrodynamical behaviour of the ALJEP with weak boundaries. Indeed, the proof in [SS18] relies on the fact that the initial profile has the same asymptotic density $\rho^* \in (0, 1)$ in both directions. This is violated here in multiple ways as the asymptotic density to the left is 0 and the asymptotic density to the right is 1.

In the long-range regime $\gamma \in (0, 1)$, we can instead use the reciprocal direction and deduce the hydrodynamical behaviour of the ALJEP on $\mathbb{Z}$ for any initial profiles ρ_0 with compact support. Indeed, this follows from the study of the ALJEP in contact with weak reservoirs in [GKX23+]. For the short-range regime $\gamma > 1$, no results are available for the case with weak reservoirs. Instead, we will make use of the finite speed of propagation of mass in the Burger's equation to extend the result from [SS18] to suitable initial profiles. This is enough to deduce the hydrodynamic behaviour of the ALJEP in contact with weak reservoirs at least when a) the initial profile ρ_0 is continuous on $[0, 1]$ and satisfies $\rho_0(0) = 0$ and $\rho_0(1) = 1$, and b) the initial configuration is distributed as a product measure with profile ρ_0.

Lemma 3.1. *Let $\rho_0 \in C(\mathbb{R}; [0, 1])$ be a continuous profile satisfying $\rho_0(x) = 0$ (resp. $\rho_0(x) = 1$) for x small (resp. large) enough, and let μ_0^N be the product measure on $\mathbb{Z}$ with marginals $\mu_0^N(\eta_x = 1) = \rho_0(x/N)$. Then, the ALJEP on $\mathbb{Z}$ with initial configuration distributed as μ_0^N satisfies a law of large numbers with hydrodynamic limit given by the solution ρ to Burger's equation [SS18, Equation (3.2)] with initial value ρ_0.*

Proof. See Appendix B. □

4 Proof of the main result

For every $N \in \mathbb{N}$, let μ_N be a measure on Ω_N. An adaptation of the usual misanthrope (or: attractive) coupling constructs processes $\eta^N, \tilde{\eta}^N$ and $\hat{\eta}^N$ with initial distribution μ_N and respective generators

$$\vartheta(N)\mathcal{L}^N_{bulk}, \quad \vartheta(N)\mathcal{L}^N_{bulk} + \theta(N)\vartheta(N)\mathcal{L}^N_{influx} \quad \text{and} \quad \vartheta(N)\mathcal{L}^N$$

satisfying

$$\eta_t^N \le \tilde{\eta}_t^N \qquad \text{and} \qquad \hat{\eta}_t^N \le \tilde{\eta}_t^N$$

for all $t \ge 0$. See [Coc85] for details of this coupling, or *e.g.* [SS18, Section 9] for an English version.

Write $\pi_t^N := \pi^N(\eta_t^N)$, $\tilde{\pi}_t^N := \pi^N(\tilde{\eta}_t^N)$ and $\hat{\pi}_t^N := \pi^N(\hat{\eta}_t^N)$ for the corresponding measure-valued processes. Since they are all atomic, the ordering implies

$$N\|\tilde{\pi}_t^N - \pi_t^N\|_{TV} = |\tilde{\eta}_t^N| - |\eta_t^N| \quad \text{and} \quad N\|\tilde{\pi}_t^N - \hat{\pi}_t^N\|_{TV} = |\tilde{\eta}_t^N| - |\hat{\eta}_t^N|,$$

where we write $|\eta| := \sum_{x\in\Lambda_N} \eta_x$ for the total mass of a configuration $\eta \in \Omega_N$.

Next, note that the Lévy-Prokhorov metric induces the topology of weak convergence and can be bounded from above by the distance in total variation, see *e.g.* [GS02, Figure 1]. In particular, using the fact that $\tilde{\pi}$ takes into account the influx of particles, we have

$$\left\{d_{J_1}(\tilde{\pi}^N, \pi^N) > \varepsilon\right\} \subseteq \left\{\sup_{t\in[0,T]} |\tilde{\eta}_t^N| - |\eta_t^N| > \varepsilon N\right\} = \left\{|\tilde{\eta}_T^N| - |\eta_T^N| > \varepsilon N\right\}$$

and similarly for $\{d_{J_1}(\tilde{\pi}^N, \hat{\pi}^N) > \varepsilon\}$. Hence, the proof reduces to the following lemma.

Lemma 4.1 (No loss of mass)**.** *The events* $\{|\tilde{\eta}_T^N| - |\eta_T^N| \ge \varepsilon N\}$ *and* $\{|\tilde{\eta}_T^N| - |\eta_T^N| \ge \varepsilon N\}$ *are superexponentially unlikely for every* $\varepsilon > 0$*.*

Proof. As both are analogous, we will concentrate on the first set only. Since the change in mass can come only from the influx of particles, it suffices to prove that the probability of εN particles entering up to time T is superexponentially small. Note that the number of particles is bounded from above by a Poisson number P with parameter

$$\lambda_N := T\|\alpha\|_\infty \cdot \theta(N)\vartheta(N) \sum_{x\in\Lambda_N, y\notin\Lambda_N} p(y,x) = o(N).$$

The usual Chernoff bound provides us with the estimate

$$\mathbb{P}(P \ge \lambda_N + x) \le \exp\left(-\frac{x^2}{\lambda_N} \cdot h\left(\frac{x}{\lambda_N}\right)\right)$$

for some function $h(u) = \frac{(1+u)\ln(1+u)-u}{u^2}$, vanishing at infinity like $\frac{\ln u}{u}$, see *e.g.* [Can19]. In particular,

$$\mathbb{P}\left(\exists t \in [0,T] \;:\; |\tilde{\eta}_t^N| - |\eta_t^N| \ge \varepsilon N\right) \le \exp\left(-\Theta\left(\varepsilon N \cdot \ln\left(\frac{\varepsilon N}{o(N)}\right)\right)\right)$$

which is superexponentially small. □

A Generalization to misanthrope processes

The proof in Section 4 relies heavily on the attractive coupling of the different processes. From [Coc85], it is known that this coupling can be constructed for the large class of *misanthrope processes*. These cover many models of interest, including the exclusion and zero range processes.

Let $k \in \mathbb{N}\cup\{\infty\}$ denote the maximal number of particles allowed at a site and set $S_k := \{0,\ldots,k\}$ or $S_\infty := \mathbb{N}_0$ accordingly.

For $N \in \mathbb{N}$, $N \ge 2$, define the *bulk* $\Lambda_N := \{1,\ldots,N-1\}$ and the space of configurations $\Omega_N := S_k^{\Lambda_N}$. For a configuration $\eta \in \Omega_N$ and sites $x, y \in \Lambda_N$, we define the three actions $\eta \mapsto \eta^{x\to y}$, $\eta \mapsto \eta^{x\uparrow}$ and $\eta \mapsto \eta^{x\downarrow}$ as follows:

1 if $\eta_x = 0$ or $\eta_y = k$, set $\eta^{x\to y} := \eta$, otherwise set $\eta^{x\to y} := \eta - \delta_x + \delta_y$, *i.e.*

$$\eta_z^{x\to y} := \begin{cases} \eta_x - 1 & \text{if } z = x \\ \eta_y + 1 & \text{if } z = y \\ \eta_z & \text{otherwise} \end{cases};$$

2 if $\eta_x = k$, set $\eta^{x\uparrow} := \eta$, otherwise set $\eta^{x\uparrow} := \eta + \delta_x$;

3 if $\eta_x = 0$, set $\eta^{x\downarrow} := \eta$, otherwise set $\eta^{x\downarrow} := \eta - \delta_x$.

For functions $f : \Omega_N \to \mathbb{R}$, define the generators

$$\mathcal{L}_{bulk}^N f(\eta) := \sum_{x,y\in\Lambda_N}^{N-1} p(x,y)\cdot b_{bulk}(\eta_x,\eta_y)\cdot\Big(f(\eta^{x,y}) - f(\eta)\Big)$$

and

$$\mathcal{L}_{influx}^N f(\eta) := \sum_{x\in\Lambda_N}\sum_{y\notin\Lambda_N} p(y,x)\cdot b_{influx}(y,\eta_x)\cdot\Big(f(\eta^{x\uparrow}) - f(\eta)\Big),$$
$$\mathcal{L}_{outflux}^N f(\eta) := \sum_{x\in\Lambda_N}\sum_{y\notin\Lambda_N} p(x,y)\cdot b_{outflux}(\eta_x,y)\cdot\Big(f(\eta^{x\downarrow}) - f(\eta)\Big),$$

where

i) p is a jump kernel,

ii) $b_{bulk} : S_k^2 \to [0,+\infty)$ is non decreasing in its first variable, non increasing in its second variable and satisfies $b(n,m) = 0$ if and only if $n = 0$ or $m = k$,

iii) $b_{influx} : \mathbb{Z}\times S_k \to (0,+\infty)$ and $b_{outflux} : S_k\times\mathbb{Z} \to [0,+\infty)$ are bounded.

We will assume that b_{bulk} is such that the following Markov processes exist. This is trivially satisfied when b_{bulk} is bounded, *e.g.* if $k \neq \infty$.

Definition A.1. *A* misanthrope process in contact with impermeable boundaries *is defined through the generator* $\mathcal{L}_{bulk}^N$*, whereas a* misanthrope process in contact with weak reservoirs *has the generator*

$$\mathcal{L}^N := \mathcal{L}_{bulk}^N + \theta(N)\left(\mathcal{L}_{influx}^N + \mathcal{L}_{outflux}^N\right)$$

for some $\theta(N) \to 0$*.*

As before, we will identify processes with values in the space of configurations Ω_N with the corresponding process with values in the space of measures via the map

$$\pi^N : \Omega_N \longrightarrow \mathcal{M}_F([0,1]), \quad \eta \mapsto \frac{1}{N}\sum_{x\in\Lambda_N}\eta_x\delta_{x/N}.$$

Theorem A.2. *If both processes are sped up by a factor* $\vartheta(N)$*, then the misanthrope process in contact with weak reservoirs is exponentially equivalent to the misanthrope process in contact with impermeable boundaries provided*

$$\vartheta(N)\theta(N)\sum_{x\in\Lambda_N, y\notin\Lambda_N}\Big(p(x,y) + p(y,x)\Big) = o(N).$$

Proof. The proof is exactly as in Section 4. The only difference is that we replace $\|\alpha\|_\infty$ by $\|b_{influx}\|_\infty$. □

Note that the proof does not depend on the underlying space. In particular, the result can be extended to misanthrope processes on any lattice, including $\mathbb{Z}^d$. Furthermore, it can be generalized to time-dependent interactions with the reservoirs as long as they are locally L^1 in time in the sense that $b_{influx} \in L^1_{loc}([0,+\infty); L^\infty(\mathcal{Z} \times S_k))$ and similarly for $b_{outflux}$. In this case $T\|b_{influx}\|_\infty$ is to be replaced by $\int_0^T \|b_{influx}(t)\|_\infty \ \mathrm{d}t$.

Although pathological counter-examples can be constructed, the statement of Appendix A.2 is sharp in most situations. This includes also the symmetric case, see for example [Sco21] (and particularly [BCG+23] therein) for the treatment of the symmetric exclusion process with long jumps in contact with reservoirs. Unfortunately, it is generally equally hard to derive the hydrodynamic behaviour of the process in contact with weak reservoirs or in contact with impermeable boundaries, so that the result only shortens proofs by providing a general argument for ignoring boundary terms.

In the context of totally asymmetric processes, however, Appendix A.2 provides a shortcut for proving the hydrodynamic behaviour (and even higher order behaviour as the fluctuations or the large deviations) as shown in Section 3.

B Proof of Lemma 3.1

For simplicity, assume that $\rho_0(x) = 0$ for $x \leq 0$ and $\rho_0(x) = 1$ for $x \geq 1$. Let $(\rho_0^{m,\ell})_{m,\ell\in\mathbb{N}}$ be a family of continuous functions satisfying $\rho_0^{m,\ell}|_{[-m,\ell]} = \rho_0|_{[-m,\ell]}$ and $\rho_0^{m,\ell}(x) = \frac{1}{2}$ on $\mathbb{R} \setminus [-2m, 2\ell]$. We may choose the family such that it is pointwise non increasing in m and pointwise non decreasing in ℓ. Write $\mu_0^{N,m,\ell}$ for the corresponding product measure on $\{0,1\}^{\mathbb{Z}}$. Furthermore, denote by $\mu_0^{N,\ell}$ the measures obtained from the pointwise limit $\lim_m \rho_0^{m,\ell}$ which is vanishes to the left of 0.

Using the attractive coupling, we may construct the ALJEPs $\eta^{N,m,\ell}$ and $\eta^{N,\ell}$ on $\mathbb{Z}$ started from $\mu_0^{N,m,\ell}$ on a common probability space such that

$$\eta_t^{N,\ell} \leq \eta_t^{N,m,\ell} \leq \eta_t^{N,m',\ell} \qquad \text{and} \qquad \eta_t^{N,\ell} \leq \eta_t^{N,\ell'} \leq \eta_t^{N},$$

for any $m' \leq m$, $\ell \leq \ell'$ and $t \geq 0$, a.s.

Similarly to the proof of Theorem 2.3, it is enough to show that both

$$\limsup_m \sup_\ell \limsup_N \mathbb{P}^N \left(\sum_{x\geq 1} \eta_T^{N,m,\ell}(x) - \eta_T^{N,\ell}(x) > \varepsilon N \right) = 0$$

and

$$\lim_\ell \limsup_N \mathbb{P}^N \left(\sum_{x=1}^{N-1} \eta_T^{N}(x) - \eta_T^{N,\ell}(x) > \varepsilon N \right) = 0.$$

As both quantities are similar, we will concentrate on the former. By construction, $\sum_{x\geq 1} \eta_t^{N,\ell}(x) =$

$\sum_{x\geq 1}\eta_0^{N,\ell}(x)=\sum_{x\geq 1}\eta_0^{N,m,\ell}(x)$. In particular, it is enough to bound

$$\begin{aligned}
&\mathbb{P}^N\left(\sum_{x\geq 1}\eta_T^{N,m,\ell}(x)-\eta_0^{N,m,\ell}(x)>\varepsilon N\right)\\
&\quad\leq\frac{\mathbb{E}^N\left[\sum_{x\geq 1}\eta_T^{N,m,\ell}(x)-\eta_0^{N,m,\ell}(x)\right]}{\varepsilon N}\\
&\quad\leq\frac{1}{\varepsilon N}\int_0^T\mathbb{E}^N\left[N\sum_{x\geq 1}\sum_{y\leq 0}p(y,x)\cdot\eta_t^{N,m,\ell}(y)\left(1-\eta_t^{N,m,\ell}(x)\right)\right]\,\mathrm{d}t\\
&\quad=\mathcal{O}\left(\frac{1}{\varepsilon}\int_0^T\sum_{y\leq 0}\frac{1}{|y|^\gamma}\cdot\mathbb{E}^N\left[\eta_t^{N,m,\ell}(y)\right]\,\mathrm{d}t\right)\\
&\quad=\mathcal{O}\left(\frac{T}{\varepsilon}\left(\frac{mN}{2}\right)^{1-\gamma}+\frac{mN}{2\varepsilon}\int_0^T\int_{-m/2}^0\rho_t^{m,\ell}(u)\,\mathrm{d}u\,\mathrm{d}t\right),
\end{aligned}$$

where $\rho^{m,\ell}$ is the hydrodynamic limit of the family $(\eta_t^{N,m,\ell})_{N\geq 2}$, which is a solution to Burger's equation with initial profile $\rho_0^{m,\ell}$, see [SS18]. The first term vanishes in N, whereas the latter becomes constantly 0 independently of N and ℓ for m large enough depending only on T.

References

[BCG+23] Cédric Bernardin, Pedro Cardoso, Patrícia Gonçalves, and Stefano Scotta. Hydrodynamic limit for a boundary driven super-diffusive symmetric exclusion. *Stochastic Processes and their Applications*, 165:43–95, 2023. DOI: 10.1016/j.spa.2023.08.002.

[Can19] Clément Canonne, 2019. URL: http://www.cs.columbia.edu/~ccanonne/files/misc/2017-poissonconcentration.pdf. Accessed: August 18, 2023.

[Coc85] Christiane Cocozza-Thivent. Processus des misanthropes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 70(4):509–523, December 1985. ISSN: 1432-2064. DOI: 10.1007/BF00531864.

[GKX23+] Patrícia Gonçalves, Julian Kern, and Lu Xu. An asymmetric exclusion process with long jumps in contact with reservoirs. Forthcoming.

[GS02] Alison L. Gibbs and Francis Edward Su. On choosing and bounding probability metrics, 2002. arXiv: math/0209021.

[QT22] Jeremy Quastel and Li-Cheng Tsai. Hydrodynamic large deviations of tasep, 2022. arXiv: 2104.04444.

[Sco21] Stefano Scotta. *Asymptotic behavior of the boundary driven exclusion with long jumps*. PhD thesis, 2021.

[Sep98] Timo Seppäläinen. Coupling the totally asymmetric simple exclusion process with a moving interface. *Markov Processes and Related Fields*, 4, January 1998.

[SS18] Sunder Sethuraman and Doron Shahar. Hydrodynamic limits for long-range asymmetric interacting particle systems. *Electronic Journal of Probability*, 23(none):1–54, 2018. DOI: 10.1214/18-EJP237.

[Var04] Srinivasa R. S. Varadhan. Large deviations for the asymmetric simple exclusion process. In *Stochastic analysis on large scale interacting systems*. Volume 39, Adv. Stud. Pure Math. Pages 1–27. Math. Soc. Japan, Tokyo, 2004. ISBN: 4-931469-24-8. DOI: `10.2969/aspm/03910001`.

[Xu22] Lu Xu. Hydrodynamics for one-dimensional asep in contact with a class of reservoirs. *Journal of Statistical Physics*, 189(1):1, July 2022. ISSN: 1572-9613. DOI: `10.1007/s10955-022-02963-x`.