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Abstract

We consider the monomer-dimer model, whose realisations are spanning sub-graphs of a given graph such that every vertex has degree zero or one. The measure depends on a parameter, the monomer activity, which rewards the total number of monomers. We consider general correlation functions including monomer-monomer correlations and dimer-dimer covariances. We show that these correlations decay exponentially fast with the distance if the monomer activity is strictly positive. Our result improves a previous upper bound from van den Berg and is of interest due to its relation to truncated spin-spin correlations in classical spin systems. Our proof is based on the cluster expansion technique.

1 Introduction

This note considers the *monomer-dimer model* [10], whose realisations are spanning sub-graphs of a given graph such that every vertex has degree zero or one. Vertices with degree zero are referred to as *monomers* and pairs of vertices connected by an edge are referred to as *dimers*. The measure depends on a parameter, the monomer activity $\rho \geq 0$, which controls the total number of monomers. In case of zero monomer activity no monomers are present and we obtain the classical *dimer model*.

By superimposing two independent realisations of the monomer-dimer model we obtain a configuration of the *double monomer-dimer model*. This model can be viewed as a random walk loop soup whose configurations are collections of self-avoiding and mutually self-avoiding paths which might be closed or open, see e.g. Figure 1. If the monomer activity is zero, all paths are closed and the double monomer-dimer model reduces to the double dimer model.

The (double) monomer-dimer model shares an intriguing similarity with the Spin $O(N)$ model, namely they both have a probabilistic reformulation as a particular random path model [11]. In this representation, the external magnetic field of the Spin $O(N)$ model plays the same role as the monomer activity in the monomer-dimer model and the two models only differ in the weight that is assigned to the number of visits of (open and closed) paths at the vertices. The qualitative behaviour of the random path model, however, is expected to not depend on the choice of such weight function.

In this note we study the rate of decay of *monomer-monomer correlations* when the monomer activity is non-zero. Through the random path representation, this question is closely related to an open problem in the Spin $O(N)$ model. Here, it is known that spin-spin correlations decay exponentially fast with the distance between the vertices when the external magnetic field is non-zero. The constant of decay in the exponent is known to be $O(h)$ as $h \to 0$ for $N = 1, 2, 3$ [7] and $O(h^2)$ for any $N \in \mathbb{N}$ [11]. It is however conjectured that the constant decays as $O(\sqrt{h})$ when $h \to 0$ for any integer value of N. The same behaviour is expected to occur in the monomer-dimer model.

Our main result shows that for any strictly positive value of the monomer activity ρ, monomer-monomer correlations decay exponentially fast with the graph distance between the vertices. For large enough

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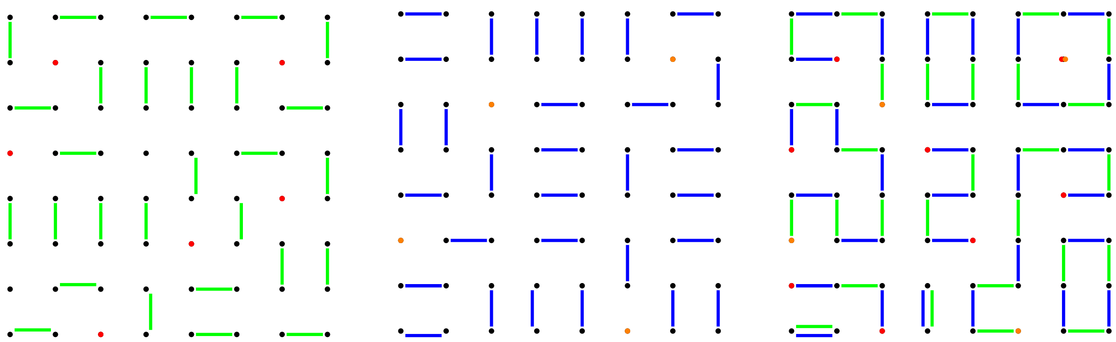

Figure 1: The first two figures show monomer-dimer configurations $\omega, \omega' \in \Omega_K$, where $K \subset \mathbb{Z}^2$. The third figure shows their superposition resulting in a collection of open and closed paths.

values of ρ, this result is derived using a cluster expansion. Applying similar analytic tools as in [7] we can then extend this result to small values of ρ and further show that the constant of decay is of order at least $O(\rho)$ as $\rho \to 0$.

It should be emphasized that our result only holds for non-zero values of the monomer activity and the behaviour of the dimer model, i.e., the monomer-dimer model at zero monomer activity, is strongly different. In dimension $d = 2$ the monomer-monomer correlation admits polynomial decay with the distance between the monomers [2, 3], while in any dimension $d \geq 3$ it exhibits long-range order [14].

Our result also holds for more general correlation functions including the *dimer-dimer covariance* as special case. It is known that this covariance decays exponentially in the distance between the edges with a constant of order $O(\rho^2)$ in the limit as $\rho \to 0$. More precisely, in [16] it is shown that the dimer-dimer covariance is upper bounded by the probability of observing a path in the double monomer-dimer model that connects these two edges. The exponential decay of such probability then follows from [15] based on an argument with disagreement percolation. We remark that for non-zero monomer activity, the connection probability behaves differently, namely it stays uniformly positive in any dimension $d \geq 3$ [13]. In this note, we improve the existing bound due to [15, 16] by showing that the constant decays as $O(\rho)$ in the limit $\rho \to 0$.

It is further interesting to compare the double monomer-dimer model with the *monomer double-dimer model* [13]. The configurations in both models are superpositions of two independently sampled monomer-dimer configurations. The difference, however, is that in the former model the two monomer-dimer configurations might have different monomer sets, while in the latter model the monomer sets have to be identical. Consequently, the paths in the double monomer-dimer model might be open, while in the monomer double-dimer model all paths are closed. In the discussion above we have seen that the double monomer-dimer model behaves drastically different if the monomer activity changes from small, but strictly positive values to zero. This, however, is not the case in the monomer double-dimer model, i.e., in the system where all the paths are closed [1, 5, 12, 13]. In particular, exponential decay of the connection probabilities only occurs for large enough values of the monomer activity.

2 Model and main result

Consider a finite undirected graph $G = (V, E)$. A dimer configuration (or perfect matching) of G is a subset $d \subset E$ of edges such that every vertex $v \in V$ is an element of precisely one edge. We let D_G be the set of all dimer configurations in G. Given a set $A \subset V$, we let G_A be the subgraph of G with vertex set $V \setminus A$ and with edge-set consisting of all the edges in E which do not touch any vertex in A. We let $D_G(A)$ be the set of dimer configurations in G_A.

In this note, we concentrate on the d-dimensional cubic lattice. We denote by $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ the graph with vertex set $\mathbb{V} = \mathbb{Z}^d$ and with edge set $\mathbb{E} = \{\{x,y\} \,:\, x, y \in \mathbb{Z}^d,\, d(x,y) = 1\}$, where $d(x,y)$ denotes the graph distance between x and y. We denote by $G_K = (K, E_K)$ the graph with vertex set $K \subset \mathbb{Z}^d$ and with edge set $E_K := \{\{x,y\} \in \mathbb{E} :\, x, y \in K\} \subset \mathbb{E}$.

Given $K \subset \mathbb{Z}^d$, the configuration space of the monomer-dimer model in G_K is denoted by Ω_K and it corresponds to the set of tupels $\omega = (M, d)$ such that $M \subset K$ and $d \in D_{G_K}(M)$. We refer to the first and second element of the tupel ω as a set of monomers and a set of dimers, respectively. We let $\mathcal{M} : \Omega_K \to K$ and $\mathcal{D} : \Omega_K \to E_K$ be the random variables defined by $\mathcal{M}(\omega) := M$ and $\mathcal{D}(\omega) := d$ for each $\omega = (M, d) \in \Omega_K$.

We define a probability measure on Ω_K,

$$\forall \omega \in \Omega_K \qquad \mathbb{P}_{K,\rho}(\omega) := \frac{\rho^{|\mathcal{M}(\omega)|}}{\mathbb{Z}_{K,\rho}}, \tag{2.1}$$

where $\rho \geq 0$ is the parameter of the model (monomer density) and $\mathbb{Z}_{K,\rho}$ is the normalizing constant (partition function).

We are interested in correlations between sets of monomers. For any $\rho \geq 0$ and any $A \subset K$ we set

$$C_{K,\rho}(A) := \mathbb{Z}_{K \setminus A, \rho}.$$

In other words, $C_{K,\rho}(A)$ corresponds to the weight of all monomer-dimer configurations in G_K with fixed monomers at all vertices in A. For any $A, B \subset K$ with $A \cap B = \emptyset$, we then introduce the correlation function

$$U_{K,\rho}(A,B) := \frac{C_{K,\rho}(A \cup B)}{C_{K,\rho}(\emptyset)} - \frac{C_{K,\rho}(A)}{C_{K,\rho}(\emptyset)} \, \frac{C_{K,\rho}(B)}{C_{K,\rho}(\emptyset)}.$$

We further set

$$U_\rho(A,B) := \lim_{K \uparrow \mathbb{Z}^d} U_{K,\rho}(A,B),$$

where the limit is in the sense of van Hove. Its existence follows from [8, Theorem 10] since our monomer-dimer model is a special case of the polymer systems studied in [8]. If $A, B \in \mathbb{E}$, then the correlation function reduces to the dimer-dimer covariance, namely

$$U_{K,\rho}(A,B) = \mathbb{P}_{K,\rho}\big(A, B \in \mathcal{D}\big) - \mathbb{P}_{K,\rho}\big(A \in \mathcal{D}\big)\, \mathbb{P}_{K,\rho}\big(B \in \mathcal{D}\big).$$

Monomer correlations, paths and $O(N)$ spin systems. We now briefly explain the relation between monomer-monomer and spin-spin correlations. Consider the Spin $O(N)$ model with $N \in \mathbb{N}$ at inverse temperature $\beta \geq 0$ and external magnetic field $h \geq 0$. In [11, Proposition 2.3] it is shown that the spin-spin correlation at $x, y \in K \subset \mathbb{V}$ is identical to the two-point function $\mathbb{G}_{G_K,N,\beta,h}(x,y)$ that is defined in a particular model of random paths. The measure of this model depends on a function $\mathcal{U} : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ which controls the number of visits of paths at the vertices. If we consider a different choice of the function $\mathcal{U}$, namely if we set $\tilde{\mathcal{U}}(r) := 1$ for any $r \in \mathbb{N}_0$, then we have that for $N = 2$, $\rho = h$ and any $\beta \geq 0$,

$$\frac{C_{K,\rho}(\{x\} \cup \{y\})}{C_{K,\rho}(\emptyset)} = \tilde{\mathbb{G}}_{G_K,N,\beta,h}(\{x,y\}),$$

where $\tilde{\mathbb{G}}_{G_K,N,\beta,h}(x,y)$ is defined as the function $\mathbb{G}_{G_K,N,\beta,h}(x,y)$, but with the choice of $\tilde{\mathcal{U}}$ instead of $\mathcal{U}$.

We now present our main theorem. It states that correlation functions between sets of monomers decay exponentially fast with the distance between their vertices. For any $A, B \subset \mathbb{V}$ we set

$$d(A,B) := \min_{(u,v):\, u\in A,\, v\in B} d(u,v).$$

Theorem 2.1. *For any $d \geq 1$ and $\rho > 0$, there exist $c, c' \in (0,\infty)$ such that for any non-empty $A, B \subset \mathbb{V}$ with $A \cap B = \emptyset$,*

$$U_\rho(A,B) \leq c'\, e^{-c\, d(A,B)}, \tag{2.2}$$

where $c = c(\rho) = \tilde{c}\,\rho$ if ρ is sufficiently small and $\tilde{c} \geq \frac{2}{\ln(2(a+1))\,(a+1)}$ with $a = e\,\sqrt{e\,|A|\,(4d-1)}$.

Remark 2.2. Concerning the decay of dimer-dimer covariances, our theorem above improves the result of [15, 16] which states that $c = O(\rho^2)$ in the limit as $\rho \to 0$. We remark, however, that it is still an open problem to show that $c = O(\sqrt{\rho})$.

3 Proof of Theorem 2.1

In this section we present the proof of Theorem 2.1. We will first prove exponential decay of monomer correlations in the regime of large ρ using a cluster expansion. Exponential decay for small ρ will then follow by applying an analytic theorem, see Theorem 3.2 below.

Proposition 3.1. *For any $d \geq 1$, any $K \subset \mathbb{Z}^d$, any non-empty $A, B \subset \mathbb{V}$ with $A \cap B = \emptyset$ and any $\rho \geq \sqrt{e\,|A|\,(4d-1)}\,e$, it holds that*

$$0 \leq U_{K,\rho}(A,B) \leq e^{-2\,d(A,B)+1}. \tag{3.1}$$

Proof. Fix $d \geq 1$, $K \subset \mathbb{Z}^d$ and two non-empty sets $A, B \subset \mathbb{V}_L$ with $A \cap B = \emptyset$. Let $\rho \geq \sqrt{e\,|A|\,(4d-1)}\,e$. To begin, we rewrite the partition function $\mathbb{Z}_{K,\rho}$ using a cluster expansion. First, we note that

$$\begin{aligned}\mathbb{Z}_{K,\rho} &= \sum_{n=0}^{|K|/2} \sum_{\substack{(M,d)\in\Omega_K:\\ |d|=n}} \rho^{|K|-2|d|}\\ &= \rho^{|K|}\Bigg(1 + \sum_{n\geq 1}\frac{\rho^{-2n}}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in E_K^n}\prod_{1\leq i<j\leq n}\big(1+\zeta(\gamma_i,\gamma_j)\big)\Bigg),\end{aligned}$$

where for $\gamma, \gamma' \in E_K$, $\zeta(\gamma,\gamma') := \mathbb{1}_{\{\gamma\cap\gamma'=\emptyset\}} - 1$. We denote by $\mathcal{G}_n$ the set of all (unoriented) connected graphs with vertex set $\mathcal{V}_n = \{1,\ldots,n\}$. We introduce the Ursell functions φ on finite ordered sequences $(\gamma_1,\ldots,\gamma_m) \in E_K^m$, which are defined by

$$\varphi(\gamma_1,\ldots,\gamma_m) := \begin{cases} 1 & \text{if } m = 1,\\ \frac{1}{m!}\sum\limits_{G\in\mathcal{G}_m}\prod\limits_{\{i,j\}\in G}\zeta(\gamma_i,\gamma_j) & \text{if } m \geq 2,\end{cases}$$

where the product is over all edges in G. For any $\gamma^* \in E_K$, using that $\rho \geq \sqrt{e\,(4d-1)}$, it holds that

$$\sum_{\gamma\in E_K}\rho^{-2}e\,|\zeta(\gamma,\gamma^*)| \leq (4d-1)\,\rho^{-2}e \leq 1.$$

By cluster expansion [6, Theorem 5.4] it thus holds that

$$\mathbb{Z}_{K,\rho} = \rho^{|K|} \exp\bigg(\sum_{m\geq 1}\sum_{(\gamma_1,\dots,\gamma_m)\in E_K^m} \varphi(\gamma_1,\dots,\gamma_m)\,\rho^{-2m}\bigg), \tag{3.2}$$

where combined sum and integrals converge absolutely. Furthermore, for any $\gamma_1 \in E_K$, we have that

$$1+\sum_{n\geq 2} n \sum_{(\gamma_2,\dots,\gamma_n)\in E_K^{n-1}} |\varphi(\gamma_1,\gamma_2,\dots,\gamma_n)|\,\rho^{-2(n-1)} \leq e. \tag{3.3}$$

Using a similar cluster expansion as above, we further obtain that for any $A\subset K$,

$$\mathbb{Z}_{K\setminus A,\rho} = \rho^{|K|-|A|} \exp\bigg(\sum_{m\geq 1}\sum_{(\gamma_1,\dots,\gamma_m)\in E_{K\setminus A}^m} \varphi(\gamma_1,\dots,\gamma_m)\,\rho^{-2m}\bigg). \tag{3.4}$$

For $m\in\mathbb{N}$ and $A\subset K$, let C_A^m denote the set of ordered sequences $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_m)\in E_K^m$ such that there exists $i\in[m]$ and $x\in A$ such that x is an endpoint of γ_i.

Fix now two disjoint subsets $A, B\subset K$. From (3.2) and (3.4) we deduce that,

$$\begin{aligned}\frac{C_{K,\rho}(A\cup B)}{C_{K,\rho}(\emptyset)} &= \frac{1}{\rho^{|A|}}\exp\bigg(-\sum_{m\geq 1}\sum_{(\gamma_1,\dots,\gamma_m)\in C_A^m}\varphi(\gamma_1,\dots,\gamma_m)\,\rho^{-2m}\bigg)\\ &\quad\times\frac{1}{\rho^{|B|}}\exp\bigg(-\sum_{m\geq 1}\sum_{(\gamma_1,\dots,\gamma_m)\in C_B^m}\varphi(\gamma_1,\dots,\gamma_m)\,\rho^{-2m}\bigg)\\ &\quad\times\exp\bigg(\sum_{m\geq 1}\sum_{(\gamma_1,\dots,\gamma_m)\in C_A^m\cap C_B^m}\varphi(\gamma_1,\dots,\gamma_m)\,\rho^{-2m}\bigg)\\ &= \frac{C_{K,\rho}(A)}{C_{K,\rho}(\emptyset)}\,\frac{C_{K,\rho}(B)}{C_{K,\rho}(\emptyset)}\exp\bigg(\sum_{m\geq 1}\rho^{-2m}\sum_{(\gamma_1,\dots,\gamma_m)\in C_A^m\cap C_B^m}\varphi(\gamma_1,\dots,\gamma_m)\bigg).\end{aligned} \tag{3.5}$$

Now observe that for any $(\gamma_1,\dots,\gamma_m)\in C_A^m\cap C_B^m$, $\varphi(\gamma_1,\dots,\gamma_m)\neq 0$ only if the graph G, which is obtained from $(\gamma_1,\dots,\gamma_m)$ by drawing an edge between i and j whenever $\zeta(\gamma_i,\gamma_j)\neq 0$, is connected. Stated differently, $\varphi(\gamma_1,\dots,\gamma_m)\neq 0$ only if there exists at least one path connecting a vertex of the set A to a vertex of the set B. In particular, it is necessary that $m\geq d(A,B)$. Thus,

$$\begin{aligned}&\sum_{m\geq 1}\rho^{-2m}\sum_{(\gamma_1,\dots,\gamma_m)\in C_A^m\cap C_B^m}|\varphi(\gamma_1,\dots,\gamma_m)|\\ &\leq e^{-2\,d(A,B)}\sum_{m\geq 1} m \sum_{\substack{(\gamma_1,\dots,\gamma_m)\in E_K^m:\\ \gamma_1\cap A\neq\emptyset}}|\varphi(\gamma_1,\dots,\gamma_m)|\left(\frac{\rho}{e}\right)^{-2m}\\ &= e^{-2\,d(A,B)}\left(\frac{\rho}{e}\right)^{-2}\sum_{\gamma_1\in C_A^1}\bigg(1+\sum_{m\geq 2} m\sum_{(\gamma_2,\dots,\gamma_m)\in E_K^{m-1}}|\varphi(\gamma_1,\dots,\gamma_m)|\left(\frac{\rho}{e}\right)^{-2(m-1)}\bigg)\\ &\leq e^{-2\,d(A,B)}\,e^3\,\rho^{-2}\,|A|\,2d \leq e^{-2\,d(A,B)},\end{aligned} \tag{3.6}$$

where in the last two steps we used (3.3) and that $\rho\geq\sqrt{e\,|A|\,(4d-1)}\,e$. From (3.5) and (3.6), we thus obtain that

$$U_{K,\rho}(A,B)\leq\frac{C_{K,\rho}(A)}{C_{K,\rho}(\emptyset)}\,\frac{C_{K,\rho}(B)}{C_{K,\rho}(\emptyset)}\left(e^{e^{-2\,d(A,B)}}-1\right)\leq e^{-2\,d(A,B)+1}.$$

□

We are now ready to prove Theorem 2.1. The proof is based on Proposition 3.1 above and on the following analytic theorem.

Theorem 3.2 ([9, Theorem A.3 and Theorem A.6])**.** *If* $f : \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ *is analytic and* $f(x) \leq 1$ *in the region* $\{z \,|\, Re(z) > 0\}$ *and if* $f(x) \leq e^{-b}$ *for* $a \leq x \leq a+2$*,* $a, b \geq 0$*, then for* $0 < x < a$*,*

$$f(x) \leq e^{-c\,x},$$

where $c = \frac{b}{\ln(2(a+1))\,(a+1)}$.

Proof of Theorem 2.1. We fix two disjoint subsets $A, B \subset \mathbb{V}$. To begin, we note that by [8, Theorem 10] for all $\rho > 0$, the function $U_\rho(A,B)$ is an analytic function of ρ on $\mathbb{R}^+ := \{x \in \mathbb{R} : \; x > 0\}$. Using Proposition 3.1 and applying Theorem 3.2 to the function $f_{A,B}(\rho) := \frac{1}{e}\, U_\rho(A,B)$ concludes the proof of the theorem. □

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