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Gelation, hydrodynamic limits and uniqueness in cluster coagulation processes

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Abstract

We consider the problem of gelation in the cluster coagulation model introduced by Norris [*Comm. Math. Phys.*, 209(2):407-435 (2000)], where clusters take values in a measure space E, and merge to form a new particle z according to a transition kernel $K(x, y, \mathrm{d}z)$. This model is general enough to incorporate various inhomogenieties in the evolution of clusters, for example, their shape, or their location in space. We derive general, sufficient criteria for stochastic gelation in this model, and for trajectories associated with this process to concentrate among solutions of a generalisation of the Flory equation; thus providing sufficient criteria for the equation to have gelling solutions. As particular cases, we extend results related to the classical Marcus-Lushnikov coagulation process and Smoluchowski coagulation equation, showing that reasonable 'homogenous' coagulation processes with exponent $\gamma > 1$ yield gelation; and also, coagulation processes with kernel $\alpha(m, n) \geq (m \wedge n) \log(m \wedge n)^{3+\varepsilon}$ for $\epsilon > 0$. In another special case, we prove a law of large numbers for the trajectory of the empirical measure of the stochastic cluster coagulation process, by means of a uniqueness result for the solution of the aforementioned generalised Flory equation. Finally, we use coupling arguments with inhomogeneous random graphs to deduce sufficient criterion for strong gelation (the emergence of a particle of size $O(N)$).

Contents

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1 Introduction

Models of coagulation arise widely in many scientific models, in areas ranging from physical chemistry (in the formation of polymers), to astrophysics (in modelling the formation of galaxies). A natural model for coagulation would have particles diffusing in space, with 'larger' particles being slower than 'smaller' ones. A well-known, classical, 'mean-field' approximation to this model from Smoluchowski [39] models a system of particles merging at a rate $\bar{K}(x,y)$ (for an appropriate function $\bar{K}(x,y)$) where x and y are masses of the particles. A finite, discrete approximation to the latter is known as the *Marcus-Lushnikov model* [29, 19, 28]. The limiting behaviour of the particle masses as the number of particles tend to infinity in the Marcus-Lushnikov model is generally expected to be encoded by a set of infinitely many differential equations (or measure-valued differential equations) known as the *Smoluchowski* or *Flory* equations.

Particular cases of the Marcus-Lushnikov model are closely related to other stochastic models: the case $\bar{K}(x,y) = 1$ corresponds to the Kingman's coalescent [27], the case $\bar{K}(x,y) = x+y$ has multiple interpretations, including being related to Aldous' continuum random tree [1, 5, 4, 9] (see also [10]), whilst the case $\bar{K}(x,y) = xy$ is closely related to the Erdős–Rényi random graph (see, for example, [25, 3, 2, 8]). We refer the reader to the review paper [6] for a more general overview (although remark that there has been a lot of progress made over the last 25 years). We also remark that in this paper, we are interested in models of *pure coagulation*; whilst a lot of work in the literature also allows for the *fragmentation* of particles.

A natural question of interest related to coagulation processes, is whether or not at some time $t > 0$, there exists the formation of *macroscopic* or *giant* particles. Motivated by the application to polymer chemistry, this is known as *gelation*. Gelation is generally defined as whether a solution of the Smoluchowski (or Flory) equation fails to 'conserve mass', which means, intuitively, that mass is lost to 'infinite' particles. This is closely linked to a loss of mass in 'large particles' in the Marcus-Lushnikov model as long as one knows that the trajectories associated with the model concentrate around the associated solution of the equation. This fact was first used by Jeon to prove existence of gelling solutions to the Smoluchowski equations [26]. In order to show this concentration, however, one needs to show that the associated trajectories of the Marcus-Lushnikov process concentrate around this equation, which may not always be the case after gelation.

Some general criteria for this concentration, including a weak law of large numbers were proved by Norris [32, 33]. A large number of related results deal with the Smoluchowski equation directly, proving sufficient conditions for mass conservation (generally when $\bar{K}(x, y) \leq x + y$), or gelation, sometimes also allowing for the fragmentation of particles; see, for example, [30, 40, 13, 14], to name a few. It may be the case, that, once gelation occurs, the 'gel' or macroscopic particles, interact with the microscopic ones, in such a way that a correction term is required in the Smoluchowkski equation, this is known as the Flory equation. A weak law of large numbers of the Marcus–Lushnikov process to this equation in a particular case was first proved by Norris [33], and later, concentration of trajectories around solutions of this equation were proved in [18, 36].

Despite this large body of literature, relatively less is known about models incorporating inhomogenieties of particles; for example, their shape, velocity, or location in space. A framework introduced by Norris in [33], he called the *cluster coagulation model*, allows one to incorporate these features in a rather general way (see also [31] for a variant that incorporates diffusion of clusters). However, apart from a particular special case of the model [22], criteria for gelation in this model are lacking, despite being of interest from both the perspective of applications in physics, and mathematically. We do note that there is a number of results related to other models incorporating the movement of particles as Brownian motions in space [20, 21, 44, 37], or as particles jumping across two sites [38, 43]; but we are not aware of any general results concerning criteria for gelation in these models, apart from the interesting *induced gelation* effect in a particular case of the two-site model [12]. More recently, from the perspective of analysis, a strain of research has focused on the study of multi-component generalisations of the Smoluchowski coagulation equations, where the mass variable is substituted by a variable in a d-dimensional Euclidean space, and there is an additional source term corresponding to the system not being in equilibrium (see for example, [17, 16, 41] and reference therein). A different approach to analyse a related spatial coagulation model and the question of gelation, using Poisson point processes and large deviations, is in progress in [7].

1.1 Overview on our contribution

Our goal in this paper, is to study gelation, and hydrodynamic limits associated with the cluster coagulation model. The main novelty of this paper consists in the following: First, in Theorem 3.2 (and following corollaries), by building upon, and generalising concepts introduced by Jeon [26] (see also [36]) we are able to provide sufficient criteria for gelation in the general setting of the cluster coagulation model; thus providing criteria that may incorporate many features of a cluster, not only the mass. In the process, we are able to provide improved criteria for gelation in the classical Markus-Lushnikov process, showing as a particular case, that reasonable 'homogenous' coagulation processes with exponent $\gamma > 1$ yield gelation; a generally accepted scientific principle, that, as far as we are aware, has not been proven rigorously (see, e.g. [6, 12, 42]). We also show, that, if for $\varepsilon > 0$, $\bar{K}(x, y) \geq (x \wedge y)(\log(x \wedge y))^{3+\varepsilon}$, gelation occurs (see Corollary 3.6). When the state space also incorporates, for example, the position of the particles, our criteria guarantees gelation for kernels that are products of a function decreasing in the distance between particles, and of a gelling kernel of the masses (see Example 3.4).

Second, we define a generalised Flory equation that is formulated in terms of a 'conserved quantity'; corresponding to an invariant associated with the cluster coagulation process. From the perspective of applications, this quantity may correspond to the total mass of the system, or, depending on the setting, for example, the 'centre of mass' or 'momentum'. Introducing this notion allows us to go beyond the setting of [32], and to prove existence of solutions for the limiting equation under weaker assumptions (Theorem 3.5). Combining this with our gelation criteria, we prove the existence of gelling solutions for such equations, under suitable assumptions on the kernel. We also obtain a uniqueness result for this *multi-type Flory equation* in a case when the kernel is 'eventually conservative' (i.e., kernels that satisfy the condition of Theorem 3.7), extending the *eventual multiplicativity* property introduced in [33]. Such a uniqueness result implies a *law of large numbers* for the paths of the stochastic cluster coagulation process (Corollary 3.8). The approach we use in this part of the paper, whilst well-established (using weak-compactness, and martingale techniques), allow for relatively few assumptions on the underlying space; we only assume that the clusters take values in a σ-compact metric space E. This means that our results encapsulate existing generalisation of the Smoluchowski equation in the literature, including those of [22, 23].

Finally, we use coupling arguments with inhomogeneous random graph models to deduce, sufficient criteria for 'strong gelation' to take place; referring to the emergence of a particle of order N. These arguments extend the one-to-one coupling used in [22].

The rest of the paper will be structured as follows.

1 In Section 2 we introduce the model, including the definition of a generalised Flory equation with a *conserved quantity* in Definition 2.1. Given the very general nature of the cluster coagulation model, in Section 2.1.2 we give examples of natural models that fit into this framework and, through the paper, we illustrate how our results apply to these examples. In Section 2.3, we revise some key concepts regarding gelation (i.e. the notions of *strong gelation* and *stochastic gelation*), slightly modifying previous definitions to our setting, if needed. We state a result, Theorem 2.2, that links the notion of gelation for the stochastic particle system to the one for the corresponding limiting equation (whose proof, almost identical to [Theorem 5, [26]], we omit in this paper).

2 In Section 3 we state the main theorems of this paper. Section 3.1 deals with results related to stochastic gelation, while Section 3.2 deals with concentration of trajectories associated with the process along solutions of the multi-type Flory equation. In this section, in particular, the notion of 'conserved quantities' is important, defined in Definition 3.1. In Section 3.2.4 we state sufficient criteria for uniqueness of solutions

of the multi-type Flory equation. Finally, Section 3.3 is concerned with strong gelation, deduced via a coupling with inhomogeneous random graphs.

3 Finally, Section 4 deals with the proofs of the main results, with Sections 4.1, 4.2 and 4.3 containing the proofs of results stated in Sections 3.1, 3.2 and 3.3 respectively.

2 The cluster coagulation process and multi-type Flory equation

2.1 Definition of the process

In this paper we consider the *cluster coagulation process*, introduced by James Norris in [33]. Recall that in the cluster coagulation process, one begins with a configuration of *clusters* in a measurable space $(E, \mathcal{B})$. Associated with a cluster $x \in E$ is a *mass function* $m : E \to (0, \infty)$. Another important quantity associated with the process is a *coagulation kernel* $K : E \times E \times \mathcal{B} \to [0, \infty)$, which satisfies the following:

1 For all $A \in \mathcal{B}$ $(x, y) \mapsto K(x, y, A)$ is measurable,

2 For all $x, y \in E$ $K(x, y, \cdot)$ is a measure on E,

3 *symmetric:* for all $A \in \mathcal{E}, x, y \in E$ $K(x, y, A) = K(y, x, A)$,

4 *finite:* for all $x, y \in E$ $\bar{K}(x, y) := K(x, y, E) < \infty$

5 *preserves mass:* for all $x, y \in E$, $m(z) = m(x) + m(y)$ for $K(x, y, \cdot)$-a.a. $z \in E$.

Suppose that we begin with a configuration of clusters labelled by an index set I. Then,

- to each pair of clusters $x, y \in E$, we associate an exponential clock (exponential random variable) with parameter $\bar{K}(x, y)$;
- upon the elapsure of the next exponential random variable in the process, corresponding to the pair x and y, say, the clusters x and y are removed and replaced by a new cluster $z \in E$, sampled according to the probability measure

$$\frac{K(x, y, \cdot)}{\bar{K}(x, y)}. \tag{1}$$

2.1.1 The infinitesimal generator associated with the process

In this paper, we will consider the process as depending on a parameter $N \in \mathbb{N}$, which one may consider as (up to random fluctuations) the total *initial mass* of the system, and analyse the process as in the limiting regime as $N \to \infty$. We consider the configuration of clusters at time t as being encoded by a *random point measure* $\mathbf{L}_t^{(N)}$ on E, so that, for any set $A \subseteq E$, $a \in (0, \infty)$, $\mathbf{L}_t^{(N)}(A \cap m^{-1}([a, \infty)))$ denotes the random number of clusters of mass at least a belonging to A. We denote by $\mathcal{M}_+(E)$ the set of finite, positive measures on E (also defining $\mathcal{M}_+(E \times E)$ in a similar manner). We may then consider the process as a measure-valued Markov process, whose infinitesimal generator $\mathcal{A}$ is defined as follows: for any bounded measurable test function $F : \mathcal{M}_+(E) \to \mathbb{R}$, we have

$$\mathcal{A}F(\mathbf{L}_t) = \frac{1}{2}\int_{E\times E\times E} \mathbf{L}_t(\mathrm{d}x)\left(\mathbf{L}_t - \delta_x\right)(\mathrm{d}y) K(x, y, \mathrm{d}z)\left(F(\mathbf{L}_t^{(x,y)\to z}) - F(\mathbf{L}_t)\right), \quad (2)$$

where $\mathbf{L}^{(x,y)\to z} := \mathbf{L} + (\delta_z - \delta_x - \delta_y)$. Note that, as $\mathbf{L}_t^{(N)}$ is assumed to be a point measure, the above integral is always with respect to a positive measure. The measure $\mathbf{L}^{(x,y)}$ describes the configuration of the system after a coagulation involving clusters the two clusters $x, y \in E$ and ending with one cluster z with $m(z) = m(x) + m(y)$, for $K(x, y, \cdot)$-a.a. $z \in E$. Note the factor $\frac{1}{2}$ in front of the generator, present to ensure that the total rate at with clusters x and y interact is $\bar{K}(x, y)$ (and not $2\bar{K}(x, y)$).

In this paper, we will be interested in the existence of gelling solutions of the following extension of the Smoluchowski equation, which we refer to as the *multi-type Flory equation*. Included in this equation, is a function ϕ, which one may regard as a *conserved quantity* of the system (see Definition 3.1 for more details).

We say that a measure valued process $(\mathbf{u}_t,)_{t\geq 0}$, taking values in $\mathcal{M}_+(E)$ is a solution of the *multi-type Flory equation* with *conserved quantity* ϕ, and initial condition $\mathbf{u}_0$ if it solves the following measure valued differential equation (see Definition 2.1 for a more formal definition):

$$\mathbf{u}_t - \mathbf{u}_0 = \int_0^t \left[Q^+(\mathbf{u}_s) - Q^-(\mathbf{u}_s)\right] \mathrm{d}s; \quad (3)$$

where $Q^+(\mathbf{u}_s)$ and $Q^-(\mathbf{u}_s)$ are measures defined such that, for appropriate test functions $J \in C_c(E; \mathbb{R})$,

$$\int_E J(y) Q^+(\mathbf{u}_s)(\mathrm{d}y) := \frac{1}{2}\int_{E\times E\times E} J(z) K(x, y, \mathrm{d}z)\mathbf{u}_s(\mathrm{d}x)\mathbf{u}_s(\mathrm{d}y), \quad (4)$$

and

$$\int_E J(y) Q^-(\mathbf{u}_s)(\mathrm{d}x) := \int_{E\times E} J(y)\bar{K}(x, y)\mathbf{u}_s(\mathrm{d}x)\mathbf{u}_s(\mathrm{d}y) + \int_E J(y) g_\infty(y)\mathbf{u}_s(\mathrm{d}y); \quad (5)$$

with g_∞ defined such that

$$g_\infty(y) := \int_E \phi(x, y)\mathbf{u}_0(\mathrm{d}x) - \int_E \phi(x, y)\mathbf{u}_s(\mathrm{d}x). \quad (6)$$

We expect, *a priori* that the Flory equation, or Smoluchowski equation encodes the behaviour of the process $(\mathbf{L}_{t/N}^{(N)}/N)_{t\geq 0}$, for N 'large'. The re-scaling of time is required to counter-balance the increase in the number of interactions as the initial mass of clusters grows with N. Thus, in general, we set

$$\bar{\mathbf{L}}_t^{(N)} := \mathbf{L}_{t/N}^{(N)}/N,$$

and generally (by abuse of notation, since such a limit may not be unique) use $(\bar{\mathbf{L}}_t^*)_{t\geq 0}$ to denote a weak limit (a limit along a subsequence) of the process.

2.1.2 Examples of cluster coagulation processes

The cluster coagulation process is general enough to encompass a large number of examples, depending on particular choices of the space E.

Example 2.1 (Classical kernel). If $E = (0,\infty)$, $K(x,y,\mathrm{d}z) = \bar{K}(x,y)\delta_{x+y}$, for a continuous symmetric function $\bar{K}(x,y)$, and the mass function $m(x) \equiv x$, the above process corresponds to the classical Marcus-Lushnikov process. In this case, if $\phi(x,y) := x\ell(y)$ for a function $\ell\colon E \to \mathbb{R}_+$, Equation 3 reduces to the classical Flory equation (see, for example, [Definition 2.2, [18]]). In this case, $g_\infty(y) = \ell(y)\left(\int_E x\mathbf{u}_0(\mathrm{d}x) - \int_E x\mathbf{u}_s(\mathrm{d}x)\right)$. When $\ell \equiv 0$, this equation corresponds to the classical Smoluchowski equation. If one interprets the term $\int_E x\mathbf{u}_0(\mathrm{d}x) - \int_E x\mathbf{u}_s(\mathrm{d}x)$ as corresponding to the mass lost to 'macroscopic' particles, the term g_∞ encodes the rate at which small clusters (of mass y) are lost in coagulation with macroscopic ones.

Example 2.2 (Historical Marcus-Lushnikov processes). One may extend E to incorporate not just the masses of clusters, but their histories. Indeed, we can take E to be a space where clusters x encode not only their mass, but the history of coagulations (a binary tree embedded in time) leading to the formation of that particle (see [23] for more details). For these processes, Jacquot in [23] proved a weak law of large numbers for the trajectories $(\bar{\mathbf{L}}_t^{(N)})_{t\in[0,T)}$ when the kernel is a function of the associated masses and it is bounded from above by a product of sublinear functions.

Example 2.3 (Bilinear coagulation processes). In the case that $E = [0,\infty)^d$, $A \in [0,\infty)^{d\times d}$ is a symmetric matrix with non-negative entries and $K(x,y,\mathrm{d}z) = (x^T Ay)\delta_{x+y}$, this model corresponds to the bilinear coagulation model studied in [22]. In that paper, the authors prove a weak law of large numbers for the particle system, showing that the trajectories converge to the unique solution of the Flory equation, and characterise explicitly the 'gelling time' (see Section 2.3), by using comparisons between this process and *inhomogeneous random graph processes.*[1]

Example 2.4 (Toy spatial coagulation models). A large number of toy models that incorporate information about the locations of clusters in 'space' fall into this framework. For example, we may take $E = \mathcal{S} \times (0,\infty)$ where $\mathcal{S} \subseteq \mathbb{R}^d$; in this case an element x of E coincides with a pair (p,n), $p \in \mathcal{S}, n \in (0,\infty)$ and we interpret p as the *location* of a cluster, and $n := m(p,n)$ as its mass. We may, then, assume that after a coagulation between clusters $x = (p,n), y = (s,o)$, the new cluster is placed at a new location, given by a measurable function of the original clusters, for example, the *centre of mass* $\frac{np+os}{n+o}$. Thus, in this case $K((p,n),(s,o),\cdot) = \delta_{\frac{np+os}{n+o},n+o}$. Another alternative would be a model in which the new particle occupies one of the locations of the previous particles with probability proportional to their mass, so that $K((p,n),(s,o),\cdot) = \frac{n}{n+o}\delta_{p,n+o} + \frac{o}{n+o}\delta_{s,n+o}$ (this is similar to the way the 'collision operator' is defined in the model of coagulating Brownian particles of [20]).

2.2 Some more notation, preliminaries and global assumptions

In this paper, we will regard E as a a σ-compact metric space with metric d. Given another space F, denote by $C_b(E;F)$, or resp. $C_c(E;F)$, the spaces of continuous functions $E \to F$ which are bounded, or resp., have compact support. In general, we write $C_b(E)$ (resp. $C_c(E)$) as a shorthand for $C_b(E;\mathbb{R})$ (resp. $C_c(E;\mathbb{R})$). We equip $\mathcal{M}_+(E)$ with a metric d that induces

[1]Actually, the model studied in [22] is slightly more general, in that clusters x belong to a metric space S, and $\bar{K}(x,y) = \pi(x)^T A\pi(y)$, where $\pi : \mathcal{S} \to \mathbb{R}^d$ is a continuous function. Clusters x may also change values according to a kernel J on S, in such a way that $\pi(x)$ is preserved.

the vague topology on $\mathcal{M}_+(E)$. [2] For any $n \in \mathbb{N}$, we denote by $\mathcal{E}_n$ the space

$$\mathcal{E}_n := \left\{ \mathbf{u} \in \mathcal{M}_+(E) : \int_E m(x)\,\mathbf{u}(\mathrm{d}x) \le n \right\}$$

and $\mathcal{E} = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$. Note that on $\mathcal{E}_n$ *weak* and *vague* topology coincides, while this is not the case on $\mathcal{E}$. We generally will regard $\mathcal{E}$ as the state space of the process, taking values in $D([0,\infty);\mathcal{E})$, the *Skorokhod space* of right-continuous functions $f : [0,\infty) \to \mathcal{E}$ with *Skorokhod metric* d_S induced by d.

For any element μ in $\mathcal{E}$, we denote $\|\mu\| := \int_E \mu(\mathrm{d}x)$. Moreover, given a measure $\mu \in \mathcal{M}_+(E)$ and a measurable function $f : E \to \mathbb{R}$, we denote by

$$\langle f, \mu \rangle := \int_E f(x)\mu(\mathrm{d}x).$$ [3]

We adapt the definition from [32, 33] of *solutions* for the generalised Flory equation to our setting.

Definition 2.1. Given a function $\phi : E \times E \to \mathbb{R}$, we say a map $t \mapsto \mathbf{u}_t \in \mathcal{M}_+(E)$, is a *solution of the multi-type Flory equation* with *conserved quantity* ϕ if the following are satisfied:

(i) for all Borel sets $A \subseteq E$ the map $t \mapsto \mathbf{u}_t(A) \colon [0,\infty) \to [0,\infty]$ is measurable;

ii) for all $f \in C_c(E)$, and $t \ge 0$, we have $\langle f, \mathbf{u}_0 \rangle < \infty$,

$$\int_0^t \int_{E\times E} f(y)\bar{K}(x,y)\mathbf{u}_s(\mathrm{d}x)\mathbf{u}_s(\mathrm{d}y)\mathrm{d}s < \infty; \quad \text{and} \quad \int_0^t \int_{E\times E} f(y)\phi(x,y)\mathbf{u}_0(\mathrm{d}x)\mathbf{u}_s(\mathrm{d}y)\mathrm{d}s < \infty; \tag{7}$$

iii) for all $f \in C_c(E)$ and $t \ge 0$, with Q^+ and Q^- as defined in (4) and (5),

$$\langle f, \mathbf{u}_t \rangle = \langle f, \mathbf{u}_0 \rangle + \int_0^t \langle f, Q^+(\mathbf{u}_s) - Q^-(\mathbf{u}_s) \rangle \,\mathrm{d}s. \tag{8}$$

iv) For each $x \in E$, $t \ge 0$ we have

$$\int_E \phi(x,u)\mu_t(\mathrm{d}u) \le \int_E \phi(x,u)\mu_0(\mathrm{d}u). \tag{9}$$

Note that point ii) ensures that the equation in iii) is well-defined (without terms of the form $\infty - \infty$).

At the level of the stochastic process, we denote by $\mathbb{P}_N(\cdot)$ and $\mathbb{E}_N[\cdot]$ probability distributions and expectations with regards to the trajectories of the process with generator $\mathcal{A}$ and (possibly random) initial condition $\bar{\mathbf{L}}_0^{(N)}$. Recall also, that the notation $\mathbf{L}_t^{(N)}$ refers to the trajectories of measures associated to the process, whilst $\bar{\mathbf{L}}_t^{(N)} = \mathbf{L}_{t/N}^{(N)}/N$ refers to its normalisation. In addition, we introduce the following notation for the regular conditional distribution and expectations when the initial condition $\bar{\mathbf{L}}_0^{(N)}$ is given by a (deterministic) measure $\boldsymbol{\pi} \in \mathcal{M}_+(E)$

$$\mathbb{P}_{N,\boldsymbol{\pi}}(\cdot) := \mathbb{P}_N\left(\cdot \,|\, \bar{\mathbf{L}}_0^{(N)} = \boldsymbol{\pi}\right) \quad \text{and} \quad \mathbb{E}_{N,\boldsymbol{\pi}}[\cdot] := \mathbb{E}_N\left[\cdot \,|\, \bar{\mathbf{L}}_0^{(N)} = \boldsymbol{\pi}\right]. \tag{10}$$

[2] We recall that the *vague* (respectively *weak*) topologies on $\mathcal{M}_+(E)$ are the smallest topologies that make the maps $\mu \to \int_E f(x)\mu(\mathrm{d}x)$ continuous for all $f \in C_c(E)$ (respectively $C_b(E)$). Note that, since E is separable and complete, $(\mathcal{M}_+(E), d)$ is a separable and complete space.

[3] We will generally only use this notation as a shorthand, and stick to the latter notation in proofs, as we believe it improves the clarity of the exposition.

Assumption 2.1. *In this paper, we will assume throughout that*

1 *as outlined above,* E *is a* σ*-compact metric space,*

2 *we have* $\bar{\mathbf{L}}_0^{(N)} = \sum_{i\in I} \frac{c_i\delta_i}{N}$ *for some finite set* $I \subseteq E$*,* $c_i \in \mathbb{N}$*, and there exists* $c' > 0$ *such that* $\sum_{i\in I} \frac{c_i}{N} \leq c'$ *almost surely,*

3 *there exists* $\mu \in \mathcal{E}$ *such that*

$$\bar{\mathbf{L}}_0^{(N)} \to \mu \tag{11}$$

weakly, in probability, and $\langle m, \mu\rangle > 0$*.*

2.3 Gelation, stochastic gelation and strong gelation

In this paper, we will generally be interested in the emergence of *gelation* in the cluster coagulation process, indicating the emergence of 'large' clusters. We first recall some definitions due to Intae Jeon [Definition 2, [26]].

For the cluster coagulation process, let τ_N^{α}, $\alpha \in (0, 1]$, denote the time of emergence of clusters of size of order N, i.e.,

$$\tau_N^{\alpha} := \inf\left\{t \geq 0 : \langle m\mathbf{1}_{m>\alpha N}, \bar{\mathbf{L}}_t^{(N)}\rangle > 0\right\}. \tag{12}$$

Definition 2.2. We have the following notions of gelation:

1 The *strong gelation time* for a cluster coagulation process is defined by

$$t_g^s := \inf\{t > 0 : \exists\, 0 < \alpha \leq 1 \text{ such that} \liminf_{N\to\infty} \mathbb{P}_N\left(\tau_N^{\alpha} \leq t\right) > 0\}.$$

2 The *gelation time* of a solution $\mathbf{u}_t$ of a multi-type Flory equation is defined by $t_g := \inf\{t \geq 0 : \langle m, \mathbf{u}_t\rangle < \langle m, \mathbf{u}_0\rangle\}$.

3 For a non-decreasing function $\psi : \mathbb{N} \to \mathbb{N}$, with $\lim_{N\to\infty} \psi(N) = \infty$, and $\delta > 0$, the (ψ, δ)-*stochastic gelation time* of the cluster coagulation process $(\bar{\mathbf{L}}^{(N)})_{N\in\mathbb{N}}$ is defined by

$$T_g^{\psi,\delta} := \inf\left\{t \geq 0 : \liminf_{N\to\infty} \mathbb{P}_N\left(\left\langle m\mathbf{1}_{m\leq\psi(N)}, \bar{\mathbf{L}}_t^{(N)}\right\rangle \leq \left\langle m, \bar{\mathbf{L}}_0^{(N)}\right\rangle - \delta\right) > 0\right\}.$$

4 We say strong gelation occurs, gelation occurs, or stochastic gelation occurs if, respectively, $t_g^s < \infty$, $t_g < \infty$ or $T_g^{\psi,\delta} < \infty$.

The following theorem, in the same flavour as [Theorem 5, [26]], provides criteria by which gelation properties of the finite cluster coagulation processes are reflected in their weak limits:

Theorem 2.2. *Suppose that* $m : E \to (0, \infty)$ *is continuous, Assumption 2.1 holds and there is a subsequence* $(\bar{\mathbf{L}}_t^{(N_k)})_{t\geq 0}$ *converging to a limit* $(\bar{\mathbf{L}}_t^{*})_{t\geq 0}$ *with continuous sample paths. For any* $t \geq 0$ *the following are equivalent:*

1 *There exists a non-decreasing function* $\psi : \mathbb{N} \to \mathbb{N}$ *with* $\psi(N) \leq N$*,* $\lim_{N\to\infty} \psi(N) = \infty$*, and* $\varepsilon > 0$ *such that*

$$\limsup_{k\to\infty} \mathbb{P}_{N_k}\left(\left\langle m\mathbf{1}_{m\leq\psi(N_k)}, \bar{\mathbf{L}}_t^{(N_k)}\right\rangle \leq \left\langle m, \bar{\mathbf{L}}_0^{(N_k)}\right\rangle - \varepsilon\right) > 0.$$

2 There exists $\varepsilon > 0$ such that

$$\mathbb{P}_N\left(\left\langle m, \bar{\mathbf{L}}_t^* \right\rangle \leq \langle m, \mu \rangle - \varepsilon\right) > 0.$$

■

Remark 2.5. The second statement of Theorem (2.2) may be trivially true if $\langle m, \mu \rangle = \infty$, but the first may not. Thus, using Theorem (2.2), the notion of stochastic gelation provides a means of extending the definition 'gelation' in the Smoluchowski/ Flory equation, to the case when $\langle m, \mu \rangle = \infty$.

We omit the proof of Theorem 2.2, as it is almost identical to the proof of [Theorem 5, [26]]. As a result of this proof, however, we are able to generally deduce criteria for the existence of gelling solutions in Flory equations by proving stochastic gelation in the cluster coagulation process.

3 Statements of main results and examples

3.1 Sufficient criteria for stochastic gelation in the coagulation process

In this section, we state general sufficient conditions for stochastic gelation in the cluster coagulation model. As this model is rather general, the conditions required are more technical than conditions for the classical Marcus-Lushnikov process. The main motivation for these results is that, in applications to non-equilibrium processes, inhomogenieties in the space E (corresponding to, for example, locations in space, the 'types' of cluster, or their velocities) may play a major role in whether or not gelation occurs. In Assumption 3.1, we can incorporate these inhomogenieties into the gelation criterion, assuming that we can 'partition' the space E (grouping together, for example, particles that are 'close', or of a similar 'type') in such a manner that we have sufficient lower bounds on the rate at which clusters belonging to a common partition interact. The techniques we use generalises those previously applied to the Marcus-Lushnikov process by Jeon [26], and Rezakhanlou [36].

In what follows, it will be helpful, to have a stopping time definition, of a 'gelling time' which is slightly different from the one in (12). Let $\psi : \mathbb{N} \to \mathbb{N}$ be non-decreasing with $\lim_{N\to\infty} \psi(N) = \infty$, and $\delta \in (0,1)$. For each $N \in \mathbb{N}$ we define the (ψ, δ)-*gelation time* $\tau_N(\psi, \delta)$ such that

$$\tau_N(\psi, \delta) := \inf\left\{t \geq 0 : \left\langle m\mathbf{1}_{m\geq\psi(N)}, \bar{\mathbf{L}}_t^{(N)} \right\rangle \geq \delta\right\},$$

i.e., the first time that the normalised total mass of clusters of size at least $\psi(N)$ exceeds δ. Note that choosing $\psi(N) = \alpha N$ we recover the stopping time defined in (12). Now, assume that the cluster coagulation process satisfies the following assumption.

Assumption 3.1. *Suppose that $(\bar{\mathbf{L}}_t^{(N)})_{t\in[0,\infty)}$ is a cluster coagulation process, with initial condition $\bar{\mathbf{L}}_0^{(N)} = \boldsymbol{\pi}$. For a function $\xi : \mathbb{N} \to \mathbb{N}$ and a non-decreasing function $\psi : \mathbb{N} \to \mathbb{N}$ with $\lim_{N\to\infty} \psi(N) = \infty$ we assume the following:*

1 *There exists a family of partitions* $\{\mathscr{P}^{(j)}\}_{j\in\mathbb{N}}$ *of* E*, where* $\forall j \in \mathbb{N}$ $\left|\mathscr{P}^{(j)}\right| \leq \xi(N)$*, and*

$$c'(P,j) > 0 \qquad \forall P \in \mathscr{P}^{(j)}, \tag{13}$$

where $c'(P,j) := \inf\{\bar{K}(x,y) : 2^j \leq m(x), m(y) < 2^{j+1}$ *and* $x, y \in P\}$.

2 *There exists a strictly decreasing sequence* $(\delta_i)_{i\in\mathbb{N}}$ *with* $\delta_i \in (0,1]$ *for all* i *and* $\delta_i \downarrow \delta > 0$ *such that:*

$$\limsup_{N\to\infty} \sum_{j=1}^{\log_2(\psi(N))} \frac{1}{(\delta_j - \delta_{j+1})^2} \left(\sum_{P\in\mathscr{P}^{(j)}} \frac{1}{c'(P,j)} \right) 2^j < \infty. \tag{14}$$

3 *We have* $\lim_{N\to\infty} \frac{\psi(N)\xi(N)}{N} = 0$.

Theorem 3.2. *Suppose that Assumption 3.1 is satisfied. Then, there exists a non decreasing function* ψ' *(depending on* $\psi, (\delta_j)_{j\in\mathbb{N}}, \xi$*), with* $\lim_{N\to\infty} \psi'(N) = \infty$ *such that:*

1 *If* $\boldsymbol{\pi}$ *is an initial condition such that* $\|\boldsymbol{\pi}\| < \infty$, $\langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle > 2\delta$*, and there exists a constant* $g_{\boldsymbol{\pi}}$*, (which may depend on* $\boldsymbol{\pi}$*) such that for all* $s \in [0,\infty)$

$$\int_{E\times E} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x)\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)\bar{K}(x,y)m(x) \leq g_{\boldsymbol{\pi}}; \tag{15}$$

then, $\limsup_{N\to\infty} \mathbb{E}_{\boldsymbol{\pi}}\left[\tau_N(\psi'(N),\delta)\right] < C$*, for a constant* C*, independent of* $\boldsymbol{\pi}$.

2 *In particular, if* (15) *is satisfied almost surely on the event* $\left\langle m\mathbf{1}_{m\geq 1}, \bar{\mathbf{L}}_0^{(N)}\right\rangle > \delta$*, and*

$$\liminf_{N\to\infty} \mathbb{P}_N\left(\left\langle m\mathbf{1}_{m\geq 1}, \bar{\mathbf{L}}_0^{(N)}\right\rangle > \delta\right) > 0$$

then stochastic gelation occurs in the process.

Remark 3.1. Since the bound in (15) may depend on $\boldsymbol{\pi}$, we expect it to be a rather weak restriction. However, even though $\bar{K}(x,y) < \infty$ for all $x, y \in E$, this condition may not always be satisfied. For example, it may be the case that $\bar{K}(x,x) = 0$, but $\lim_{y\to x}\bar{K}(x,y) = \infty$, and the coagulation dynamics may allow particles to become 'arbitrarily close' together.

Remark 3.2. In the conditions of Theorem 3.2, the condition on $\left\langle m\mathbf{1}_{m\geq 1}, \bar{\mathbf{L}}_0^{(N)}\right\rangle$ ensures that there is enough mass bounded from below to form a gel. The indicators $\mathbf{1}_{m\geq 1}$ may be replaced by any $\mathbf{1}_{m\geq c}$ for any $c > 0$. However, as we can re-scale the mass function in a cluster coagulation process to obtain another mass function; and the property of gelation is invariant under this re-scaling, we use the indicator $\mathbf{1}_{m\geq 1}$ without loss of generality.

The following corollary applies these conditions to the classical Marcus-Lushnikov process; providing criterion for stochastic gelation that improves those appearing in [26, Corollary 1] and [36, Theorem 1.3].

Corollary 3.3. *Suppose we are in the setting of Example 2.1. Then, if* $\bar{K}(x,y) \geq c'_j > 0$ *on* $2^j \leq x, y < 2^{j+1}$*, for any* $\delta \in (0,1)$ *there exists a sequence* $(\delta_i)_{i\in\mathbb{N}} \in (0,1]^{\mathbb{N}}$ *with* $\delta_i \downarrow \delta$ *such that*

$$\sum_{j=1}^{\infty} \frac{2^j}{c'_j(\delta_j - \delta_{j+1})^2} < \infty; \tag{16}$$

and $\langle m\mathbf{1}_{m\geq 1}, \mu\rangle > 0$*, stochastic gelation occurs in the coagulation process. In particular, when* $\langle m\mathbf{1}_{m\geq 1}, \mu\rangle > 0$*, stochastic gelation occurs if*

1. *We have* $\inf_{i\in[1,2]} \bar{K}(1,i) > 0$ *and for all* x, y *sufficiently large* $\bar{K}(cx, cy) = c^{\gamma}\bar{K}(x, y)$, *with* $\gamma > 1$

or, alternatively,

2. *there exists* $\varepsilon > 0$ *such that, for all* x, y *sufficiently large* $\bar{K}(x, y) \geq (x\wedge y) \log{(x \wedge y)}^{3+\varepsilon}$.

The following provides another example where the gelation criteria may be applied to a toy spatial model:

Example 3.3. Consider the toy spatial coagulation models, introduced in Example 2.4, where $E = \mathcal{S} \times (0, \infty)$. Suppose that we begin with N clusters (of mass 1, say), sampled i.i.d from the uniform distribution on the hypercube $\mathcal{S} = [0, 1]^d$, and consider the kernel,

$$\bar{K}((p, n), (s, o)) := \begin{cases} \frac{\kappa_0}{(||p-s||)^{\alpha}}, & \text{if } p \neq s \\ 0, & \text{otherwise;} \end{cases}$$

with $\kappa_0 \in (0, \infty)$ a constant and $\alpha > 0$. Note that if particles merge at a constant rate κ_0 (without influence of distance), it is well-known that gelation does not occur, however, in this model, the distance now plays an important role, and one may readily verify that if $\alpha/d > 1$, stochastic gelation occurs. Indeed, one may readily verify (for example, by using induction), that almost surely, the initial configuration of clusters is such that the distance between any two points is positive throughout the dynamics of the coagulation process, and hence Equation (15) is satisfied almost surely. Now, for each $j \in \{1, 2, \dots, \log_2(\psi(N))\}$ we take a partition of $\mathcal{S}$ that consists of $\xi(N)$ hypercubes with side-length $\frac{1}{(\xi(N))^{1/d}}$. Note that $c'(P, j) = \kappa_0 \min_{p,s\in P} \frac{1}{(||p-s||)^{\alpha}} \geq \xi(N)^{\alpha/d}$, for all P. Fix, for example, $\delta_j := \delta + 2^{-j}$; then the sum in (14) reads

$$\limsup_{N\to\infty} \frac{\xi(N)}{(\xi(N))^{\alpha/d}} \sum_{j=1}^{\log_2 \psi(N)} 2^{3j} \leq C \limsup_{N\to\infty} (\xi(N))^{1-\alpha/d}(\psi(N))^3 = \limsup_{N\to\infty} C\frac{\psi(N)^3}{\xi(N)^{\alpha/d-1}},$$

for some $C \in \mathbb{R}^+$. Now if $\frac{\alpha}{d} > 1$ and we set $\psi(N) := \xi(N)^{\frac{\alpha/d-1}{3}}$; we can choose any $\xi(N)$ such that $\lim_{N\to\infty} \xi(N) = \infty$, and that fulfils the third condition in Assumption 3.1.

Example 3.4. Consider the toy spatial coagulation models, introduced in Example 2.4, where $E = \mathcal{S} \times (0, \infty)$. A natural choice of kernel may be to choose a function that is a product of a non-increasing function of the distance d between clusters (clusters interact more quickly if they are closer together), and a function of their mass. In this manner, suppose we choose kernel of product form $\bar{K}((p, n), (s, o)) := h(d(p, s))W(n, o)$ where $h : [0, \infty) \to [0, \infty)$ is non-increasing and non-zero, and W is continuous and satisfies the conditions under which Corollary 3.3 applies. As h is bounded from below (by c_0 say) on an interval $[0, \varepsilon)$, for ε sufficiently small, and $\mathcal{S}$ is compact, we can choose a finite partition $\mathscr{P}$ consisting of open balls of radius ε, such that, if $P \in \mathscr{P}$ for all $x, y \in P$ we have $\bar{K}((p, n), (s, o)) \geq c_0W(n, o)$. By choosing $\mathscr{P}^{(j)} = \mathscr{P}$ for each j, one readily verifies (in a similar manner to the proof of Corollary 3.3) that all conditions in Assumption 3.1 are verified. Finally, if we begin, for example, with $|\pi| = N$ clusters of mass 1, the maximum value of $m(x)$ equals N for all $x \in E$, and we can easily verify condition (15) for $\bar{K}$, by setting $g_{\pi} = N \max_{(x,y)\in(\mathcal{S}\times[0,N])^2} \bar{K}(x, y)$.

3.2 Concentration of trajectories on solutions of multi-type Flory equations and uniqueness

As alluded in the Definition 2.1, an important feature of a multi-type Flory equation is a *conserved quantity* ϕ. This as a function $\phi : E \times E \to \mathbb{R}$ such that, for any x, the quantity $\langle \phi(\cdot, x), \bar{\mathbf{L}}_t^{(N)} \rangle$ is fixed for each $t > 0$. Perhaps the most natural conserved quantity is a coagulation process is *mass*, which corresponds to the choice of function $\phi(x, y) = m(x)$; as masses add upon coagulation, this is fixed along trajectories of the process. However, one may imagine, in models encoding more information about clusters, that there are other quantities conserved, reflecting, for example, the *centre of mass* of clusters in space, or the *momentum* of the system (see examples in Section 2.1.2).

Definition 3.1 (Conserved or sub-conserved quantities)**.** A function $\phi : E \times E \to \mathbb{R}$ is said to be *conservative* or a *conserved quantity* if for all $x, y, q \in E$, for $K(x, y, \cdot)$ a.a. z,

$$\phi(z, q) = \phi(x, q) + \phi(y, q). \tag{17}$$

It is, similarly, said to be *sub-conservative* if for all $x, y, q \in E$, for $K(x, y, \cdot)$ a.a. z,

$$\phi(z, q) \leq \phi(x, q) + \phi(y, q). \tag{18}$$

It is said to be *doubly sub-conservative* (similarly *doubly conservative*), if it is sub-conservative in the second argument in addition to the first, so that in addition for all $x, y, q \in E$, for $K(x, y, \cdot)$ a.a. z,

$$\phi(q, z) = \phi(q, x) + \phi(q, y). \tag{19}$$

Finally, if a function $\xi : E \to \mathbb{R}$ is such that $\phi(x, y) = \xi(x)$ is conservative (resp. sub-conservative), we also say ξ is conservative (resp. sub-conservative).

Remark 3.5. If ϕ is symmetric and conservative, it is also doubly conservative. Thus, some examples of doubly conservative functions include $\phi(x, y) = m(x)m(y)$, or bilinear functions for coagulation kernels of the type described in Example 2.3. Indeed, when $x, y \in [0, \infty)^d$ and we have $K(x, y, \mathrm{d}z) = \bar{K}(x, y)\delta_{x+y}$, then for any matrix A the function $\phi(x, y) = x^T A y$ is a doubly conservative function. In Example 2.3 this means that $\bar{K}$ itself is a doubly conservative function.

3.2.1 Conditions for tightness

Lemma 3.4. *Assume that the following hold.*

1 *There exists a doubly sub-conservative ϕ' such that $\bar{K} \leq \phi'$ pointwise.*

2 *We have*

$$\limsup_{N \to \infty} \mathbb{E}_N \left[\int_{E \times E} \bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N} \right) \phi'(x, y) \right] < \infty. \tag{20}$$

3 *There exists a doubly sub-conservative function $\phi'' : E \times E \to [0, \infty)$, such that for all $n \in \mathbb{N}$ the set*

$$\mathcal{E}_n^* := \left\{ \mathbf{u} \in \mathcal{M}_+(E \times E) : \int_E \mathbf{u}(\mathrm{d}x \times \mathrm{d}y)\phi''(x, y) \leq n \right\} \tag{21}$$

is compact, and ϕ'' satisfies (20)

Then the sequence of probability measures $(\mathbb{P}_N)_{N\in\mathbb{N}}$ is a tight family of probability measures on the Skorokhod space $D([0,\infty);\mathcal{E})$.

Example 3.6. A simple case in which Lemma 3.4 may be satisfied is when $\phi'(x,y) = \phi''(x,y) = m(x)m(y)$. In this case, if m is continuous, an argument applying Markov's inequality and Prokhorov's theorem shows that (21) is satisfied if the sets $\{x : m(x) \leq k\}$ are compact. Alternatively, in the setting of Example 2.3 (and Remark 3.5), bilinear kernels can be made to satisfy the second assumption, if, for example, the matrix A has non-zero entries, since the sets $\{x, y \in [0,\infty)^d : x^T A y \leq k\}$ are compact.

Example 3.7. Not that, if $\xi' : [0,\infty) \to [0,\infty)$ is continuous and sub-additive [4], the function $\phi'(x,y) = (\xi'(m(x)))(\xi'(m(y)))$ is doubly sub-conservative. This is the analogue of 'sublinear' function used by Norris in [33].

Lemma 3.4 shows that the probability measures $(\mathbb{P}_N)_{N\in\mathbb{N}}$ on the space $D([0,\infty);\mathcal{E})$ induced by the processes $(\bar{\mathbf{L}}_t^{(N)})_{t\in[0,\infty)}$ are tight; and thus, by Prokhorov's theorem, the collection of random trajectories $\left\{(\bar{\mathbf{L}}_t^{(N)})_{t\in[0,\infty)}, N \in \mathbb{N}\right\}$ contains weakly convergent subsequences. The following theorem gives criteria, under which any limit point of such a subsequence is concentrated on trajectories that solve the multi-type Flory equation.

3.2.2 Concentration of trajectories on the multi-type Flory equation

Theorem 3.5. *Assume that, the hypotheses of Lemma 3.4 are satisfied; and in addition:*

1 *The functions $\bar{K}$ and ϕ' appearing in Lemma 3.4 are continuous.*

2 *There exists a continuous, conservative function ϕ satisfying* (20)*, such that one of the following hold:*

   2.1 *For an increasing collection of sets $(C_k)_{k\in\mathbb{N}} \subseteq E$, with $\bigcap_{k\in\mathbb{N}} \overline{C_k^c} = \varnothing$ we have, for any compact $C' \subseteq E$*

$$\limsup_{k\to\infty} \sup_{x\in C_k^c,\, y\in C'} \left|\bar{K}(x,y) - \phi(x,y)\right| < \infty. \tag{22}$$

   2.2 *There exists a continuous doubly sub-conservative function ϕ^* satisfying Equation* (20)*, such that, for a collection of sets $(C_k)_{k\in\mathbb{N}} \subseteq E$, we have, for any compact $C' \subseteq E$*

$$\lim_{k\to\infty} \sup_{x\in C_k^c,\, y\in C'} \frac{\left|\bar{K}(x,y) - \phi(x,y)\right|}{\phi^*(x,y)} = 0. \tag{23}$$

3 *The limiting initial condition μ from* (11) *is such that for any compact set $C' \subseteq E$*

$$\lim_{N\to\infty} \sup_{y\in C'} \left| \int_E \bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x)\phi(x,y) - \int_E \mu(\mathrm{d}x)\phi(x,y) \right| = 0 \quad \textit{almost surely.} \tag{24}$$

Then, the limit $\bar{\mathbf{L}}^*$ of any weakly convergent subsequence of an asymptotically conservative coagulation process $(\bar{\mathbf{L}}_t^{(N)})_{t\geq 0}$ is, almost surely, a solution of the multi-type Flory equation with conserved quantity ϕ and initial condition μ (according to Definition 2.1).

[4] Recall that ξ' is sub-additive if $\xi'(x+y) \leq \xi'(x) + \xi'(y)$.

Examples 3.8. Some particular instances where Theorem 3.5 applies, are the following:

- We can take $\phi' = \phi'' = \phi^*$ in Theorem 3.5 and Lemma 3.4, with ϕ' satisfying (20). In this case, if there exist a nested sequence of compact sets $(C_k)_{k\in\mathbb{N}}$ such that $\bigcup_{k\in\mathbb{N}} C_k = E$ and $\phi'(x,y) > k$ on $(C_k \times C_k)^c$, one readily verifies the compactness in (21), by Prokhorov's theorem and Markov's inequality. Often, these sets $(C_k)_{k\in\mathbb{N}}$ can also be used in (22) or (23).

- For example, assuming continuity of $\bar{K}$ we can take

$$\phi'(x,y) = \xi'(m(x))\xi'(m(y)) \tag{25}$$

  for some continuous, sub-linear, positive function ξ', with $\lim_{k\to\infty} \xi'(k) = \infty$, and, assuming $m^{-1}([0,k])$ is compact, choose $C_k = m^{-1}([0,k])$.

- In a similar setting, we can choose $\phi'(x,y) = m(x) \wedge m(y)$, in which case (22) is satisfied. It may the case that

$$\int_{E\times E} m(x) \wedge m(y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) < \infty,$$

  but $\langle m, \mu\rangle = \infty$; and, as far as we are aware, this case is not covered by previous results appearing in the literature. In a similar vein, we can choose $\phi'(x,y) = \xi'(m(x) \wedge m(y))$ continuous, sub-linear, positive function ξ'.

- Alternatively, we may choose $\phi(x,y) = m(x)\ell(y)$ for some measurable function $\ell : E \to \mathbb{R}_+$, $\phi^*(x,y) = m(x)$, and $\phi'(x,y) = \xi'(m(x))\xi'(m(y))$, and $\lim_{N\to\infty}\langle m, \bar{\mathbf{L}}_0^{(N)}\rangle = \langle m, \mu\rangle$ almost surely (which may be used to show (24)). In the setting of the classical Marcus-Lushnikov process (Example 2.1), this example includes a mild strengthening of [Theorem 2.3, [18]], showing concentration of trajectories around the classical Flory equation. However, we allow for random initial conditions $(\bar{\mathbf{L}}_0^{(N)})_{N\in\mathbb{N}}$, and do not require $\xi'(x) \geq 1$.

Remark 3.9. If one considers the initial condition $\bar{\mathbf{L}}_0^{(N)} := \frac{\sum_{i=1}^N \delta_{X_i}}{N}$, where X_i are i.i.d samples from the limiting measure μ, if we choose ϕ' according to (25), by applying the strong law of large numbers, one can readily verify that Equations (20) is satisfied when $\langle \xi' \circ m, \mu\rangle < \infty$. In this case, the term δ_x/N appearing in Equation (20) is crucial for this argument to work, since it may be the case, for example, that $\langle \xi' \circ m\rangle < \infty$, but $\langle (\xi' \circ m)^2\rangle = \infty$.

Example 3.10. Consider the two toy spatial coagulation models introduced in Example 2.4 where $E = \mathcal{S} \times \mathbb{N}_0$ and $\mathcal{S} \subset \mathbb{R}^d$. Moreover fix $\alpha \in (0,1)$ and take $\bar{K}((p,n),(s,o)) := (no)^\alpha f(||p-s||)$, where $||\cdot||$ denotes the euclidean norm and $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a continuous function such that f is uniformly bounded. Then we have that

$$\lim_{m\to\infty} \frac{\bar{K}((p,n),(s,o))}{n} = 0$$

for any $p, s \in \mathcal{S}$ and $o \in \mathbb{N}_0$, and the function $\phi = 0$ satisfies the properties in (23) and (17).

Example 3.11. We consider again the spatial coagulation models introduced in Example 2.4. Suppose in addition that

$$\lim_{m\to\infty} \frac{\bar{K}((p,n),(s,o))}{n} = \ell(s,o) < \infty, \tag{26}$$

where ℓ is a continuous function that does not depend on the position p of the mass going to infinity; then, the function $\phi := n\ell(s,o)$ satisfies the properties in (23) and (17) for any choice of the location of the newly formed particle, which we can indicate by $X((p,m),(s,n))$. Indeed (26) implies condition (23), and we can also check that also (17) is satisfied:

$$\phi\left((X((p,m),(s,n)), m+n), (v,j)\right) = (m+n)\ell(v,j) = \phi((p,m)),(v,j)) + \phi((s,n),(v,j)). \tag{27}$$

3.2.3 Existence of gelling solutions to the multi-type Flory equation

The following is an immediate implication of Theorem 2.2, Theorem 3.2 and Theorem 3.5 (hence we omit providing an explicit proof):

Corollary 3.6. *Suppose a cluster coagulation process $(\bar{\mathbf{L}}_t^{(N)})_{t\geq 0}$ satisfies Assumption 3.1 and the conditions of Theorem 3.5, and that m is continuous. Then, there exists a gelling solution to the multi-type Flory equation (Definition 2.1), with conserved quantity ϕ and initial condition μ.*

Remark 3.12. Note that it may be the case that Assumption 3.1 is satisfied, but $\langle m, \mu\rangle = \infty$. In this case, if the conditions of Theorem 3.5 apply, trajectories of the cluster coagulation process still concentrate around solutions of a Smoluchowski equation; and thus it is natural to use the notion of *stochastic gelation* to define gelling solutions of such equations.

Remark 3.13. One of the novelties of Corollary 3.6 comes from the criterion for gelation under the conditions of Corollary 3.3. In particular, this confirms that a large class of homogeneous kernels, with exponent $\gamma > 1$ have gelling solutions, a well-known conjecture from scientific modelling literature [6, 12]. Previously, Wagner showed that the *mass flow process* associated with such models is *explosive*, a property conjectured to hold for all coagulation kernels with gelling solutions [12, 42].

Remark 3.14. If we have uniqueness of the solution of the associated Flory equation, then Theorem 3.5 also implies a weak law of large numbers for the cluster coagulation process, (since weak convergence to a constant implies convergence in probability). This has been shown by Norris [33] in the cases that either $\bar{K}(x,y) \leq m(x) + m(y)$, $\bar{K}(x,y) = \xi'(m(x))\xi'(m(y))$ where $\xi' : [0,\infty) \to [0,\infty)$ is continuous, sub-linear and $(\xi')^2$ is sub-linear, or if the kernel is 'eventually multiplicative' (meaning that, for some $R > 0$, $\bar{K}(x,y) = m(x)m(y)$ on $(m^{-1}([0,R]) \times m^{-1}([0,R]))^c$; see [page 411, [33]]). Notably, Norris also shows an exponential rate of convergence of the coagulation process to the limit, in a the restricted case of 'polymer models' when the limit is unique (see [Theorem 4.2 and Theorem 4.3, [33]]).

We note, however, that the setting we consider here is more general, and there are a wide range of kernels satisfying Equation (23) which are not eventually multiplicative. The natural extension of this result is therefore to consider kernels that are, in some sense, "eventually conservative".

3.2.4 Criteria for uniqueness for eventually conservative kernels

The following theorem applies to certain kernels $\bar{K}(x,y)$ which coincide with a conservative function outside a compact set, and are thus, in a sense, 'eventually conservative':

Theorem 3.7. *Suppose that $(\mu_t)_{t\geq 0}$ is a solution to the multi-type Flory equation (according to Definition 2.1) with conserved quantity ϕ and initial condition μ_0 such that $\|\mu_0\| < \infty$. Also, assume that, for each $k \in \mathbb{N}$ sufficiently large, the set*

$$D_k := \left\{x \in E : \int_E \phi(x,y)\mu_0(\mathrm{d}y) \leq k\right\}$$

is compact, with $\bigcup_{k\in\mathbb{N}} D_k = E$; we have $\bar{K}(x,y) \leq c'\phi(x,y)$ for some $c' > 0$ and for some R sufficiently large, $\phi(x,y) = \bar{K}(x,y)$ on $(D_R \times D_R)^c$. Then, the solution $(\mu_t)_{t\geq 0}$ is unique.

Corollary 3.8. *Under the hypotheses of Lemma 3.4, Theorem 3.5 and Theorem 3.7; if $(\mu_t)_{t\geq 0}$ denotes the associated unique solution to the multi-type Flory equation, we have*

$$(\bar{\mathbf{L}}^{(N)})_{t\geq 0} \longrightarrow (\mu_t)_{t\geq 0} \quad \textit{in probability, in } D([0,\infty);\mathcal{E}).$$

Remark 3.15. Note that, unlike in the context of Theorem 3.5, the condition that $\phi(x,y) = \bar{K}(x,y)$ on $(D_R \times D_R)^c$, combined with the symmetry of $\bar{K}$ implies that $\phi(x,y)$ is symmetric.

Remark 3.16. If we take $\phi(x,y) = m(x)m(y)$, one readily verifies that $\bar{K}(x,y)$ is eventually conservative in the sense of Theorem 3.7 if the sets $m^{-1}([0,R])$ as in Remark 3.14 are compact.

Remark 3.17. Note that, in the setting of Example 2.3, the kernel $\bar{K}(x,y) = x^T Ay$ is a symmetric conservative function, hence the uniqueness result, and weak law of large numbers from Theorem 3.7 and Corollary 3.8 extends the results of [22].

Remark 3.18. It is possible to also apply these results to more general spaces E; as long as E is a σ-compact metric space. This means that we can take E to be, for example, an appropriate (restricted) space of functions and the kernel to be of the form

$$K(f,g,\mathrm{d}h) = \delta_{\frac{m(f)f+m(g)g}{m(f)+m(g)}}\bar{K}(f,g).$$

With any kernel of this form, a symmetric, bilinear form gives rise to a symmetric, conservative function ϕ; for example, if $E = C([0,1];[0,1])$ we see that $\phi(f,g) = \int f(x)g(x)\mathrm{d}x$ is symmetric and conservative.

3.3 Strong gelation: couplings with inhomogeneous random graphs

In certain cases, we may deduce properties about the coagulation model by coupling with auxiliary processes. It is well-known that for the classical multiplicative kernel $K(x,y) = m(x)m(y)$, the cluster masses at time $t > 0$ are in one-to-one correspondence with the sizes of components of the Erdős–Rényi random graph with N vertices and $1-e^{-t/N}$ edge probability. Likewise, the work by Patterson and Heydecker [22] shows that for cluster coagulation models of a 'bilinear' type, the cluster masses are in one-to-one correspondence with an *inhomogenous random graph*. In both these cases, gelation coincides with the emergence of a 'giant component' in the associated random graph, and as the giant component is always of order N, strong gelation occurs, with an explicit description of the gelation time.

In this section, we extend these correspondences to produce, under certain assumptions on the coagulation kernel, a monotone coupling with an associated inhomogeneous graph model, hence providing sufficient criteria for strong gelation (resp. stochastic gelation), with an explicit lower bound (resp. upper bound) on the gelation time.

Definition 3.2. A coagulation process $(\mathbf{L}_t)_{t\geq 0}$ with kernel $K : E \times E \times \mathcal{B} \to\to [0,\infty)$ is *graph dominating* if it satisfies the following identity: for all $x, y, q \in E$, for $K(x, y, \cdot)$ a.a. z, we have

$$\bar{K}\,(z,q) \geq \bar{K}(x,q) + \bar{K}(y,q). \tag{28}$$

Conversely, a coagulation process is *graph dominated* if the opposite inequality holds: for all $x, y, q \in E$, for $K(x, y, \cdot)$ a.a. z, we have

$$\bar{K}\,(z,q) \leq \bar{K}(x,q) + \bar{K}(y,q). \tag{29}$$

Remark 3.19. Note the connection between graph dominated/dominating kernels and conserved quantities in Definition 3.1. Indeed, if a kernel is conservative, the cluster sizes are in one-to-one correspondence with the component sizes of an inhomogenous random graph.

We will also consider *mono-dispersed* coagulation processes, i.e. kernels with initial configuration consisting of clusters with mass 1. More precisely, we say a coagulation process $(\mathbf{L}_t^{(N)})_{t\geq 0}$ is *mono-dispersed*, if for each N

$$\mathrm{Supp}\Big(\mathbf{L}_0^{(N)}\Big) \subseteq \{x \in E : m(x) = 1\}. \tag{30}$$

Now, in order to state our theorem, recalling μ from (11), we define the operator $T_{\bar{K},\mu}$ such that, for any $f \in L^2(\mu)$, we have

$$T_{\bar{K},\mu} f(x) := \int_E f(y)\bar{K}(x,y)\mu(\mathrm{d}y), \tag{31}$$

and its operator norm:

$$\Sigma(\bar{K},\mu) := \|T_{\bar{K},\mu}\|_{L^2(\mu)} = \sup_{f\in L^2(\mu), \|f\|_{L^2(\mu)}=1} \left\|T_{\bar{K},\mu} f\right\|_{L^2(\mu)}. \tag{32}$$

We then have the following result.

Theorem 3.9. *Suppose $(\bar{\mathbf{L}}_t^{(N)})_{t\geq 0}$ is a coagulation process with mono-dispersed initial configuration, and that $t^* := \inf\{t > 0 : t\Sigma(K,\mu) > 1\}$. Then, recalling the definitions of t_g^s and $T_g^{\psi,\delta}$ from Definition 2.2*

1. *If $(\bar{\mathbf{L}}_t^{(N)})_{t\geq 0}$ is graph dominating, then strong gelation occurs, with $t_g^s \leq t^*$.*
2. *If $(\bar{\mathbf{L}}_t^{(N)})_{t\geq 0}$ is graph dominated then, for any $\psi : \mathbb{N} \to \mathbb{N}$ non-decreasing with $\lim_{N\to\infty} \psi(N) = \infty$, and for any $\delta > 0$ we have $T_g^{\psi,\delta} \geq t^*$.*

In fact, as we will see in Section 4.3, we can say a bit more: that the cluster sizes in the coagulation model *dominate* or are *dominated* by the component sizes of an associated inhomogeneous random graph model when the coagulation process is graph dominating or graph dominated respectively.

Example 3.20. Suppose that we are in the same setting as Example 2.4 with particles moving to the centre of mass and $\rho : \mathcal{S} \to [0,\infty)$ is an even (i.e., $\rho(p) = \rho(-p)$), concave function. Then, the coagulation process with kernel $\bar{K} : (\mathcal{S} \times \mathbb{N}_0)^2 \to [0,\infty)$ defined by

$$\bar{K}((p,m),(s,n)) := mn\rho(p-s)$$

is *graph dominating* . Indeed, one may readily compute that for all $q \in \mathcal{S}$:

$$\begin{aligned} \bar{K}\left(\left(\frac{mp+ns}{m+n}, m+n\right), (q,\ell)\right) &= (m+n)\ell\rho\left(\frac{mp+ns}{m+n} - q\right) \\ &= (m+n)\ell\rho\left(\frac{m(p-q)+n(s-q)}{m+n}\right) \\ &\geq (m+n)\ell\left(\frac{m}{m+n}\rho\,(p-q) + \frac{n}{m+n}\rho(s-q)\right) \\ &= m\ell\rho(p-q) + n\ell\rho(s-q) \\ &= \bar{K}((p,m),(q,\ell)) + \bar{K}((s,n),(q,\ell)). \end{aligned} \tag{33}$$

Conversely, if $\rho' : \mathcal{S} \to [0,\infty)$ is an even, convex function, then the kernel defined by $\bar{K}((p,m),(s,n)) := mn\rho'(p-s)$ is *graph dominated*.

Example 3.21. If $\mathcal{S}$ is a compact normed vector space with norm $\|\cdot\|$, then the function $\rho(\cdot) := \max_{p\in\mathcal{S}} \|p\| - \|\cdot\|$ is concave. Thus, in this case, the kernel $mn\rho(p-s)$ is a decreasing function of the distance between two points.

4 Proofs of main results

4.1 Criteria for gelation: proof of Theorem 3.2 and Corollary 3.3

This section is dedicated to the proofs of Theorem 3.2 and Corollary 3.3.

In the remainder of this section, we will often assume that we work with a fixed N with initial condition $\bar{\mathbf{L}}_0^{(N)} = \boldsymbol{\pi}$, with $|\boldsymbol{\pi}| < \infty$ such that, as in the statement of Theorem 3.2, $\langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle > \delta$, and Assumption 3.1 is satisfied. We may further assume, by making δ_0 smaller if necessary, that we have a sequence $(\delta_i)_{i\in\mathbb{N}}$ satisfying the requirement of Assumption 3.1 such that each $\delta_i < \langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle$. Moreover, we choose ψ' such that

$$\psi'(N) = \psi(N) \wedge \max\left\{k : \frac{2^{k+2}}{\delta_k - \delta_{k+1}} \leq \frac{N}{\xi(N)}\right\}; \tag{34}$$

one readily verifies that ψ' is non-decreasing, and $\lim_{N\to\infty}\psi'(N) = \infty$. In order to prove Theorem 3.2, we define the family of functions $F_k : \mathcal{E} \to \mathbb{R}$ such that, for each $t \in [0,\infty)$

$$F_k(\bar{\mathbf{L}}_t^{(N)}) := \left\langle m\mathbf{1}_{m\geq 2^{k+1}}, \bar{\mathbf{L}}_t^{(N)}\right\rangle / \langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle \tag{35}$$

and an associated family of stopping times $(\mathcal{T}_k)_{k\in\mathbb{N}}$ defined such that

$$\mathcal{T}_k := \inf\left\{t > 0 : \left\langle m\mathbf{1}_{m\geq 2^i}, \bar{\mathbf{L}}_t^{(N)}\right\rangle \geq \delta_i \text{ for } i = 0,1,\ldots,k\right\},$$

i.e., the first time that the normalised total mass of clusters with mass at least 2^i exceeds δ_i for each $i = 0,\ldots,k$. Note that the functions F_k depend on $\boldsymbol{\pi}$ and the times $\mathcal{T}_k$ depend on N, but for brevity of notation, we will exclude this dependence in the remainder of the section.

Lemma 4.1. *For $N \in \mathbb{N}$ sufficiently large, for each $k \leq \log_2(\psi(N))$, we have*

$$\mathbb{E}_{N,\boldsymbol{\pi}}[\mathcal{T}_k] < \infty.$$

Proof. In this proof, we denote by $\sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \ldots$, the times of coagulations (jumps) in the normalised coagulation process $(\bar{\mathbf{L}}_t^{(N)}, t \geq 0)$ in increasing order, with $\sigma_0 := 0$. We also set

$$c'_{\min} := \min_{i=1,\ldots,|\mathscr{P}^{(0)}|} \{c'(i,0)\},$$

with $c'_{\min} > 0$ by Assumption 3.1.

Note, that, the time re-scaling $t \mapsto Nt$ in the normalised process corresponds to re-scaling the coagulation kernel $K \mapsto K/N$, as if X is exponentially distributed with rate K, $N \times X$ is exponentially distributed with rate K/N.

Fix $k \leq \log_2(\psi(N))$ and $i < \|\boldsymbol{\pi}\|$ such that $\sigma_i < \mathcal{T}_k$ (recall that $\|\boldsymbol{\pi}\|$ encodes the initial number of clusters). Then, if $x_1, \ldots, x_{\|\boldsymbol{\pi}\|-i}$ denote the clusters at time σ_i (i.e. the points in the point measure $\bar{\mathbf{L}}_{\sigma_i}$), we have

$$\sigma_{i+1} - \sigma_i \sim \operatorname{Exp}\left(\sum_{j=1}^{\|\boldsymbol{\pi}\|-i} \sum_{k=j+1}^{\|\boldsymbol{\pi}\|-i} \frac{\bar{K}(x_j, x_k)}{N}\right).$$

By Equation (13), as long as there exists j, j' such that $x_j, x_{j'} \in P_\ell^{(0)}$ for some $P_\ell^{(0)} \in \mathscr{P}^{(0)}$, then $\sigma_{i+1} - \sigma_i$ is stochastically bounded above by X_i, where $X_i \sim \operatorname{Exp}\left(\frac{c'_{\min}}{N}\right)$. Now, note that if no such ℓ exists, it must be the case that each set $P_j^{(0)}$ contains at most 1 cluster of mass larger or equal than 1. If each element of the partition contains exactly one cluster of mass at most $\psi(N)$, since $\xi(N)$ denotes the maximum size of the partition, $\langle m\mathbf{1}_{m \leq \psi(N)}, \mathbf{L}_{\sigma_i}\rangle$ is at most $\xi(N)\psi(N)$. Consequentially, the total proportion of mass in clusters at least $\psi(N)$ is

$$\left\langle m\mathbf{1}_{m \geq \psi(N)}, \bar{\mathbf{L}}_{\sigma_i}^{(N)}\right\rangle \geq \langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle \left(1 - \frac{\xi(N)\psi(N)}{\langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle N}\right) > \delta_0,$$

for N sufficiently large (using the third assertion in Assumption 3.1). But this is not possible, since $\sigma_i < \mathcal{T}_k \leq \mathcal{T}_{\log_2(\psi(N))}$, and the above would imply that $\sigma_i \geq \mathcal{T}_{\log_2(\psi(N))}$. Note also, that as there can be at most $\|\boldsymbol{\pi}\| - 1$ coagulation events, by a similar argument it must be the case that $\sigma_{\|\boldsymbol{\pi}\|-1} \geq \mathcal{T}_{\log_2(\psi(N))}$. As a result, for $k \leq \log_2(\psi(N))$,

$$\mathbb{E}_{N,\boldsymbol{\pi}}\left[\mathcal{T}_k\right] < \mathbb{E}\left[\sum_{i=1}^{\|\boldsymbol{\pi}\|-1} X_i\right] = (\|\boldsymbol{\pi}\| - 1)N/c'_{\min} < \infty.$$

□

We note that the upper bound on the expectation of $\mathcal{T}_k$ that we get in the proof of Lemma 4.1 may be a large over-estimate. However, since we need it to apply Doob's optional sampling theorem on such a stopping time in the proof of Theorem 3.2, it is enough to know that the expectation is finite.

Lemma 4.2. *Suppose that $\vec{v} = (v_1, \ldots, v_n)$, $\vec{c} = (c_1, \ldots, c_n)$ are such that $v_i \in \mathbb{N}$ and $c_i > 0$, for all $i = 1, \ldots, n$. Then, if $\sum_{i=1}^n v_i \geq k_1 > n$ and $\sum_{i=1}^n \frac{1}{c_i} = k_2$ then*

$$\sum_i c_i \left(v_i^2 - v_i\right) \geq \frac{(k_1 - n)^2}{2k_2}. \tag{36}$$

Proof. Re-writing the left side of Equation (36) we get

$$\sum_i c_i(v_i^2 - v_i)\mathbf{1}_{\{v_i>1\}} \geq \sum_i c_i \frac{v_i^2}{2}\mathbf{1}_{\{v_i\geq 2\}}. \tag{37}$$

We seek to optimise the right hand side of Equation (37) subject to the constraint

$$\sum_i v_i \mathbf{1}_{\{v_i\geq 2\}} \geq k_1 - n.$$

When doing so, we may remove the condition that $v_i \geq 2$, and that the v_i are integers, as this can only make the minimum smaller. Exploiting Lagrange multiplier techniques, we seek to minimise $J(\vec{v}, \vec{c}, \lambda) := \sum_i c_i v_i^2/2 + \lambda v_i$. The Hessian is the diagonal matrix, with entries given by c_i, so any extreme point is a minimiser. Setting $\frac{\partial J}{\partial v_i} = 0$, we have $v_i = \frac{-\lambda}{c_i}$. Solving for λ we get $\lambda = \frac{n-k_1}{k_2}$. Substituting into v_i yields the result. □

4.1.1 Proof of Theorem 3.2

Proof of Theorem 3.2. First, we fix $k < \log_2(\psi'(N)) \leq \log_2(\psi(N))$ and note that, given the initial condition $\boldsymbol{\pi}$, the functions F_k are always bounded above by 1, and are clearly measurable. Now, recalling the generator in (2), we may consider the normalised process $(\bar{\mathbf{L}}_t^{(N)}, t \geq 0)$ as a Markov process taking values in $\mathcal{E}$ with generator $\mathcal{A}_N$ (now dependent on N), defined such that for any bounded measurable test function $F : \mathcal{E} \to \mathbb{R}$, we have

$$\mathcal{A}_N F(\bar{\mathbf{L}}_t^{(N)}) \tag{38}$$

$$= \frac{N}{2}\int_{E\times E\times E} (\bar{\mathbf{L}}_t^{(N)}(\mathrm{d}x)\left(\bar{\mathbf{L}}_t^{(N)} - \frac{\delta_x}{N}\right)(\mathrm{d}y) K(x, y, \mathrm{d}z)\left(F\left(\bar{\mathbf{L}}_t^{(N)} + \frac{(\delta_z - \delta_x - \delta_y)}{N}\right) - F\left(\bar{\mathbf{L}}_t^{(N)}\right)\right), \tag{39}$$

where the N scaling somes from dividing the kernel by N (the time rescaling) and multiplying the measures $\bar{\mathbf{L}}_t^{(N)}$ by N (corresponding to normalising the cluster masses). As F_k only considers the masses of clusters larger than 2^{k+1}, and ignores other features, and the mass of the coagulated cluster z is $m(z) = m(x) + m(y)$, we deduce that

$$\mathcal{A}_N F_k(\bar{\mathbf{L}}_t) = \frac{1}{2\langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle}\int_{E\times E} \bar{\mathbf{L}}_t(\mathrm{d}x)\left(\bar{\mathbf{L}}_t - \frac{1}{N}\delta_x\right)(\mathrm{d}y)\bar{K}(x, y) \tag{40}$$

$$\times\left[(m(x) + m(y))\mathbf{1}(m(x) + m(y) \geq 2^{k+1}) - m(x)\mathbf{1}(m(x) \geq 2^{k+1}) - m(y)\mathbf{1}(m(y) \geq 2^{k+1})\right]. \tag{41}$$

Note that $\mathcal{A}_N$ corresponds to the generator $\mathcal{A}$ in (2) when we scale time and $\mathbf{L}$ with N as described when we introduced $\bar{\mathbf{L}}_t^{(N)}$. Now,

$$M_{F_k}(t) := F_k(\bar{\mathbf{L}}_t^{(N)}) - F_k(\boldsymbol{\pi}) - \int_0^t A_N F_k(\bar{\mathbf{L}}_s^{(N)})\mathrm{d}s^5$$

is a martingale with respect to the natural filtration of the process initiated by the measure $\boldsymbol{\pi}$. Now, thanks to hypothesis (15), we may bound the martingale:

$$|M_{F_k}(t)| \leq 2 + \frac{1}{\langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle} g_{\boldsymbol{\pi}} t.$$

[5] The fact that $M_{F_k}(t)$ is a martingale follows from the definition of the infinitesimal generator of the process, see, for example, [Proposition 7.1.6, [35]]

Since, by Lemma 4.1, $\mathcal{T}_{k+1}$ has finite expectation, for any fixed N the collection $(M_{F_k}(t \wedge \mathcal{T}_{k+1}))_{t\geq 0}$ is pointwise dominated by the integrable function $2 + g_{\boldsymbol{\pi}}\mathcal{T}_{k+1}/N$, hence uniformly integrable. Thus, by Doob's optional sampling theorem (see, for example, [Corollary 2.3.6 and Theorem 2.3.2, [35]]) $M_{F_k}(t \wedge \mathcal{T}_{k+1})$ is also a martingale, and

$$\mathbb{E}_{N,\boldsymbol{\pi}}\left[F_k(\bar{\mathbf{L}}_{\mathcal{T}_{k+1}})\right] = \mathbb{E}_{N,\boldsymbol{\pi}}\left[F_k(\bar{\mathbf{L}}_{\mathcal{T}_k})\right] + \mathbb{E}_{N,\boldsymbol{\pi}}\left[\int_{\mathcal{T}_k}^{\mathcal{T}_{k+1}} \mathcal{A}_N F_k(\bar{\mathbf{L}}_t)\right] \mathrm{d}t. \tag{42}$$

Note, that, if $\mathcal{T}_k \leq t < \mathcal{T}_{k+1}$, by definition, we have

$$\left\langle m\mathbf{1}_{m\geq 2^k}, \bar{\mathbf{L}}_t^{(N)}\right\rangle \geq \delta_k \quad \text{and} \quad \left\langle m\mathbf{1}_{m\geq 2^{k+1}}, \bar{\mathbf{L}}_t^{(N)}\right\rangle < \delta_{k+1}, \tag{43}$$

so that

$$\left\langle m\mathbf{1}_{2^k\leq m<2^{k+1}}, \bar{\mathbf{L}}_t^{(N)}\right\rangle \geq (\delta_k - \delta_{k+1}). \tag{44}$$

Now, we consider the term in (40), and, using the fact that

$$m(x)\mathbf{1}(m(x) \geq 2^{k+1}) + m(y)\mathbf{1}(m(y) \geq 2^{k+1}) \leq (m(x) + m(y))\mathbf{1}(m(x) \text{ or } m(y) \geq 2^{k+1}),$$

we bound this from below by

$$\begin{aligned}
&\frac{1}{2\langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle}\int_{E\times E} \bar{\mathbf{L}}_t^{(N)}(\mathrm{d}x)\left(\bar{\mathbf{L}}_t^{(N)} - \frac{1}{N}\delta_x\right)(\mathrm{d}y)\bar{K}(x,y) \\
&\qquad\qquad \times (m(x)+m(y))\mathbf{1}(m(x)+m(y) \geq 2^{k+1} > m(x), m(y)) \\
&\overset{(13)}{\geq} \frac{1}{2\langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle}\sum_{n_1,n_2=2^k}^{2^{k+1}-1}\sum_{P\in\mathscr{P}^{(k)}} c'(P,k)(n_1+n_2)\left\langle \mathbf{1}_{n_1\leq m(x)<n_1+1, x\in P}, \bar{\mathbf{L}}_t^{(N)}\right\rangle \\
&\qquad\qquad \times \left(\left\langle \mathbf{1}_{n_2\leq m(x)<n_2+1, x\in P}, \bar{\mathbf{L}}_t^{(N)}\right\rangle - \mathbf{1}_{\{n_1=n_2\}}\right) \\
&\geq \frac{2^k}{\langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle}\sum_{n_1,n_2=2^k}^{2^{k+1}-1}\sum_{P\in\mathscr{P}^{(k)}} c'(P,k)\left\langle \mathbf{1}_{n_1\leq m(x)<n_1+1, x\in P}, \bar{\mathbf{L}}_t^{(N)}\right\rangle \\
&\qquad\qquad \times \left(\left\langle \mathbf{1}_{n_2\leq m(x)<n_2+1, x\in P}, \bar{\mathbf{L}}_t^{(N)}\right\rangle - \mathbf{1}_{\{n_1=n_2\}}\right),
\end{aligned} \tag{45}$$

where the first inequality comes by restricting the integral over $E\times E$ to the space $\bigcup_{P\in\mathscr{P}^{(k)}} P\times P$ and then using the inequality from assumption (13).

By exchanging the order of summation in (45), we may lower bound this further by applying Lemma 4.2 to the integer valued random variables $\sum_{n=2^k}^{2^{k+1}-1} N\left\langle \mathbf{1}_{n\leq m(x)<n+1, x\in P}, \bar{\mathbf{L}}_t^{(N)}\right\rangle$, indexed by the elements of $\mathscr{P}^{(k)}$. Note that these are almost surely integer valued as they count the number of particles at time t in each element P of the partition $\mathscr{P}^{(k)}$. Also note that the assumptions of Lemma 4.2 may be applied since

$$\sum_{P\in\mathscr{P}^{(k)}}\sum_{n=2^k}^{2^{k+1}-1} N\left\langle \mathbf{1}_{n\leq m(x)<n+1, x\in P}, \bar{\mathbf{L}}_t^{(N)}\right\rangle = N\left\langle \mathbf{1}_{2^k\leq m<2^{k+1}}, \bar{\mathbf{L}}_t^{(N)}\right\rangle \geq N(\delta_k - \delta_{k+1})/2^{k+1}, \tag{46}$$

where the last inequality follows from the fact that $\left\langle m\mathbf{1}_{2^k\leq m<2^{k+1}}, \bar{\mathbf{L}}_t^{(N)}\right\rangle \geq (\delta_k - \delta_{k+1})$ for all $t \in [\mathcal{T}_k, \mathcal{T}_{k+1})$. We obtain:

$$\begin{aligned}
&\frac{2^k}{N^2 \langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle} \\
&\times \sum_{P \in \mathscr{P}^{(k)}} c'(P,k) \left(\left(\sum_{n=2^k}^{2^{k+1}-1} N \left\langle \mathbf{1}_{n \leq m(x) < n+1, x\in P}, \bar{\mathbf{L}}_t^{(N)} \right\rangle \right)^2 - \sum_{n=2^k}^{2^{k+1}-1} N \left\langle \mathbf{1}_{n \leq m(x) < n+1, x\in P}, \bar{\mathbf{L}}_t^{(N)} \right\rangle \right) \\
&\qquad \overset{(36)}{\geq} \frac{2^k((\delta_k - \delta_{k+1})N/2^{k+1} - \xi(N))^2}{\langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle\, 2N^2 (\sum_{P\in\mathscr{P}^{(k)}} c'(P,k)^{-1})}.
\end{aligned}$$

If $\xi(N) \leq \frac{(\delta_k - \delta_{k+1})N}{2^{k+2}}$, as guaranteed by (34) and the fact that $k < \log_2(\psi'(N))$, we may bound the previous further by

$$\frac{(\delta_k - \delta_{k+1})^2}{2^{k+5} \langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle \left(\sum_{P\in\mathscr{P}^{(k)}} c'(P,k)^{-1}\right)};$$

Now, observing that $0 \leq F_k(\bar{\mathbf{L}}_t^{(N)}) \leq 1$, by applying (42) we have

$$\begin{aligned}
1 &\geq \mathbb{E}_{N,\boldsymbol{\pi}}\left[F_k(\bar{\mathbf{L}}_{\mathcal{T}_{k+1}}^{(N)})\right] \\
&\geq \mathbb{E}_{N,\boldsymbol{\pi}}\left[\int_{\mathcal{T}_k}^{\mathcal{T}_{k+1}} \mathcal{A}F_k(\bar{\mathbf{L}}_t^{(N)})\mathrm{d}t\right] \geq \frac{(\delta_k - \delta_{k+1})^2}{2^{k+5} \langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle \left(\sum_{P\in\mathscr{P}^{(k)}} c'(P,k)^{-1}\right)} \mathbb{E}_{N,\boldsymbol{\pi}}\left[(\mathcal{T}_{k+1} - \mathcal{T}_k)\right];
\end{aligned}$$

which in turn yields

$$\mathbb{E}_{N,\boldsymbol{\pi}}\left[(\mathcal{T}_{k+1} - \mathcal{T}_k)\right] \leq \frac{2^{k+5} \langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle \left(\sum_{P\in\mathscr{P}^{(k)}} c'(P,k)^{-1}\right)}{(\delta_k - \delta_{k+1})^2}. \tag{47}$$

Now, summing over k, up until $\ell = \log_2(\psi'(N))$, we have

$$\begin{aligned}
\mathbb{E}_{N,\boldsymbol{\pi}}\left[\tau(\psi'(N), \delta)\right] \leq \mathbb{E}_{N,\boldsymbol{\pi}}\left[\tau(\psi'(N), \delta_\ell)\right] &= \sum_{k=1}^{\ell-1} \mathbb{E}_{N,\boldsymbol{\pi}}\left[(\mathcal{T}_{k+1} - \mathcal{T}_k)\right] \\
&\leq \sum_{k=1}^{\log_2(\psi'(N))} \frac{2^{k+5} \langle m\mathbf{1}_{m\geq 1}, \boldsymbol{\pi}\rangle \left(\sum_{P\in\mathscr{P}^{(k)}} c'(P,k)^{-1}\right)}{(\delta_k - \delta_{k+1})^2}.
\end{aligned} \tag{48}$$

Taking limits superior of both sides, the right-hand side is bounded by Equation (14).

To complete the proof of the theorem, we prove the second statement. Suppose that, by assumption, $\varepsilon > 0$ is such that

$$\liminf_{N\to\infty} \mathbb{P}_N\left(\left\langle m\mathbf{1}_{m\geq 1}, \bar{\mathbf{L}}_0^{(N)}\right\rangle > \varepsilon\right) = p_0 > 0.$$

Now, if we choose a sequence $(\delta_i)_{i\in\mathbb{N}}$ with each $\delta_i < \varepsilon$, (so that $\delta = \lim_{i\to\infty}\delta_i < \varepsilon$), as the upper-bound $C = C((\delta_i))$ is independent of the initial condition $\boldsymbol{\pi}$, we may bound the expected value

$$\mathbb{E}_{N,\boldsymbol{\pi}}\left[\tau(\psi'(N), \delta) \,\middle|\, \left\langle m\mathbf{1}_{m\geq 1}, \bar{\mathbf{L}}_0^{(N)}\right\rangle > \delta\right] \leq C,$$

hence, by Markov's inequality, for each N,

$$\mathbb{P}_N\left(\tau(\psi'(N), \delta) \leq 2C \,\middle|\, \left\langle m\mathbf{1}_{m\geq 1}, \bar{\mathbf{L}}_0^{(N)}\right\rangle > \delta\right) \geq \frac{1}{2},$$

and therefore,

$$\liminf_{N\to\infty} \mathbb{P}_N\left(\tau(\psi'(N),\delta) \le 2C\right) \ge \frac{p_0}{2}.$$

This implies that

$$\liminf_{N\to\infty} \mathbb{P}_N\left(\left\langle m\mathbf{1}_{m\le\psi'(N)}, \bar{\mathbf{L}}_{2C}^{(N)}\right\rangle \le \left\langle m, \bar{\mathbf{L}}_0^{(N)}\right\rangle - \delta\right) \ge 1 - \frac{p_0}{2},$$

hence stochastic gelation occurs (with (ψ',δ) gelation time at most $2C$). □

4.1.2 Proof of Corollary 3.3

We finish off this section with the proof of Corollary 3.3.

Proof of Corollary 3.3. In order to apply Theorem 3.2, we first need to show that the conditions of Assumption 3.1 are satisfied. We first note that in Assumption 3.1, applying (16) we can take $\xi(N) \equiv 1$, and $\psi(N)$ any non-decreasing function such that $\lim_{N\to\infty}\psi(N) = \infty$ and $\lim_{N\to\infty}\frac{\psi(N)}{N} = 0$. Now, suppose that $\boldsymbol{\pi}$ is a given initial condition with $\|\boldsymbol{\pi}\| < \infty$ and $\langle m\mathbf{1}_{m\ge 1}, \boldsymbol{\pi}\rangle > \delta$. Now, to show the final assumption, given $\boldsymbol{\pi}$, we define the following set of *possible* clusters:

$$\mathcal{G}_{\boldsymbol{\pi}} := \left\{ x \in [0,\infty) : x = \sum_{i=1}^{j} x_i, \quad x_1,\dots,x_j \in \operatorname{Supp}(\boldsymbol{\pi}), j \in \mathbb{N} \right\}.$$

Since $\|\boldsymbol{\pi}\| < \infty$, we have $|\mathcal{G}_{\boldsymbol{\pi}}| < \infty$. Moreover, recalling that for all $x, y \in [0,\infty)$ we have $\bar{K}(x,y) < \infty$, and for each $s \in [0,\infty)$ we have $\|\bar{\mathbf{L}}_s^{(N)}\| \le \|\boldsymbol{\pi}\|$, we deduce that

$$\int_{[0,\infty)\times[0,\infty)} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x)\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)\bar{K}(x,y)(x+y) \le \max_{x,y\in\mathcal{G}_{\boldsymbol{\pi}}} \left\{(x+y)\bar{K}(x,y)\right\} \|\boldsymbol{\pi}\|^2.$$

Thus, in this case, (15) is satisfied almost surely on $\left\langle m\mathbf{1}_{m\ge1}, \bar{\mathbf{L}}_0^{(N)}\right\rangle > \delta$. But now, since by (11), $\bar{\mathbf{L}}_0^{(N)} \to \mu$ weakly, in probability, it converges almost surely along a sub-sequence. Moreover, as the function $m(x) = x\mathbf{1}_{x\ge1}$ in this case, it is continuous and bounded from below. By the Portmanteau theorem we deduce that

$$\liminf_{N\to\infty} \left\langle m\mathbf{1}_{m\ge1}, \bar{\mathbf{L}}_0^{(N)}\right\rangle \ge \langle m\mathbf{1}_{m\ge1}, \mu\rangle > 0$$

almost surely, and thus, for any sub-sequence $(N_k)_{k\in\mathbb{N}}$

$$\limsup_{k\to\infty} \mathbb{P}_{N_k}\left(\left\langle m\mathbf{1}_{m\ge1}, \bar{\mathbf{L}}_0^{(N_k)}\right\rangle \ge \frac{\langle m\mathbf{1}_{m\ge1},\mu\rangle}{2}\right) \ge \frac{1}{2}.$$

Thus, if we set $\varepsilon = \frac{\langle m\mathbf{1}_{m\ge1},\mu\rangle}{2}$, we have $\liminf_{N\to\infty}\mathbb{P}_N\left(\left\langle m\mathbf{1}_{m\ge1}, \bar{\mathbf{L}}_0^{(N)}\right\rangle\right) > 0$. This proves the first assertion of the corollary.

We now prove the first of the final two assertions. Set $\kappa^* := \inf_{i\in[1,2]}\bar{K}(1,i)$. By the homogeneity assumption there exists $j_0 \in \mathbb{N}$ such that, for all $j \ge j_0$ we have (assuming w.l.o.g. $x < y$), $\bar{K}(x,y) = x^\gamma \bar{K}\left(1, \frac{y}{x}\right) \ge \kappa^* 2^{\gamma j}$ whenever $2^j \le x, y < 2^{j+1}$. Finally, setting (for example) $\delta_j = \delta + \frac{1}{1-\delta}2^{-(\gamma-1)/8j}$, we also deduce condition (16) with $c_j' := \kappa^* 2^{\gamma j}$.

For the final assertion, by assumption there exists a $j_0 \in \mathbb{N}$ such that, for all $j \geq j_0$ we have $\bar{K}(x,y) \geq 2^j (j \log 2)^{3+\varepsilon}$ for $2^j \leq x, y < 2^{j+1}$. Note that, if we set $\delta_0 = 1$, since $\delta_i \downarrow \delta$ for all i, we have

$$\lim_{i \to \infty} \sum_{j=1}^{i} (\delta_j - \delta_{j+1}) \leq 1 - \delta,$$

and the terms $\delta_j - \delta_{j+1}$ need to be decreasing. By choosing $\delta_j - \delta_{j+1} = \kappa_0^\delta j^{-(1+\varepsilon/3)}$, say, for an appropriate normalising constant κ_0^δ, we deduce the result. □

4.2 Concentration of trajectories along solutions of the multi-type Flory equation

The normalised cluster coagulation process $(\bar{\mathbf{L}}_t^{(N)})_{t \geq 0}$ induces a probability measure $\mathbb{P}_N$ on the space $D([0,\infty); \mathcal{E})$. We wish to show that the family of probability measures $(\mathbb{P}_N)_{N \in \mathbb{N}}$ is tight, which implies by Prokhorov's theorem [34, Theorem IV.29, page 82] that $(\mathbb{P}_N)_{N \in \mathbb{N}}$ has a weakly convergent subsequence. We then show that any such subsequence concentrates on solutions of the *multi-type Flory equation*. Throughout this section, it will be beneficial to have associated "conserved" quantities, preserved by the dynamics of the process. This motivates the following lemma:

Lemma 4.3. *For any cluster coagulation process $(\bar{\mathbf{L}}_t^{(N)})_{t \geq 0}$, given any doubly sub-conservative $\phi' : E \times E \to \mathbb{R}$, almost surely for all $t \geq 0$ we have, for each $y \in E$*

$$\int_E \left(\bar{\mathbf{L}}_t^{(N)}(\mathrm{d}x) \right) \phi'(x,y) \leq \int_E \left(\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x) \right) \phi'(x,y) \tag{49}$$

and

$$\int_{E \times E} \bar{\mathbf{L}}_t^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_t^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N} \right) \phi'(x,y) \leq \int_{E \times E} \bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N} \right) \phi'(x,y). \tag{50}$$

Proof. For Equation (49) if $\tau_1 < \tau_2$ denote times of two consecutive coagulation events, with τ_2 involving the coagulation of clusters x' and y' to a new cluster z, we have

$$\int_E \left(\bar{\mathbf{L}}_{\tau_2}^{(N)}(\mathrm{d}x) - \bar{\mathbf{L}}_{\tau_1}^{(N)}(\mathrm{d}x) \right) \phi'(x,y) = \left(\phi'(z,y) - \phi'(x',y) - \phi'(y',y) \right).$$

The right-hand side is 0 for $K(x',y',\mathrm{d}z)$—a.a. z, hence almost surely. Now, for Equation (50), note that integrals of $\phi'(x,y)$ with respect to the product measure

$$\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N} \right)$$

are nothing but sums of $\phi'(x,y)$ across all the distinct pairs of clusters x, y in the process at time s. Thus, if $\tau_1 < \tau_2$ denote times of two consecutive coagulation events, with τ_2 involving the coagulation of clusters x' and y' to a new cluster z, we have

$$\begin{aligned}
&\int_{E \times E} \bar{\mathbf{L}}_{\tau_2}^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_{\tau_2}^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N} \right) \phi'(x,y) - \int_{E \times E} \bar{\mathbf{L}}_{\tau_1}^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_{\tau_1}^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N} \right) \phi'(x,y) \\
&= -\frac{\phi(x',y') + \phi(y',x')}{N^2} + \int_E \left(\bar{\mathbf{L}}_{\tau_1}^{(N)} - \frac{\delta_{x'}}{N} - \frac{\delta_{y'}}{N} \right) (\mathrm{d}u) \frac{(\phi'(z,u) - \phi'(x',u) - \phi'(y',u))}{N^2} \\
&\quad + \int_E \left(\bar{\mathbf{L}}_{\tau_1}^{(N)} - \frac{\delta_{x'}}{N} - \frac{\delta_{y'}}{N} \right) (\mathrm{d}u) \frac{(\phi'(u,z) - \phi'(u,x') - \phi'(u,y'))}{N^2} \leq 0,
\end{aligned}$$

for $K(x', y', \mathrm{d}z)-$ a.a. z. Note that the first term in the second line comes from the contribution to the integral from the pair x', y' involved with the coagulation, and the other integrals in the second and third line, comes from the difference in the contributions to the integrals from pairs (v, u) where $v \in \{x', y', z\}$, and u is a cluster not involved in the coagulation event. The result follows by iterating over the jumps in the process. □

4.2.1 Tightness: the proof of Lemma 3.4

In order to prove Lemma 3.4, we apply some well established tightness criterion, stated in the appendix.

Proof of Lemma 3.4. We will apply Lemma A.2. For the first compact containment criterion, first recall that by (21), the set

$$\mathcal{E}_n^* := \left\{ \mathbf{u} \in \mathcal{M}_+(E \times E) : \int_E \mathbf{u}(\mathrm{d}x \times \mathrm{d}y)\phi''(x, y) \leq n \right\}$$

is compact. Now, note that, by Lemma 4.3, if we have

$$\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N} \right) \in \mathcal{E}_n^*, \quad \text{then, for all } s \geq 0, \quad \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N} \right) \in \mathcal{E}_n^*.$$

Suppose that we denote by $\mathcal{D}_N$ the set

$$\mathcal{D}_N := \left\{ \mathbf{u} \in \mathcal{E} : \mathbf{u} = \sum_{i \in I} \frac{c_i \delta_i}{N},\ c_i \in \mathbb{N},\ I \subseteq E \right\},$$

and $\iota_N : \mathcal{D}_N \to \mathcal{M}_+(E \times E)$ denotes the map $\mathbf{u} \mapsto \mathbf{u}(\mathrm{d}x) \left(\mathbf{u}(\mathrm{d}y) - \frac{\delta_x}{N} \right)$; and extend this map to a map $\iota : \bigcup_{N \in \mathbb{N}} \mathcal{D}_N \to \mathcal{M}_+(E \times E)$ such that $\iota \equiv \iota_N$ on $\mathcal{D}_N$. We now note that for any $n \in \mathbb{N}$ the set

$$B_n := \left\{ \mathbf{u} \in \bigcup_{N \in \mathbb{N}} \mathcal{D}_N : \iota(\mathbf{u}) \in \mathcal{E}_n^* \right\}$$

is relatively compact. Indeed, by the compactness of $\mathcal{E}_n^*$, any sequence $(\iota(\mathbf{u}_i))_{i \in \mathbb{N}}$ has a convergent subsequence $(\iota(\mathbf{u}_{i_k}))_{k \in \mathbb{N}}$. Suppose ν denotes a limit of this subsequence. There, are two cases: we can either find a further subsequence (which we also denote $(\iota(\mathbf{u}_{i_k}))_{k \in \mathbb{N}}$), such that, for some $N' \in \mathbb{N}$ we have $(\iota(\mathbf{u}_{i_k}))_{k \in \mathbb{N}} = (\iota_{N'}(\mathbf{u}_{i_k}))_{k \in \mathbb{N}}$, or it is the case that for any $N' \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\iota(\mathbf{u}_{i_k}) = \iota_j(\mathbf{u}_{i_k})$ for some $j \geq N'$. In the latter case, (since the co-efficient of the δ_x term tends to 0), we readily verify that

$$\mathbf{u}_{i_k} \otimes \mathbf{u}_{i_k} \to \nu,$$

hence $(\mathbf{u}_{i_k})_{k \in \mathbb{N}}$ also converges weakly. We may similarly deduce the result in the first case, when

$$\mathbf{u}_{i_k}(\mathrm{d}x) \left(\mathbf{u}_{i_k}(\mathrm{d}y) - \frac{\delta_x}{N'} \right) \to \nu(\mathrm{d}x \times \mathrm{d}y).$$

Now, since ϕ'' satisfies (20) we know $\mathbb{E}_N \left[\int_{E \times E} \bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N} \right) \phi''(x, y) \right] < c_0$, for some $c_0 \in \mathbb{N}$. Therefore, by Markov's inequality, for any $c_1 \in \mathbb{N}$,

$$\liminf_{N \to \infty} \mathbb{P}_N \left(\forall t \geq 0\ \bar{\mathbf{L}}_t^{(N)} \in B_{c_1} \right) = \liminf_{N \to \infty} \mathbb{P}_N \left(\left\langle \phi'', \iota(\bar{\mathbf{L}}_0^{(N)}) \right\rangle \leq c_1 \right) \geq 1 - \frac{c_0}{c_1}.$$

Thus, by fixing $c_1 > c_0/\varepsilon$ and choosing the closure of $B_{c_1} \subseteq \mathcal{E}$ as the required compact set, we have the required compact containment condition (150).

For the second criterion, we will define an appropriate family of test functions $\mathbb{F}$, then apply Lemma A.1. In particular, we choose the family of functions $\mathbb{F}$ from $\mathcal{E}$ to $\mathbb{R}$ such that

$$\mathbb{F} := \left\{ \tilde{J} : \; \tilde{J}(\mathbf{u}) = \int_E J(x)\mathbf{u}(\mathrm{d}x); \; J \in C_c(E;\mathbb{R}) \right\},$$

where $C_c(E;\mathbb{R})$ denotes the set of continuous functions on E with compact support. By the definition of the weak topology, this family consists of continuous functions and it is straightforward to see that it is closed under addition. Moreover, since E is σ-compact, a measure μ is uniquely determined by the values of $\langle f, \mu \rangle$, where $f \in C_c(E;\mathbb{R})$, thus this family separates points. Now, let $\tilde{J} \in \mathbb{F}$ be given, with associated function $J : E \mapsto \mathbb{R}$, so that

$$\tilde{J}(\bar{\mathbf{L}}_t) = \langle J, \bar{\mathbf{L}}_t \rangle = \int_E J(x)\bar{\mathbf{L}}_t(\mathrm{d}x). \tag{51}$$

We seek to apply Lemma A.1 to the family of pushforward measures

$$\left\{ \tilde{J}_* \mathbb{P}_N : \; N \in \mathbb{N} \right\}.$$

Note that these are measures on the space $D([0,\infty),\mathbb{R})$ which a separable, and complete metric space, hence Lemma A.1 applies. Also note, that as the continuous image of a compact set is compact, we can take $\tilde{J}(\mathcal{E}_{c_1})$ as the compact set for the first condition, and thus we need only verify the second condition of Lemma A.1.

We note that, for fixed T, and $\eta = \eta(T)$ sufficiently small, we can find an integer $K \in \mathbb{N}$ such that $\eta < T/K =: \eta' \leq 2\eta$. Therefore, we can define a partition $\{t_i\}$ of $[0,T]$ such that $t_{i+1} - t_i = \eta' > \eta$, so that

$$\begin{aligned} \tilde{J}_*(\mathbb{P}_N)\left(\{f : w'(f,\eta,T) \geq \varepsilon\}\right) &= \mathbb{P}_N\left(w'((\tilde{J}(\bar{\mathbf{L}}_t))_{t\in[0,T]},\eta,T) \geq \varepsilon\right) \\ &\leq \mathbb{P}_N\left(\sup_{s,t\in[0,T],|s-t|\leq\eta'} \left|\tilde{J}(\bar{\mathbf{L}}_s) - \tilde{J}(\bar{\mathbf{L}}_t)\right| \geq \varepsilon\right), \end{aligned} \tag{52}$$

where w' denotes the modulus of continuity defined in (149). We now have the following claim:

Claim 4.3.1. *For some constant $C = C(T)$, we have*

$$\limsup_{N\to\infty} \mathbb{E}_N\left[\sup_{s,t\in[0,T],|s-t|\leq\eta} \left|\tilde{J}(\bar{\mathbf{L}}_t) - \tilde{J}(\bar{\mathbf{L}}_s)\right|\right] < C\eta, \tag{53}$$

To complete the proof of Lemma 3.4 using Claim 4.3.1, observe that by Markov's inequality, for all $n \geq N$, we have

$$\lim_{\eta\to 0}\limsup_{N\to\infty} \mathbb{P}_N\left(\sup_{s,t\in[0,T],|s-t|\leq\eta} \left|\tilde{J}(\bar{\mathbf{L}}_t) - \tilde{J}(\bar{\mathbf{L}}_s)\right| \geq \varepsilon\right) < \lim_{\eta\to 0}\frac{C\eta}{\varepsilon} = 0,$$

implying (151). □

Proof of Claim 4.3.1. We first note some relevant facts: since $\bar{\mathbf{L}}_t$ is a pure jump Markov process, and $\tilde{J}$ is bounded and measurable, it is well-known that

$$M_N(t) := \tilde{J}(\bar{\mathbf{L}}_t) - \tilde{J}(\bar{\mathbf{L}}_0) - \int_0^t \mathcal{A}_N \tilde{J}(\bar{\mathbf{L}}_s)\mathrm{d}s \tag{54}$$

and its quadratic variation

$$Q_N(t) := M_N(t)^2 - \int_0^t (\mathcal{A}_N \tilde{J}^2 - 2\tilde{J}\mathcal{A}_N\tilde{J})(\bar{\mathbf{L}}_s)\mathrm{d}s \tag{55}$$

are both martingales under $\mathbb{P}_N$ (see, for example, the proofs of Proposition 7.1.6, and Proposition 8.3.3 in [35]). From Equation (54), the triangle inequality, and sub-additivity of taking suprema, it follows that

$$\mathbb{E}_N\left[\sup_{s,t\in[0,T],|s-t|\leq\eta}\left|\tilde{J}[\bar{\mathbf{L}}(t)] - \tilde{J}[\bar{\mathbf{L}}_s]\right|\right] \tag{56}$$

$$\leq \mathbb{E}_N\left[\sup_{s,t\in[0,T],|s-t|\leq\eta}|M_N(t)-M_N(s)|\right] + \mathbb{E}_N\left[\sup_{s,t\in[0,T],|s-t|\leq\eta}\left|\int_s^t \mathcal{A}_N\tilde{J}(\bar{\mathbf{L}}_\theta)\mathrm{d}\theta\right|\right] \tag{57}$$

$$\leq 2\mathbb{E}_N\left[\sup_{0\leq t\leq T}|M_N(t)|\right] + \mathbb{E}_N\left[\sup_{s,t\in[0,T],t-s\leq\eta}\left|\int_s^t \mathcal{A}_N\tilde{J}(\bar{\mathbf{L}}_\theta)\mathrm{d}\theta\right|\right]. \tag{58}$$

We will complete the proof by bounding the two terms on the right side of Equation (58). For the first, observe that by Doob's maximal inequality,

$$\mathbb{E}_N\left[\left(\sup_{0\leq t\leq T}|M(t)|\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}_N\left[|M(_NT)|^p\right],$$

so that, by setting $p=2$, and recalling that $\mathbb{E}_N\left[Q(t)\right]=0$, we deduce from Equation (55) that

$$\mathbb{E}_N\left[\sup_{0\leq t\leq T}|M_N(t)|\right]^2 \leq \mathbb{E}_N\left[\left(\sup_{0\leq t\leq T}|M_N(t)|\right)^2\right] \tag{59}$$

$$\leq 4\mathbb{E}_N\left[M_N(T)^2\right] = 4\mathbb{E}_N\left[\int_0^T (\mathcal{A}_N\tilde{J}^2 - 2\tilde{J}\mathcal{A}_N\tilde{J})(\bar{\mathbf{L}}_s)\mathrm{d}s\right]. \tag{60}$$

Now, we recall that the generator $\mathcal{A}_N$ of the normalised process $(\bar{\mathbf{L}}_t^{(N)})_{t\geq 0}$ may be written as follows: for bounded measurable test functions F, we have

$$\mathcal{A}_N F(\bar{\mathbf{L}}_t^{(N)}) = \frac{N}{2}\int_{E\times E\times E}(\bar{\mathbf{L}}_t^{(N)}(\mathrm{d}x)\left(\bar{\mathbf{L}}_t^{(N)} - \frac{\delta_x}{N}\right)(\mathrm{d}y)K(x,y,\mathrm{d}z)\left(F\left(\bar{\mathbf{L}}_t^{(N)} + \frac{(\delta_z-\delta_x-\delta_y)}{N}\right) - F\left(\bar{\mathbf{L}}_t^{(N)}\right)\right).$$

For simplicity, for the remainder of this section, whenever unambiguous, we drop the superscript (or subscript) (N) when referring to $\bar{\mathbf{L}}_t^{(N)}$, $M_N(t)$ and $\mathcal{A}_N$. Abusing notation, for each t we denote by $\bar{\mathbf{L}}_t^{(x,y)\to z} := \bar{\mathbf{L}}_t + (\delta_z - \delta_x - \delta_y)/N$

Thus,

$$
\begin{aligned}
&(\mathcal{A}\tilde{J}^2 - 2\tilde{J}\mathcal{A}\tilde{J})(\bar{\mathbf{L}}_s) \\
&= \frac{N}{2}\int_{E\times E\times E} \bar{\mathbf{L}}_s(\mathrm{d}x)\left(\bar{\mathbf{L}}_s - \frac{\delta_x}{N}\right)(\mathrm{d}y)K(x,y,\mathrm{d}z)\left(\tilde{J}(\bar{\mathbf{L}}_s^{(x,y)\to z})^2 - \tilde{J}(\bar{\mathbf{L}})^2\right) \\
&\qquad - N\tilde{J}(\bar{\mathbf{L}}_s)\int_{E\times E\times E} \bar{\mathbf{L}}_s(\mathrm{d}x)\left(\bar{\mathbf{L}}_s - \frac{\delta_x}{N}\right)(\mathrm{d}y)K(x,y,\mathrm{d}z)\left(\tilde{J}(\bar{\mathbf{L}}_s^{(x,y)\to z}) - \tilde{J}(\bar{\mathbf{L}})\right) \\
&= \frac{N}{2}\int_{E\times E\times E} \bar{\mathbf{L}}_s(\mathrm{d}x)\left(\bar{\mathbf{L}}_s - \frac{\delta_x}{N}\right)(\mathrm{d}y)K(x,y,\mathrm{d}z)\left(\tilde{J}(\bar{\mathbf{L}}_s^{(x,y)\to z})^2 - \tilde{J}(\bar{\mathbf{L}}_s)^2\right) \\
&\quad - N\int_{E\times E\times E} \bar{\mathbf{L}}_s(\mathrm{d}x)\left(\bar{\mathbf{L}}_s - \frac{\delta_x}{N}\right)(\mathrm{d}y)K(x,y,\mathrm{d}z)\left(\tilde{J}(\bar{\mathbf{L}}_s)\tilde{J}(\bar{\mathbf{L}}_s^{(x,y)\to z}) - \tilde{J}(\bar{\mathbf{L}}_s)^2\right) \\
&= \frac{N}{2}\int_{E\times E\times E} \bar{\mathbf{L}}_s(\mathrm{d}x)\left(\bar{\mathbf{L}}_s - \frac{\delta_x}{N}\right)(\mathrm{d}y)K(x,y,\mathrm{d}z)\left(\tilde{J}(\bar{\mathbf{L}}_s^{(x,y)\to z}) - \tilde{J}(\bar{\mathbf{L}}_s)\right)^2, \qquad (61)
\end{aligned}
$$

where, by definition (51) of $\tilde{J}$, we have

$$
\tilde{J}(\bar{\mathbf{L}}_s^{(x,y)\to z}) - \tilde{J}(\bar{\mathbf{L}}_s) = \left(J(z) - J(y) - J(x)\right)/N. \qquad (62)
$$

Combining this with Equations (61) and (59), we get

$$
\begin{aligned}
&\mathbb{E}_N\left[\sup_{0\le t\le T}|M(t)|\right]^2 \qquad (63) \\
&\quad \le \mathbb{E}_N\left[\frac{2}{N}\int_0^T \mathrm{d}s\int_{E\times E\times E}\bar{\mathbf{L}}_s(\mathrm{d}x)\left(\bar{\mathbf{L}}_s - \frac{\delta_x}{N}\right)(\mathrm{d}y)K(x,y,\mathrm{d}z)\times\left(J(z)-J(y)-J(x)\right)^2\right]. \qquad (64)
\end{aligned}
$$

Moreover, recalling that J is continuous with compact support, by the extreme value theorem, it is bounded. Therefore, bounding $(J(z)-J(y)-J(x))^2$ by a constant c_J, and recalling that, by assumption, $\bar{K} \le \phi'$ pointwise, we have

$$
\begin{aligned}
\mathbb{E}_N\left[\sup_{0\le t\le T}|M(t)|\right]^2 &\le \mathbb{E}_N\left[\frac{2c_J}{N}\int_0^T \mathrm{d}s\int_{E\times E}\bar{\mathbf{L}}_s(\mathrm{d}x)\left(\bar{\mathbf{L}}_s(\mathrm{d}y) - \frac{\delta_x}{N}\right)\bar{K}(x,y)\right] \qquad (65) \\
&\le \frac{2c_J}{N}\mathbb{E}_N\left[\int_0^T \mathrm{d}s\int_{E\times E}\bar{\mathbf{L}}_s(\mathrm{d}x)\left(\bar{\mathbf{L}}_s(\mathrm{d}y) - \frac{\delta_x}{N}\right)\phi'(x,y)\right] \qquad (66) \\
&= \frac{2c_JT}{N}\mathbb{E}_N\left[\int_{E\times E}\bar{\mathbf{L}}_0(\mathrm{d}x)\left(\bar{\mathbf{L}}_0(\mathrm{d}y) - \frac{\delta_x}{N}\right)\phi'(x,y)\right]. \qquad (67)
\end{aligned}
$$

The last step is possible since we now observe, that, as ϕ' is doubly sub-conservative, for each $s \in [0,\infty)$ we have

$$
\int_{E\times E}\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x)\left(\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N}\right)\phi'(x,y) \le \int_{E\times E}\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x)\left(\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N}\right)\phi'(x,y), \qquad (68)
$$

almost surely.

Thus, by Equations (67) and (68), we have

$$
\mathbb{E}_N\left[\sup_{0\le t\le T}|M(t)|\right] \le \sqrt{\frac{2c_JT}{N}\mathbb{E}_N\left[\int_{E\times E}\bar{\mathbf{L}}_0(\mathrm{d}x)\left(\bar{\mathbf{L}}_0(\mathrm{d}y) - \frac{\delta_x}{N}\right)\phi'(x,y)\right]}. \qquad (69)
$$

In order to bound the second term on the right-side of (58), we apply a similar argument:

$$\begin{aligned}
&\left|\int_s^t \mathcal{A}\tilde{J}(\bar{\mathbf{L}}_\theta)\mathrm{d}\theta\right| \\
&= \left|\frac{N}{2}\int_s^t \mathrm{d}\theta \int_{E\times E\times E} \bar{\mathbf{L}}_t(\mathrm{d}x)\left(\bar{\mathbf{L}}_t^{(N)} - \frac{\delta_x}{N}\right)(\mathrm{d}y)K(x,y,\mathrm{d}z)\tilde{J}(\bar{\mathbf{L}}_\theta^{(x,y)\to z}) - \tilde{J}(\bar{\mathbf{L}}_\theta)\right| \qquad (70)\\
&\leq \frac{N}{2}\int_s^t \mathrm{d}\theta \int_{E\times E\times E} \bar{\mathbf{L}}_t(\mathrm{d}x)\left(\bar{\mathbf{L}}_t - \frac{\delta_x}{N}\right)(\mathrm{d}y)K(x,y,\mathrm{d}z)\left|\tilde{J}(\bar{\mathbf{L}}_\theta^{(x,y)\to z}) - \tilde{J}(\bar{\mathbf{L}}_\theta)\right|
\end{aligned}$$

As before, using Equation (62), and the fact that $|x| = \sqrt{x^2}$, we may bound the previous by

$$\begin{aligned}
&\frac{\sqrt{c_J}}{2}\int_s^t \mathrm{d}\theta \int_{E\times E} \bar{\mathbf{L}}_s(\mathrm{d}x)\left(\bar{\mathbf{L}}_s(\mathrm{d}y) - \frac{\delta_x}{N}\right)\bar{K}(x,y) &(71)\\
&\qquad \leq \frac{\sqrt{c_J}(t-s)}{2}\int_{E\times E} \bar{\mathbf{L}}_0(\mathrm{d}x)\left(\bar{\mathbf{L}}_0(\mathrm{d}y) - \frac{\delta_x}{N}\right)\phi'(x,y), &(72)
\end{aligned}$$

where the final inequality follows from (68). Thus, we have obtained the upper bound

$$\mathbb{E}_N\left[\sup_{s,t\in[0,T],|s-t|\leq\eta}\left|\int_s^t \mathcal{A}\tilde{J}(\bar{\mathbf{L}}_\theta)\mathrm{d}\theta\right|\right] \leq \frac{\sqrt{c_J}}{2}\eta\mathbb{E}_N\left[\int_{E\times E} \bar{\mathbf{L}}_0(\mathrm{d}x)\left(\bar{\mathbf{L}}_0(\mathrm{d}y) - \frac{\delta_x}{N}\right)\phi'(x,y)\right]. \qquad (73)$$

Thus, combining Equation (58) with Equations (69) and (73), and passing to the limit as $N \to \infty$,

$$\begin{aligned}
&\limsup_{N\to\infty}\mathbb{E}_N\left[\sup_{s,t\in[0,T],|s-t|\leq\eta}\left|\tilde{J}(\bar{\mathbf{L}}_t) - \tilde{J}(\bar{\mathbf{L}}_s)\right|\right] &(74)\\
&\qquad \leq \frac{\sqrt{c_J}}{2}\eta\limsup_{N\to\infty}\mathbb{E}_N\left[\int_{E\times E} \bar{\mathbf{L}}_0(\mathrm{d}x)\left(\bar{\mathbf{L}}_0(\mathrm{d}y) - \frac{\delta_x}{N}\right)\phi'(x,y)\right] &(75)
\end{aligned}$$

the latter bound being finite by (20). Setting

$$C := \frac{\sqrt{c_J}}{2}\limsup_{N\to\infty}\mathbb{E}_N\left[\int_{E\times E} \bar{\mathbf{L}}_0(\mathrm{d}x)\left(\bar{\mathbf{L}}_0(\mathrm{d}y) - \frac{\delta_x}{N}\right)\phi'(x,y)\right]$$

concludes the proof of (53). □

4.2.2 Accumulation of trajectories on solutions of the multi-type Flory equation

Now, let $\mathbb{P}^*$ denote an accumulation point of the tight sequence of measures $(\mathbb{P}_N)_{N\in\mathbb{N}}$, and assume, by passing to a subsequence, and re-indexing, that $\mathbb{P}_N \to \mathbb{P}^*$ with respect to the weak topology on the space of measures on $D([0,\infty);\mathcal{E})$. Following our previous notation, we denote by $\bar{\mathbf{L}}^{(N)}$ and $\bar{\mathbf{L}}^*$ random trajectories sampled from these distributions. Mostly out of convenience of notation, applying the Skorokhod representation theorem [34, Theorem IV.13, page 71],[6] we assume that $(\bar{\mathbf{L}}^{(N)})_{N\in\mathbb{N}}$ converges to $\bar{\mathbf{L}}^*$ pointwise for all $\omega \in \Omega$ with respect to the Skorokhod topology on $D([0,\infty);\mathcal{E})$ on some enlarged probability space $(\Omega, \mathscr{F}, \mathbb{P}(\cdot))$. For the rest of the section, we use the notation $\mathbb{E}[\cdot]$ to denote expectations with respect to this enlarged probability space. This will allow us to more easily draw conclusions about the limiting trajectory $\bar{\mathbf{L}}^*$, and thus the limiting measure $\mathbb{P}_*$. We now have the following proposition:

[6]Noting that as a tight probability measure on a metric space, $\mathbb{P}^*$ concentrates on a separable set.

Proposition 4.4. *For any $t \in [0,\infty)$ we have $\bar{\mathbf{L}}_t^{(N)} \to \bar{\mathbf{L}}_t^*$ almost surely in the weak topology. In addition, $\bar{\mathbf{L}}_t^{(N)} \otimes \bar{\mathbf{L}}_t^{(N)} \to \bar{\mathbf{L}}_t^* \otimes \bar{\mathbf{L}}_t^*$ almost surely in the weak topology, where the symbol $\otimes$ denotes the product measure on the space $(\Omega, \mathscr{F}, \mathbb{P}(\cdot))$.*

Proof of Proposition 4.4. The proof is the result of the following observations:

(I) First note that, for any $J \in C_c(E;\mathbb{R})$, the operator $\tilde{J} : D([0,\infty);\mathcal{E}) \to D([0,\infty);\mathbb{R})$ is continuous, as the function $\tilde{J} : \mathcal{E} \to \mathbb{R}$ defined by $\tilde{J}(u) = \langle J, u\rangle$ is continuous (see for example, [Theorem 4.3, [24]]). This implies that, if $\tilde{J}(\bar{\mathbf{L}}^{(N)})$ denotes the map $t \mapsto \tilde{J}(\bar{\mathbf{L}}_t^{(N)})$, for any $J \in C_c(E;\mathbb{R})$, we have $\tilde{J}(\bar{\mathbf{L}}^{(N)}) \to \tilde{J}(\bar{\mathbf{L}}^*)$ almost surely in $D([0,\infty);\mathbb{R})$.

(II) Applying (53), and observing that $\sup_{s,t\in[0,T],|s-t|\le\eta} \left|\tilde{J}(\bar{\mathbf{L}}_t) - \tilde{J}(\bar{\mathbf{L}}_s)\right|$ is monotone decreasing in η, we have

$$\lim_{N\to\infty} \mathbb{E}\left[\lim_{\eta\to 0} \sup_{s,t\in[0,T],|s-t|\le\eta} \left|\tilde{J}(\bar{\mathbf{L}}_t) - \tilde{J}(\bar{\mathbf{L}}_s)\right|\right] = 0. \tag{76}$$

In addition, one may readily verify that, for any $T \in [0,\infty)$ the functional

$$x \mapsto \lim_{\eta\to 0} \sup_{s,t\in[0,T],|s-t|\le\eta} \|x(t) - x(s)\| \tag{77}$$

is a continuous functional with respect to the Skorokhod topology. Consequentially, by bounded convergence,

$$0 = \mathbb{E}\left[\lim_{N\to\infty}\lim_{\eta\to 0} \sup_{s,t\in[0,T],|s-t|\le\eta} \left|\tilde{J}(\bar{\mathbf{L}}_t^{(N)}) - \tilde{J}(\bar{\mathbf{L}}_s^{(N)})\right|\right] \overset{(77)}{=} \mathbb{E}\left[\lim_{\eta\to 0} \sup_{s,t\in[0,T],|s-t|\le\eta} \left|\tilde{J}(\bar{\mathbf{L}}_t^*) - \tilde{J}(\bar{\mathbf{L}}_s^*)\right|\right] \tag{78}$$

for any $J \in C_c(E;\mathbb{R})$, where in the final equality we have used the continuity of (77). Therefore, the function $\tilde{J}(\bar{\mathbf{L}}^*) : [0,\infty) \to \mathbb{R}$ such that $t \mapsto \tilde{J}(\bar{\mathbf{L}}_t^*)$ is almost surely continuous (i.e., $\tilde{J}(\bar{\mathbf{L}}^*) \in C([0,\infty),\mathbb{R})$ almost surely).

(III) This continuity implies that for any sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to t$, for any $J \in C_c(E;\mathbb{R})$ we have

$$\tilde{J}(\bar{\mathbf{L}}_{t_n}^*) = \int_E J(x)\bar{\mathbf{L}}_{t_n}^*(\mathrm{d}x) \to \int_E J(x)\bar{\mathbf{L}}_t^*(\mathrm{d}x) \quad \text{almost surely.}$$

But, since, by assumption on the initial condition we have $\bar{\mathbf{L}}_0^* = \mu$, where μ denotes the limiting measure from (11), and the dynamics of the process ensure that $\|\bar{\mathbf{L}}_t^{(N)}\|$ is non-increasing for each N, we readily verify that each for each $t \in [0,\infty)$ we have $\bar{\mathbf{L}}_t^*(E) \le \|\mu\|$. Thus, by approximating any $F \in C_b(E;\mathbb{R})$ by compactly supported functions, for any $F \in C_b(E;\mathbb{R})$ we have

$$\int_E F(x)\bar{\mathbf{L}}_{t_n}^*(\mathrm{d}x) \to \int_E F(x)\bar{\mathbf{L}}_t^*(\mathrm{d}x) \quad \text{almost surely.}$$

This implies that $\bar{\mathbf{L}}^*$ is, almost surely, a continuous trajectory of measures, i.e., $\bar{\mathbf{L}}^* \in C([0,\infty);\mathcal{E})$.

(IV) It is well-known that in a Skorokhod space the projection map $\pi_t : D([0,\infty); E) \to E$ is a continuous functional at any trajectory $x \in D([0,\infty); E)$ for which t is a continuity point. Since every $t \in [0,\infty)$ is a continuity point of $\bar{\mathbf{L}}^*_t$, this implies that for any $t \in [0,\infty)$ we have $\bar{\mathbf{L}}^{(N)}_t \to \bar{\mathbf{L}}^*_t$ almost surely in the weak topology, as required. Now, by a similar approach to the proof of Lemma 3.4, the family of measures $\left\{\bar{\mathbf{L}}^{(N)}_t \otimes \bar{\mathbf{L}}^{(N)}_t, N \in \mathbb{N}\right\}$, is (almost surely) tight, and by assumption uniformly bounded in norm, thus almost surely relatively compact by [34, Theorem IV.29, page 82]; and any accumulation point must be $\bar{\mathbf{L}}^*_t \otimes \bar{\mathbf{L}}^*_t$. Thus, $\bar{\mathbf{L}}^{(N)}_t \otimes \bar{\mathbf{L}}^{(N)}_t \to \bar{\mathbf{L}}^*_t \otimes \bar{\mathbf{L}}^*_t$ almost surely.

□

Proof of Theorem 3.5

Parts of the proof of Theorem 3.5 rely on equations from the proof of Lemma 3.4, hence we recommend that the reader be acquainted with this proof first.

Proof of Theorem 3.5. In order to simplify some expressions, we make some shorthands. Recall that for any compactly supported $J \in C_c(E;\mathbb{R})$, we denote by $\tilde{J}$ the functional such that

$$\tilde{J}(\bar{\mathbf{L}}^{(N)}_s) = \langle J, \bar{\mathbf{L}}^{(N)}_s \rangle = \int_E \bar{\mathbf{L}}^{(N)}_s(\mathrm{d}x) J(x).$$

We also define the following functionals:

$$G^+(\bar{\mathbf{L}}^{(N)}_s, J) := \frac{1}{2}\int_{E\times E\times E} \bar{\mathbf{L}}^{(N)}_s(\mathrm{d}x)\bar{\mathbf{L}}^{(N)}_s(\mathrm{d}y) K(x,y,\mathrm{d}z) J(z). \tag{79}$$

We recall that, by assumption, there exists a continuous function $\phi : E \times E \mapsto \mathbb{R}^+$ that satisfies Equations (17), (23) and (20). We then define $\hat{G}$ by

$$\hat{G}(\bar{\mathbf{L}}^{(N)}_s, J) := \int_{E\times E} \bar{\mathbf{L}}^{(N)}_s(\mathrm{d}x)\bar{\mathbf{L}}^{(N)}_s(\mathrm{d}y) \left[\bar{K}(x,y) - \phi(x,y)\right] J(y). \tag{80}$$

Finally, we define the functional

$$H(\bar{\mathbf{L}}^{(N)}_s, J) := \int_E \bar{\mathbf{L}}^{(N)}_s(\mathrm{d}y) \left\langle \phi(\cdot,y), \bar{\mathbf{L}}^{(N)}_0 \right\rangle J(y) = \int_{E\times E} \bar{\mathbf{L}}^{(N)}_s(\mathrm{d}y)\bar{\mathbf{L}}^{(N)}_0(\mathrm{d}x)\phi(x,y)J(y). \tag{81}$$

We now have the following claim:

Claim 4.4.1. *Almost surely, for any $t \in [0,\infty)$, $J \in C_c(E;\mathbb{R})$ we have*

$$\tilde{J}(\bar{\mathbf{L}}^*_t) - \tilde{J}(\mu) = \int_0^t G^+(\bar{\mathbf{L}}^*_s, J) - \hat{G}(\bar{\mathbf{L}}^*_s, J) - H(\bar{\mathbf{L}}^*_s, J)\,\mathrm{d}s. \tag{82}$$

Note that the truth of Equation (82) for any $J \in C_c(E;\mathbb{R})$ implies that, almost surely, $(\bar{\mathbf{L}}^*_t)_{t\geq 0}$ satisfies (8) in Definition 2.1. Now, note that, as an application of Proposition 4.4, for any $t \geq 0$ we have

$$\bar{\mathbf{L}}^{(N)}_t(\mathrm{d}x)\left(\bar{\mathbf{L}}^{(N)}_t(\mathrm{d}y) - \frac{\delta_x}{N}\right) \to \bar{\mathbf{L}}^*_t \otimes \bar{\mathbf{L}}^*_t \tag{83}$$

almost surely in the weak topology. Recall that we have $\bar{K} \leq \phi'$, where ϕ' is doubly sub-conservative and continuous by the first assumption of Theorem 3.5, and ϕ (which is also continuous, and ϕ' both satisfy (20). Thus, exploiting weak convergence, and Lemma 4.3, we deduce that $(\bar{\mathbf{L}}^{*}_{t})_{t\geq 0}$ also satisfies Equations (7) and (9), thus is a solution of the multi-type Flory equation in the sense given by Definition 2.1. Hence the claim completes the proof of the theorem. □

It thus suffices to prove the claim.

Proof of Claim 4.4.1. First note that by recalling the martingale from Equation (54), together with the bound from (69) in the proof of Lemma 3.4, we obtain

$$\lim_{N\to\infty} \mathbb{E}\Bigg[\Bigg|\tilde{J}(\bar{\mathbf{L}}^{(N)}_{t}) - \tilde{J}(\bar{\mathbf{L}}^{(N)}_{0}) - \frac{1}{2}\int_{0}^{t} \mathrm{d}s \int_{E\times E\times E} \bar{\mathbf{L}}^{(N)}_{s}(\mathrm{d}x)\left(\bar{\mathbf{L}}^{(N)}_{s}(\mathrm{d}y) - \frac{\delta_x}{N}\right) K(x,y,\mathrm{d}z) \tag{84}$$

$$\times\,(J(z) - J(y) - J(x))\Bigg|\Bigg] = 0. \tag{85}$$

Now, we define

$$G^{-}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J) := \frac{1}{2}\int_{E\times E} \bar{\mathbf{L}}^{(N)}_{s}(\mathrm{d}x)\left(\bar{\mathbf{L}}^{(N)}_{s}(\mathrm{d}y) - \frac{\delta_x}{N}\right)\bar{K}((x,y)(J(x)+J(y)), \tag{86}$$

$$G^{+}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J) := \frac{1}{2}\int_{E\times E\times E} \bar{\mathbf{L}}^{(N)}_{s}(\mathrm{d}x)\left(\bar{\mathbf{L}}^{(N)}_{s}(\mathrm{d}y) - \frac{\delta_x}{N}\right) K(x,y,\mathrm{d}z)J(z), \tag{87}$$

$$\hat{G}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J) := \int_{E\times E} \bar{\mathbf{L}}^{(N)}_{s}(\mathrm{d}x)\left(\bar{\mathbf{L}}^{(N)}_{s}(\mathrm{d}y) - \frac{\delta_x}{N}\right)\left[\bar{K}(x,y) - \phi(x,y)\right] J(y). \tag{88}$$

We may thus re-write the inner integral appearing in Equation (84) as $G^{+}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J) - G^{-}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J)$, so that

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\tilde{J}(\bar{\mathbf{L}}^{(N)}_{t}) - \tilde{J}(\bar{\mathbf{L}}^{(N)}_{0}) - \int_{0}^{t} G^{+}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J) - G^{-}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J)\,\mathrm{d}s\right|\right] = 0. \tag{89}$$

Now, we seek to exploit the convergence of $\bar{\mathbf{L}}^{(N)}_{s}$ to $\bar{\mathbf{L}}^{*}_{s}$, but note that as the integrand appearing in Equation (86) is in general unbounded, and G^{-} is, in general, not continuous. However, it is possible to show that this functional coincides with a continuous functional on the trajectories $s \mapsto \bar{\mathbf{L}}^{(N)}_{s}$. Indeed, since ϕ is conservative, by Lemma 4.3 the quantity $\left\langle \phi(\cdot, y), \bar{\mathbf{L}}^{(N)}_{s}\right\rangle$ is fixed, by adding and subtracting the term corresponding to $\int_{E\times E} \bar{\mathbf{L}}^{(N)}_{s}(\mathrm{d}y)\bar{\mathbf{L}}^{(N)}_{0}(\mathrm{d}x)\phi(x,y)J(y)$, we have

$$G^{-}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J) = \hat{G}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J) + H(\bar{\mathbf{L}}^{(N)}_{s}, J) - \mathcal{E}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J),$$

with

$$\mathcal{E}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J) := \frac{1}{N}\int_{E} \phi(x,x)J(x)\bar{\mathbf{L}}^{(N)}_{s}(\mathrm{d}x).$$

Thus, re-writing Equation (89), we get

$$\lim_{N\to\infty} \mathbb{E}\left[\left|\tilde{J}(\bar{\mathbf{L}}^{(N)}_{t}) - \tilde{J}(\bar{\mathbf{L}}^{(N)}_{0}) - \int_{0}^{t} G^{+}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J) - \hat{G}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J) - H(\bar{\mathbf{L}}^{(N)}_{s}, J) + \mathcal{E}_{N}(\bar{\mathbf{L}}^{(N)}_{s}, J)\,\mathrm{d}s\right|\right] = 0. \tag{90}$$

Now, in order to complete the proof of Equation (82), we need to argue that we can pass the limit inside the expectation, and exploit weak convergence to replace the terms corresponding to $\bar{\mathbf{L}}^{(N)}$ with $\bar{\mathbf{L}}^{*}$.We can pass the limit inside if the term $\hat{G}(\bar{\mathbf{L}}_s^{(N)}, J)$ was bounded, and then, need to argue continuity of the operators $\hat{G}$ and H. Consequentially, we first approximate the functional $\hat{G}$ by truncations $(\hat{G}^{(k)})_{k\in\mathbb{N}}$, such that, with compact sets as defined in Equation (23)

$$\hat{G}^{(k)}(\bar{\mathbf{L}}_s^{(N)}, J) := \int_{C_k} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) \int_E \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) \left(\bar{K}(x,y) - \phi(x,y)\right) J(y). \tag{91}$$

We will now finish the proof with another claim.

Claim 4.4.2. *Almost surely, uniformly in $s \in [0,\infty)$, $J \in C_c(E)$ we have*

$$\lim_{k\to\infty} \limsup_{N\to\infty} \mathbb{E}\left[\left|\int_0^t \hat{G}_N(\bar{\mathbf{L}}_s^{(N)}, J) - \hat{G}^{(k)}(\bar{\mathbf{L}}_s^{(N)}, J)\,\mathrm{d}s\right|\right] = 0, \tag{92}$$

$$\limsup_{N\to\infty} \mathbb{E}\left[\left|\int_0^t G_N^{+}(\bar{\mathbf{L}}_s^{(N)}, J) - G^{+}(\bar{\mathbf{L}}_s^{(N)}, J)\,\mathrm{d}s\right|\right] = 0, \quad \limsup_{N\to\infty} \mathbb{E}\left[\left|\int_0^t \mathcal{E}_N(\bar{\mathbf{L}}_s^{(N)}, J)\,\mathrm{d}s\right|\right] = 0, \tag{93}$$

$$\lim_{k\to\infty} \mathbb{E}\left[\left|\int_0^t \hat{G}(\bar{\mathbf{L}}_s^{*}, J) - \hat{G}^{(k)}(\bar{\mathbf{L}}_s^{*}, J)\,\mathrm{d}s\right|\right] = 0, \tag{94}$$

and

$$\lim_{N\to\infty} H(\bar{\mathbf{L}}_s^{(N)}, J) = H(\bar{\mathbf{L}}_s^{*}, J) \quad \textit{almost surely.} \tag{95}$$

Indeed, if Equations (92), (93), (94) and (95) are satisfied, by approximating $\hat{G}$ by $\hat{G}^{(k)}$ (using the triangle inequality) in the second equality, and using bounded convergence for the third, we have

$$0 = \lim_{N\to\infty} \mathbb{E}\left[\left|\tilde{J}(\bar{\mathbf{L}}_t^{(N)}) - \tilde{J}(\bar{\mathbf{L}}_0^{(N)}) - \int_0^t G_N^{+}(\bar{\mathbf{L}}_s^{(N)}, J) - \hat{G}_N(\bar{\mathbf{L}}_s^{(N)}, J) - H_N(\bar{\mathbf{L}}_s^{(N)}, J) + \mathcal{E}_N(\bar{\mathbf{L}}_s^{(N)}, J)\,\mathrm{d}s\right|\right] \tag{96}$$

$$\overset{(92),(93)}{=} \lim_{k\to\infty}\lim_{N\to\infty} \mathbb{E}\left[\left|\tilde{J}(\bar{\mathbf{L}}_t^{(N)}) - \tilde{J}(\bar{\mathbf{L}}_0^{(N)}) - \int_0^t G^{+}(\bar{\mathbf{L}}_s^{(N)}, J) - \hat{G}^{(k)}(\bar{\mathbf{L}}_s^{(N)}, J) - H(\bar{\mathbf{L}}_s^{(N)}, J)\,\mathrm{d}s\right|\right] \tag{97}$$

$$= \lim_{k\to\infty} \mathbb{E}\left[\lim_{N\to\infty}\left|\tilde{J}(\bar{\mathbf{L}}_t^{(N)}) - \tilde{J}(\bar{\mathbf{L}}_0^{(N)}) - \int_0^t G^{+}(\bar{\mathbf{L}}_s^{(N)}, J) - \hat{G}^{(k)}(\bar{\mathbf{L}}_s^{(N)}, J) - H(\bar{\mathbf{L}}_s^{(N)}, J)\,\mathrm{d}s\right|\right] \tag{98}$$

$$\overset{(95)}{=} \lim_{k\to\infty} \mathbb{E}\left[\left|\tilde{J}(\bar{\mathbf{L}}_t^{*}) - \tilde{J}(\mu) - \int_0^t G^{+}(\bar{\mathbf{L}}_s^{*}, J) - \hat{G}^{(k)}(\bar{\mathbf{L}}_s^{*}, J) - H(\bar{\mathbf{L}}_s^{*}, J)\,\mathrm{d}s\right|\right] \tag{99}$$

$$\overset{(94)}{=} \mathbb{E}\left[\left|\tilde{J}(\bar{\mathbf{L}}_t^{*}) - \tilde{J}(\mu) - \int_0^t G^{+}(\bar{\mathbf{L}}_s^{*}, J) - \hat{G}(\bar{\mathbf{L}}_s^{*}, J) - H(\bar{\mathbf{L}}_s^{*}, J)\,\mathrm{d}s\right|\right]. \tag{100}$$

□

Finally, we finish the proof of Claim 4.4.2.

Proof of Claim 4.4.2. Note that we have

$$\mathbb{E}\left[\left|\int_0^t \hat{G}_N(\bar{\mathbf{L}}_s^{(N)}, J) - \hat{G}^{(k)}(\bar{\mathbf{L}}_s^{(N)}, J)\,\mathrm{d}s\right|\right] \tag{101}$$

$$\leq \frac{1}{N}\mathbb{E}\left[\left|\int_0^t \int_{C_k} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) |\bar{K}(x,x) - \phi(x,x)||J(x)|\,\mathrm{d}s\right|\right] \tag{102}$$

$$+ \mathbb{E}\left[\int_0^t \int_{C_k^c} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) \int_E \left(\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N}\right) \left|\bar{K}(x,y) - \phi(x,y)\right| |J(y)|\,\mathrm{d}s\right], \tag{103}$$

where we immediately see that

$$\limsup_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\left|\int_0^t \int_{C_k} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x)|\bar{K}(x,x)-\phi(x,x)||J(x)|\,\mathrm{d}s\right|\right] = 0, \tag{104}$$

since $\bar{K}$, ϕ and J are continuous, they are bounded on the support of J and thus so is the expectation. Now, if Equation (22) applies, then the integrand of the second term in (101) is bounded by some constant $c' > 0$, thus by (83) and the Portmanteau theorem, we have

$$\limsup_{N\to\infty} \mathbb{E}\left[\int_0^t \int_{C_k^c} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) \int_E \left(\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N}\right) \left|\bar{K}(x,y)-\phi(x,y)\right| |J(y)|\,\mathrm{d}s\right] \tag{105}$$

$$\leq c'\mathbb{E}\left[\int_0^t \bar{\mathbf{L}}_s^*\left(\overline{C_k^c}\right)\bar{\mathbf{L}}_s^*(E)\mathrm{d}s\right]; \tag{106}$$

and applying bounded convergence, using the fact that $\bigcap_{k\in\mathbb{N}} \overline{C_k^c} = \varnothing$, we deduce (92). Otherwise, in the case that Equation (23) applies, we bound the second term in (101) as follows:

$$\mathbb{E}\left[\int_0^t \int_{C_k^c} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) \int_E \left(\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N}\right) \left|\bar{K}(x,y)-\phi(x,y)\right| |J(y)|\,\mathrm{d}s\right], \tag{107}$$

$$\leq \mathbb{E}\left[\int_0^t \int_{C_k^c} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) \int_E \left(\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N}\right) \phi^*(x,y)\,\mathrm{d}s\right] \|J\|_\infty \tag{108}$$

$$\times \sup_{x\in C_k^c,\, y\in \mathrm{Supp}(J)} \left|\frac{\bar{K}(x,y)-\phi(x,y)}{\phi^*(x,y)}\right|, \tag{109}$$

Since ϕ^* is doubly sub-conservative, and satisfies Equation (20), we have

$$\limsup_{N\to\infty} \mathbb{E}\left[\int_0^t \int_{C_k^c\times E} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N}\right) \phi^*(x,y)\,\mathrm{d}s\right] \tag{110}$$

$$\leq \limsup_{N\to\infty} \mathbb{E}\left[\int_0^t \int_{E\times E} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N}\right) \phi^*(x,y)\,\mathrm{d}s\right] \tag{111}$$

$$\leq \limsup_{N\to\infty} t\,\mathbb{E}\left[\int_{E\times E} \bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N}\right) \phi^*(x,y)\right] \overset{(20)}{<} \infty. \tag{112}$$

Combining Equation (110), with (101), (104) and (107), we deduce Equation (92).

Equation (93) is proved in an analogous manner to (104), exploiting the compact support of J. For (94), by the monotone convergence theorem, (83) and Fatou's lemma we have

$$\mathbb{E}\left[\int_0^t \int_{E\times E} \bar{\mathbf{L}}_s^*(\mathrm{d}x)\bar{\mathbf{L}}_s^*(\mathrm{d}y)\phi^*(x,y)\,\mathrm{d}s\right] \tag{113}$$

$$= \mathbb{E}\left[\int_0^t \lim_{j\to\infty} \int_{E\times E} \bar{\mathbf{L}}_s^*(\mathrm{d}x)\bar{\mathbf{L}}_s^*(\mathrm{d}y)\left(\phi^*(x,y)\wedge j\right)\,\mathrm{d}s\right] \tag{114}$$

$$= \mathbb{E}\left[\int_0^t \lim_{j\to\infty}\lim_{N\to\infty} \int_{E\times E} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N}\right) \left(\phi^*(x,y)\wedge j\right)\,\mathrm{d}s\right] \tag{115}$$

$$\leq \limsup_{N\to\infty} \mathbb{E}\left[\int_0^t \int_{E\times E} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}x) \left(\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y) - \frac{\delta_x}{N}\right) \phi^*(x,y)\,\mathrm{d}s\right] \overset{(110)}{<} \infty. \tag{116}$$

Applying Equation (113) we deduce Equation (94) in a similar manner to Equation (92). Finally, recalling the definition of the functional H from Equation (81), we have

$$\lim_{N\to\infty} H(\bar{\mathbf{L}}_s^{(N)}, J) = \lim_{N\to\infty} \int_{E\times E} \bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x)\phi(x,y)J(y).$$

Thus,

$$\limsup_{N\to\infty}\left|H(\bar{\mathbf{L}}_s^{(N)},J)-H(\bar{\mathbf{L}}_s^*,J)\right| \tag{117}$$

$$=\limsup_{N\to\infty}\left|\int_{E\times E}\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x)\phi(x,y)J(y)-\int_{E\times E}\bar{\mathbf{L}}_s^*(\mathrm{d}y)\mu(\mathrm{d}x)\phi(x,y)J(y)\right| \tag{118}$$

$$\le\limsup_{N\to\infty}\left|\int_{E\times E}\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x)\phi(x,y)J(y)-\int_{E\times E}\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)\mu(\mathrm{d}x)\phi(x,y)J(y)\right| \tag{119}$$

$$+\limsup_{N\to\infty}\left|\int_{E\times E}\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)\mu(\mathrm{d}x)\phi(x,y)J(y)-\int_{E\times E}\bar{\mathbf{L}}_s^*(\mathrm{d}y)\mu(\mathrm{d}x)\phi(x,y)J(y)\right|. \tag{120}$$

The second term in the upper bound of (117) is 0, since the map $y\mapsto\int_E\mu(\mathrm{d}x)\phi(x,y)J(y)$ is bounded and continuous (because J has compact support and ϕ is continuous), and $\bar{\mathbf{L}}_s^{(N)}\to\bar{\mathbf{L}}_s^*$. On the other hand, by applying Equation (24), with the compact set C' chosen to be the support of J, for any $\varepsilon>0$, there exists N_0 such that, for all $N\ge N_0$ we have

$$\left|\int_{E\times E}\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x)\phi(x,y)J(y)-\int_{E\times E}\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)\mu(\mathrm{d}x)\phi(x,y)J(y)\right|\le\varepsilon\left|\int_E\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)J(y)\right|,$$

and thus, taking limits superior of both sides, since J is bounded and continuous,

$$\limsup_{N\to\infty}\left|\int_{E\times E}\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)\bar{\mathbf{L}}_0^{(N)}(\mathrm{d}x)\phi(x,y)J(y)-\int_{E\times E}\bar{\mathbf{L}}_s^{(N)}(\mathrm{d}y)\mu(\mathrm{d}x)\phi(x,y)J(y)\right|\le\varepsilon\left|\int_E\bar{\mathbf{L}}_s^*(\mathrm{d}y)J(y)\right|.$$

Sending $\varepsilon\to0$, we deduce that the first term in the upper bound of (117) is also 0, hence conclude the proof of (95). □

4.2.3 Uniqueness of eventually conservative solutions to the multi-type Flory equation

Proof of Theorem 3.7. Suppose that $(\mu_s)_{s\ge0}$ and $(\hat{\mu}_s)_{s\ge0}$ denote two solutions to the Flory equation, with a given initial condition μ_0, with $\|\mu_0\|<\infty$. Suppose that $(\mu_s-\hat{\mu}_s)\,|_{D_R}$ denotes the measure $(\mu_s-\hat{\mu}_s)$ restricted to D_R. By a well-known property of the total variation distance, we may write

$$\|(\mu_s-\hat{\mu}_s)\,|_{D_R}>0\|=\sup_{f:\|f\|_\infty=1}\left\langle f\mathbf{1}_{D_R},(\mu_s-\hat{\mu}_s)\right\rangle.$$

Note that, by a straightforward approximation argument (approximating a measurable function pointwise by continuous functions), if $(\mu_t)_{t\ge0}$ is a solution to the multi-type Flory equation as in Definition 2.1, Equation (8) is satisfied for all bounded measurable functions supported on the compact set D_k, for $k \in \mathbb{N}$. Thus, for f such that $\|f\|_\infty = 1$, we have

$$\langle f\mathbf{1}_{D_R}, (\mu_s - \hat{\mu}_s))\rangle = \frac{1}{2}\int_0^s \int_{E\times E\times E} f(z)\mathbf{1}_{D_R}(z)K(u,v,\mathrm{d}z)\left(\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right)\mathrm{d}r \tag{121}$$

$$- \int_0^s \int_{E\times E} f(u)\mathbf{1}_{D_R}(u)\bar{K}(u,v)\left(\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right)\mathrm{d}r$$
$$- \int_0^s \int_{E\times E} f(u)\mathbf{1}_{D_R}(u)\phi(u,v)\left(\mu_r(\mathrm{d}u)\mu_0(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\mu_0(\mathrm{d}v)\right)\mathrm{d}r$$
$$+ \int_0^s \int_{E\times E} f(u)\mathbf{1}_{D_R}(u)\phi(u,v)\left(\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right)\mathrm{d}r. \tag{122}$$

We now bound the values of each of the terms in the above display. For the first, since ϕ is conservative, we have $\mathbf{1}_{D_R}(z) \leq \mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)$ for $K(u,v,\mathrm{d}z)$-a.a. z. Moreover, bounding $f(z)$ above by 1, we have

$$\begin{aligned}&\frac{1}{2}\int_0^s \int_{E\times E\times E} f(z)\mathbf{1}_{D_R}(z)K(u,v,\mathrm{d}z)\left(\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right)\mathrm{d}r\\ &\leq \frac{1}{2}\int_0^s \int_{E\times E} \mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)\bar{K}(u,v)\left|\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right|\mathrm{d}r. \end{aligned}\tag{123}$$

We now have the following claim:

Claim 4.4.3. *We have*

$$\int_{E\times E} \mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)\phi(u,v)\left|\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right| \leq 2R\int_E \mathbf{1}_{D_R}(v)\left|\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}v)\right|. \tag{124}$$

By applying Claim 4.4.3, and bounding $\bar{K}(x,y) \leq c'\phi(x,y)$, we may now bound the right-side of (123):

$$\frac{1}{2}\int_0^s \int_{E\times E} \mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)\bar{K}(u,v)\left|\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right|\mathrm{d}r \tag{125}$$
$$\leq \frac{c'}{2}\int_0^s \int_{E\times E} \mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)\phi(u,v)\left|\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right|\mathrm{d}r \tag{126}$$
$$\leq c'\int_0^s R\int_E \mathbf{1}_{D_R}(v)\left|\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}v)\right|\mathrm{d}r \leq c'R\int_0^s \left\|(\mu_r - \hat{\mu}_r)\,|_{D_R}\right\|\mathrm{d}r. \tag{127}$$

Next, re-writing, and combining the second and fourth terms in (121), recalling that $\phi(x,y)$ coincides with $\bar{K}(x,y)$ on $(D_R \times D_R)^c$ (so in particular $D_R \times D_R^c$) we get

$$\int_0^s \int_{E\times E} f(u)\mathbf{1}_{D_R}(u)\left(\phi(u,v) - \bar{K}(u,v)\right)\left(\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right)\mathrm{d}r \tag{128}$$
$$= \int_0^s \int_{E\times E} f(u)\mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)\left(\phi(u,v) - \bar{K}(u,v)\right)\left(\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right)\mathrm{d}r \tag{129}$$
$$\leq (c'+1)\int_0^s \int_{E\times E} \phi(u,v)\mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)\left|\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right|\mathrm{d}r \tag{130}$$
$$\overset{(124)}{\leq} 2R(c'+1)\int_0^s \int_E \mathbf{1}_{D_R}(v)\left|\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}v)\right|\mathrm{d}r \leq 2R(C+1)\int_0^s \left\|(\mu_r - \hat{\mu}_r)\,|_{D_R}\right\|\mathrm{d}r, \tag{131}$$

where in the second to last inequality, we use the bound $K(u,v) \leq C\phi(u,v)$. Finally, for the third term in (121), observing that μ_0 is a positive measure, we make a similar computation:

$$-\int_0^s \int_{E\times E} f(u)\mathbf{1}_{D_R}(u)\phi(u,v)\left(\mu_r(\mathrm{d}u)\mu_0(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\mu_0(\mathrm{d}v)\right)\mathrm{d}r \tag{132}$$

$$\leq \int_0^s \int_{E\times E} \mathbf{1}_{D_R}(u)\phi(u,v)\left|\mu_r(\mathrm{d}u)\mu_0(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\mu_0(\mathrm{d}v)\right|\mathrm{d}r \tag{133}$$

$$= \int_0^s \int_E \mathbf{1}_{D_R}(u)\int_E \phi(u,v)\mu_0(\mathrm{d}v)\left|\mu_r(\mathrm{d}u) - \hat{\mu}_r(\mathrm{d}u)\right|\mathrm{d}r \leq R\int_0^s \left\|(\mu_r - \hat{\mu}_r)\,|_{D_R}\right\|\mathrm{d}r \tag{134}$$

Combining Equations (125), (128) and (132), to bound (121) we deduce that

$$\left\|(\mu_s - \hat{\mu}_s)\,|_{D_R}\right\| = \sup_{f:\|f\|_\infty=1} \langle f\mathbf{1}_{D_R}, (\mu_s - \hat{\mu}_s))\rangle \leq 3R(c'+1)\int_0^s \left\|(\mu_r - \hat{\mu}_r)\,|_{D_R}\right\|\mathrm{d}r.$$

Claim 4.4.4. *Suppose that $(\mu_t)_{t\geq 0}$ is a solution to the multi-type Flory equation. Then, if $\xi : E \to \mathbb{R}_+$ is a sub-conservative function, for each $t \geq 0$*

$$\int_E \xi(x)\mu_t(\mathrm{d}x) \leq \int_E \xi(x)\mu_0(\mathrm{d}x). \tag{135}$$

By Claim 4.4.4 applied to the function $\xi(x) \equiv 1$, we know that for each $t \geq 0$, we have

$$\left\|(\mu_s - \hat{\mu}_s)\,|_{D_R}\right\| \leq 2\|\mu_0\|.$$

We can thus apply Gronwall's lemma to deduce that $\|(\mu_s - \hat{\mu}_s)\,|_{D_R}\| = 0$. As $\bigcup_{k\in\mathbb{N}} D_k = E$, it must be the case that $\|(\mu_s - \hat{\mu}_s)\| = 0$, showing uniqueness. □

We finish with the proofs of Claim 4.4.3 and Claim 4.4.4:

Proof of Claim 4.4.3. We bound

$$\int_{E\times E} \mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)\phi(u,v)\left|\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right| \tag{136}$$

$$\leq \int_{E\times E} \mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)\phi(u,v)\left|\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \mu_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right| \tag{137}$$

$$+ \int_{E\times E} \mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)\phi(u,v)\left|\mu_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right|. \tag{138}$$

For the first term on the right-hand side of (136), we integrate the variable u, applying (9)

$$\int_{E\times E} \mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)\phi(u,v)\left|\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v) - \mu_r(\mathrm{d}u)\hat{\mu}_r(\mathrm{d}v)\right|\mathrm{d}r$$
$$\leq \int_E \mathbf{1}_{D_R}(v)\int_E \phi(u,v)\mu_r(\mathrm{d}u)\left|\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}v)\right|\mathrm{d}r$$
$$\leq \int_E \mathbf{1}_{D_R}(v)\int_E \phi(u,v)\mu_0(\mathrm{d}u)\left|\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}v)\right|\mathrm{d}r \leq R\int_E \mathbf{1}_{D_R}(v)\left|\mu_r(\mathrm{d}v) - \hat{\mu}_r(\mathrm{d}v)\right|\mathrm{d}r;$$

and applying a similar argument for the second term in (136), we deduce the result. □

Proof of Claim 4.4.4. Suppose first that ξ is bounded. Then, applying (8) to the function $\xi \mathbf{1}_{D_k}$, we have

$$\langle \xi \mathbf{1}_{D_k}, \mu_s - \mu_0 \rangle = \frac{1}{2}\int_0^s \int_{E\times E\times E} \left(\xi(z)\mathbf{1}_{D_k}(z) - \xi(u)\mathbf{1}_{D_k}(u) - \xi(v)\mathbf{1}_{D_k}(v)\right) K(u,v,\mathrm{d}z)\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v)\mathrm{d}r \tag{139}$$

$$+ \int_0^s \int_{E\times E} \xi(u)\mathbf{1}_{D_k}(u)\phi(u,v)\left(\mu_r(\mathrm{d}u)(\mu_r(\mathrm{d}v) - \mu_0(\mathrm{d}v))\right)\mathrm{d}r. \tag{140}$$

By (9), since $\xi(u) \geq 0$, and μ_r is a positive measure, we deduce that the second term on the right-side above is non-positive. In addition, since ϕ is conservative, so is the function $x \mapsto \langle \phi(x,\cdot), \mu_0\rangle$, and we deduce that $\mathbf{1}_{D_R}(z) \leq \mathbf{1}_{D_R}(u)\mathbf{1}_{D_R}(v)$ for $K(u,v,\mathrm{d}z)$-a.a. z. In addition, since ξ is sub-conservative,

$$\begin{aligned}\int_{E\times E\times E} &\left(\xi(z)\mathbf{1}_{D_k}(z)\right) K(u,v,\mathrm{d}z)\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v)\\ &\leq \int_{E\times E\times E} \left(\xi(u)+\xi(v)\right)\mathbf{1}_{D_k}(u)\mathbf{1}_{D_k}(u) K(u,v,\mathrm{d}z)\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v)\\ &\leq \int_{E\times E\times E} \left(\xi(u)\mathbf{1}_{D_k}(u)+\xi(v)\mathbf{1}_{D_k}(v)\right) K(u,v,\mathrm{d}z)\mu_r(\mathrm{d}u)\mu_r(\mathrm{d}v).\end{aligned}$$

Thus, $\langle \xi\mathbf{1}_{D_k}, \mu_s\rangle \leq \langle \xi\mathbf{1}_{D_k}, \mu_0\rangle$, and we deduce the result from monotone convergence. Finally, we can extend the result to unbounded ξ, again from monotone convergence (approximating ξ from below by the sub-conservative functions $\xi \wedge j$, for $j \in \mathbb{N}$). □

4.3 Coupling with inhomogeneous random graph models

In this section, we recall that the notation $(\mathbf{L}_t^{(N)})_{t\geq 0}$ refers to the non-normalised coagulation process, whilst the notation $(\bar{\mathbf{L}}_t^{(N)})_{t\geq 0} = (\mathbf{L}_{t/N}^{(N)}/N)_{t\geq 0}$ refers to the normalised process. In order to construct couplings of the coagulation process with random graphs, it will be useful to label clusters (i.e. the points in the point measure $\mathbf{L}_0^{(N)}$).[7] Suppose we start with $\|\mathbf{L}_0^{(N)}\| \in \mathbb{N}$ labelled initial clusters, $x_1, \dots, x_{\|\mathbf{L}_0^{(N)}\|}$. As the coagulation process involves clusters 'merging' of the clusters that we have at time zero, we can consider those clusters as building blocks for clusters that arrive later in the process. Therefore, on the coupling level, it will be helpful to identify clusters at any time with subsets of $[\|\mathbf{L}_0^{(N)}\|]$, indicating which of the initial blocks constitute each cluster.

Furthermore, as we begin with a mono-dispersed initial condition, all of the clusters have mass one, and thus the mass of a cluster at time t is given by the cardinality of the subset that identifies it. Consequentially, we say that two subsets $I, J \subseteq [\|\mathbf{L}_0^{(N)}\|]$ have *merged* by time t if there exists a cluster at time t, identified by a set S, such that $I \cup J \subseteq S$. We use the same labelling to define the associated random graph process and we identify connected components with subsets of $[\|\mathbf{L}_0^{(N)}\|]$ that represents the labelling of the vertices in such a component. The graph process is defined in terms of the initial locations and masses of the coagulation process, such that, given any pair of vertices $i < j \in [\|\mathbf{L}_0^{(N)}\|]$, there is an exponential clock $Z_{i,j}$ with parameter $\kappa(i,j)$ as in (141), independent of all the others. When this clock rings, we include the edge $\{i,j\}$ in the graph.

[7]More formally, such a labelling involves expanding the underlying space E, to a space $E \times \mathbb{N}$, including labels, but we omit this, for brevity.

Definition 4.1. Given a graph comparable coagulation process $(\mathbf{L}_t^{(N)})_{t\geq 0}$, the *associated graph* process is the random graph process $(G_t^{(N)})_{t\geq 0}$ on vertex set $[\|\mathbf{L}_0^{(N)}\|]$, where:

1 $G_0^{(N)}$ is the empty graph on $\|\mathbf{L}_0^{(N)}\|$ vertices, i.e, containing no edges.

2 Given any pair $i, j \in [\|\mathbf{L}_0^{(N)}\|]$ with $i < j$, let $Z_{i,j}$ be an exponential random variable with parameter

$$\kappa(i,j) := \bar{K}(x_i, x_j). \tag{141}$$

Moreover, assume that all the $\{Z_{i,j}\}_{i<j}$ are independent and include the edge $(i,j) \in G_t^{(N)}$ if and only if $t \geq Z_{ij}$.

With regards to the graph process, we say that two subsets $I, J \subseteq [\|\mathbf{L}_0^{(N)}\|]$ are *connected* at time t if they belong to the same connected component. We identify the connected component of a vertex $i \in [\|\mathbf{L}_0^{(N)}\|]$ with the connected subgraph $\mathcal{C}(i)$ containing i.

Proposition 4.5. *For a graph dominating coagulation kernel, or a graph dominated coagulation kernel, one may couple the coagulation process with its associated random graph process such that the following hold.*

1 If the coagulation kernel is graph dominating, *for all $t > 0$, if the vertices in a pair of subsets $I, J \subseteq [\|\mathbf{L}_0^{(N)}\|]$ belong to the same connected component in the graph process at time t, the correspondent clusters merged in the coagulation process by time t.*

2 If the coagulation kernel is graph dominated, *for all $t > 0$, if clusters with indexes in a pair of subsets $I, J \subseteq [\|\mathbf{L}_0^{(N)}\|]$ merged in the coagulation process by time t, the corresponding vertices are connected in the graph process at time t.*

Suppose that, for each $n \in \mathbb{N}$, $M_n^{(N)}(t)$ denotes the number of connected components of size n at time t in $(G_t^{(N)})_{t\geq 0}$. The above proposition then yields the following corollary (a reformulation of the proposition).

Corollary 4.6. *For any graph comparable coagulation process $(\mathbf{L}_t^{(N)})_{t\geq 0}$, there exists a coupling of the coagulation process and its associated random graph process such that, on the coupling,*

1 if the coagulation process is graph dominating, for all $t \geq 0, j \in \mathbb{N}$, we have

$$\langle m\mathbf{1}_{m\geq j}, \mathbf{L}_t^{(N)} \rangle \geq \sum_{n=j}^{\infty} n M_n^{(N)}(t) \qquad \textit{almost surely;} \tag{142}$$

2 if the coagulation process is graph dominated, for all $t \geq 0, j \in \mathbb{N}$, we have

$$\langle m\mathbf{1}_{m\geq j}, \mathbf{L}_t^{(N)} \rangle \leq \sum_{n=j}^{\infty} n M_n^{(N)}(t) \qquad \textit{almost surely.} \tag{143}$$

The proposition and corollary may be used to transfer existing results related to the associated graph process (an inhomogeneous random graph) to the coagulation process. In the remainder of the section, we first work towards a proof of Proposition 4.5 (leaving the proof of Corollary 4.6, a reformulation of this result, to the reader). Then, we finish the section with the proof of Theorem 3.9.

4.3.1 Proof of Proposition 4.5

Suppose in this section, for brevity of notation, we begin with an initial condition $\mathbf{L}_0^{(N)} = \boldsymbol{\pi}$. As we will consider the clusters as being labelled, and identify clusters in the coagulation process with the labels of the initial clusters that have merged into it, we use $x_I \in E$ to denote the random variable representing the cluster created by merging initial clusters with index in I. Abusing notation, in order to describe rates of exponential clocks for the coupling, denote the rate at which particles x_I and x_J merge by

$$\bar{K}(I, J) := \bar{K}(x_I, x_J). \tag{144}$$

Likewise, in the graph process, we write

$$\kappa(I, J) := \sum_{k \in I, \ell \in J} \kappa(k, \ell). \tag{145}$$

We then have the following claim, which is what allows a pathwise coupling to work.

Claim 4.6.1. *For disjoint sets $S_1, S_2, S_3, S_4 \subseteq [\|\boldsymbol{\pi}\|]$, given $x_{S_1}, x_{S_2}, x_{S_3}, x_{S_4} \in E$, if a coagulation process is graph dominating, almost surely (with respect to the dynamics of the coagulation process) we have*

$$\bar{K}(S_1 \cup S_2, S_3 \cup S_4) \geq \bar{K}(S_1, S_3) + \bar{K}(S_1, S_4) + \bar{K}(S_2, S_3) + \bar{K}(S_2, S_4). \tag{146}$$

Alternatively, if a labelled coagulation process is graph dominated, almost surely we have

$$\bar{K}(S_1 \cup S_2, S_3 \cup S_4) \leq \bar{K}(S_1, S_3) + \bar{K}(S_1, S_4) + \bar{K}(S_2, S_3) + \bar{K}(S_2, S_4). \tag{147}$$

Proof. First notice that from the notation introduced in (144), we have $\bar{K}(S_1 \cup S_2, S_3 \cup S_4) = \bar{K}(x_{S_1 \cup S_2}, x_{S_3 \cup S_4})$; recall from (1) that, given x_{S_1} and x_{S_2}, $x_{S_1 \cup S_2}$ has distribution $\frac{K(x_{S_1}, x_{S_2}, \cdot)}{\bar{K}(x_{S_1}, x_{S_2})}$ (and the analogous property holds for $x_{S_3 \cup S_4}$). Therefore, we can apply inequality (28), thus obtaining

$$\bar{K}(S_1 \cup S_2, S_3 \cup S_4) \geq \bar{K}(S_1, S_3 \cup S_4) + \bar{K}(S_2, S_3 \cup S_4) \quad \text{almost surely;}$$

we only need to exploit symmetry of $\bar{K}$ and iterate this inequality to prove (146). The proof of (147) works in the same way. □

Proof of Proposition 4.5. In the proof of this lemma, we will denote by $Z(s)$ an independent copy of an exponentially distributed random variable, with parameter s. We will prove only the first statement, since the second is similar.

Suppose that $\sigma_0, \sigma_1, \sigma_2, \ldots$ denote the coagulation times associated with the coagulation process; where we set $\sigma_0 := 0$. Note that, associating clusters with subsets of $[\|\boldsymbol{\pi}\|]$ induces a partition of $[\|\boldsymbol{\pi}\|]$ which we denote by $\mathscr{P}(t)$. Likewise, the connected components of the associated random graph form a partition of $[\|\boldsymbol{\pi}\|]$, which we denote by $\mathscr{H}(t)$. Now, we construct a coupling $(\hat{\mathscr{P}}(t), \hat{\mathscr{H}}(t))$ of the two processes such that, for all $t \geq 0$, $\hat{\mathscr{H}}(t)$ is a refinement of $\hat{\mathscr{P}}(t)$. At time $\tau_0 = 0$ this is trivial; both partitions are identical. Now, assume that this is true for all $t \leq \tilde{\tau}_i$. We seek to construct $\tilde{\tau}_{i+1}$ such that this is also true for $\tilde{\tau}_{i+1}$. In order to do so, we let the graph evolve independently according to its dynamics, so that any two distinct connected components $J_i, J_j \subseteq \hat{\mathscr{H}}(\tilde{\tau}_i)$ merge after $Z(\kappa(J_i, J_j))$ (where we recall the definition of $\kappa(J_i, J_j)$ from Equation (145)). Now, since by induction hypothesis $\hat{\mathscr{H}}(\tilde{\tau}_i)$ is a refinement of $\hat{\mathscr{P}}(\tilde{\tau}_i)$, for any two sets $S_\ell, S_{\ell'} \in \hat{\mathscr{P}}(\tilde{\tau}_i)$ there exist disjoint sets $I_1, \dots, I_k, J_1, \dots, J_{k'} \in \hat{\mathscr{H}}(\tilde{\tau}_i)$ such that $S_\ell = \bigcup_{i=1}^k I_i$ and $S_{\ell'} = \bigcup_{i=1}^{k'} J_i$. Then, by iterated application of Claim 4.6.1 we have

$$r := K(S_\ell, S_{\ell'}) - \sum_{i<j} \kappa(I_i, J_j) \geq 0 \quad \text{almost surely.}$$

We now merge the clusters S_ℓ and $S_{\ell'}$ at time

$$\min \{Z(\kappa(I_i, J_j)) : i \in [k], j \in [k']\} \cup \{Z(r)\}. \tag{148}$$

By the minimum property of exponential random variables, this is distributed like $Z(K(S_\ell, S_{\ell'}))$ as required. We now verify that $\hat{\mathscr{H}}(\tilde{\tau}_{i+1})$ is a refinement of $\hat{\mathscr{P}}(\tilde{\tau}_{i+1})$. Indeed, suppose that clusters S_ℓ and $S_{\ell'}$ merge at time $\tilde{\tau}_{i+1}$. Then, if this occurs because the minimum in Equation (148) is given by $Z(r)$, there is nothing to prove: the sets $I_1, \dots, I_k, J_1, \dots, J_{k'}$ form a disjoint partition of $S_\ell \cup S_{\ell'}$. Otherwise, for some m, m', the sets $I_m, J_{m'}$ merge to form a set V, say. In this case, the sets $I_1, \dots, I_{m-1}, I_{m+1}, \dots, I_k, J_1, \dots, J_{m'-1}, J_{m'+1}, \dots, J_{k'}, V$ form a disjoint partition of $S_\ell \cup S_{\ell'}$. The result follows. □

4.3.2 Proof of Theorem 3.9

Proof of Theorem 3.9. In this proof, for each $N \in \mathbb{N}$ we use results related to the emergence of a 'giant component' in the inhomogenous random graph [11], to show that, with probability tending to 1 the associated random graph process $(G_t^{(N)})_{t\geq 0}$ has a component of order N at a certain time $t/N > 0$. In particular, note that at any time $t/N > 0$, the probability of an edge between two nodes, i, j associated with initial clusters x_i, x_j in the associated graph process is

$$\xi_{N,t}(x_i, x_j) := 1 - e^{-\bar{K}(x_i, x_j)t/N},$$

and thus, by [Remark 2.4, Theorem 3.1, Theorem 3.5 [11]], there exists $\alpha \in [0, 1]$ such that[8] at time t^*/N, with probability $1 - o(1)$, there exists a unique component of size αN in $G_{t^*/N}^{(N)}$. Thus, by applying the first statement of Corollary 4.6, (recalling the definition of τ_N^α from (12) and the identity $\bar{\mathbf{L}}_t^{(N)} = \mathbf{L}_{t/N}^{(N)}/N$), for any graph dominating coagulation process $(\bar{\mathbf{L}}_s^{(N)})_{s\geq 0}$ we have

$$\mathbb{P}\left(\tau_N^\alpha \leq t^*\right) \geq \mathbb{P}\left(\langle m\mathbf{1}_{m\geq \alpha N}, \mathbf{L}_{t^*/N}^{(N)}\rangle \geq \alpha N\right) \geq \mathbb{P}\left(\sum_{n\geq N\alpha} nM_n^{(N)}(t^*/N) \geq \alpha N\right) = 1 - o(1).$$

This implies that $\liminf_{N\to\infty} \mathbb{P}\left(\tau_N^\alpha \leq t^*\right) = 1$, thus, $t_g^s \leq t^*$.

For the second statement, note that by [Theorem 3.5 and Theorem 3.6, [11]], for any $t < t^*$ in the associated graph $G_{t/N}^{(N)}$ we have

$$\sum_{n\geq \psi(N)} \frac{nM_n^{(N)}(t/N)}{N} \xrightarrow{N\to\infty} 0 \qquad \text{in probability.}$$

[8]This α would correspond to $\xi(\kappa)$ appearing in [11]

By the second statement of Corollary 4.6, this implies that, for any graph dominated coagulation process $(\bar{\mathbf{L}}_s^{(N)})_{s\geq 0}$, for any $\delta > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\left\langle m\mathbf{1}_{m\leq\psi(N)}, \bar{\mathbf{L}}_t^{(N)}\right\rangle \leq \left\langle m, \bar{\mathbf{L}}_0^{(N)}\right\rangle - \delta\right) = \lim_{N\to\infty} \mathbb{P}\left(\left\langle m\mathbf{1}_{m>\psi(N)}, \bar{\mathbf{L}}_t^{(N)}\right\rangle \geq \delta\right) = 0,$$

hence, for any $\delta > 0$ we have $T_g^{\psi,\delta} > t$. As $t < t^*$ was arbitrary, this completes the proof. □

A General criteria for relative compactness

Recall that for each N, the cluster coagulation process $(\bar{\mathbf{L}}_t^{(N)})_{t\in[0,\infty)}$ is defined as taking values in the space

$$\mathcal{E} = \bigcup_{n\in\mathbb{N}} \{\mathbf{u} \in \mathcal{M}_+(\mathcal{E}) : \langle m, \mathbf{u}\rangle \leq n\}$$

Recall also that equip $\mathcal{E}$ with the Prokhorov metric, which metrises the topology of weak convergence. We may interpret $(\bar{\mathbf{L}}_t)_{t\in[0,\infty)}$ as a trajectory in $D([0,\infty);\mathcal{E})$, the space of right-continuous functions $f : [0,\infty) \to \mathcal{E}$ with left-limits. We equip $D([0,\infty);\mathcal{E})$ with the *Skorokhod metric* d. Recall that for a separable, complete metric space $(\mathcal{E}, \delta)$ with $q := \delta \wedge 1$, the Skorokhod metric on $D([0,\infty);\mathcal{E})$ is defined as follows: Let Λ denotes the set of all strictly increasing functions mapping $[0,\infty)$ onto $[0,\infty)$, and $\Lambda' \subseteq \Lambda$ the subset of Lipschitz functions. Then, for $\lambda \in \Lambda'$, define

$$\gamma(\lambda) := \sup_{s>t\geq 0} \left|\log \frac{\lambda(s) - \lambda(t)}{s-t}\right| < \infty$$

Then, for $f, g \in D([0,\infty);\mathcal{E})$, we define

$$d(f,g) := \inf_{\lambda\in\Lambda} \left(\gamma(\lambda) \vee \int_0^\infty e^{-tu} \left(\sup_{t\geq 0} q(f(t\wedge u), g(t\wedge u))\right) \mathrm{d}u\right). \tag{149}$$

It is well-established that $D([0,\infty);\mathbb{R})$ is a separable and complete metric space see, for example, [Theorem 5.6, [15]]. In this paper, we use the following, well-known criterion for tightness in Skorokhod spaces. The first, from [15], has been slightly reformulated for out purposes. First, we define the following modulus of continuity: for $f \in D([0,\infty);\mathbb{R})$, $\eta > 0$, $T \in [0,\infty)$, we define

$$w'(f,\eta,T) := \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} |f(s) - f(t)|;$$

where $\{t_i\}$ ranges over all partitions of $[0,T]$, such that $0 = t_0 < t_1 < \cdots < t_n = T$, with $t_{i+1} - t_i > \eta$ and $n \geq 1$.

Lemma A.1 (Corollary 7.4, page 129 [15]). *A collection of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ on the metric space $D([0,\infty);\mathbb{R})$ is tight if and only if the following criteria are satisfied:*

1 *For all $t \in [0,\infty) \cap \mathbb{Q}$ and $\varepsilon > 0$, there exists a compact set $K(t,\varepsilon) \subseteq \mathcal{E}$ such that, for all $n \in \mathbb{N}$*

$$\liminf_{n\to\infty} \mu_n(\{f : f(t) \in K(t,\varepsilon)\}) \geq 1 - \varepsilon, \tag{150}$$

2 For any $T \in [0, \infty)$, there exists $\eta > 0$ such that, for all $n \in \mathbb{N}$

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \mu_n \left(\{ f : w'(f, \eta, T) \geq \varepsilon \} \right) = 0. \tag{151}$$

■

In literature surrounding stochastic processes, the first condition is often known as *compact containment*. The following well-known tightness criterion due to Jakubowski applies more generally to $D([0, \infty); F)$, where F is a completely regular Hausdorff spaces with metrisable compacts. Since we assume $\mathcal{E}$ is a metric space, it applies to $D([0, \infty); \mathcal{E})$:

Lemma A.2 (Theorem 4.6, [24])**.** *A collection of probability measures $\{\mu_i\}_{i \in I}$ on $D([0, \infty); \mathcal{E})$ is tight if and only if the following criteria are satisfied:*

1 For any $t > 0$ and $\varepsilon > 0$ there is a compact set $K(t, \varepsilon) \subseteq \mathcal{E}$ such that, for all $i \in I$,

$$\mu_i \left(\{ f : \forall\, s \in [0, t]\, f(s) \in K(t, \varepsilon) \} \right) \geq 1 - \varepsilon.$$ [9]

2 There exists a family of continuous functions $\mathbb{F}$ from $\mathcal{E}$ to $\mathbb{R}$ such that

2.1 The family $\mathbb{F}$ separates points, i.e., for any $x, y \in \mathcal{E}$ there exists $f \in \mathbb{F}$ such that $f(x) \neq f(y)$.

2.2 The family $\mathbb{F}$ is closed under addition, i.e., if $f, g \in \mathbb{F}$ then $f + g \in \mathbb{F}$.

2.3 Let, for $f \in \mathbb{F}$, $\tilde{f} : D([0, \infty); \mathcal{E}) \to D([0, \infty); \mathbb{R})$ denote the map such that $\tilde{f}(x) = f \circ x$, for $x \in D([0, \infty); \mathcal{E})$. Then, for each $f \in \mathbb{F}$ the family of pushforward measures $\left\{ \tilde{f}_*(\mu_i) \right\}_{i \in I}$ is a tight family on $D([0, \infty); \mathbb{R})$.

■

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[9] If $I = \mathbb{N}$ then we can replace this condition with $\liminf_{n \to \infty} \mu_n \left(\{ f : \forall\, s \in [0, t]\, f(s) \in K(t, \varepsilon) \} \right) \geq 1 - \varepsilon$, see, for example, the proof of [Corollary 7.4, page 130 [15]].

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