Convergence to a non-trivial equilibrium for two-dimensional catalytic super-Brownian motion

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Abstract

In contrast to the classical super-Brownian motion (SBM), the SBM $(X_t^\varrho)_{t\geq 0}$ in a super-Brownian medium ϱ (constructed in [DF96a]) is known to be persistent in all three dimensions of its non-trivial existence: The full intensity is carried also by all longtime limit points ([DF96a, DF96b, EF96]). Uniqueness of the accumulation point, however, has been shown so far only in dimensions $d = 1$ and $d = 3$ ([DF96a, DF96b]). Here we fill this gap and show that convergence also holds in the critical dimension $d = 2$. We identify the limit as a random multiple of Lebesgue measure.

Our main tools are a self-similarity of X^ϱ in $d = 2$ and the fact that the medium has "gaps" in the space-time picture. The self-similarity implies that persistent convergence of X_t^ϱ as $t \to \infty$ is equivalent to the absolute continuity of X_t^ϱ at a fixed time $t > 0$. Absolute continuity however will be obtained via the fact that in absence of the catalytic medium, X^ϱ is smoothed according to the heat flow.

Contents

1 Introduction

Consider continuous super-Brownian motion $(\varrho_t)_{t\geq 0}$ in R^d. Roughly speaking, the *super-Brownian motion* X^ϱ *in the super-Brownian medium* ϱ is a continuous super-Brownian motion in R^d with local branching rate "proportional to" ϱ. A construction can be found in [DF96a].

As a rule, both the *catalyst* process ϱ and the *reactant* process X^ϱ will be started at time zero from multiples $\ell_c = i_c\ell$, $i_c > 0$, and $\ell_r = i_r\ell$, $i_r > 0$, respectively, of the normed Lebesgue measure ℓ on R^d.

In [DF96a] also the study of longtime behavior of X^ϱ was initiated, and then continued in [DF96b] and [EF96]. From these papers it is known that X^ϱ is *persistent* in all three dimensions $d \leq 3$ of its non-trivial existence. (In $d = 3$ the catalyst process ϱ was actually started from its stationary distribution rather than from ℓ_c at time zero; this simplification is of course not possible in lower dimensions where ϱ clusters in the longtime limit, hence dies out locally.) Here persistence means that all weak limit points of X_T^ϱ as $T \to \infty$ have the full intensity measure ℓ_r again. In dimensions one and three the stronger result of persistent *convergence* has been shown in ([DF96a, DF96b]). For dimension $d = 2$ persistence of X^ϱ has been shown in ([EF96]). The approach of Etheridge and Fleischmann is to show the relative compactness of the set

$$\left\{ \mathbb{P}_{\ell_c} \left((P^\varrho_{0,\ell_r}(X^\varrho_T)^2) \in (\cdot) \right) : \; T > 0 \right\}$$

of laws of random second moments by p.d.e. methods. However, the uniqueness of the limit point, and hence convergence, has not been established yet. The *main purpose* of this paper is to show (persistent) convergence in dimension $d = 2$ (Theorem 6 on p.14) with a proof that is independent of [EF96]. We do so by first showing that for fixed time $T > 0$, the random measure X_T^ϱ has a.s. a density ζ_T^ϱ w.r.t. the Lebesgue measure (Theorem 5). We make use of the self-similarity of X^ϱ in dimension $d = 2$ to infer the convergence of X_T^ϱ to a random multiple of Lebesgue measure as $T \to \infty$.

Heuristically, the *catalyst* process ϱ can be thought of as a large number of small particles, all moving around in R^d according to independent Brownian motions. The particles die with a high rate and are replaced (independently) at the location of their death by zero or two offspring, each possibility occurring with probability $\frac{1}{2}$ (critical binary branching). The offspring continue to evolve in the same manner as their parent. In this "classical" setting, the rate at which a particle dies is proportional to some positive constant γ called the *branching rate*. In other words, ϱ arises as a diffusion approximation to a critical binary branching Brownian motion with constant branching rate.

Concerning the *reactant* X^ϱ, the heuristic picture is the same except that the reactant particles die only when they are in contact with a catalyst. Compare with a *chemical reaction diffusion system* where the catalyst itself may vary in time and space and may only be present in some localized regions such as networks of filaments. Here we are interested in the case when the catalyst itself is a super-Brownian motion (SBM) ϱ with constant branching rate $\gamma > 0$. As mentioned above, we will take both initial states ϱ_0 and X_0^ϱ to be multiples of the Lebesgue measure, ℓ_c and ℓ_r, respectively.

A strong approach to this *"one-way interaction"* model can be made by means of Dynkin's additive functional approach to superprocesses ([Dyn91]). In fact, given the medium ϱ, an intrinsic X^ϱ–particle (reactant) following a Brownian path W branches according to the clock given by the *collision local time,* $L_{[W,\varrho]}(\mathrm{d}s)$, of W with ϱ ([BEP91]). Somewhat more formally:

$$L_{[W,\varrho]}(\mathrm{d}s) := \mathrm{d}s \int \varrho_s(\mathrm{d}y)\, \delta_y(W_s). \tag{1}$$

These collision local times $L_{[W,\varrho]}$ make sense non-trivially in dimensions $d \leq 3$ ([EP94]), although the measures $\varrho_s(\mathrm{d}y)$ are singular for $d \geq 2$ ([DH79]). For this reason, in dimensions $d \leq 3$ the *catalytic SBM* X^ϱ could successfully be constructed in [DF96a] as a continuous measure-valued (time-inhomogeneous) Markov process $(X^\varrho, P^\varrho_{r,\mu})$, given the catalyst process ϱ (*quenched approach*). In particular, $P^\varrho_{0,\ell_\mathrm{r}}$ denotes the law of the process X^ϱ (for ϱ fixed) and $\mathbb{P}_{\ell_\mathrm{c}}$ the distribution of ϱ if at time zero we start X^ϱ in ℓ_r and ϱ in ℓ_c a.s. Averaging the random laws $P^\varrho_{0,\ell_\mathrm{r}}$ by means of $\mathbb{P}_{\ell_\mathrm{c}}$ gives the *annealed* distribution

$$\mathcal{P}_{\ell_\mathrm{c},\ell_\mathrm{r}} := \mathbb{P}_{\ell_\mathrm{c}} P^\varrho_{0,\ell_\mathrm{r}}$$

of X^ϱ. (In $d > 3$, the model X^ϱ can be thought of as being only the heat flow.)

We are interested mainly in the *critical dimension* $d = 2$. Here the catalyst ϱ_T dies out locally as $T \to \infty$, and in the large regions without catalyst only the smoothing heat flow acts on the reactant X^ϱ. On the other hand, a finite window of observation will be visited by increasingly large catalytic clumps at arbitrarily large times (recall that the time averaged two-dimensional catalyst ϱ has a proper random limit despite local extinction, see, e.g., [FG86]). These clumps mainly act as killers for the reactant (recall that critical binary branching with infinite rate degenerates to a pure killing). But according to the main result of [EF96, Theorem 1], the smoothing effect in catalyst free regions wins this competition with the killing, leading to persistence: The intensity measure ℓ_r of X^ϱ_T is preserved also for all accumulation points (in law) as $T \to \infty$.

The *main result* (Theorem 6) of the present note is that, in this critical dimension two, X^ϱ_T has a *unique* limit law as $T \to \infty$, which moreover is given by the distribution of some random multiple $\zeta^\varrho_1 \ell$ of Lebesgue measure ℓ, where ζ^ϱ_1 has non-zero variance. (Note that by persistence the expectation of ζ^ϱ_1 is given by the intensity i_r of the reactant X^ϱ_T at any finite time T.) This convergence statement is in fact true with respect to the *annealed* law $\mathcal{P}_{\ell_\mathrm{c},\ell_\mathrm{r}} = \mathbb{P}_{\ell_\mathrm{c}} P^\varrho_{0,\ell_\mathrm{r}}$. But concerning the *quenched* model, the convergence also holds formulated in terms of convergence in distribution of the *random laws* $P^\varrho_{0,\ell_\mathrm{r}}$. However, for fixed medium ϱ one cannot expect convergence (since the mentioned random ergodic limit theorem for ϱ is not an almost sure statement).

Recall that in dimension *one*

$$X^\varrho_T \underset{T\uparrow\infty}{\longrightarrow} \ell_\mathrm{r}, \quad \text{in } P^\varrho_{0,\ell_\mathrm{r}}\text{–probability}, \quad \text{for } \mathbb{P}_{\ell_\mathrm{c}}\text{–almost all } \varrho, \tag{2}$$

([DF96a, Theorem 51]). Thus in dimension two we have additionally some randomness in the limit *and* the convergence type, caused by the fact that the catalyst process ϱ dies out locally only in probability (opposed to dimension one where with $\mathbb{P}_{\ell_\mathrm{c}}$–probability one any finite window is eventually vacant). In particular, the limit law of X^ϱ_T is *not* spatially ergodic.

Our proof is based on the *self-similarity* of the two-dimensional catalytic

SBM $X = X^{\varrho}$:

$$K^{-1} X_{KT}(K^{1/2}\ \cdot\ ) \stackrel{\mathcal{L}}{=} X_T\,, \qquad T, K > 0, \tag{3}$$

with respect to the annealed distribution $\mathcal{P}_{\ell_c,\ell_r}$ and also w.r.t. the random laws P^{ϱ}_{0,ℓ_r} ([DF96b, Proposition 13]). Here coincidence w.r.t. the random laws P^{ϱ}_{0,ℓ_r} formally means that

$$\mathsf{P}_{\ell_c}\Big[P^{\varrho}_{0,\ell_r}\Big[K^{-1}X_{KT}(K^{1/2}\ \cdot\ ) \in (\cdot)\Big] \in (\cdot)\Big] \;=\; \mathsf{P}_{\ell_c}\Big[P^{\varrho}_{0,\ell_r}[X_T \in (\cdot)] \in (\cdot)\Big]\,.$$

In particular, the self-similarity implies that

$$X_T \;\stackrel{\mathcal{L}}{=}\; T X_1(T^{-1/2}\ \cdot\ ). \tag{4}$$

But as $T \to \infty$, the r.h.s. is heuristically the *asymptotic density* $\xi^{\varrho}_1(0)$ of the random measure X_1 at the origin 0, times the Lebesgue measure ℓ. Hence, as noted in [DF96b, Remark 14], the self-similarity of X relates the question of persistent convergence of X_T as $T \to \infty$ and absolute continuity of X_1.

The *key* to our results is indeed the fact (Proposition 3) that in dimensions two and three, *for* P_{ℓ_c}*–almost all* ϱ and Lebesgue almost all space points z, there is an (infinitely divisible) element $\xi^{\varrho}_1(z) \geq 0$ in the Lebesgue space $L^2 := L^2(P^{\varrho}_{0,\ell_r})$ satisfying $P^{\varrho}_{0,\ell_r}\xi^{\varrho}_1(z) \equiv i_r$ and such that the L^2–convergence

$$\Big\langle T^{d/2} X^{\varrho}_1\big(z + T^{-1/2}\ \cdot\ \big), \varphi\Big\rangle \underset{T\uparrow\infty}{\longrightarrow} \xi^{\varrho}_1(z)\|\varphi\|_1, \qquad \varphi \in \mathcal{C}^{\mathrm{comp}}_+, \tag{5}$$

takes place. Here $\mathcal{C}^{\mathrm{comp}}_+$ is the cone of all non-negative continuous functions on R^d with compact support, $\langle \mu, \varphi\rangle$ abbreviates the integral $\int \mu(\mathrm{d}x)\,\varphi(x)$, and $\|\cdot\|_1$ denotes the $L^1(\ell)$–norm. The reason behind this is that in dimensions $d \geq 2$, for P_{ℓ_c}–almost all ϱ and almost all z we find a $\delta = \delta(\varrho, z) > 0$ such that

$$\varrho_s\left(B_\delta(z)\right) = 0, \qquad 1 - \delta \leq s \leq 1, \tag{6}$$

(Proposition 1), where $B_\delta(z)$ denotes an open ball in R^d of radius δ centered at z. Consequently, in a "backward neighborhood" of the time-space point $[1, z]$, the catalyst is absent. Roughly speaking, this implies that in this region only the heat flow acts on the reactant X^{ϱ}. The smoothing effect of the heat flow makes sure that the asymptotic density $\xi^{\varrho}_1(z)$ of the random measure X^{ϱ}_1 at z exists and that it has full expectation i_r.

Now the L^2–convergence (5) implies convergence in P^{ϱ}_{0,ℓ_r}–law of the random measures $T^{d/2}X^{\varrho}_1(z + T^{-1/2}\ \cdot)$ towards the random multiple $\xi^{\varrho}_1(z)\,\ell$ of Lebesgue measure. Since $\xi^{\varrho}_1(z)$ carries the full expectation, the random measure X^{ϱ}_1 is *absolutely continuous* with P^{ϱ}_{0,ℓ_r}–probability one (see [DF95, Basic Lemma 2.7.1]). This is our *second result* (Theorem 5).

Finally, we show in Theorem 9 that for $\mathbb{P}_{\ell_c}$–a.a. ϱ there exists an open set $\mathbf{Z}_t^\varrho \subset \mathsf{R}^d$ such that the field $\{\xi_t^\varrho(z) : z \in \mathbf{Z}_t^\varrho\}$ is locally $L^2(P_{0,\ell_r}^\varrho)$–Lipschitz continuous.

The rest of the note is laid out as follows. In Section 2 we recall the formal characterization of the catalytic SBM X^ϱ, establish the fact that around "most" time space point $[t, z]$ there is no catalytic mass, and provide our key step, the existence of an asymptotic spatial density at those $[t, z]$. Our main results are formulated and proved in the final section.

For background on SBM we recommend [Daw93].

2 Preparations

2.1 Catalytic SBM

First we want to recall the formal characterization of the catalytic SBM in terms of its Laplace transition functional.

Fix a number $p > d$ with d the dimension of R^d, and introduce the reference function

$$\phi_p(x) := \frac{1}{(1 + |x|^2)^{p/2}}, \qquad x \in \mathsf{R}^d. \tag{7}$$

Write $\mathcal{B}_+^p$ for the set of all functions φ on R^d such that $0 \le \varphi \le c_\varphi \, \phi_p$ for some (finite) constant c_φ. Let $\mathcal{M}_p$ denote the set of all (non-negative) measures μ defined on R^d such that $\langle \mu, \phi_p \rangle = \int \mu(\mathrm{d}x)\, \phi_p(x) < \infty$ (*p–tempered measures*). $\mathcal{M}_p$ is endowed with the weakest topology such that the map $\mu \mapsto \langle \mu, \varphi \rangle$ is continuous for $\varphi = \phi_p$ and for each $\varphi \in \mathcal{C}_+^{\mathrm{comp}}$.

Fix a constant $\gamma > 0$. By definition, the *catalyst process* ϱ *with branching rate* γ is a continuous $\mathcal{M}_p$–valued time-homogeneous Markov process $(\varrho, \mathbb{P}_\mu)$ with Laplace transition functional

$$\mathbb{P}_\mu \exp \langle \varrho_t , -\varphi \rangle = \exp \langle \mu, -u(t) \rangle, \qquad t \ge 0, \quad \mu \in \mathcal{M}_p, \quad \varphi \in \mathcal{B}_+^p. \tag{8}$$

Here $u = \{u(t) : t \ge 0\} = \{u(t,x) : t \ge 0,\ x \in \mathsf{R}^d\}$ is the unique non-negative solution to the *basic cumulant equation*

$$\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u - \gamma\, u^2 \quad \text{on } (0, \infty) \times \mathsf{R}^d \tag{9}$$

with *initial* condition $u(0,x) = \varphi(x)$, $x \in \mathsf{R}^d$. (Where needed, 'solution' has to be understood in a *mild* sense.) In other words, ϱ is a continuous (critical) SBM with constant branching rate γ. It serves as our *random medium* (*catalyst*). From now on *assume* $\varrho_0 = i_c \, \ell = \ell_c$ where $i_c > 0$ is a fixed constant, the catalyst's initial density. Note that under $\mathbb{P}_{\ell_c}$ the expectation

$$\mathbb{P}_{\ell_c} \varrho_s \equiv \ell_c \tag{10}$$

of ϱ is constant in time.

In order to characterize the *catalytic* SBM we have, roughly speaking, to replace the constant rate γ in (9) by the (randomly) varying rate $\varrho_t(x)$, where $\varrho_t(x)$ is the *generalized* derivative $\frac{\varrho_t(\mathrm{d}x)}{\mathrm{d}x}(x)$ of the measure $\varrho_t(\mathrm{d}x)$. Because of time-inhomogeneity, it is convenient to write the related formal equation in a *backward setting:*

$$-\frac{\partial}{\partial r}\, v_t^{\varrho}(r,x) \;=\; \frac{1}{2}\,\Delta v_t^{\varrho}(r,x) - \varrho_r(x)\, v_t^{\varrho}(r,x)^2, \tag{11}$$

$0 \le r \le t,\;\; x \in \mathsf{R}^d$. Now the initial condition becomes a *terminal* condition: $v_t^{\varrho}(t) = \varphi$. After a formal integration, we can rewrite (11) rigorously and probabilistically as

$$v_t^{\varrho}(r,x) \;=\; \Pi_{r,x}\left[\varphi(W_t) - \gamma \int_r^t L_{[W,\varrho]}(\mathrm{d}s)\, v_t^{\varrho}(s,W_s)^2\right], \tag{12}$$

$0 \le r \le t,\;\; x \in \mathsf{R}^d$, where $\Pi_{r,x}$ is the law of (standard) Brownian motion W starting at time r from x, and $L_{[W,\varrho]}$ denotes the *collision local time* of W with ϱ, formally introduced in (1). Based on the finite measure case [EP94], in [DF96a, Theorem 42] it was demonstrated that this collision local time $L_{[W,\varrho]}$ makes non-trivially sense for P_{ℓ_c}–a.a. ϱ, namely as a continuous additive functional of Brownian motion W, provided that $d \le 3$. Moreover ([DF96a, Proposition 6]), for t, φ fixed, and P_{ℓ_c}–almost all ϱ, there is a unique non-negative solution v_t^{ϱ} to (12). Finally ([DF96a, § 5.4]), for P_{ℓ_c}–a.a. ϱ, there exists a continuous $\mathcal{M}_p$–valued time-inhomogeneous Markov process $(X^{\varrho}, P^{\varrho}_{r,\mu})$ with Laplace transition functional

$$P^{\varrho}_{r,\mu} \exp\langle X_t^{\varrho}, -\varphi\rangle \;=\; \exp\langle \mu, -v_t^{\varrho}(r)\rangle\,, \tag{13}$$

$0 \le r \le t,\;\; \mu \in \mathcal{M}_p\,,\;\; \varphi \in \mathcal{B}_+^p\,,$ and v_t^{ϱ} the solution to (12). This is the *catalytic SBM* X^{ϱ} with catalyst ϱ, which was intuitively introduced in Section 1. As a rule, we will start X^{ϱ} at time 0 from $X_0^{\varrho} = i_{\mathrm{r}}\,\ell = \ell_{\mathrm{r}}$ with $i_{\mathrm{r}} > 0$ a fixed constant, the reactant's initial density.

Since the branching mechanism is critical, X^{ϱ} has *expectation*

$$P^{\varrho}_{r,\mu}\,\langle X_t^{\varrho}, \psi\rangle \;\equiv\; \langle \mu, S_{t-r}\psi\rangle\,, \qquad \psi \in \mathcal{B}^p, \tag{14}$$

independent of the catalytic medium ϱ. Here $S = \{S_t : t \ge 0\}$ is the semigroup of Brownian motion. In particular,

$$P^{\varrho}_{0,\ell_{\mathrm{r}}}\, X_t^{\varrho} \;\equiv\; \ell_{\mathrm{r}}\,, \qquad \mathsf{P}_{\ell_c}\text{–a.a. } \varrho, \tag{15}$$

independently of time. The *variances* (given ϱ) related to (14) can be expressed by

$$\mathrm{Var}^{\varrho}_{r,\mu}\,\langle X_t^{\varrho}, \psi\rangle \;=\; 2\int \mu(\mathrm{d}x) \int_r^t \mathrm{d}s \int \varrho_s(\mathrm{d}y)\, \mathrm{p}_{s-r}(y-x)\, \big(S_{t-s}\psi\big)^2(y)\,, \tag{16}$$

see [DF96a, formula (95)] (the extension to signed functions ψ is easy to justify). Here p denotes the Brownian transition density function:

$$\mathrm{p}_s(x) \;:=\; (2\pi s)^{-d/2} \exp\Big[-\frac{|x|^2}{2s}\Big], \qquad s>0, \quad x\in \mathsf{R}^d. \tag{17}$$

In particular,

$$\mathrm{Var}_{0,\ell}^{\varrho}\,\langle X_t^{\varrho},\psi\rangle \;=\; 2\,i_{\mathrm{r}}\int_0^t \mathrm{d}s\,\Big\langle \varrho_s\,,(S_{t-s}\psi)^2\Big\rangle\;. \tag{18}$$

2.2 Catalyst free regions close to time-space points

The starting point for our development is the following observation:

Proposition 1 (catalyst free regions close to time-space points) *Let $d\geq 2$, and fix $t>0$. Denote by $\mathbf{Z}_t^{\varrho}$ the open set of all those $z\in\mathsf{R}^d$ such that there exists a $\delta=\delta(\varrho,z)\in(0,t)$ with*

$$\sup_{s\in[t-\delta,t]} \varrho_s\left(B_\delta(z)\right) \;=\; 0. \tag{19}$$

Then

$$\mathbb{P}_{\ell_c}\Big(\ell\left(\mathsf{R}^d\setminus \mathbf{Z}_t^{\varrho}\right)=0\Big) \;=\; 1. \tag{20}$$

In particular, by spatial translation invariance of $\mathbb{P}_{\ell_c}$, for fixed $z\in\mathsf{R}^d$, there is a $\delta=\delta(\varrho,z)$ such that (19) holds for $\mathbb{P}_{\ell_c}$–almost all ϱ.

Proof By a well-known scaling property of SBM (see, e.g., [FK94, Lemma 4.5.1]) we may take $\gamma=\frac{1}{2}$ for the catalyst's branching rate. Also, since the $\mathbb{P}_{\ell_c}$–distribution of ϱ is translation invariant in the space variable, it suffices to show

$$\mathbb{P}_{\ell_c}\Big(\ell\left((\mathbf{Z}_t^{\varrho})^{\mathrm{c}}\cap B_1(0)\right)=0\Big) \;=\; 1. \tag{21}$$

We may reformulate this as

$$\mathbb{P}_{\ell_c}\bigg(\sup_{s\in[t-\delta,t]} \varrho_s\left(B_\delta(z)\right)>0 \text{ for } \ell\text{–a.a. } |z|<1 \text{ and all } \delta\in(0,t)\bigg) \;=\; 0. \tag{22}$$

We want to distinguish between the contributions to X_t^{ϱ} from different initial regions.

1° (*decomposition*) For $n,N\geq 1$, write

$$A^{n,N} \;:=\; \Big\{x\in\mathsf{R}^d:\; N(n-1)\leq |x| < Nn\Big\},$$

and $\ell_{\mathrm{c}}^{n,N}$ for the 'restriction' $\mathbf{1}_{A^{n,N}}\ell_{\mathrm{c}}$ of ℓ_{c} to the ring $A^{n,N}$. Then, for N fixed, ϱ can be represented as the sum of independent SBM $\varrho^{n,N}$, $n\geq 1$, on R^d, where $\varrho^{n,N}$ starts from the finite measure $\ell_{\mathrm{c}}^{n,N}$.

2° (*negligible contribution from outside*) Fix $t > 0$. First we want to show that for each $\eta > 0$,

$$\sum_{n\geq 2} \mathbb{P}_{\ell_c^{n,N}}\Big(\sup_{s\leq t} \varrho_s\big(B_\eta(0)\big) > 0\Big) \underset{N\uparrow\infty}{\longrightarrow} 0. \tag{23}$$

For this purpose, we may assume that $\eta > 2\sqrt{t}$. Applying [DIP89, Theorem 3.3 (a)] (with R there replaced by η), we find a constant $c = c(d,t)$ such that for $n \geq 2$ and $N > (\eta + 2)$,

$$\begin{aligned}
&\mathbb{P}_{\ell_c^{n,N}}\Big(\sup_{s\leq t} \varrho_s\big(B_\eta(0)\big) > 0\Big)\\
&\leq\ c\int \ell_c^{n,N}(\mathrm{d}x)\,\big(|x| - (\eta+1)\big)^{d-2} \exp\Bigg[-\frac{\big(|x|-(\eta+1)\big)^2}{2t}\Bigg].
\end{aligned}$$

Hence, for the sum in (23) we get the upper bound

$$c\int_{|x|\geq N} \ell(\mathrm{d}x)\,\big(|x| - (\eta+1)\big)^{d-2} \exp\Bigg[-\frac{\big(|x|-(\eta+1)\big)^2}{2t}\Bigg] \underset{N\uparrow\infty}{\longrightarrow} 0,$$

proving (23).

3° (*main term*) By the previous step, it suffices to show that for $N \geq 1$ fixed, (21) holds with $\mathbb{P}_{\ell_c}$ replaced by $\mathbb{P}_{\ell_c^{1,N}}$.

Recall that $t > 0$ is fixed. From [Per89, Corollary 1.3] we know that the closed support S_t (say) of the measure ϱ_t is a Lebesgue zero set, with $\mathbb{P}_{\ell_c^{1,N}}$–probability one. Therefore almost every point of R^d does, with probability one, not belong to the closed support S_t of ϱ_t.

If now ϱ is a sample such that $\varrho_t = 0$, then $\mathbb{P}_{\ell_c^{1,N}}$–almost surely, $\varrho_s = 0$ for all $s < t$ sufficiently close to t, since the extinction time of Feller's branching diffusion has a continuous law. Hence it remains to deal with the case $\varrho_t \neq 0$.

Now let $z \in B_1(0) \cap (S_t)^c$ and let $\delta' = \delta'(\varrho, z)$ be such that $B_{\delta'}(z) \cap S_t = \emptyset$. Fix $0 < \varepsilon < 1/2$. By Theorem 9.3.2.2 of [Daw93] there is a constant $c = c(d,t)$ such that $\mathbb{P}_{\ell_c^{1,N}}$–a.s. there exists a $\delta'' = \delta''(\varrho) \in (0,t)$ such that for $s \in [t-\delta'', t]$,

$$\mathrm{S}_s\ \subseteq\ \mathrm{S}_t + B_{c(t-s)^\varepsilon}(0) \tag{24}$$

on the event $\{\varrho_t \neq 0\}$. We can reduce $\delta'' > 0$ such that even $c\,(\delta'')^\varepsilon \leq \frac{\delta'}{2}$ holds. Then (19) is true with $\delta := \left(\frac{\delta'}{2} \wedge \delta''\right)$, finishing the proof. ■

Remark 2 (dimension one) A property as in Proposition 1 is *not* valid in dimension one since there ϱ has a jointly continuous density field (see, e.g., [KS88]). ◇

2.3 Key step: asymptotic density of X_t^ϱ at points in $\mathbf{Z}_t^\varrho$

Recall the open set $\mathbf{Z}_t^\varrho$ of 'full' Lebesgue measure introduced in Proposition 1. Here we want to prove the following result.

Proposition 3 (asymptotic density at points in $\mathbf{Z}_t^\varrho$) *Let $d = 2$ or 3, and fix $t > 0$. For $\mathbb{P}_{\ell_c}$–almost all ϱ the following holds. For each $z \in \mathbf{Z}_t^\varrho$, there is an element $\xi_t^\varrho(z) \geq 0$ in the Lebesgue space $L^2 = L^2(P_{0,\ell_r}^\varrho)$ with expectation*

$$P_{0,\ell_r}^\varrho\, \xi_t^\varrho(z) \,\equiv\, i_r \,>\, 0 \tag{25}$$

and variance

$$0 \,<\, \mathrm{Var}_{0,\ell}^\varrho\, \xi_t^\varrho(z) \,=\, 2\, i_r \int_0^t \mathrm{d}s \int \varrho_s(\mathrm{d}y)\, \mathrm{p}_{t-s}^2(z-y) \,<\, \infty \tag{26}$$

and such that the L^2–convergence

$$\Big\langle K^{d/2} X_t^\varrho\big(z + K^{-1/2}(\,\cdot\,)\big), \varphi \Big\rangle \underset{K\uparrow\infty}{\longrightarrow} \xi_t^\varrho(z)\, \|\varphi\|_1, \qquad \varphi \in \mathcal{C}_+^{\mathrm{comp}}, \tag{27}$$

takes place. In particular, for $z \in \mathsf{R}^d$ fixed, $\xi_t^\varrho(z)$ exists with those properties, for $\mathbb{P}_{\ell_c}$–a.a. ϱ.

The *idea of proof* of Proposition 3 is to define $\xi_t^\varrho(z)$ as the L^2–limit of $S_\varepsilon X_{t-\varepsilon}(z)$ as $\varepsilon \to 0$, where, by an abuse of notation, the continuous density function of the absolutely continuous measure $S_\varepsilon X_{t-\varepsilon}$ is denoted by the same symbol $S_\varepsilon X_{t-\varepsilon}$. However we first have to show the *existence* of the L^2–limit, more precisely, of the L^2–limit in (27).

For $\varphi \in \mathcal{C}_+^{\mathrm{comp}}$, define the contractions

$$\varphi_K(y) \,:=\, K^{d/2} \varphi(K^{1/2} y), \qquad y \in \mathsf{R}^d, \quad K \geq 1, \tag{28}$$

and shifts

$$\theta_z \varphi(y) \,:=\, \varphi(y - z), \qquad y, z \in \mathsf{R}^d. \tag{29}$$

Lemma 4 (existence of the L^2–limit) *For $d = 2, 3,$ and $t > 0$ fixed, for $\mathbb{P}_{\ell_c}$–almost all ϱ the following holds. For each $\varphi \in \mathcal{C}_+^{\mathrm{comp}}$ and $z \in \mathbf{Z}_t^\varrho$, there exists an element $Y_{t,z}^\varrho(\varphi)$ in $L^2 = L^2(P_{0,\ell_r}^\varrho)$ such that*

$$\langle X_t^\varrho, \theta_z \varphi_K \rangle \underset{K\uparrow\infty}{\longrightarrow} Y_{t,z}^\varrho(\varphi) \quad \text{in } L^2$$

holds. $Y_{t,z}^\varrho(\varphi)$ has expectation

$$P_{0,\ell_r}^\varrho\, Y_{t,z}^\varrho(\varphi) \,=\, i_r\, \|\varphi\|_1 \tag{30}$$

and variance

$$\mathrm{Var}_{0,\ell}^\varrho\, Y_{t,z}^\varrho(\varphi) \,=\, 2\, i_r\, \|\varphi\|_1^2 \int_0^t \mathrm{d}s\, \big\langle \varrho_s\,, \theta_z \mathrm{p}_{t-s}^2 \big\rangle \,<\, \infty. \tag{31}$$

Proof By the expectation and variance formulas (15) and (18), respectively,

$$P^{\varrho}_{0,\ell_{\mathrm{r}}} \langle X^{\varrho}_t, \psi\rangle^2 \;=:\; \|\langle X^{\varrho}_t, \psi\rangle\|^2_2 \;=\; \langle \ell_{\mathrm{r}}\,, \psi\rangle^2 + 2\, i_{\mathrm{r}} \int_0^t \mathrm{d}s \;\Big\langle \varrho_s\,, (S_{t-s}\psi)^2\Big\rangle\,, \qquad (32)$$

$\psi \in \mathcal{C}^{\mathrm{comp}}$. Later we want to apply this to $\psi = \theta_z(\varphi_K - \varphi_L)$, for $K, L \geq 1$ and $z \in \mathbf{Z}^{\varrho}_t$, in which case $\langle \ell_{\mathrm{r}}\,, \psi\rangle = 0$.

1° (*choice of samples* ϱ) Fix $t > 0$. First we want to specify a set $R = R(t)$ of samples ϱ for which the lemma holds. Set

$$R' \;:=\; \Big\{\varrho : \int_0^t \mathrm{d}s\, \langle \varrho_s\,, \mathrm{p}_{16(t-s)}\rangle < \infty\Big\}, \quad R''_n \;:=\; \Big\{\varrho : \int_0^t \mathrm{d}s\, \varrho_s(B_n(0)) < \infty\Big\},$$

$n \geq 1$. Note that $\mathbb{P}_{\ell_{\mathrm{c}}}(R') = 1 = \mathbb{P}_{\ell_{\mathrm{c}}}(R''_n)$, $n \geq 1$, since

$$\mathbb{P}_{\ell_{\mathrm{c}}} \int_0^t \mathrm{d}s\, \langle \varrho_s\,, \mathrm{p}_{16(t-s)}\rangle \;=\; t\, i_{\mathrm{c}}\,, \qquad \mathbb{P}_{\ell_{\mathrm{c}}} \int_0^t \mathrm{d}s\, \varrho_s(B_n(0)) \;=\; t\, \ell_{\mathrm{c}}(B_n(0)),$$

where we used (10). Now let

$$R \;:=\; \Big\{\varrho : \ell\left((\mathbf{Z}^{\varrho}_t)^{\mathrm{c}}\right) = 0\Big\} \;\cap\; R' \cap \bigcap_{n=1}^{\infty} R''_n \qquad (33)$$

with $\mathbf{Z}^{\varrho}_t$ from Proposition 1. Clearly $\mathbb{P}_{\ell_{\mathrm{c}}}(R) = 1$. We will show that the assertions of the lemma hold for $\varrho \in R$.

2° (*preliminary estimates*) For the moment, fix $N \geq 1$. For $\delta > 0$ and $K > 4N^2/\delta^2$, there is a constant $c(\delta)$ such that

$$\mathrm{p}_s\left(K^{-1/2}y + z - x\right) \;\leq\; (2\pi s)^{-d/2} \exp\Big[-\frac{\delta^2}{8s}\Big] \;\leq\; c(\delta), \qquad (34)$$

$s > 0,\; |z - x| \geq \delta,\; |y| \leq N$. This implies for $K > 4N^2/\delta^2$ that

$$\sup_{|z-x|\,\geq\,\delta} S_s\theta_z\varphi_K\,(x) \;\leq\; c(\delta)\, \|\varphi\|_1\,, \qquad s > 0. \qquad (35)$$

On the other hand, for $|z - x| \geq \delta,\; K > 4N^2/\delta^2$, and $s > 0$,

$$\begin{aligned}(S_s\theta_z\varphi_K)\,(x) \;&\leq\; \|\varphi\|_1 \sup\Big\{\mathrm{p}_s(y + z - x) :\; |y| < \delta/2\Big\} \\ &\leq\; \|\varphi\|_1\, 2^d\, \mathrm{p}_{4s}(z - x),\end{aligned} \qquad (36)$$

since for x and y in that range

$$|y + z - x| \;\geq\; |z - x| - \frac{\delta}{2} \;\geq\; \frac{|z - x|}{2}\,.$$

3° (*an ε–error term*) Now fix $\varrho \in R$, $z \in \mathbf{Z}_t^{\varrho}$, and $\delta = \delta(\varrho, z)$ according to Proposition 1 such that (19) holds. Take $\varepsilon = \varepsilon(\varrho) \in (0, \delta)$, and assume $K > 4N^2/\delta^2$. Write A_ε^ϱ for the closed support of the measure $\mathrm{d}s|_{[t-\varepsilon,t]}\, \varrho_s(\mathrm{d}x)$ on $[t-\varepsilon, t] \times \mathsf{R}^d$. Note that $[t-\varepsilon, t] \times B_\delta(z)$ and A_ε^ϱ are disjoint. By Proposition 1, (35), and (36), for $(s, x) \in A_\varepsilon^\varrho$,

$$S_{t-s}\theta_z \varphi_K\,(x) \;\le\; \Big[c(\delta)\,\|\varphi\|_1\Big] \;\wedge\; \Big[2^d\,\|\varphi\|_1\,\mathrm{p}_{4(t-s)}(z-x)\,\mathbf{1}\{|z-x| \ge \delta\}\Big]. \quad (37)$$

Therefore, using both parts of the r.h.s.,

$$\begin{aligned} &\int_{t-\varepsilon}^{t} \mathrm{d}s \;\big\langle \varrho_s\,, (S_{t-s}\theta_z\varphi_K)^2 \big\rangle \\ &\qquad \le\; 2^d\, c(\delta)\, \|\varphi\|_1^2 \int_{t-\varepsilon}^{t} \mathrm{d}s \int_{|z-x| \ge \delta} \varrho_s(\mathrm{d}x)\, \mathrm{p}_{4(t-s)}(z-x). \end{aligned} \quad (38)$$

In the interior integral we distinguish between $|z-x| \ge \frac{|x|}{2}$ and the opposite. In the first case, the (restricted) double integral can be estimated from above by

$$\int_0^t \mathrm{d}s \int \varrho_s(\mathrm{d}x)\, \mathrm{p}_{4(t-s)}(x/2). \quad (39)$$

In the other case, using that $|z-x| < \frac{|x|}{2}$ implies $|x| < 2|z|$, we get the upper bound

$$\begin{aligned} &\int_0^t \mathrm{d}s \int_{|x| \le 2|z|} \varrho_s(\mathrm{d}x)\; \big(8\pi(t-s)\big)^{-d/2} \exp\left[-\frac{\delta^2}{8(t-s)}\right] \\ &\qquad \le\; c(\delta) \int_0^t \mathrm{d}s \int_{|x| \le 2|z|} \varrho_s(\mathrm{d}x), \end{aligned} \quad (40)$$

where we used the second inequality in (34). By the assumption $\varrho \in R$, the double integrals in (39) and in the second line of (40) are finite. Hence, the second line in (38) is finite which implies that

$$\lim_{\varepsilon \downarrow 0} \limsup_{K \uparrow \infty} \int_{t-\varepsilon}^{t} \mathrm{d}s \;\big\langle \varrho_s\,, (S_{t-s}\theta_z\varphi_K)^2 \big\rangle \;=\; 0, \qquad \varrho \in R. \quad (41)$$

Therefore it suffices to deal with the remaining part of the integral in (32), for the originally chosen $\varepsilon = \varepsilon(\varrho)$.

4° (*further preliminary estimates*) Let $x, y, z \in \mathsf{R}^d$, and $0 \le s \le t - \varepsilon$ with $\varepsilon > 0$. If $|y + z - x| \ge \frac{|x|}{2}$, then

$$\mathrm{p}_{t-s}(y+z-x) \;\le\; 2^d\, \mathrm{p}_{4(t-s)}(x). \quad (42)$$

In the opposite case we get $|x| \leq 2|y+z|$ implying

$$|y+z-x|^2 \geq |x|^2 - 2|x|\,|y+z| \geq |x|^2 - 4|y+z|^2,$$

hence

$$\mathrm{p}_{t-s}(y+z-x) \leq 2^d\,\mathrm{p}_{4(t-s)}(y+z-x) \leq \exp\Big[\frac{|y+z|^2}{2(t-s)}\Big] 2^d\,\mathrm{p}_{4(t-s)}(x). \quad (43)$$

If we additionally assume that $|y| \leq N$, then we can (42) and (43) combine to

$$\mathrm{p}_{t-s}(y+z-x) \leq \exp\Big[\frac{N^2+|z|^2}{\varepsilon}\Big] 4^d\,\mathrm{p}_{16(t-s)}(x), \quad (44)$$

$0 \leq s \leq t-\varepsilon,\ \ \varepsilon > 0.$

5° (*domination for the main term*) Let $\varphi \in \mathcal{C}_+^{\mathrm{comp}}$ and choose $N \geq 1$ such that $\mathrm{supp}(\varphi_K) \subseteq B_N(0)$, $K \geq 1$. Then, for $x, z \in \mathsf{R}^d$, $0 \leq s \leq t-\varepsilon$, and $\varepsilon > 0$, from (44) and the trivial estimate $\mathrm{p}_{t-s}(x) \leq (2\pi\varepsilon)^{-d/2}$ we infer that for $K \geq 1$,

$$\left(S_{t-s}\theta_z\varphi_K\right)^2(x) \leq \|\varphi\|_1^2\,(2\pi\varepsilon)^{-d/2} \exp\Big[\frac{N^2+|z|^2}{\varepsilon}\Big] 4^d\,\mathrm{p}_{16(t-s)}(x).$$

Note that for $\varrho \in R$ this upper bound is integrable w.r.t. $\mathrm{d}s\big|_{[0,t-\varepsilon]}\,\varrho_s(\mathrm{d}x)$.

6° (*Cauchy sequence*) From the pointwise convergence

$$S_{t-s}\theta_z(\varphi_K - \varphi_L)(x) \xrightarrow[K,L\uparrow\infty]{} 0, \qquad t-s>0, \quad x,z \in \mathsf{R}^d,$$

by dominated convergence we conclude that for $t-\varepsilon > 0$, $\varrho \in R$, and $z \in \mathsf{R}^d$,

$$\int_0^{t-\varepsilon} \mathrm{d}s \left\langle \varrho_s, \left(S_{t-s}\theta_z(\varphi_K - \varphi_L)\right)^2 \right\rangle \xrightarrow[K,L\uparrow\infty]{} 0. \quad (45)$$

Together with (41), $\|\langle X_t^\varrho, \theta_z(\varphi_K - \varphi_L)\rangle\|_2^2 \to 0$ as $K, L \to \infty$, for the fixed $t > 0$, $\varrho \in R$, and $z \in \mathbf{Z}_t^\varrho$. That is, there exists the L^2–limit of $\langle X_t^\varrho, \theta_z\varphi_K\rangle$ as $K \to \infty$, denoted by $Y_{t,z}^\varrho(\varphi)$.

7° (*moment formulas*) By (32) (with $\psi = \theta_z\varphi_K$), and again dominated convergence,

$$\begin{aligned}\|Y_{t,z}^\varrho(\varphi)\|_2^2 &= \langle \ell_{\mathrm{r}}, \varphi\rangle^2 + 2\,i_{\mathrm{r}} \int_0^t \mathrm{d}s\ \left\langle \varrho_s\,, \lim_{K\to\infty} \left(S_{t-s}\theta_z\varphi_K\right)^2 \right\rangle \\ &= i_{\mathrm{r}}^2\,\|\varphi\|_1^2 + 2\,i_{\mathrm{r}}\,\|\varphi\|_1^2 \int_0^t \mathrm{d}s\ \left\langle \varrho_s\,, \theta_z \mathrm{p}_{t-s}^2\right\rangle < \infty.\end{aligned}$$

Since L^2–convergence implies $L^1(P^{\varrho}_{0,\ell_r})$–convergence, from (15) we conclude for the claimed expectation (30) of $Y^{\varrho}_{t,z}(\varphi)$. Combined with the just derived second moment formula, the variance formula (31) follows too, finishing the proof. ■

Proof of Proposition 3 Let $t > 0$, and fix ϱ such that the assertions in Lemma 4 hold. Take $z \in \mathbf{Z}^{\varrho}_t$, $\varphi \in \mathcal{C}^{\mathrm{comp}}_+$, and recall the notation φ_K of (28). It is easy to see that for $0 < \varepsilon < t$,

$$\left\langle S_\varepsilon X^{\varrho}_{t-\varepsilon}, \theta_z \varphi_K \right\rangle \underset{K\uparrow\infty}{\longrightarrow} (S_\varepsilon X^{\varrho}_{t-\varepsilon})(z)\, \|\varphi\|_1 \quad \text{in } L^2. \tag{46}$$

By the Markov property (applied at time $t-\varepsilon$), and the expectation and variance formulas (14) and (16), respectively,

$$\begin{aligned} &\left\| \left\langle S_\varepsilon X^{\varrho}_{t-\varepsilon}, \theta_z \varphi_K \right\rangle - \left\langle X^{\varrho}_t, \theta_z \varphi_K \right\rangle \right\|_2^2 \\ &= P^{\varrho}_{0,\ell_r}\, 2 \int X^{\varrho}_{t-\varepsilon}(\mathrm{d}x) \int_{t-\varepsilon}^{t} \mathrm{d}s \int \varrho_s(\mathrm{d}y)\, \mathrm{p}_{s-(t-\varepsilon)}(y-x)\, (S_{t-s}\theta_z\varphi_K)^2(y) \\ &= 2\, i_{\mathrm{r}} \int_{t-\varepsilon}^{t} \mathrm{d}s\, \left\langle \varrho_s, (S_{t-s}\theta_z\varphi_K)^2 \right\rangle. \end{aligned}$$

But the latter integral converges to 0 as first $K \uparrow \infty$ and then $\varepsilon \to 0$ (recall (41)). Hence, by (46), $\|\varphi\|_1\, S_\varepsilon X^{\varrho}_{t-\varepsilon}(z)$ is a Cauchy sequence in L^2 as $\varepsilon \to 0$ with the limit denoted by $\|\varphi\|_1\, \xi^{\varrho}_t(z)$. Clearly $\|\varphi\|_1 \xi^{\varrho}_t(z) = Y^{\varrho}_{t,z}(\varphi)$ a.s., with $Y^{\varrho}_{t,z}(\varphi)$ from Lemma 4. Thus the moment formulas (25) and (26) follow from (30) and (31), respectively. This finishes the proof. ■

3 Main results

3.1 Absolutely continuous states

According to the results in [DFR91], the one-dimensional catalytic SBM X^{ϱ} has *absolutely continuous states* (recall that the one-dimensional ϱ has $\mathbb{P}_{\ell_c}$–almost surely a jointly continuous density field). In contrast to the classical higher-dimensional SBM, the catalytic SBM X^{ϱ} shares this property also in the other dimensions:

Theorem 5 (absolutely continuous states) *In dimensions $d = 2, 3$, for $t > 0$ fixed and $\mathbb{P}_{\ell_c}$–almost all ϱ, the random measure X^{ϱ}_t is absolutely continuous, P^{ϱ}_{0,ℓ_r}–a.s. Hence, X^{ϱ}_t is absolutely continuous $\mathcal{P}_{\ell_c,\ell_r}$–a.s.*

Proof Fix $t > 0$. According to Propositions 1 and 3, $\mathbb{P}_{\ell_c}$–almost surely the asymptotic density $\xi^{\varrho}_t(z)$ exists for all z in the complement $\mathbf{Z}^{\varrho}_t$ of a Lebesgue zero set:

$$\langle X^{\varrho}_t, \theta_z \varphi_K \rangle \underset{K\uparrow\infty}{\longrightarrow} \xi^{\varrho}_t(z) \text{ in } P^{\varrho}_{0,\ell_r}\text{–law}, \qquad z \in \mathbf{Z}^{\varrho}_t, \tag{47}$$

for a fixed $\varphi \in \mathcal{C}_+^{\text{comp}}$ with $\|\varphi\|_1 = 1$. Moreover, $P^{\varrho}_{0,\ell_r}\,\xi^{\varrho}_t(z) \equiv i_r$. Hence with Lemma 2.7.1 from [DF95], it follows that X^{ϱ}_t is absolutely continuous with P^{ϱ}_{0,ℓ_r}–probability one (note that the finiteness assumption on the initial measure imposed in [DF95] can easily be removed).

The second claim follows immediately by mixing the random laws P^{ϱ}_{0,ℓ_r} by means of $\mathbb{P}_{\ell_c}$. ■

3.2 Persistent convergence in dimension two

Recalling Proposition 3, for convenience, we set $\zeta^{\varrho} := \xi^{\varrho}_1(0)$.

Theorem 6 (persistent convergence in $d = 2$) *In dimension $d = 2$, the convergence*

$$X^{\varrho}_T \underset{T\uparrow\infty}{\longrightarrow} \zeta^{\varrho}\ell \tag{48}$$

holds in law with respect to both the annealed distribution $\mathcal{P}_{\ell_c,\ell_r}$ and with respect to the random laws P^{ϱ}_{0,ℓ_r}. Moreover, $\mathbb{P}_{\ell_c}$–almost surely, ζ^{ϱ} has expectation

$$P^{\varrho}_{0,\ell_r}\,\zeta^{\varrho} \equiv i_r \tag{49}$$

(persistence) and variance

$$0 < \mathit{Var}^{\varrho}_{0,\ell_r}\,\zeta^{\varrho} = 2\,i_r \int_0^1 \mathrm{d}s\,\big\langle \varrho_s\,,\mathrm{p}^2_{1-s}\big\rangle < \infty, \tag{50}$$

whereas the $\mathcal{P}_{\ell_c,\ell_r}$–variance of $\zeta = \zeta^{\varrho}$ is infinite.

Here the phrase

$$X^{\varrho}_T \underset{T\uparrow\infty}{\longrightarrow} \zeta^{\varrho}\ell \text{ with respect to the } \textit{random} \text{ laws } P^{\varrho}_{0,\ell_r} \tag{51}$$

can be expressed more formally as follows. Given ϱ, let Q^{ϱ}_T and Q^{ϱ}_{∞} denote the laws of the random measures X^{ϱ}_T and $\zeta^{\varrho}\ell$, respectively. Let $\mathbf{Q}_T$ and $\mathbf{Q}_{\infty}$ refer to the distributions of the random laws Q^{ϱ}_T and Q^{ϱ}_{∞}, respectively, (the randomness coming from the medium ϱ distributed by $\mathbb{P}_{\ell_c}$). Then the desired more formal expression for (51) is

$$\mathbf{Q}_T \text{ converges } \textit{weakly} \text{ to } \mathbf{Q}_{\infty} \text{ as } T \to \infty. \tag{52}$$

Convergence in law w.r.t. the annealed law instead means the weak convergence

$$\mathbb{P}_{\ell_c} Q^{\varrho}_T \underset{T\uparrow\infty}{\longrightarrow} \mathbb{P}_{\ell_c} Q^{\varrho}_{\infty}\,.$$

Remark 7 (convergence concepts) Whereas in Proposition 3 the convergence claim (27) is true for *almost all* ϱ, this strong convergence concept is *lost* in establishing (48) by the transition in law during exploiting self-similarity. ◇

Proof of Theorem 6 By Proposition 3, for $\mathbb{P}_{\ell_c}$–almost all ϱ,

$$\langle X_1^\varrho, \varphi_K \rangle \underset{K\uparrow\infty}{\longrightarrow} \zeta^\varrho \, \|\varphi\|_1, \qquad \varphi \in \mathcal{C}_+^{\rm comp}, \tag{53}$$

in P_{0,ℓ_r}^ϱ–law, where ζ^ϱ has expectation (49) and P_{0,ℓ_r}^ϱ–variance (50). Hence (53) holds in $\mathcal{P}_{\ell_c,\ell_r}$–law. By self-similarity in the variant (4),

$$\langle X_T^\varrho, \varphi \rangle \underset{T\uparrow\infty}{\longrightarrow} \zeta^\varrho \|\varphi\|_1, \qquad \varphi \in \mathcal{C}_+^{\rm comp},$$

in $\mathcal{P}_{\ell_c,\ell_r}$–law and with respect to the random laws P_{0,ℓ_r}^ϱ. (Recall that the latter statement can more formally expressed in the spirit of the reformulation of (51) to (52).) Since $\mathcal{C}_+^{\rm comp}$ is convergence determining, and the exceptional set of ϱ concerning (53) is independent of φ, statement (48) holds.

We still have to show that the $\mathcal{P}_{\ell_c,\ell_r}$–variance of ζ^ϱ is infinite. By definition, it is given by

$$\mathcal{P}_{\ell_c,\ell_r}(\zeta^\varrho)^2 - (\mathcal{P}_{\ell_c,\ell_r}\zeta^\varrho)^2 \;=\; \mathbb{P}_{\ell_c} P_{0,\ell_r}^\varrho(\zeta^\varrho)^2 - (\mathbb{P}_{\ell_c} P_{0,\ell_r}^\varrho \zeta^\varrho)^2. \tag{54}$$

But by (49), the second term at the r.h.s. can be written as $\mathbb{P}_{\ell_c}(P_{0,\ell_r}^\varrho \zeta^\varrho)^2$. Therefore (54) can be continued with

$$= \; \mathbb{P}_{\ell_c} {\rm Var}_{0,\ell_r}^\varrho \zeta^\varrho \;=\; 2\, i_{\rm r} \int_0^1 {\rm d}s \; \langle \ell_{\rm c}, {\rm p}_{1-s}^2 \rangle$$

where we exploited (50) and (10). Using the identity ${\rm p}_r^2 = \frac{1}{4\pi r}\, {\rm p}_{r/2}$, we arrive at

$$= \; \frac{i_{\rm r}}{2\pi} \int_0^1 {\rm d}s \; \frac{1}{1-s} \; \big\langle \ell_{\rm c}, {\rm p}_{(1-s)/2} \big\rangle \;=\; \frac{i_{\rm r} i_{\rm c}}{2\pi} \int_0^1 {\rm d}s \; \frac{1}{1-s} \;=\; \infty.$$

This completes the proof. ■

Remark 8 (lattice model) It can be expected that in the two-dimensional simple branching random walk in the simple branching random medium the analogous statement to Theorem 6 holds with the limit population $\zeta^\varrho \ell$ replaced by a *mixed Poisson system* (homogeneous Poisson point process with random intensity). ◇

3.3 Local L^2–Lipschitz continuity of ξ_t^ϱ

In this final subsection we establish the following L^2–continuity property of the asymptotic densities $\xi_t^\varrho(z)$ (from Proposition 3) in the space coordinate z running in $\mathbf{Z}_t^\varrho$ (from Proposition 1).

Theorem 9 (Local L^2–Lipschitz continuity of ξ_t^ϱ) *Let $d = 2$ or 3. Fix $t > 0$. For $\mathbb{P}_{\ell_c}$–a.a. ϱ, the field $\{\xi_t^\varrho(z) : z \in \mathbf{Z}_t^\varrho\}$ of asymptotic densities is* *locally* $L^2(P^{\varrho}_{0,\ell_r})$*–Lipschitz continuous: For* $\mathbb{P}_{\ell_c}$*–a.a.* ϱ *and any compact set* $D \subset \mathbf{Z}^{\varrho}_t$ *there exists a constant* $c = c(\varrho, D)$ *such that*

$$\|\xi^{\varrho}_t(z_2) - \xi^{\varrho}_t(z_1)\|_2 \le c\,|z_2 - z_1|, \qquad z_1, z_2 \in D. \tag{55}$$

Proof 1° (*reduction*) Note that by compactness it suffices to show that for $\mathbb{P}_{\ell_c}$–a.a. ϱ and each $z_0 \in \mathbf{Z}^{\varrho}_t$ there exists an $\varepsilon = \varepsilon(\varrho, z_0) > 0$ and a constant $c = c(\varrho, z_0, \varepsilon)$ such that

$$\|\xi^{\varrho}_t(z_2) - \xi^{\varrho}_t(z_1)\|_2 \le c\,|z_2 - z_1|, \qquad z_1, z_2 \in B_\varepsilon(z_0). \tag{56}$$

2° (*estimates for* grad p_s) For $s > 0$ and $x \in \mathsf{R}^d$,

$$|\mathrm{grad}\ \mathrm{p}_s(x)| = \frac{|x|}{s}\,\mathrm{p}_s(x),$$

hence

$$|\mathrm{grad}\ \mathrm{p}_s(x)| \le 2\,\mathrm{p}_s(x), \quad \text{if} \quad |x| \le 2s. \tag{57}$$

On the other hand, for $|x| \ge s/2$ (that is $1 \le 2|x|/s$),

$$|\mathrm{grad}\ \mathrm{p}_s(x)| \le 4s^{-1}\frac{|x|^2}{2s}\,\mathrm{p}_s(x) \le 4s^{-1}2^{d/2}\,\mathrm{p}_{2s}(x), \tag{58}$$

since $re^{-r} \le e^{-r/2}$ for $r \ge 0$.

3° (*a heat kernel estimate*) We will also need the following simple estimate. Consider $x \in \mathsf{R}^d$ and $s > 0$. Let $0 < \delta \le |x| \vee s$ and $y \in B_{\delta/2}(0)$. If $|x| \ge \delta$ then $|x+y| \ge |x| - |y| \ge |x| - \frac{\delta}{2} \ge \frac{|x|}{2}$, hence

$$\mathrm{p}_s(x+y) \le 2^d\,\mathrm{p}_{4s}(x).$$

In the opposite case, $|x| < \delta$ (hence $s \ge \delta$),

$$|x+y|^2 \ge \Big(|x| - \frac{\delta}{2}\Big)^2 \ge |x|^2 - |x|\,\delta \ge |x|^2 - \delta^2,$$

which together with $s \ge \delta$ implies (as in (43))

$$\mathrm{p}_s(x+y) \le 2^d\,\mathrm{e}^{\delta/8}\,\mathrm{p}_{4s}(x). \tag{59}$$

Therefore (59) holds for s, δ, x, y satisfying $0 < \delta < |x| \vee s$ and $y \in B_{\delta/2}(0)$.

4° (*suitable catalyst samples*) Denote by Q the set of rational numbers, and put $\mathsf{Q}_{++} := \{q \in \mathsf{Q} : q > 0\}$. Fix $t > 0$. For $q \in \mathsf{Q}_{++}$ and $z \in \mathsf{Q}^d$, set

$$R_{q,z} := \Big\{\varrho : \int_0^t \mathrm{d}s\ \big\langle \varrho_s, \theta_z \mathrm{p}_{q(t-s)} \big\rangle < \infty\Big\}, \qquad R := \bigcap_{z \in \mathsf{Q}^d,\ q \in \mathsf{Q}_{++}} R_{q,z}.$$

Since $\mathbb{P}_{\ell_c}\int_0^t \mathrm{d}s\, \left\langle \varrho_s\,, \theta_z \mathrm{p}_{q(t-s)} \right\rangle = t\, i_c < \infty$, each $R_{q,z}$ has the full $\mathbb{P}_{\ell_c}$–measure, hence $\mathbb{P}_{\ell_c}(R) = 1$. From now on in this proof, we restrict to $\varrho \in R$.

Note that $\mathbf{Z}_t^\varrho$ is open in R^d and Q^d dense in R^d. Hence by (59) we get that for $\varrho \in R$, $z_0 \in \mathbf{Z}_t^\varrho$, and $q \in \mathsf{Q}_{++}$,

$$\int_0^t \mathrm{d}s\ \left\langle \varrho_s\,, \theta_{z_0} \mathrm{p}_{q(t-s)} \right\rangle \ <\ \infty. \tag{60}$$

In fact, for $z_0 \in \mathbf{Z}_t^\varrho$, let δ' be as in Proposition 1 and choose $z \in \mathsf{Q}^d$ with $|z - z_0| < \delta'/3$. Hence z fulfills (19) with $\delta := \frac{2}{3}\delta'$. Thus by (59) for $\varrho \in R$,

$$\begin{aligned} C(\varrho, z_0, q) \ &:=\ \int_0^t \mathrm{d}s\ \left\langle \varrho_s\,, \theta_{z_0} \mathrm{p}_{q(t-s)} \right\rangle \qquad (61)\\ &=\ \int_0^t \mathrm{d}s\ \int \varrho_s(\mathrm{d}x)\ \theta_z \mathrm{p}_{q(t-s)}(x + (z - z_0))\, \mathbf{1}\{\delta \leq s \vee |x|\} \\ &\leq\ 2^d e^{\delta/8} \int_0^t \mathrm{d}s\ \left\langle \varrho_s\,, \theta_z \mathrm{p}_{4q(t-s)} \right\rangle \ <\ \infty. \end{aligned}$$

5° (*an estimate away from* z_0) Fix ϱ and $z_0 \in \mathbf{Z}_t^\varrho$ as well as $\delta = \delta(\varrho, z_0)$ according to Proposition 1. Define $\varepsilon = \varepsilon(\varrho, z_0) := \frac{1}{4}\delta(\varrho, z_0)$. For the rest of this proof we will consider $z_1, z_2 \in B_\varepsilon(z_0)$.

Take $\varphi \in \mathcal{C}_+^{\mathrm{comp}}$ with $\|\varphi\|_1 = 1$, and closed support contained in $B_1(0)$. Recall the notation φ_K of (28). We want to estimate $|S_s\,(\theta_{z_2}\varphi_K - \theta_{z_1}\varphi_K)|$ for $K > 16/\delta^2$ (implying $\mathrm{supp}(\varphi_K) \subset B_{\delta/4}(0)$).

Start with

$$\begin{aligned} &\Big| S_s\,(\theta_{z_2}\varphi_K - \theta_{z_1}\varphi_K) \Big|\,(x) \\ &\leq\ \sup\Big\{ \Big| \mathrm{p}_s(y + z_2 - x) - \mathrm{p}_s(y + z_1 - x) \Big| :\ y \in B_{\delta/4}(0) \Big\}, \end{aligned} \tag{62}$$

$s \in [0, t]$, $x \in \mathsf{R}^d$. By the mean value theorem, this inequality can be continued with

$$\leq\ |z_2 - z_1|\, \sup\Big\{ |\mathrm{grad}\ \mathrm{p}_s(y)| :\ y \in B_{\delta/2}(x - z_0) \Big\}. \tag{63}$$

If our consideration is now restricted to $|x - z_0| \geq \delta$, we arrive at

$$\leq\ |z_2 - z_1|\, \sup\Big\{ |\mathrm{grad}\ \mathrm{p}_s(y)| :\ |y| \geq \frac{|x - z_0|}{2} \Big\},$$

and, by using (58), an additional restriction to $|x - z_0| \geq s$ gives

$$\leq\ |z_2 - z_1|\, 4s^{-1} 2^{d/2}\, \mathrm{p}_{2s}\Big(\frac{x - z_0}{2}\Big)\ =\ |z_2 - z_1|\, 4s^{-1} 8^{d/2}\, \mathrm{p}_{8s}(x - z_0).$$

Set

$$c_1(\delta)\ :=\ 16 \cdot 8^d \sup\Big\{ s^{-2}\, \mathrm{p}_{8s}(y) :\ \ s > 0,\ |y| \geq \delta \Big\}.$$

Note that $c_1(\delta) < \infty$. Then, altogether, for $s \in [0,t]$ and $|x - z_0| \geq (\delta \vee s)$,

$$\Big| S_s \left(\theta_{z_2} \varphi_K - \theta_{z_1} \varphi_K \right) \Big|^2 (x) \;\leq\; |z_2 - z_1|^2 \, c_1(\delta) \, \mathrm{p}_{8s}(x - z_0). \tag{64}$$

6° (*an estimate close to* z_0) Recall that by (62) and (63),

$$\Big| S_s \left(\theta_{z_2} \varphi_K - \theta_{z_1} \varphi_K \right) \Big| (x) \;\leq\; |z_2 - z_1| \, \sup \Big\{ |\mathrm{grad}\ \mathrm{p}_s(y)| : \; y \in B_{\delta/2}(x - z_0) \Big\}.$$

Further, by (57), for $s \in [\delta, t]$ and $|x - z_0| \leq s$ (note that $|y| \leq 2s$), the r.h.s. of this expression is dominated by

$$|z_2 - z_1| \, 2 \sup \Big\{ \mathrm{p}_s(y) : \; y \in B_{\delta/2}(x - z_0) \Big\}.$$

From here we get the trivial estimate

$$\Big| S_s \left(\theta_{z_2} \varphi_K - \theta_{z_1} \varphi_K \right) \Big| (x) \;\leq\; |z_2 - z_1| \, \delta^{-d/2}.$$

Further, by (59) we get the inequality

$$\exp \Big[- \frac{|y|^2}{2s} \Big] \;\leq\; \mathrm{e}^{\delta/8} \exp \Big[- \frac{|x - z_0|^2}{8s} \Big].$$

Hence

$$\Big| S_s \left(\theta_{z_2} \varphi_K - \theta_{z_1} \varphi_K \right) \Big| (x) \;\leq\; |z_2 - z_1| \, 2^d \, \mathrm{e}^{\delta/8} \, \mathrm{p}_{4s}(x - z_0).$$

If we let $c_2(\delta) := 2^d \, \delta^{-d/2} \mathrm{e}^{\delta/8}$, we can combine both estimates to

$$\Big| S_s \left(\theta_{z_2} \varphi_K - \theta_{z_1} \varphi_K \right) \Big|^2 (x) \;\leq\; |z_2 - z_1|^2 \, c_2(\delta) \, \mathrm{p}_{4s}(x - z_1), \tag{65}$$

$s \in [\delta, t]$, $|x - z_0| \leq s$.

7° (*final steps*) Recall $C(\varrho, z_0, q)$ from (61). Set

$$c(\varrho, z_0, \varepsilon) \;:=\; \Big(c_1(\delta) \, C(\varrho, z_0, 8) + c_2(\delta) \, C(\varrho, z_0, 4) \Big)^{1/2}. \tag{66}$$

By the expectation and variance formulas (15) and (18), respectively,

$$\Big\| \Big\langle X_t^{\varrho}, \theta_{z_2} \varphi_K - \theta_{z_1} \varphi_K \Big\rangle \Big\|_2^2 \;=\; 2 \, i_{\mathrm{r}} \int_0^t \mathrm{d}s \; \Big\langle \varrho_s, \Big(S_{t-s} \left(\theta_{z_2} \varphi_K - \theta_{z_1} \varphi_K \right) \Big)^2 \Big\rangle.$$

We distinguish between $|x - z_1| \geq (t - s) \vee \delta$ and the opposite. In the latter case we additionally distinguish between $s \in [t - \delta, t]$ and the remaining case.

Then we obtain

$$\begin{aligned} &= 2\, i_{\mathrm{r}} \int_0^t \mathrm{d}s \, \left\langle \varrho_s, \left(S_{t-s} \left(\theta_{z_2}\varphi_K - \theta_{z_1}\varphi_K \right) \right)^2 \mathbf{1}_{\left(B_{(t-s)\vee\delta}(z_0)\right)^c} \right\rangle \\ &\quad + 2\, i_{\mathrm{r}} \int_0^{t-\delta} \mathrm{d}s \, \left\langle \varrho_s, \left(S_{t-s} \left(\theta_{z_2}\varphi_K - \theta_{z_1}\varphi_K \right) \right)^2 \mathbf{1}_{B_{t-s}(z_0)} \right\rangle \\ &\le |z_2 - z_1|^2 \, c(\varrho, z_1, \varepsilon)^2. \end{aligned}$$

Hence

$$\begin{aligned} \|\xi_t(z_2) - \xi_t(z_1)\|_2 &\le \limsup_{K\to\infty} \left\| \left\langle X_t^{\varrho}, \theta_{z_2}\varphi_K - \theta_{z_1}\varphi_K \right\rangle \right\|_2 \\ &\le |z_2 - z_1| \, c(\varrho, z_1, \varepsilon). \end{aligned}$$

This finishes the proof. ■

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