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Decomposition and diagonalization in solving large systems

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Abstract

Consider the nonlinear equation $(*)$ $x = Tx + f$ with a strictly contractive operator T in some real separable Hilbert space. A well–known procedure to approximate the unique solution $x^*(f)$ of $(*)$ is the projection-iteration method which can be characterized as a method of diagonalization. In case that $(*)$ is a large system which can be represented as a system of weakly coupled subsystems, an efficient method to approximate $x^*(f)$ is the decomposition method which is a block iteration scheme. One realization of this method is the waveform relaxation method. In this note we combine the diagonalization technique with the decomposition method and derive conditions for the convergence of the resulting iteration scheme.

1 Introduction

Let X be a Banach space, let T be a strictly contractive operator mapping X into X. To given $f \in X$ we consider the nonlinear equation

$$x = Tx + f. \tag{1.1}$$

Well–known procedures to approximate the unique solution $x^*(f)$ of (1.1) are iteration, projection and projection–iteration schemes [7]. The basic idea of a projection–iteration scheme is to approximate the underlying infinite dimensional problem by a finite dimensional one where the dimension of which increases during each iteration step. This method was developed in the late sixties and can be characterized as a diagonalization procedure [3, 7].

In this paper we additionally assume that X can be represented as the product space $X = X_1 \times X_2 \times \ldots \times X_n$ where X_i is a Banach space with the norm $||.||_i$ for $i = 1, \ldots n$. Thus, (1.1) can be rewritten in the form

$$\begin{aligned} x_1 &= T_1(x_1, \ldots, x_n) + f_1, \\ x_2 &= T_2(x_1, \ldots, x_n) + f_2, \\ &\cdots\cdots\cdots\cdots \\ x_n &= T_n(x_1, \ldots, x_n) + f_n. \end{aligned} \tag{1.2}$$

The representation (1.2) means that (1.1) can be considered as a system of interacting subsystems where the behavior of the subsystems can be quite different, that is, (1.1) is a large scale system.

An iteration scheme which approximates the unique solution x^* and takes into account the different nature of the subsystems is the decomposition method [6, 11]. There are different levels on which decomposition can be taken into consideration: the level of approximating linear systems, the level of approximating nonlinear systems, and the level of the original system. In case that (1.1) is equivalent to a

system of differential algebraic equations the latter decomposition is called waveform relaxation. This approach was proposed by Lelarasmee et al [4] for the time-domain analysis of large scale integrated circuits, recently there is a large interest for such procedures (see [1] and references therein), also in dynamic process simulation [12, 13]. The efficiency of the waveform relaxation method strongly depends on a suitable decomposition of (1.1) into weakly interacting subsystems.
The goal of this paper is to combine the idea of waveform relaxation technique (decomposition) with the projection–iteration scheme (diagonalization). We derive conditions under which the resulting procedure converges.

2 The Method of Diagonalization

In case X is a real separable Hilbert space H and T is strictly contractive on H, projection-iteration methods represent an important tool to solve (1.1) [3, 7, 8]. Let $w_1 \dots w_n$ be linearly independent elements of H, let H_n be the linear hull spanned by these elements. Let $P_n : H \to H_n$ be a projection, $T_n := P_n T P_n$, $f_n := P_n f$. Let x_n^* be the unique solution of

$$x_n = T_n x_n + f_n. \tag{2.1}$$

Under the assumptions above, the sequence $\{x_n^*\}$ strongly converges to x^*. In general, the solution of (2.1) requires an iterative procedure, that is, for $n = 1, 2, \dots$ we get the following sequences

$$\begin{array}{c} x_1^1, x_1^2, x_1^3, \dots \to x_1^*(f_1) \\ x_2^1, x_2^2, x_2^3, \dots \to x_2^*(f_2) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_k^1, x_k^2, x_k^3, \dots \to x_k^*(f_k) \\ \hfill \downarrow \\ \hfill x^*(f) \end{array} \tag{2.2}$$

Considering this scheme it is obvious to ask for the convergence of the sequence

$$x_n = T_n x_{n-1} + f_n, \quad n = 1, 2, \dots. \tag{2.3}$$

which can be interpreted as a diagonalization of the sequences (2.2). The following general theorem yields a sufficient condition for an affirmative answer to this question [3].

Proposition 2.1 *Let $\{T_i\}$ be a sequence of strictly contractive operators mapping X into itself, let x_i^* be the fixed point of T_i, let k_i be the contraction constant of T_i. Assume the following hypotheses are valid* *(i)* $x_i^* \to y^*$ *as* $i \to \infty$.

(ii) $k_i \le k < 1 \quad \forall\ i$.

Then the iteration scheme (2.3) *strongly converges to* y^*.

In case T_n is generated by means of a projection, the property $T_i \to T$ immediately implies $x_i^* \to x^* = Tx^*$.
In what follows we construct an iterative procedure of the type (2.3) to solve (1.1) which however cannot be interpreted as a projection-iteration method.

3 The Waveform-Relaxation-Method

Another iterative scheme which can be used under some conditions to solve equation (1.2) is the so–called waveform relaxation method. Its main features are the following [1, 4, 5, 9, 14]:

1. Decomposition of a given large system into subsystems.
2. Independent solution of the subsystems taking into account inputs from other subsystems.

The application of this method exhibits the following advantages: the decomposed systems are smaller, each subsystem can be solved by an appropriate method, the method is highly parallelizable. Its efficiency depends on the fact how weak the subsystems are coupled. To formulate a sufficient condition for its convergence we consider only a decomposition into two subsystems, the extension to the general case can be easily done.

Let us suppose that the Banach space X can be represented as $X = X_1 \times X_2$ where X_1 and X_2 are Banach spaces with the norms $||.||_1$ and $||.||_2$, resp. such that equation (1.1) can be rewritten in the form

$$\begin{aligned} x_1 &= T_1(x_1, x_2) + f_1, \\ x_2 &= T_2(x_1, x_2) + f_2. \end{aligned} \tag{3.1}$$

The problem how to determine an appropriate norm in X will be trated later. Concerning the operators T_1 and T_2 we assume

(H$_1$). $T_1 : X_1 \times X_2 \to X_1$ *and* $T_2 : X_1 \times X_2 \to X_2$ *are globally Lipschitz continuous, that is, there are nonnegative constants* k_{ij}, $1 \le i,\ j \le 2$, *such that* $\forall\ x_1 \bar{x}_1 \in X_1$, $\forall\ x_2 \bar{x}_2 \in X_2$

$$
\begin{aligned}
||T_1(x_1,x_2)-T_1(\bar{x}_1,\bar{x}_2)||_1 &\leq k_{11}||x_1-\bar{x}_1||_1-k_{12}||x_2-\bar{x}_2||_2,\\
||T_2(x_1,x_2)-T_2(\bar{x}_1,\bar{x}_2)||_2 &\leq k_{21}||x_1-\bar{x}_1||_1-k_{22}||x_2-\bar{x}_2||_2
\end{aligned} \tag{3.2}
$$

where the Lipschitz constants k_{ij} satisfy

$$
k_{11} < 1, \; k_{22} + \frac{k_{21}k_{12}}{1-k_{11}} < 1. \tag{3.3}
$$

Theorem 3.1 *Assume the hypothesis* (H_1) *is valid. Then the equation* (1.1) *has for any* $f \in X$ *a unique solution.*

We give a proof of this theorem since it suggests several iteration schemes to solve (3.1).

Proof. $k_{11} < 1$ implies the existence of a function $g_f : X_2 \to X_1$ such that the first equation in (3.1) is equivalent to

$$
x_1 = g_f(x_2). \tag{3.4}
$$

It is easy to show that g_f satisfies $\forall\, x_2, \bar{x}_2 \in X_2$

$$
||g_f(x_2) - g_f(\bar{x}_2)||_1 \leq \frac{k_{12}}{1-k_{11}}||x_2-\bar{x}_2||_2. \tag{3.5}
$$

Substituting (3.4) into the second equation of (3.1) we get

$$
x_2 = T_2(g_f(x_2), x_2) + f_2.
$$

By (3.2) and (3.5) we have $\forall\, x_2, \bar{x}_2 \in X_2$

$$
||T_2(g(x_2),x_2) - T_2(g(x_2),x_2)||_2 \leq \left(\frac{k_{12}k_{21}}{1-k_{11}} + k_{22}\right)||x_2-\bar{x}_2||.
$$

From (3.3) we can conclude that $\tilde{T}_2^f(x_2) := T_2(g(x_2),x_2) + f_2$ is stricly contractive. Therefore, to given f, by Banach's fixed point theorem, $\tilde{T}_2^f$ has a unique fixed point $x_2^*(f)$. Hence, $x^*(f) = (g_f(x_2^*(f), x_2^*(f))$ is the unique solution of (1.1), q.e.d. □

From the proof of Theorem 3.1 we get that the following (nonlinear) waveform iteration scheme is convergent

$$
\begin{aligned}
x_1^k &= T_1(x_1^k, x_2^{k-1}) + f_1,\\
x_2^k &= T_2(x_1^k, x_2^k) + f_2
\end{aligned}
$$

In general, this scheme requires at each step to solve a nonlinear equation in some Banach space. Under our conditions, this can be done by an iterative procedure

start: $k := 0$, choose any initial guess (x_1^0, x_2^0).

loop: $k := k+1$, determine x_1^k and x_2^k as limits of

$$\begin{aligned} x_{1,j}^k &= T_1(x_{1,j-1}^k, x_2^{k-1}) + f_1, \ j = 1,2,\ldots, \text{ with } x_{1,0}^k = x_1^{k-1}, \\ x_{2,j}^k &= T_1(x_1^{k-1}, x_{2,j-1}^k,) + f_2, \ j = 1,2,\ldots, \text{ with } x_{2,0}^k = x_2^{k-1}. \end{aligned}$$

In correspondence with the idea of diagonalization in the frame of projection–iteration methods we can ask for the convergence of the diagonalized procedure

$$\begin{aligned} x_1^k &= T_1(x_1^{k-1}, x_2^{k-1}) + f_1, \\ x_2^k &= T_2(x_1^k, x_2^{k-1}) + f_2. \end{aligned} \tag{3.6}$$

Another essential feature of the projection–iteration method is to approximate the underlying operator and the given element, and to improve this approximation in each iteration step. In our case this means to ask for the convergence of the iteration scheme

$$\begin{aligned} x_1^k &= T_1^{(k)}(x_1^{k-1}, x_2^{k-1}) + f_1^{(k)}, \\ x_2^k &= T_2^{(k)}(x_1^k, x_2^{k-1}) + f_2^{(k)} \end{aligned} \tag{3.7}$$

where $T_1^{(k)}, T_2^{(k)}, f_1^{(k)}, f_2^{(k)}$ are approximations of T_1, T_2, f_1, f_2 respectively. The following section is devoted to the convergence of the iteration schemes (3.6) and (3.7).

4 Convergence of the Diagonalized Procedures

The question for the convergence of the iteration scheme (3.6) is equivalent to the convergence of the procedure

$$x_2^k = T_{2,k}(x_2^{k-1}) + f_2 \tag{4.1}$$

where the operator $T_{2,k} : X_2 \to X_2$ is defined by

$$T_{2,k}(x_2) := T_2(x_1^k, x_2). \tag{4.2}$$

The iteration scheme (4.1) has the same structure as the procedure (2.3).

Under the validity of hypothesis (H_1) it is easy to prove that the operators $T_{2,k}$ are contractions with a uniform contraction constant. Since the operators $T_{2,k}$ are not generated by means of projections, the convergence of their fixed points are not so obvious. From that reason we prove the convergence of the diagonalization procedure (3.7) in a straightforward manner without using Proposition 2.1.

Theorem 4.1 *Assume that hypothesis (H_1) hold. Then the diagonalized procedure (3.6) converges.*

Proof. From (3.6) and (3.2) we get

$$||x_1^k - x_1^{k-1}||_1 \le k_{11}||x_1^{k-1} - x_1^{k-2}||_1 + k_{12}||x_2^{k-1} - x_2^{k-2}||_2, \tag{4.3}$$

$$\begin{aligned} &||x_2^k - x_2^{k-1}||_2 \le k_{21}||x_1^k - x_1^{k-1}||_1 + k_{22}||x_2^{k-1} - x_2^{k-2}||_2 \\ &\le k_{11}k_{21}||x_1^{k-1} - x_1^{k-2}||_1 + (k_{22} + k_{21}k_{12})||x_2^{k-1} - x_2^{k-2}||_2. \end{aligned} \tag{4.4}$$

Uptil now we have not fixed a norm in X. We define a norm $|||.|||$ in X by

$$|||(x_1, x_2)||| := a_1||x_1||_1 + a_2||x_2||_2 \tag{4.5}$$

where a and b are positive numbers which will be choosen in the sequel. From (4.3) – (4.5) we obtain

$$\begin{aligned} |||(x_1^k, x_2^k) - (x_1^{k-1}, x_2^{k-1})||| \;\le\; & (a_1k_{11} + a_2k_{11}k_{21})||x_1^k - x_2^{k-1}||_1 + \\ & +(a_1k_{12} + a_2(k_{21}k_{12} + k_{22}))||x_2^k - x_2^{k-1}||_2. \end{aligned} \tag{4.6}$$

Let K be the matrix

$$K = \begin{pmatrix} k_{11} & k_{21}k_{11} \\ k_{12} & k_{12}k_{21} + k_{22} \end{pmatrix}$$

Without loss of generality we may assume that all entries of K are positive. We denote by κ the spectral radius of K. A simple calculation shows that the relations (3.3) imply

$$0 < \kappa < 1. \tag{4.7}$$

Accoording to the Perron–Frobenius theory, κ is a simple eigenvalue with a strictly positive eigenfunction $e := (e_1, e_2)^T$. Therefore we have

$$\begin{aligned} e_1k_{11} + e_2k_{11}k_{21} &= \kappa e_1 \\ e_1k_{12} + e_2(k_{21}k_{12} + k_{22}) &= \kappa e_2. \end{aligned} \tag{4.8}$$

Setting

$$a_1 := e_1, \ a_2 := e_2 \tag{4.9}$$

we obtain from (4.6) and (4.8)

$$|||(x_1^k, x_2^k) - (x_1^{k-1}, x_2^{k-1})||| \leq \kappa |||(x_1^{k-1}, x_2^{k-1}) - (x_1^{k-2}, x_2^{k-2})|||$$

Thus, by (4.7) the sequence (3.7) converges in the norm $|||.|||$ defined in (4.5) where a_1 and a_2 satisfy (4.9) , q.e.d. □

Remark 4.2 *It is easy to see that hypothesis* (H_1) *also implies the convergence of the procedure*

$$\begin{aligned} x_1^k &= T_1(x_1^{k-1}, x_2^{k-1}) + f_1, \\ x_2^k &= T_2(x_1^{k-1}, x_2^{k-1}) + f_2. \end{aligned} \tag{4.10}$$

which can be used for parallel computing technique.

To prove the convergence of the scheme (3.7) let us assume

$(\mathbf{H_2})$. *For* $j = 1, 2$ *there exists a sequence of operators* $T_j^{(i)}, i = 1, 2, \ldots$ *mapping* $X_1 \times X_2$ *into* X_j *such that the following properties hold*

(i) $T_j^{(i)}$ *is globally lipschitzian* $\forall i$, *i.e., there are nonnegative constants* $k_{jl}^{(i)}$, $j, l = 1, 2$ *such that* $\forall x_k, \bar{x}_k \in X_k, k = 1, 2$

$$||T_j^{(i)}(x_1, x_2) - T_j^{(i)}(\bar{x}_1, \bar{x}_2) \leq \sum_{l=1}^{2} k_{jl}^{(i)} ||x_l - \bar{x}_l||_l. \tag{4.11}$$

(ii) To the matrix $K^{(i)}$ *defined by*

$$K^{(i)} = \begin{pmatrix} k_{11}^{(i)} & k_{21}^{(i)} k_{11}^{(i)} \\ k_{12}^{(i)} & k_{12}^{(i)} k_{21}^{(i)} + k_{22}^{(i)} \end{pmatrix}$$

there is a strictly positive matrix K *such that*

$$K^{(i)} \leq K \tag{4.12}$$

where "≤" denotes the usial partial ordering, and

$$\sigma(K) = \kappa < 1. \tag{4.13}$$

Let $e := (e_1, e_2)$ be a strictly positive eigenvector of K to the eigenvalue κ. In what follows we introduce the norm $|||.|||$ in $X = X_1 \times X_2$ by (4.5) where $a_i = e_i$.

Let $T^{(i)} := (T_1^{(i)}, T_2^{(i)})$. From Theorem 4.1 it follows that under the hypothesis (H_2) each operator $T^{(i)}$ is strictly contractive in X with respect to the norm $|||.|||$. Let $x_*^{(i)}$ be the corresponding fixed point. Concerning the sequence $x_*^{(i)}$ of fixed points of $T^{(i)}$ we suppose

(H$_3$). *There is a $y_* \in X$ such that*

$$|||y_* - x_*^{(i)}||| \to 0 \quad \text{as} \quad i \to \infty.$$

Theorem 4.3 *Assume the hypotheses (H_2) and (H_3) to be valid. Then the sequence x^k defined by* (3.7) *satisfies*

$$|||x^k - x_*||| \to 0. \tag{4.14}$$

Proof. By the inequality

$$|||x^k - y_*||| \le |||x^k - x_*^{(i)}||| + |||x_*^{(i)} - y_*||| \tag{4.15}$$

and by hypothesis (H_3) it sufficies to prove

$$|||x^k - x_*^{(i)}||| \to \quad \text{as} \quad k \to \infty. \tag{4.16}$$

From (3.7) and (4.11) we get

$$\begin{aligned} ||x_1^k - x_{1,*}^{(k)}||_1 &= ||T_1^{(k)}(x_1^{k-1}, x_2^{k-1}) - T_1^{(k)}(x_{1,*}^{(k)}, x_{2,*}^{(k)})||_1 \\ &\le k_{11}^{(k)}||x_1^{k-1} - x_{1,*}^{(k)}||_1 + k_{12}^{(k)}||x_2^{k-1} - x_{2,*}^{(k)}||_2, \end{aligned} \tag{4.17}$$

$$\begin{aligned} ||x_2^k - x_{2,*}^{(k)}||_2 &\le k_{21}^{(k)}||x_1^k - x_{1,*}^{(k)}||_1 + k_{22}^{(k)}||x_2^{k-1} - x_{2,*}^{(k)}||_2 \\ &\le k_{11}^{(k)}k_{21}^{(k)}||x_1^{k-1} - x_{1,*}^{(k)}||_1 \\ &+ (k_{22}^{(k)} + k_{12}^{(k)}k_{22}^{(k)})||x_2^{k-1} - x_{2,*}^{(k)}||_2. \end{aligned} \tag{4.18}$$

Introducing the notation

$$\delta^k := \begin{pmatrix} ||x_1^k - x_{1,*}^{(k)}||_1 \\ ||x_2^k - x_{2,*}^{(k)}||_2 \end{pmatrix} \qquad \delta_*^k := \begin{pmatrix} ||x_{1,*}^{(k)} - x_{1,*}^{(k-1)}||_1 \\ ||x_{2,*}^{(k)} - x_{2,*}^{(k-1)}||_2 \end{pmatrix}$$

and taking into account (4.12) we may rewrite the inequalities (4.17), (4.18) in the form

$$\delta^k \le K\delta^{k-1} + \delta_*^k. \tag{4.19}$$

From (4.19) and (4.13) we obtain

$$|||\delta^k||| \le \kappa|||\delta^{k-1}||| + |||\delta_*^k||| \le \kappa^{k-1}|||\delta^1||| + \sum_{\nu=1}^{k-1} \kappa^{k-\nu}|||\delta_*^\nu|||. \tag{4.20}$$

According to (4.13), to given $\varepsilon > 0$ there is a $k_0(\varepsilon)$ such that

$$\kappa^{k-1}|||\delta^1||| \leq \varepsilon/3 \text{ for } k \leq k_0(\varepsilon). \tag{4.21}$$

From hypothesis (H_3) it follows that $|||\delta_*^k|||$ is uniformly bounded, that is, there is a positive constant c such that

$$|||\delta_*^k||| \leq c \text{ for all } k,$$

moreover, there is a $k_1(\varepsilon) \geq k_0(\varepsilon)$ such that

$$|||\delta_*^k||| \leq \varepsilon \left(3 \sum_{\mu=1}^{\infty} k^{\mu}\right)^{-1} \quad \text{for } k \leq k_1(\varepsilon).$$

Therefore, for $k > k_1(\varepsilon)$ we have

$$\begin{aligned}
\sum_{\nu-1}^{k-1} \kappa^{k-\nu}|\ ||\delta_*^\nu||\ | &= \sum_{\nu=1}^{k_1} \kappa^{k-\nu}|\ ||\delta_*^\nu||\ | + \sum_{\nu=k+1}^{k-1} \kappa^{k-\nu}|\ ||\delta_*^\nu||\ | \leq \\
&\leq k_1 c \kappa^{k-k}1 + \varepsilon \sum_{\nu=k+1}^{k-1} \kappa^{k-\nu} \left(3 \sum_{\mu=1}^{\infty} k^{\mu}\right)^{-1} \leq \\
&\leq k_1 c \kappa^{k-k}1 + \varepsilon/3.
\end{aligned} \tag{4.22}$$

Consequently, there is a $k_2(\varepsilon) \geq k_1(\varepsilon)$ such that

$$k_1 c \kappa^{k-k}1 \leq \varepsilon/3. \tag{4.23}$$

Finally, for $k \geq k_{21}(\varepsilon)$ we get from (4.20) - (4.23)

$$|||\delta^k||| \leq \varepsilon.$$

Hence, (4.14) is valid, q.e.d. □

5 Applications

As a first application we consider the integral–algebraic system

$$\begin{aligned}
x_1(t) &= \int_0^1 K(s,t)\ f(x_1(s), x_2(s), s), \\
x_2(t) &= g(x_1(t), x_2(t), t)
\end{aligned} \tag{5.1}$$

assuming the following assumptions:

(V$_1$). $K : [0,1] \times [0,1] \to L(R^n, R^k)$ is measurable and fulfills

$$m^2 := \sup_{s \in [0,1]} \int_0^1 ||K(s,t)||^2 dt < \infty$$

where $||.||$ is the induced matrix norm to any norm $|.|$ in R^k.

(V2). $f : R^n \times R^m \times [0,1] \to R^k$ and $g : R^n \times R^m \times [0,1] \to R^m$ satisfy the Caratheodory condition.

(V3). There are nonnegative constants c_1, c_2 and functions $\mu_1, \mu_2 \in L^2([0,1], R)$ such that $\forall$ $(x_1, x_2) \in R^n \times R^m$ and f.a.a. $t \in [0,1]$

$$\begin{aligned} |f(x_1,x_2,t)| &\leq \mu_1(t) + c_1(|x_1| + |x_2|) \\ |g(x_1,x_2,t)| &\leq \mu_2(t) + c_2(|x_1| + |x_2|). \end{aligned}$$

(V4). There are nonnegative constants k_1, k_2, l_1, l_2 such that $\forall$ $x_1, x_2 \in R^n, y_1, y_2 \in R^m, t \in [0,1]$

$$\begin{aligned} |f(x_1,x_2,t) - f(\bar{x}_2,\bar{x}_2,t)| &\leq k_1|x_1 - \bar{x}_1| + k_2|x_2 - \bar{x}_2|, \\ |g(x_1,x_2,t) - g(\bar{x}_1,\bar{x}_2,t)| &\leq l_1|x_1 - \bar{x}_1| + l_2|x_2 - \bar{x}_2| \end{aligned} \tag{5.2}$$

(V5). $2mk_1 < 1, \quad 2l_2 - 4m(l_2k_1 - l_1k_2) < 1.$

Using Theorem 3.1 we can establish the following result.

Theorem 5.1 *Assume the hypotheses* $(V_1) - (V_5)$ *hold. Then system* (5.1) *has a unique solution.*

Proof. Let $X_1 := L^2([0,1], R^n)$, $X_1 := L^2([0,1], R^m)$ equipped with the usual norm

$$||x_i||_i := \int_0^1 |x_i(t)|^2 dt, i = 1, 2.$$

The hypotheses (V_2) and (V_3) imply that the Nemyzkii operators F and G defined by

$$\begin{aligned} F(x_1,x_2)(t) &:= f(x_1(t), x_2(t), t), \\ G(x_1,x_2)(t) &:= g(x_1(t), x_2(t), t) \end{aligned}$$

are continuous mappings from $X_1 \times X_2$ into X_1 and X_2 resp. [2]. Let us introduce the operator H by

$$H(x_1,x_2)(t) := \int_0^1 K(s,t)\ F(x_1,x_2)(s)ds.$$

Then the system (5.1) can be represented in the form

$$\begin{aligned} x_1 &= H(x_1,x_2), \\ x_2 &= G(x_1,x_2). \end{aligned}$$

From (V_1) and from the property of F mentioned above we get $H : X_1 \times X_2 \to X_1$.

After some straightforward calculations we have

$$\begin{aligned} \|H(x_1,x_2)-H(\bar{x}_1,\bar{x}_2)\|_1 &\leq 2mk_1\|x_1-\bar{x}_1\|_1+2mk_2\|x_2-\bar{x}_2\|_2, \\ \|G(x_1,x_2)-G(\bar{x}_1,\bar{x}_2)\|_2 &\leq 2l_1\|x_1-\bar{x}_1\|_1+2l_2\|x_2-\bar{x}_2\|_2. \end{aligned}$$

It is easy to verify that under the assumptions $(V_1)-(V_5)$ the hypothesis (H_1) of Theorem 3.1 holds. Thus, system (5.1) has a unique solution under the hypotheses above, q.e.d. □

Concerning the approximation of (x_1^*, x_2^*) we obtain from Theorem 4.1

Theorem 5.2 *Assume the hypothese of Theorem 5.1 hold. Then the diagonalized iterative scheme*

$$\begin{aligned} x_1^n &= H(x_1^{n-1}, x_2^{n-1}), \\ x_2^n &= G(x_1^n, x_2^{n-1}) \end{aligned}$$

converges to $(x_1^*, x_2^*) \in X_1 \times X_2$ *for any initial guess* $(x^0, y^0) \in X_1 \times X_2$.

The following application shows that under some additional conditions the global lipschitz condition (3.2) can be relaxed. We consider the nonautonomous differential system

$$\begin{aligned} \frac{dx_1}{dt} &= A_1(t)x_1 + f_1(x_1, x_2, t) \\ \frac{dx_2}{dt} &= A_2(t)x_2 + f_2(x_1, x_2, t). \end{aligned} \tag{5.3}$$

under the following assumptions

$(\mathbf{A_1})$. For $i = 1, 2$, $A_i : R \to L(R^{n_i}, R^{n_i})$ is continuous and 2π - periodic.

Let $X_i(t)$ be the fundamental matrix of the system

$$\frac{dx_i}{dt} = A_i(t)x_i, \quad i = 1, 2$$

satisfying $X_i(0) = I_{n_i}$ where I_{n_i} represents the $n_i \times n_i$ unit matrix.

$(\mathbf{A_2})$. For $i = 1, 2$, $X_i(2\pi) - I_{n_i}$ is invertible, i.e., 1 is not an eigenvalue of the monodromy matrix $X_i(2\pi)$.

Hypothesis $(\mathbf{A_2})$ implies the existence of the *Green's* matrix

$$G_i(t,s) := \begin{cases} X_i(t)[I_{n_i} - X_i(2\pi)]^{-1}X_i^{-1}(s) & \text{for } 0 \leq s \leq t \leq 2\pi, \\ X_i(t)[I_{n_i} - X_i(2\pi)]^{-1}]X(2\pi)X_i^{-1}(s) & \text{for } 0 \leq t \leq s \leq 2\pi. \end{cases} \tag{5.4}$$

Let D_1 and D_2 are open bounded regions in R^{n_1} and R^{n_2} resp., $D := D_1 \times D_2$.

(A_3). For $i = 1, 2$, $f_i : D \times R \to D_i$ is continuous, 2π - periodic in t and such that the initial problem to (1.1) has a unique solution.

For $i = 1, 2$, let $C_i^{2\pi}$ be the metric space of 2π-periodic continuous functions mapping R into G_i where the metric is defined by means of the usual maximum norm $||.||_i$. We define $C^{2\pi}$ by $C^{2\pi} := C_1^{2\pi} \times C_2^{2\pi}$, in the sequel we will define an appropriate metric in $C^{2\pi}$. For $x \in C^{2\pi}$ we define for $i = 1, 2$ the operator T_i by

$$T_i x(t) := \int_0^{2\pi} G_i(t,s) f_i(x(s), s) ds, \quad i = 1, 2.$$

According to the assumptions $(A_1) - (A_3)$ we have $T_i : C^{2\pi} \to C_i^{2\pi}$ for $i = 1, 2$. Let $T := (T_1, T_2)$. Then, the problem of the existence of a harmonic solution of (1.1) is equivalent to the existence of a fixed point of the operator T in $C^{2\pi}$. If we know an approximation q of some 2π-periodic solution of (5.3) then we can derive a result on the existence of a 2π-periodic solution which yields at the same time an information about the location of this solution. From that reason we assume

(A_4). There is a function $q \in C^{2\pi}$ and a positive number r such that

(i). $Tq \in C^{2\pi}$.

(ii). $\forall t \in [0, 2\pi]$, the ball K_r^t defined by $K_r^t := \{x \in R^{n_1+n_2} : |x - Tq(t)| \leq r\}$ belongs to D, Let $K_r := \cup_t K_r^t$. $\forall x, \bar{x} \in K_r$ $f := (f_1, f_2)$ satisfies

$$\begin{aligned} |f_1(x_1, x_2, t) - f_1(\bar{x}_1, \bar{x}_2, t)| &\leq l_{11}|x_1 - \bar{x}_1| + l_{12}|x_2 - \bar{x}_2| \\ |f_2(x_1, x_2, t) - f_2(\bar{x}_1, \bar{x}_2, t)| &\leq l_{21}|x_1 - \bar{x}_1| + l_{22}|x_2 - \bar{x}_2| \end{aligned} \tag{5.5}$$

where $|.|$ is the euclidian norm and all l_{ij} are assumed to be strictly positive.

(iii). Let m_1, m_2 be positive numbers such that for $t \in [0, 1]$

$$||G_i(t,s)|| := \max_{s \in [0,1]} \sqrt{\sum_{j,k=1}^{n_i} g_{i,jk}^2(t,s)} \leq m_i.$$

The spectral radius ϱ of the matrix

$$L = \begin{pmatrix} l_{11}m_1 & l_{21}m_1 \\ l_{21}m_2 & l_{22}m_2 \end{pmatrix}$$

is less than 1.

(iv). We denote by $C^{2\pi}(K_r)$ the subset of all functions in $C^{2\pi}$ mapping into K_r and introduce in $C^{2\pi}(K_r)$ a norm $|||.|||$ by

$$|||.||| := \eta_1||.||_1 + \eta_2||.||_2$$

where (η_1, η_2) is the eigenvector of L to the eigenvalue ϱ.

$$|||Tq - q||| \leq \frac{1 - \varrho}{\varrho} r.$$

Theorem 5.3 *Assume the hypotheses* $(A_1) - (A_4)$ *are valid. Let* $S_r := \{x \in C^{2\pi} : |||x - Tq||| \leq r\}$. *Then (5.3) has a unique harmonic solution* p *in* S_r *and the iteration scheme (4.10) with* $x^0 \in S_r$ *converges to* p.

Proof. Let $x \in S_r$. Then we can prove under the assumptions above

$$|||Tx - Tq||| \leq \varrho|||x - q||| \leq \varrho(|||x - Tq||| + |||Tq - q|||) \leq r,$$

that is, T maps S_r into itself. Analogously we can prove that T is strictly contractive in S_r. □

Remark 5.4 *If we replace* L *by the matrix* $\tilde{L}$ *where*

$$\tilde{L} = \begin{pmatrix} l_{11}m_1 & l_{21}l_{11}m_1 \\ l_{21}m_2 & (l_{22} + l_{21}l_{12})m_2 \end{pmatrix}$$

then the iteration scheme (3.6) converges to the harmonic solution p.

Remark 5.5 *If we represent* p *as a trigonometric series and approximate* p *by a truncated series (projection to a finite sum), then we can use the iteration scheme (3.7) to approximate* p. *The realization of that procedure consists in solving systems of nonlinear equations (computation of Fourier coefficients) of increasing order.*

References

[1] K. Burrage, *Parallel and Sequential Methods for Ordinary Differential Equations.* Oxford 1995.

[2] S. Fučik, *Solvability of nonlinear equations and boundary value problems.* Dordrecht, 1980.

[3] H. Gajewski, *Über die Konvergenz von Diagonalfolgen bei nichtlinearen Funktionalgleichungen.* Monatsber. DAW 10, 657–666 (1968).

[4] E. Lelarasmee, A.E. Ruehli, A.L. Sangiovanni-Vincentelli, *The waveform relaxation method for the time-domain analysis of large scale integrated circuits.* IEEE Trans. Computer-aided design, vol CAD-1, 131–145 (1982).

[5] U. Miekkala, O. Nevanlinna, *Convergence of dynamic iteration methods for initial value problems.* Siam J. Sci. Stat. Comput., 459–482 (1987).

[6] J.M. Ortega, W.C. Rheinboldt, *Iterative solution of nonlinear equations in several variables.* Academic Press, SanDiego, 1970.

[7] V.V. Petryshyn, *On the iteration, projection and projection-iteration methods in solution of nonlinear functional equations.* J. Math. Anal. Appl. 21, 575–607 (1968).

[8] K.R. Schneider, *Näherungsverfahren zur Ermittlung periodischer Lösungen von Systemen nichtlinearer periodischer Differentialgleichungen.* Computing 10, 63–82 (1972).

[9] K.R. Schneider, *A remark on the waveform relaxation method.* Intern. J. Circuit Theory and Appl. 19, 101-104 (1991).

[10] K.R. Schneider, *A note on the waveform relaxation method.* Syst. Anal. Modelling Simulation 8, 599–605 (1991).

[11] K.R. Schneider, I. Bremer, *A remark on solving large systems in function spaces.* Aplikace Mathematiky 35, 494–498 (1990).

[12] A.R. Secchi, M. Morari, E.C. Biscaia Jr., *The waveform relaxation method in the concurrent dynamic process simulation.* Computers chem. Enng. 17, S453–S465 (1993).

[13] A. Skjellum, S. Mattison, *Waveform relaxation for concurrent dynamic of distillation columns.* Third Conf. Hyp. Con. Computers at Appl., G. Fox (Ed.), N.Y. ACM, 1062–1071 (1988).

[14] S. Vandewalle, D. Roose, *The parallel waveform relaxation multigrid method.* Proc. Third SIAM Conf. on Parallel Process. for Sci. Comp., Los Angeles, Dec. 1987.

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