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Rough PDEs for local stochastic volatility models

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Abstract

In this work, we introduce a novel pricing methodology in general, possibly non-Markovian local stochastic volatility (LSV) models. We observe that by conditioning the LSV dynamics on the Brownian motion that drives the volatility, one obtains a time-inhomogeneous Markov process. Using tools from rough path theory, we describe how to precisely understand the conditional LSV dynamics and reveal their Markovian nature. The latter allows us to connect the conditional dynamics to so-called rough partial differential equations (RPDEs), through a Feynman-Kac type of formula. In terms of European pricing, conditional on realizations of one Brownian motion, we can compute conditional option prices by solving the corresponding linear RPDEs, and then average over all samples to find unconditional prices. Our approach depends only minimally on the specification of the volatility, making it applicable for a wide range of classical and rough LSV models, and it establishes a PDE pricing method for non-Markovian models. Finally, we present a first glimpse at numerical methods for RPDEs and apply them to price European options in several rough LSV models.

1 Introduction

In mathematical finance, a large class of asset-price models can be described by dynamics of the form

$$X_t^{t,x} = x, \quad dX_s^{t,x} = f(s, X_s^{t,x})v_s dW_s + g(s, X_s^{t,x})v_s dB_s, \quad 0 \leq t < s \leq T \tag{1}$$

where W and B are independent Brownian motions on some filtered probability space $(\Omega, (\mathcal{F}_t), P)$, v is adapted to the smaller filtration $(\mathcal{F}_t^W)$ generated just by W and where the functions f, g are sufficiently regular for (1) to have a unique strong solution for any starting point (t, x). Let us also write $(\mathcal{F}_t^B)$ for the filtration generated by B and assume for conciseness that $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^B$. We view v as one's preferred "backbone" stochastic volatility model. Throughout this work, we make

Assumption 1 *The process $(v_t : 0 \leq t \leq T)$ is $(\mathcal{F}_t^W)$-progressive with bounded sample paths.*

This includes all $(\mathcal{F}_t^W)$-adapted, continuous volatility specification, and thus includes virtually all classical SV models in use, Heston, Bergomi, Stein-Stein, as well as recent *rough volatility* variants thereof, see [3] and references therein.

The function pair (f, g) is chosen to account for correlation, leverage effects and to calibrate to the market's implied volatility surface: the classical *local stochastic volatility* (LSV) specification $f(t, x) \equiv \sigma(t, x)\rho$, $g(t, x) \equiv \sigma(t, x)\sqrt{1-\rho^2}$, with *leverage function* σ and correlation parameter ρ, is not only contained as special case, but it in fact equivalent upon allowing for local correlation $\rho = \rho(t, x)$. For more information on LSV modelling, see e.g. [18].

A typical pricing problem in such models then amounts to computing an expectation like

$$E[\Phi(X^{0,x})] = E[\Phi(X_t^{0,x} : 0 \leq t \leq T)]$$

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for some sufficiently regular payoff functional Φ on path space $C[0,T]$, leaving aside here (for notational simplicity only) the impact of interest rates and discounting future pay-flows. Monte Carlo simulation yields the probably most flexible approach to this problem. It fails, however, to make use of the specific structure of the dynamics (1): A key observation here is that the Brownian motion W suffices to fully specify the possibly rather involved dynamics of v and so, conditioning on the full evolution of W, one should be able to view the dynamics (1) as that of a time-inhomogeneous diffusion driven by the independent Brownian motion B. As a result, the computation of prices such as $E[\Phi(X^{0,x})]$ should be possible by generating Monte Carlo simulations of v and its integral against W and then exploit the $(\mathcal{F}_t^B)$-Markov property of the conditional dynamics to efficiently compute for each such realization a sample of the conditional expectation $E[\Phi(X^{0,x})|\mathcal{F}_T^W]$; the desired unconditional expectation $E[\Phi(X^{0,x})]$ would then be obtained by averaging over these samples.

From a *financial-engineering and modelling point of view*, a conceptual appealing feature of this approach is that it allows one to disentangle the volatility model generating v (and its dW-integral) from the nonlinear local vol-functions f, g and the second Brownian motion B: As soon as we can sample from our preferred stochastic volatility model, we will be able to work out the corresponding conditional local-vol like prices, with all the advantages such a (conditionally) Markovian specification affords. For vanilla options payoffs $\Phi(X^{0,x}) = \phi(X_T^{0,x})$ for instance, a Feynman-Kac-like PDE description should be possible and even for some path-dependent options such as barriers the conditional pricing-problem is easier to handle in such a conditional local-vol model. Moreover, merely requiring the realizations of v and its dW-integral as an interface, the local-vol pricer can be implemented using Markov techniques without any knowledge of the possibly highly non-Markovian volatility model it is going to be connected to.

Obviously, the heuristic view of the dynamics (1) given W is challenged immediately by the need to interpret the dW-part in its dynamics in a mathematically rigorous way. The present paper explains how Lyons' *rough path theory* (e.g. [13] and references therein) can not only be used efficiently to obtain such an interpretation, but also to give precise meaning to each step outlined in the previous paragraphs and that goes hand in hand with numerical recipes, first implementations of which are integral part of this work.

We should emphasize that the emergence of *rough* paths methods is *not* triggered by some *rough* volatility specification and in fact our results are novel even in the classical situation with diffusive SV dynamics. That said, we do cover local *rough* stochastic volatility models, even beyond Hurst parameter $H \in (0,1/2)$, such as recent log-modulated SV models (think "$H = 0$"). This is in contrast to [4] where a robust view of $\int F(W,B,W^H)dW$, similarly against dB, is made possible using regularity structures or, equivalently, generalized rough path considerations [12, 17]. The reason we can do this is an important idea of this work: rather than viewing Brownian motion W, B (or W^H) as Brownian (or Gaussian) rough path, we take a new perspective in recognizing the integral of $dI_t = v_t dW_t$ as fundamental object, bypassing all difficulties coming from rigid controlledness[1] assumptions central to rough paths and regularity structures. More precisely, we lift $I(\omega) = (\int_0^\cdot v_s dW_s)(\omega)$ to a local martingale rough path given by

$$\mathbf{I}(\omega) \equiv (I(\omega), \mathbb{I}(\omega)) = \left(\int_0^\cdot v_s dW_s(\omega), \int_0^\cdot \int_0^t v_s v_t dW_s dW_t(\omega)\right) \in \mathscr{C}^{1/2^-} \text{ a.s.}$$

which is the key to turn the above heuristics into rigorous mathematics. In the language of rough path theory, a.e. realization of $\mathbf{I}(\omega)$ constitutes a rough path which typically is non-geometric. To distinguish

[1] Leaving technical definitions to [13], this would require $v_t - v_s \approx C_t(W_t - W_s)$, with precise Hölder or p-variation estimates.

between random and analytical objects we will denote by $\mathbf{Y} = (Y, \mathbb{Y})$ any such deterministic rough path. The *rough path* bracket $[\mathbf{Y}] := Y^2 - 2\mathbb{Y}$, if applied to $\mathbf{I}(\omega)$, agrees a.s. with the familiar bracket process from stochastic analysis, given by $d[I]_t = v_t^2 dt$. A deterministic analogue of instantaneous volatility v_t is then given by the square-root of $d[\mathbf{Y}]_t/dt$, whenever well-defined, denoted by $\mathbf{v}_t^{\mathbf{Y}}$. We are thus led to study the rough stochastic differential equation (RSDE)

$$X_t^{t,x,\mathbf{Y}} = x, \quad dX_s^{t,x,\mathbf{Y}} = f(s, X_s^{t,x,\mathbf{Y}})d\mathbf{Y}_s + g(s, X_s^{t,x})\mathbf{v}_s^{\mathbf{Y}} dB_s, \quad 0 \le t < s \le T. \tag{2}$$

Our first main result is given as follows and a precise formulation can be found in Section 3.1.

Theorem 1.1 *Under Assumption 1, for sufficiently nice (f, g) and (deterministic) rough path $\mathbf{Y} = (Y, \mathbb{Y})$, there exists a unique solution $X^{t,x,\mathbf{Y}}$ to the RSDE* (2)*, which is a time-inhomogeneous Markov process with respect to the filtration $(\mathcal{F}_t^B)$. Moreover, for the solution $X^{t,x}$ to* (1) *we have*

$$\mathrm{Law}(X^{t,x} | \mathcal{F}_T^W \vee \mathcal{F}_t^B) = \mathrm{Law}(X^{t,x,\mathbf{I}} | \mathcal{F}_T^W \vee \mathcal{F}_t^B).$$

If we additionally assume that $v \ge 0$, then $X^{t,x}$ and $X^{t,x,\mathbf{I}}$ are indistinguishable.

By independence of $\mathbf{I}$ and B, Theorem 1.1 makes rigorous what the heuristic conditioning on W means for the dynamics in (1) and it clarifies in what sense exactly they become Markovian when conditioned on W. The conditional dynamics $X^{t,x,\mathbf{Y}}$ provides an example of a rough semimartingale as introduced in [16].

To delineate the usefulness of this conditional Markov property, we focus on a vanilla option and discuss how to compute its price $E[\phi(X_T^{t,x})]$. When $X^{t,x}$ is Markovian, such an expectation can be computed by solving the associated Feynman-Kac PDE. When, however, the dynamics of v in (1) are generated by a rough volatility model, say, this classical approach is no longer available. Yet, as explained above, we still have that the conditioned dynamics are Markovian. We thus condition on a realization of v and its dW-integral I as a rough path with Lipschitz bracket $\mathbf{I}(\omega) = \mathbf{Y}$ and show that for the associated Markov process $X^{t,x,\mathbf{Y}}$ of (2) the function

$$u^{\mathbf{Y}}(t, x) := E[\phi(X_T^{t,x,\mathbf{Y}})] \tag{3}$$

is a Feynman-Kac-like solution to the *rough partial differential equation* (RPDE)

$$\begin{cases} -d_t u^{\mathbf{Y}} &= L_t[u^{\mathbf{Y}}](\mathbf{v}_t^{\mathbf{Y}})^2 dt + \Gamma_t[u^{\mathbf{Y}}] d\mathbf{Y}_t^g \\ u^{\mathbf{Y}}(T, x) &= \phi(x) \end{cases} \tag{4}$$

where we call $\mathbf{Y}^g = \mathbf{Y} + [\mathbf{Y}]/2 := (Y, Y^2/2)$ the *geometrification* of the rough path $\mathbf{Y}$; the reader may think of $\mathbf{Y}$ (resp. $\mathbf{Y}^g$) as deterministic counterparts of dI (resp. $\circ dI$) in the Itô (resp. Stratonovich) sense in a semimartingale setting. Here, the spatial differential operators L_t and Γ_t are given by

$$L_t := \frac{1}{2} g^2(t, x)\partial_{xx}^2 + f_0(t, x)\partial_x \quad \text{and} \quad \Gamma_t := f(t, x)\partial_x$$

with $f_0(t, x) := -\frac{1}{2}\partial_x f(t, x) f(t, x)$, from the non-geometric/geometric conversion. Leaving a more precise formulation to Sections 3.2 and 3.3, our second main result can be stated as follows.

Theorem 1.2 *Under Assumption 1, for sufficiently nice (f, g) and continuous bounded terminal data ϕ, $\phi \in C_b^0(\mathbb{R}, \mathbb{R}_+)$, we have a unique solution $u^{\mathbf{Y}} = u^{\mathbf{Y}}(t, x)$ to the RPDE* (4) *for a suitable class of rough paths $\mathbf{Y} = (Y, \mathbb{Y})$. For all $(t, x) \in [0, T] \times \mathbb{R}$ we have the a.s. identities*

$$E[\phi(X_T^{t,x}) | \mathcal{F}_T^W](\omega) = u^{\mathbf{I}(\omega)}(t, x) \qquad \text{and} \qquad E[\phi(X_T^{0,x}) | \mathcal{F}_t] = E[u^{\mathbf{I}}(t, z) | \mathcal{F}_t^W]\big|_{z = X_t^{0,x}}$$

where $X^{t,x}$ *denotes the unique strong solution to* (1). *In particular,*

$$E[\phi(X_T^{0,x})] = E[u^{\mathbf{I}}(0,x)].$$

It is natural to ask how the RPDE (4) is related to the SPDE when formally one replaces $d\mathbf{Y}^g$ by $\circ dI$, and we provide a short discussion at the end of Section 3.2 for the interested reader.

By way of illustration, we first consider the pure SV case with constant $f \equiv \rho$ and $g \equiv \sqrt{1-\rho^2}$, so that the dynamics (1) amount to an additive stochastic volatility model in Bachelier form. In this case, the explicit solution to (4) is given by

$$u^{\mathbf{Y}}(t,x) = \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{ -\frac{(y-x-\rho Y_{t,T})^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}} \right\} dy, \qquad [\mathbf{Y}]_{t,T} = \int_t^T (\mathbf{v}_s^{\mathbf{Y}})^2 ds$$

which precisely expresses the fact that the regular conditional distribution of $X_T^{t,x}$, given $\mathcal{F}_T^W$, is Gaussian with mean $\rho(I_T - I_t)$ and variance $(1-\rho^2)\int_t^T v_r^2 dr$. Similar formulae hold for multiplicative stochastic volatility models in Black-Scholes form, when $f(x) \equiv \rho x$ and $g(x) \equiv \sqrt{1-\rho^2}x$, in this case the conditional distribution is log-normal, also with explicit mean and variance, known as *Romano–Touzi formula*, [27]. We should also point out that our approach is not limited to 2-factor models (i.e. built on two Brownian motions B, W), cf. the discussion in Section 3.5.

The approach of this paper can be compared with an existing branch in the literature, related to so-called *backward stochastic partial differential equations* (BSPDE), which have been studied extensively in [24, 25, 23, 22]. Put in our context, one can consider the random field

$$E[\phi(X_T^{t,x})|\mathcal{F}_t](\omega) = \tilde{u}(t,x;\omega)$$

and view it as part of the *stochastic Feynman-Kac solution* $(\tilde{u}, \psi)$ to the BSPDE

$$\begin{aligned} -d_t\tilde{u}(t,x) &= \left[\frac{v_t^2}{2}\left(g^2(t,x) + f^2(t,x)\right)\partial_{xx}^2 \tilde{u}(t,x) + f(t,x)v_t\partial_x\psi(t,x)\right] dt - \psi(t,x)dW_t, \\ \tilde{u}(T,x) &= \phi(x). \end{aligned}$$

In the classical SV case, that is for multiplicative fields $f(x) \equiv \rho x$ and $g(x) \equiv \sqrt{1-\rho^2}x$, this has recently been studied in [6]. This approach has very mild requirements on the process $v_t(\omega)$, somewhat similar to our Assumption 1.

Another related pricing approach in non-Markovian frameworks comes from *path-dependent PDEs* (PPDEs), with path-derivatives ∂_ω as in functional Itô calculus. The necessary extensions to treat singular Volterra dynamics, as relevant for rough volatility, were obtained in [28], to which we also refer for selected pointers to the vast PPDE literature. In [20] the authors consider LSV-type dynamics of the form

$$dX_t = F(t, X_t, V_t)\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right) \text{ with } V_t = V_0 + \int_0^t K(t-s)(b(V_r)dr + \xi(V_r))dW_r.$$

and see that $P_t = \mathbb{E}[\phi(X_T^{0,x})|\mathcal{F}_t]$ is the unique solution to the terminal value PPDE problem

$$(\partial_t + \mathcal{L}_{xx} + \mathcal{L}_{x\omega} + \mathcal{L}_\omega + \mathcal{L}_{\omega\omega})P(t, X_t, \Theta^t) = 0, \quad P(T, x, \Theta^T) = \phi(x),$$

where, leaving all definitions and notations to the afore-mentioned papers,

$$\mathcal{L}_{x\omega} = \rho F(t,x,v)\xi(\Theta_t^t)\langle \partial_{x,\omega}, K^t\rangle, \qquad \mathcal{L}_{xx} = \frac{1}{2}F(t,x,\Theta_t^t)\partial_x^2,$$
$$\mathcal{L}_{\omega\omega} = \frac{1}{2}\xi(\Theta_t^t)^2\langle \partial_\omega^2, (K^t,K^t)\rangle, \qquad \mathcal{L}_{\omega} = b(\Theta_t^t)\langle \partial_\omega, K^t\rangle.$$

In a recent paper [7], such PPDEs were used to analyse weak discretization schemes for rough volatility models. A comparison of all these different approaches would be desirable, but this is not the purpose of this paper. (To the best of our knowledge, even a systematic comparison between BSPDEs and PPDEs, numerically and otherwise, is not available in the literature.)

As final contribution of this paper, we also give a first glimpse at a numerical approach useful in the general setting. Specifically, we discuss both a first and a second order scheme towards approximating solutions to the RPDE. For the analytic benchmarks described in the previous paragraph, the ensuing Monte Carlo simulation turn out to be very accurate.

For the goal of computing option prices, we have to compare our method with plain Monte Carlo simulation, i.e., the direct approximation of $E[\phi(X_T^{0,x})]$ vs. simulation of $E[u^{\mathbf{I}}(t,x)]$. By construction, the variance of $u^{\mathbf{I}}(t,x)$ is smaller than the variance of $\phi(X_T^{0,x})$, however, at the cost of increasing computational time per sample. Apart from reducing the variance, taking conditional expectations also increases regularity of the payoff ϕ, making the use of efficient deterministic quadrature methods (such as quasi Monte Carlo or sparse grids quadrature) possible, but even multi-level Monte Carlo. We refer to [5, 1] in the context of a very different method. Along the same lines, we note that greeks such as the Delta $\frac{\partial}{\partial x}E[\phi(X_T^{0,x})]$ can now be computed by differentiating inside the expectation, i.e., as $E[\frac{\partial}{\partial x}u^{\mathbf{I}}(t,x)]$, thanks to the increased regularity. Similarly, our approach may be useful for computing the density of $X_T^{0,x}$.

2 Rough preliminaries

We begin by introducing some fundamental concepts of rough path theory, which are essential to understand the techniques henceforth. The technical details will be discussed in Appendix A for the interested reader, and for more details about rough path theory we refer to [13] and [15]. Unless stated otherwise, we consider V to be any Banach space. Consider the set $\Delta_{[0,T]} := \{(s,t) \in [0,T]^2 : 0 \le s \le t \le T\}$, and for any path $Y : [0,T] \to V$, we make use of the two increment notations $(\delta Y)_{s,t} = Y_{s,t} := Y_t - Y_s$ for $s \le t$.

Definition 2.1 *For $\alpha \in (1/3, 1/2]$, the pair $\boldsymbol{Y} := (Y, \mathbb{Y})$ is called α-Hölder rough path on V, where $Y : [0,T] \longrightarrow V$ and $\mathbb{Y} : \Delta_{[0,T]} \longrightarrow V \otimes V$ such that*

$$\|Y\|_\alpha := \sup_{0\le s<t\le T} \frac{|Y_t - Y_s|}{|t-s|^\alpha} < \infty, \quad \|\mathbb{Y}\|_{2\alpha} := \sup_{0\le s<t\le T} \frac{|\mathbb{Y}_{s,t}|}{|t-s|^{2\alpha}} < \infty$$

and Chen's relation holds true:

$$\mathbb{Y}_{s,t} = \mathbb{Y}_{s,u} + \mathbb{Y}_{u,t} + Y_{s,u} \otimes Y_{u,t}, \quad 0 \le s \le u \le t \le T. \tag{5}$$

We denote the space of α-Hölder rough paths by $\mathscr{C}^\alpha([0,T],V)$.

Remark 2.2 *If $Y : [0,T] \to V$ is smooth, one can check that for the choice $\mathbb{Y}_{s,t} := \int_s^t Y_{s,r} \otimes dY_r$, where the integral is defined in a Riemann-Stieltjes sense, $\boldsymbol{Y} = (Y, \mathbb{Y})$ defines an α-Hölder rough path, and Chen's relation is a direct consequence of the additivity of the integral. If Y is only α-Hölder continuous, this integral might not have any meaning, but we can think of $\mathbb{Y}$ as postulating the value of this integral. The additional structure $\mathbb{Y}$ added to the path Y, gives rise to a notion of integration against dY for a large class of integrands, extending Riemann-Stieljes integration. This is one of the key features of rough path theory.*

For two α-Hölder rough paths $\boldsymbol{Y} = (Y, \mathbb{Y})$ and $\boldsymbol{Z} = (Z, \mathbb{Z})$, we introduce the rough path distance

$$\varrho_{\alpha,2\alpha}(Y, \mathbb{Y}; Z, \mathbb{Z}) := \|Y - Z\|_\alpha + \|\mathbb{Y} - \mathbb{Z}\|_{2\alpha}.$$

It is not difficult to see that $\mathscr{C}^\alpha([0,T], V)$ together with the map $(\mathbf{Y}, \mathbf{Z}) \mapsto |Y_0 - Z_0| + \varrho_{\alpha,2\alpha}(\mathbf{Y}, \mathbf{Z})$ defines a complete metric space.

Definition 2.3 *For $\alpha \in (1/3, 1/2]$, we call an element $\boldsymbol{Y} = (Y, \mathbb{Y}) \in \mathscr{C}^\alpha([0,T], V)$ weakly geometric rough path, if*

$$\mathrm{Sym}(\mathbb{Y}_{s,t}) = \frac{1}{2} Y_{s,t} \otimes Y_{s,t}, \quad 0 \le s \le t \le T. \tag{6}$$

We denote the space of weakly geometric rough paths by $\mathscr{C}_g^\alpha([0,T], V)$.

Remark 2.4 *If $Y : [0,T] \to V$ is smooth, we can check that* (6) *is nothing else than integration by parts for $\mathbb{Y}_{s,t} := \int_s^t Y_{s,r} \otimes dY_r$. If the latter has no initial meaning, we can think of the condition* (6) *as imposing this important property for the postulated value $\mathbb{Y}$.*

When discussing rough partial differential equations in Section 3, a crucial technique involves approximating weakly geometric rough paths using rough path lifts of more regular paths. Consider $V = \mathbb{R}^d$, and Lipschitz paths $Y^\epsilon : [0,T] \longrightarrow \mathbb{R}^d$. As already mentioned in the remarks above, for all $\alpha \in (1/3, 1/2]$ we have the canonical lift

$$\mathbf{Y}^\epsilon = (Y^\epsilon, \mathbb{Y}^\epsilon) := \left(Y^\epsilon, \int_0^\cdot Y^\epsilon_{0,t} \otimes dY^\epsilon_t \right) \in \mathscr{C}_g^\alpha([0,T], \mathbb{R}^d),$$

where the integration can be understood in a Riemann-Stieltjes sense. Now using so-called *geodesic-approximations*[2], it is possible to prove the following result, see [13, Proposition 2.8] or [15, Proposition 8.12].

Proposition 2.5 *Let $\alpha \in (1/3, 1/2]$ and $\mathbf{Y} = (Y, \mathbb{Y}) \in \mathscr{C}_g^\alpha([0,T], \mathbb{R}^d)$. Then there exist Lipschitz continuous paths $Y^\epsilon : [0,T] \longrightarrow \mathbb{R}^d$, such that*

$$\mathbf{Y}^\epsilon := \left(Y^\epsilon, \int_0^\cdot Y^\epsilon_{0,t} \otimes dY^\epsilon_t \right) \longrightarrow \mathbf{Y} = (Y, \mathbb{Y}) \text{ uniformly on } [0,T] \text{ as } \epsilon \to 0, \tag{7}$$

and we have uniform estimates

$$\sup_\epsilon \left(\|Y^\epsilon\|_\alpha + \|\mathbb{Y}^\epsilon\|_{2\alpha} \right) < \infty. \tag{8}$$

[2] See [15, Chapter 5.2] for instance.

Remark 2.6 *Motivated by the last proposition, we say* $\mathbf{Y}^\epsilon$ *converges weakly to* $\mathbf{Y}$ *in* $\mathscr{C}^\alpha([0,T],\mathbb{R}^d)$*, in symbols* $\mathbf{Y}^\epsilon \rightharpoonup \mathbf{Y}$ *in* $\mathscr{C}^\alpha$*, if and only if* (7) *and* (8) *hold true. Using an interpolation argument for* α*-Hölder norms, see for instance [15, Proposition 5.5],* $\mathbf{Y}^\epsilon \rightharpoonup \mathbf{Y}$ *in* $\mathscr{C}^\alpha$ *then implies that* $\mathbf{Y}^\epsilon \to \mathbf{Y}$ *in* $\mathscr{C}^{\alpha'}$ *for all* $\alpha' \in (1/3,\alpha)$*, that is* $\varrho_{\alpha',2\alpha'}(\mathbf{Y}^\epsilon,\mathbf{Y}) \to 0$ *as* $\epsilon \to 0$*.*

A crucial concept in rough path theory, particularly relevant for this paper, is the notion of *rough brackets*. Rough brackets play a similar role for rough path lifts as quadratic variation does in stochastic integration.

Definition 2.7 *For any* $\boldsymbol{Y} \in \mathscr{C}^\alpha([0,T],V)$ *with* $\alpha \in (1/3,1/2]$*, we define the rough brackets as*

$$[\boldsymbol{Y}]_t := Y_{0,t} \otimes Y_{0,t} - 2\,\mathrm{Sym}(\mathbb{Y}_{0,t}).$$

It is not difficult to check that $t \mapsto [\mathbf{Y}]_t \in C^{2\alpha}([0,T],\mathrm{Sym}(V\otimes V))$. The main reason to introduce rough brackets is the following result, which is discussed in more detail in Appendix A.2.

Lemma 2.8 *Let* $\alpha \in (1/3,1/2]$*. Then every* α*-Hölder rough path* $\mathbf{Y} = (Y,\mathbb{Y}) \in \mathscr{C}^\alpha([0,T],V)$ *can be identified uniquely with a pair* $(\mathbf{Y}^g,[\mathbf{Y}])$*, where* $\mathbf{Y}^g = (Y,\mathbb{Y}^g)$ *is a weakly geometric* α*-Hölder rough path with second level* $\mathbb{Y}^g = \mathbb{Y} + \frac{1}{2}\delta[\mathbf{Y}]$*. In particular, we have the following bijection*

$$\mathscr{C}^\alpha([0,T],V) \longleftrightarrow \mathscr{C}^\alpha_g([0,T],V) \oplus C^{2\alpha}_0([0,T],\mathrm{Sym}(V\otimes V)),$$

where $C^{2\alpha}_0$ *denotes the space of* 2α*-Hölder continuous paths starting from* 0*.*

The weakly geometric rough path $\mathbf{Y}^g$ is sometimes called *geometrification* of the rough path $\mathbf{Y}$. Notice that the expression $\mathbb{Y}^g = \mathbb{Y} + \frac{1}{2}\delta[\boldsymbol{Y}]$ is reminiscient of the Itô-Stratonovich relation in stochastic integration theory, and we will demonstrate how it generalizes this concept.

In the remaining part of this section, we focus on the case where $V = \mathbb{R}^d$. There are certain cases where the brackets of a rough path $\mathbf{Y}$ are Lipschitz continuous, rather than only 2α-Hölder as in the general case. This motivates to introduce the space of rough paths with Lipschitz brackets.

Definition 2.9 *Let* $\alpha \in (1/3,1/2)$ *and* $\mathbf{Y} \in \mathscr{C}^\alpha([0,T],\mathbb{R}^d)$*. We say* $\boldsymbol{Y}$ *is a rough path with Lipschitz brackets, in symbols* $\mathbf{Y} \in \mathscr{C}^{\alpha,1}([0,T],\mathbb{R}^d)$*, if* $t \mapsto [\boldsymbol{Y}]_t$ *is Lipschitz continuous. In this case, we set*

$$\mathbf{V}^{\mathbf{Y}}_t := \frac{d[\mathbf{Y}]_t}{dt} \in \mathbb{S}^d,$$

for almost every t*. Moreover, we say that* $\mathbf{Y}$ *has non-decreasing Lipschitz brackets, in symbols* $\mathbf{Y} \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R}^d)$*, if additionally we have* $\mathbf{V}^{\mathbf{Y}}_t \in \mathbb{S}^d_+$*. In this case we denoted by* $\mathbf{v}^{\mathbf{Y}}$ *the square-root of* $\mathbf{V}^{\mathbf{Y}}$*, that is* $\mathbf{v}^{\mathbf{Y}}(\mathbf{v}^{\mathbf{Y}})^T = \mathbf{V}^{\mathbf{Y}}$*.*

We refer the interested reader to Appendix A.2 for more details about the spaces of rough paths with Lipschitz brackets.

An important class of rough paths comes from enhancing stochastic processes. Consider a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ fulfilling the usual conditions. The most basic, but probably most important examples are the following two lifts of standard Brownian motion, see for instance [13, Chapter 3] for details.

Example 2.10 *Consider a d-dimensional standard Brownian motion B. It is well-known that we can define the two rough path lifts $\boldsymbol{B}^{\mathrm{Ito}} = (B, \mathbb{B}^{\mathrm{Ito}})$ and $\boldsymbol{B}^{\mathit{Strat}} = (B, \mathbb{B}^{\mathit{Strat}})$, where*

$$\mathbb{B}^{\mathrm{Ito}}_{s,t} := \int_s^t B_{s,r} dB_r, \quad \mathbb{B}^{\mathit{Strat}}_{s,t} = \int_s^t B_{s,r} \circ dB_r,$$

where dB denotes the Itô-, and $\circ dB$ the Stratonovich-integration. Using standard Itô-calculus, one can check that for any $\alpha \in (1/3, 1/2)$ we have $\boldsymbol{B}^{\mathrm{Ito}}(\omega) \in \mathscr{C}^{\alpha}([0,T],\mathbb{R}^d)$ and $\boldsymbol{B}^{\mathit{Strat}}(\omega) \in \mathscr{C}^{\alpha}_g([0,T],\mathbb{R}^d)$ almost surely.

It is not difficult to extend the last example to continuous local martingales M of the form

$$M_t = M_0 + \int_0^t \sigma_s dB_s, \quad 0 \le t \le T,$$

where $\sigma \in \mathbb{R}^{d\times d}$ is an $(\mathcal{F}^B_t)$-progressively measurable process such that almost surely

$$\|\sigma\|_{\infty;[0,T]} := \sup_{0\le t\le T} |\sigma_t| < \infty.$$

Indeed, we can define $\boldsymbol{M}^{\text{Itô}} = (M, \mathbb{M}^{\text{Itô}})$ and $\boldsymbol{M}^{\text{Strat}} = (M, \mathbb{M}^{\text{Strat}})$ by

$$\mathbb{M}^{\text{Itô}}_{s,t} = \int_s^t M_{s,r} \otimes dM_r, \quad \mathbb{M}^{\text{Strat}}_{s,t} = \int_s^t M_{s,r} \otimes \circ dM_r.$$

The proof of the following result can be found in Appendix A.1.

Proposition 2.11 *For any $\alpha \in (1/3, 1/2)$, we have $\boldsymbol{M}^{\mathit{Itô}} \in \mathscr{C}^{\alpha}([0,T],\mathbb{R}^d)$ and $\boldsymbol{M}^{\mathit{Strat}} \in \mathscr{C}^{\alpha}_g([0,T],\mathbb{R}^d)$ almost surely.*

Remark 2.12 *Using Itô-Stratonovich correction, we can notice that $\mathbb{M}^{\mathit{Itô}}_{s,t} = \mathbb{M}^{\mathit{Strat}}_{s,t} - \frac{1}{2}\delta[M]_{s,t}$, where $[M]$ denotes the matrix of covariations $[M^i, M^j]_{1\le i,j\le d}$. From Lemma 2.8 we know that we can uniquely identify $\boldsymbol{M}^{\mathit{Itô}}$ with a pair $(\boldsymbol{M}^g, [\boldsymbol{M}^{\mathit{Itô}}])$. By uniqueness it readily follows that $\mathbb{M}^g = \mathbb{M}^{\mathit{Strat}}$ and $[\boldsymbol{M}^{\mathit{Itô}}]_{s,t} = [M]_{s,t}$ almost surely. This motivates to use the following notation $\boldsymbol{M} := \boldsymbol{M}^{\mathit{Itô}}$ and $\boldsymbol{M}^g := \boldsymbol{M}^{\mathit{Strat}}$. Finally, since $[M]_t = \int_0^t \sigma_s\sigma_s^T ds$, we have almost surely that the map $t \mapsto [M]_t$ is Lipschitz continuous, and $\frac{d[M]_t}{dt} = \sigma_t\sigma_t^T$ is bounded almost surely, thus by Definition 2.9, we have $\mathbf{M} \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R}^d)$.*

3 Pricing in local stochastic volatility models using RPDEs

In this section, we present the main results of this paper that establish a connection between general local stochastic volatility models and rough partial differential equations (RPDEs).

3.1 Conditioning in local stochastic volatility dynamics

Consider two standard Brownian motions W and B, an $(\mathcal{F}^W_t)$-adapted volatility process $(v_t)_{t\in[0,T]}$, such that Assumption 1 holds true, and set $V = v^2$. For $f, g \in C^3_b([0,T]\times\mathbb{R},\mathbb{R})$ and $I_t := \int_0^t v_s dW_s$, we are interested in the local stochastic volatility model

$$X^{t,x}_t = x, \quad dX^{t,x}_s = f(s, X^{t,x}_s) dI_s + g(s, X^{t,x}_s) v_s dB_s, \quad 0 \le t < s \le T. \tag{9}$$

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Thanks to [26, Chapter 3 Theorem 7] it is clear that the SDE (9) has unique strong solutions for all $(t,x) \in [0,T] \times \mathbb{R}$. It should be noted that we focus on one-dimensional dynamics in this presentation for the sake of clarity. However, there is no obstacle in generalizing the approach to multivariate dynamics, as discussed in Section 3.5.

By conditioning on the Brownian motion W in (9), we wish to disintegrate the randomness arising from the $(\mathcal{F}_t^W)$-adapted pair (I,v), and obtain a Markovian nature in B for the conditional dynamics. To make this rigorous, we use the notion of rough paths with non-decreasing Lipschitz brackets introduced in Definition 2.9. For such a rough path $\mathbf{Y} \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R})$, with Lipschitz continuous rough brackets $t \mapsto [\mathbf{Y}]_t$, see Definition 2.7, we introduced the notation

$$\left([\mathbf{Y}]_t, \frac{d[\mathbf{Y}]_t}{dt}, \sqrt{\frac{d[\mathbf{Y}]_t}{dt}}\right) = \left(\int_0^t \mathbf{V}_s^{\mathbf{Y}} ds, \mathbf{V}_t^{\mathbf{Y}}, \mathbf{v}_t^{\mathbf{Y}}\right).$$

Recall that the Itô rough path lift $\mathbf{I} = (I, \int \delta I dI)$ of the martingale I constitutes a (random) rough path in $\mathscr{C}^{\alpha,1+}([0,T],\mathbb{R})$ with brackets $[\mathbf{I}] = [I] = \int V_t dt$, see Proposition 2.11 and Remark 2.12. Thus we have the following consistency

$$\left([I]_t, \frac{d[I]_t}{dt}, \sqrt{\frac{d[I]_t}{dt}}\right) = \left(\int_0^t V_s ds, V_t, |v_t|\right) = \left(\int_0^t \mathbf{V}_s^{\mathbf{I}} ds, \mathbf{V}_t^{\mathbf{I}}, \mathbf{v}_t^{\mathbf{I}}\right). \tag{10}$$

Replacing the pair (I,v) in the SDE (9) with the deterministic pair $(\mathbf{Y}, \mathbf{v}^{\mathbf{Y}})$ for some $\mathbf{Y} \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R})$ yields the rough stochastic differential equation (RSDE)[3]

$$X_t^{t,x,\mathbf{Y}} = x, \quad dX_s^{t,x,\mathbf{Y}} = f(s, X_s^{t,x,\mathbf{Y}}) d\mathbf{Y}_s + g(s, X_s^{t,x,\mathbf{Y}}) \mathbf{v}_s^{\mathbf{Y}} dB_s, \quad t < s \leq T. \tag{11}$$

At least formally, considering the consistency (10), when choosing $\mathbf{Y} = \mathbf{I}(\omega)$ one expects the solution to equation (11) to coincide in some sense with the solution to equation (9). In order to treat (11) as a genuine rough differential equation (RDE), we define the martingale $M^{\mathbf{Y}} := \int \mathbf{v}^{\mathbf{Y}} dB$ and we want to define a joint rough path lift of $(M^{\mathbf{Y}}(\omega), Y)$[4]. The following definition explains how to lift $(M(\omega), Y)$ for *any* $(\mathcal{F}_t^B)$-local martingale M.

Definition 3.1 *Let* $\alpha \in (1/3, 1/2]$ *and* $\mathbf{Y} = (Y, \mathbb{Y}) \in \mathscr{C}^\alpha([0,T],\mathbb{R})$. *For any* $(\mathcal{F}_t^B)$*-local martingale* M*, we define the joint lift* $\mathbf{Z}^{\mathbf{Y}}(\omega) = (Z^{\mathbf{Y}}(\omega), \mathbb{Z}^{\mathbf{Y}}(\omega))$ *by*

$$Z^{\mathbf{Y}}(\omega) := (M(\omega), Y), \qquad \mathbb{Z}_{s,t}^{\mathbf{Y}}(\omega) := \begin{pmatrix} \int_s^t M_{s,r} dM_r & \int_s^t M_{s,r} dY_r \\ \int_s^t Y_{s,r} dM_r & \mathbb{Y}_{s,t} \end{pmatrix} (\omega), \tag{12}$$

where the first entry of $\mathbb{Z}^{\mathbf{Y}}$ *is the (canonical) Itô rough path lift of the local martingale* M*, see Proposition 2.11,* $\int_s^t Y_{s,r} dM_r$ *is a well-defined Itô integral, and we set* $\int_s^t M_{s,r} dY_r := M_{s,t} Y_{s,t} - \int_s^t Y_{s,r} dM_r$*, imposing integration by parts.*

The following theorem demonstrates that we can use the joint lift to ensure the well-posedness of equation (11). Furthermore, it establishes that the unique solution is an $(\mathcal{F}_t^B)$-Markov process, and when $\mathbf{Y} = \mathbf{I}(\omega)$, the conditional distributions, given $\mathcal{F}_T^W \vee \mathcal{F}_t^B$, of the solutions to (9) and (11) coincide. The proof of this result is discussed in Appendix A.3.

[3] A comprehensive theory about more general RSDEs can be found in [14].

[4] The joint-lift method for RSDEs has been studied in [10] for the case of Brownian motion, that is $v^{\mathbf{Y}} \equiv 1$ in our setting, and more recently for general càdlàg and p-rough paths in [16].

Theorem 3.2 *Let $\alpha \in (1/3, 1/2]$ and assume that $f, g \in C_b^3([0,T]\times\mathbb{R},\mathbb{R})$. For any $\boldsymbol{Y} = (Y, \mathbb{Y}) \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R})$ and $M_t^{\mathbf{Y}} = \int_0^t \mathbf{v}_s^{\mathbf{Y}} dB_s$, the joint lift $\boldsymbol{Z}^{\mathbf{Y}}$ almost surely defines an α'-Hölder rough path for any $\alpha' \in (1/3, \alpha)$. Moreover, there exists a unique solution to the rough differential equation*

$$X_t^{t,x,\mathbf{Y}} = x, \quad dX_s^{t,x,\boldsymbol{Y}}(\omega) = (g,f)(s, X_s^{t,x,\boldsymbol{Y}}(\omega))d\boldsymbol{Z}_s^{\mathbf{Y}}(\omega), \quad 0 \le t < s \le T, \tag{13}$$

for almost every ω, and $X^{t,x,\mathbf{Y}}$ defines a time-inhomogeneous Markov process. Under Assumption 1, for the unique solution $X^{t,x}$ to (9) *it holds that*

$$\mathrm{Law}\left(X^{t,x}\middle|\, \mathcal{F}_T^W \vee \mathcal{F}_t^B\right) = \mathrm{Law}\left(X^{t,x,\mathbf{I}}\middle|\, \mathcal{F}_T^W \vee \mathcal{F}_t^B\right).$$

If we additionally assume that $v \geq 0$, then we even have indistinguishability $X^{t,x}(\omega) = X^{t,x,\mathbf{I}(\omega)}$ for almost every $\omega \in \Omega$.

3.2 Feynman-Kac representation and rough stochastic differential equations

The goal of this section is to exploit the Markovian nature of the unique solution $X^{t,x,\mathbf{Y}}$ to the RDE (13), by relating it to *rough partial differential equations* (RPDEs). As we will explore more detailed in the next section, thinking of $\mathbf{Y} = \mathbf{I}(\omega)$ for some fixed ω, the $(\mathcal{F}_t^B)$-Markov process $X^{t,x,\mathbf{Y}}$ can be interpreted as the LSV price dynamics (9) conditioned on the Brownian motion W. Thus, it is desirable to characterize expected values of the form $E[\phi(X_T^{t,x,\mathbf{Y}})]$ for European payoff functions ϕ, which for simplicity we assume to be continuous and bounded. It turns out that $(t,x) \mapsto E[\phi(X_T^{t,x,\mathbf{Y}})]$ can be obtained from solutions to certain RPDEs, through a Feynman-Kac type of formula.

More specifically, let $\phi \in C_b^0(\mathbb{R},\mathbb{R})$ and fix $\mathbf{Y} = (Y, \mathbb{Y}) \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R})$, and recall the unique corresponding pair $(\mathbf{Y}^g, [\mathbf{Y}])$ as described in Lemma 2.8. We are interested in the linear second order RPDE

$$\begin{cases} -d_t u^{\mathbf{Y}} &= L_t[u^{\mathbf{Y}}]\mathbf{V}_t^{\mathbf{Y}} dt + \Gamma_t[u^{\mathbf{Y}}] d\mathbf{Y}_t^g, \quad (t,x) \in [0,T[\times\mathbb{R} \\ u^{\mathbf{Y}}(T,x) &= \phi(x), \quad x \in \mathbb{R}, \end{cases} \tag{14}$$

where

$$\begin{aligned} L_t[u^{\mathbf{Y}}](x) &:= \frac{1}{2} g^2(t,x)\partial_{xx}^2 u^{\mathbf{Y}}(t,x) + f_0(t,x)\partial_x u(t,x) \\ \Gamma_t[u^{\mathbf{Y}}](x) &:= f(t,x)\partial_x u^{\mathbf{Y}}(t,x), \end{aligned}$$

and $f_0(t,x) := -\frac{1}{2} f(t,x)\partial_x f(t,x)$. Let us quickly describe how we define solutions to (14). By Proposition 2.5, we can find $Y^\epsilon \in \mathrm{Lip}([0,T],\mathbb{R})$, such that its (canonical) rough path lift $\mathbf{Y}^{g,\epsilon} \rightharpoonup \mathbf{Y}^g$ in $\mathscr{C}^\alpha$ as $\epsilon \to 0$, see also Remark 2.6. Replacing $d\mathbf{Y}_t^g$ with $dY_t^\epsilon = \dot{Y}_t^\epsilon dt$ in the RPDE (14), we obtain a classical (backward) PDE of the form

$$\begin{cases} -\partial_t u &= L_t[u]\mathbf{V}_t^{\mathbf{Y}} + \Gamma_t[u]\dot{Y}_t^\epsilon, \\ u(T,x) &= \phi(x). \end{cases} \tag{15}$$

From the Feynman-Kac theorem, we know that any bounded solution $u^\epsilon \in C^{1,2}([0,T]\times\mathbb{R})$ to this PDE has the unique representation

$$u^\epsilon(t,x) = E[\phi(X_T^{t,x,\epsilon})], \tag{16}$$

where $X^{t,x,\epsilon}$ is the unique strong solution to the SDE

$$X_t^{t,x,\epsilon} = x, \quad dX_s^{t,x,\epsilon} = g(s, X_s^{t,x,\epsilon})\mathbf{v}_s^{\mathbf{Y}} dB_s + \left(f_0(s, X_s^{t,x,\epsilon})\mathbf{V}_s^{\mathbf{Y}} + f(s, X_s^{t,x,\epsilon})\dot{Y}_s^\epsilon\right) ds.$$

Remark 3.3 *It is worth to recall that there are situations where we cannot expect solutions to* (15) *to be in $C^{1,2}$, and one needs a notion for weak-solutions. However, if u^ϵ defined in* (16) *is bounded and continuous on $[0,T] \times \mathbb{R}$, then it still represents a weak-solution to the PDE* (15)*, and it is in fact the unique viscosity solution.We refer to [8] for an exposition of viscosity solutions, and details of the stochastic representation of viscosity solutions can be found in [11].*

For the purpose of this paper, it is enough to know that the Feynman-Kac representation in (16) is the suitable notion for a general solution to the PDE (15). In the following theorem we prove that there exists a unique limit of u^ϵ as $\epsilon \to 0$, which we define to be the unique solution to the RPDE (14).

Theorem 3.4 *Let $\alpha \in (1/3, 1/2]$ and $\mathbf{Y} \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R})$. Consider a sequence $Y^\epsilon \in \mathrm{Lip}([0,T],\mathbb{R})$, with (canonical) rough path lift $\mathbf{Y}^{g,\epsilon}$, such that $\boldsymbol{Y}^{g,\epsilon} \rightharpoonup \boldsymbol{Y}^g$ in $\mathscr{C}^\alpha$. Moreover, let $f, g \in C_b^3([0,T]\times\mathbb{R})$ and $\phi \in C_b^0(\mathbb{R},\mathbb{R})$. Let u^ϵ be the unique bounded Feynman-Kac (viscosity) solution* (16) *with respect to Y^ϵ. Then there exists a function $u^{\mathbf{Y}} = u^{\mathbf{Y}}(t,x) \in C_b^0([0,T] \times \mathbb{R}, \mathbb{R})$, only depending on $\mathbf{Y}$ but not on its approximation, such that $u^\epsilon \to u^{\mathbf{Y}}$ pointwise, with Feynman-Kac representation*

$$u^{\mathbf{Y}}(t,x) = E[\phi(X_T^{t,x,\mathbf{Y}})].$$

Moreover, the solution map

$$\begin{aligned} S : C_b^0(\mathbb{R},\mathbb{R}) \times \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R}) &\longrightarrow C_b^0([0,T] \times \mathbb{R}, \mathbb{R}) \\ (\phi, \mathbf{Y}) &\longmapsto u^{\mathbf{Y}} \end{aligned}$$

is continuous.

Proof First denoted by $X^{t,x,\mathbf{Y}}$ unique solution to the RDE (13), see Theorem 3.2. It is discussed in Appendix A.3 Remark A.9, that we can equivalently write the RSDE (11) as

$$X_t^{t,x,\boldsymbol{Y}} = x, \quad dX_s^{t,x,\boldsymbol{Y}} = g(s, X_s^{t,x,\boldsymbol{Y}})\mathbf{v}_s^{\mathbf{Y}} dB_s + f_0(s, X_s^{t,x,\boldsymbol{Y}})\mathbf{V}_s^{\mathbf{Y}} ds + f\left(s, X_s^{t,x,\boldsymbol{Y}}\right) d\mathbf{Y}_s^g$$

where $f_0(t,x) := -\frac{1}{2}f(t,x)\partial_x f(t,x)$. Denote by $\mathbf{Y}^\epsilon$ the rough path in $\mathscr{C}^{\alpha,1+}$ such that $\mathbf{Y}^\epsilon = (Y^\epsilon, \mathbb{Y}^{g,\epsilon} - \frac{1}{2}\delta[\mathbf{Y}]) = (Y^\epsilon, \mathbb{Y}^{g,\epsilon} - \frac{1}{2}\delta \int \mathbf{V}_t^{\mathbf{Y}} dt)$, see Lemma A.1. From the same lemma we know that $\|\mathbf{Y}^\epsilon - \mathbf{Y}\|_{\alpha',1+} \longrightarrow 0$ as $\epsilon \to 0$. From [26, Chapter 3 Theorem 7] we know that the following (classical) SDE

$$X_t^{t,x,\epsilon} = x, \quad dX_s^{t,x,\epsilon} = g(s, X_s^{t,x,\epsilon})\mathbf{v}_s^{\mathbf{Y}} dB_s + \left(f_0(s, X_s^{t,x,\epsilon})\mathbf{V}_s^{\mathbf{Y}} + f\left(s, X_s^{t,x,\epsilon}\right)\dot{Y}_s^\epsilon\right) ds,$$

has a unique strong solution. Applying Lemma A.8, we have $X^{t,x,\epsilon} = X^{t,x,\mathbf{Y}^\epsilon}$ almost surely, where $X^{t,x,\mathbf{Y}^\epsilon}$ is the unique solution to the RDE (13), where we replace $\mathbf{Y}$ by $\mathbf{Y}^\epsilon$, and we have

$$X^{t,x,\boldsymbol{Y}^\epsilon} \longrightarrow X^{t,x,\boldsymbol{Y}} \,\mathrm{ucp\,as}\,\epsilon \downarrow 0. \tag{17}$$

Since ϕ is bounded, the random variables $\phi(X_T^{t,x,\boldsymbol{Y}^\epsilon})$ and $\phi(X_T^{t,x,\boldsymbol{Y}})$ are integrable (uniformly in ϵ), and we can define

$$u^{\mathbf{Y}} = u^{\mathbf{Y}}(t,x) := E[\phi(X_T^{t,x,\mathbf{Y}})].$$

Combining this with (17), it readily follows that $u^\epsilon \longrightarrow u^{\mathbf{Y}}$ pointwise.
Finally, consider the solution map $S(\phi, \mathbf{Y}) = u^{\mathbf{Y}}$. Then for any sequence $(\phi^n, \boldsymbol{Y}^n)$ converging to $(\phi, \boldsymbol{Y})$, we want to show that

$$u^n = E[\phi^n(X_T^{\cdot,\cdot,\boldsymbol{Y}^n})] \longrightarrow u^{\mathbf{Y}} = E[\phi(X_T^{\cdot,\cdot,\boldsymbol{Y}})]$$

with respect to the supremum norm on $C_b^0([0,T] \times \mathbb{R}, \mathbb{R})$. But this follows from exactly the same arguments as above, namely by constructing the RDE (13) for both $\mathbf{Y}^n$ and $\mathbf{Y}$, and using the same stability arguments. □

Remark 3.5 *The pointwise convergence $u^\epsilon \to u^{\mathbf{Y}}$ in the last theorem may be improved to locally uniform convergence. This could be achieved by applying an Arzelà-Ascoli argument, together with local Lipschitz estimates of $(t, x, \mathbf{Y}) \mapsto X^{t,x,\mathbf{Y}}$. The latter, however, needs certain uniform integrability estimates for the joint-lift $\mathbf{Z}^{\mathbf{Y}}$, appearing from the Lipschitz constants in RDE stability results. For the joint-lift with Brownian motion (B, Y), this was for instance done in [10, Theorem 8], see also [13, Theorem 9].*

We end this section with a brief discussion about the comparison of RPDEs and SPDEs for the interested reader.

Remark 3.6 *Formally, replacing $d\mathbf{Y}^g$ in* (14) *with $\circ dI$, the Stratonovich differential of our local martingale I, leads to (terminal value) SPDE of the form*

$$-d_t u = L_t[u]v_t^2 dt + \Gamma_t[u] \circ dI, \qquad u(T,x) = \phi(x). \tag{18}$$

Remarkably, the SPDE (18) *is, to the best of our knowledge, beyond existing SPDEs theory – despite the fact that the Markovian case, say $v_t \equiv 1$ (otherwise absorb $v = v(t,x)$ into the coefficients of L_t) is very well-understood. Indeed, in this case* (18) *simplifies to*[5]

$$-d_t u = L_t[u]dt + \Gamma_t[u] \circ dW, \qquad u(T,x) = \phi(x); \tag{19}$$

Linear SPDEs of similar type ("Zakai equation") have been studied for decades in non-linear filtering theory. By using backward *(Stratonovich) integration against W, one can give honest meaning to* (19)*, with* backward *adapted solution, meaning that $u = u(t,x;\omega)$ is measurable w.r.t. $\mathcal{F}_{t,T}^W = \sigma(W_v - W_u : t \le u \le v \le T)$. The issue with* (18) *is, complications from* forward *adapted (v_t) aside, that there is no general backward integration against local martingales*[6]*, rendering previous approaches useless. Thus, to the best of our knowlede, the rough paths approach of* (14) *is the only way to make sense of this SPDE* (18).

As is common in mathematical finance, the SDE dynamics of (9) *is naturally given in Itô, not in Stratonovich, form. Yet, all SPDEs (resp. RPDEs) have been written with noise in Stratonovich (resp. geometric rough path) type. At least when $v \equiv 1$, the reason is easy to appreciate. If one rewrites* (19) *in backward Itô form,*

$$-d_t u = \tilde{L}_t[u]dt + \Gamma_t[u]d\overleftarrow{W}, \qquad u(T,x) = \phi(x),$$

its well-posedness would require for $\tilde{L}$ to satisfy a so-called stochastic parabolicity *condition. Having Stratonovich, resp. geometric rough path, noise in* (19)*, resp.* (14)*, bypasses such complications.*

[5]Strictly speaking, one should write $\circ d\overleftarrow{W}$ in (19) to emphasize the use of *backward* Stratonovich interpretation.
[6]... owing to the fact that the class of semimartingales is not stable under time-reversal ...

3.3 European pricing with RPDEs

In this section we use the Feynman-Kac representation obtained in Theorem 3.4, to price European options in local stochastic volatility models. More precisely, for some $(t,x) \in [0,T] \times \mathbb{R}$, let $X^{t,x}$ be the unique strong solution to the SDE (9), and consider a bounded and continuous payoff function $\phi \in C_b(\mathbb{R},\mathbb{R})$. Our goal is to compute the prices

$$(t,x) \mapsto E[\phi(X_T^{t,x})].$$

Since the $(\mathcal{F}_t^W)$-adapted volatility process v is possibly non-Markovian, we consider the conditional LSV-dynamics, conditioned on the Brownian motion W, to disentagle v from the dynamics X. In Section 3.1 we discussed how to rigorously do so, and we derived an $(\mathcal{F}_t^B)$-Markov property for the conditional dynamics, see Theorem 3.2. To make use of the Markovian nature in B, that is to apply the Feynman-Kac result Theorem 3.4, we study *conditional prices*

$$u(t,x,\omega) = E[\phi(X_T^{t,x})|\mathcal{F}_T^W](\omega).$$

The following theorem is the main result of this section, and it shows that u is the pathwise solution to the RPDE (14).

Theorem 3.7 *Let $g,f \in C_b^3([0,T]\times\mathbb{R},\mathbb{R})$, $\phi \in C_b^0(\mathbb{R},\mathbb{R}_+)$ and let Assumption 1 hold. Moreover, for any $\mathbf{Y} = (Y,\mathbb{Y}) \in \mathcal{C}^{\alpha,1+}([0,T],\mathbb{R})$, we denote by $u^{\mathbf{Y}} = u^{\mathbf{Y}}(t,x)$ the solution to the RPDE* (14)*, given by Theorem 3.4. Then we have $u(t,x,\omega) = u^{\mathbf{Y}}(t,x)|_{\mathbf{Y}=\boldsymbol{I}(\omega)}$ almost surely.*

Proof From Theorem 3.2 we know that $\mathrm{Law}(X^{t,x,\boldsymbol{I}}|\mathcal{F}_T^W) = \mathrm{Law}(X^{t,x}|\mathcal{F}_T^W)$, where $X^{t,x,\boldsymbol{Y}}$ is the unique solution to the RDE (13). Thus we have $u(t,x,\omega) = E[\phi(X_T^{t,x,\boldsymbol{I}})|\mathcal{F}_T^W](\omega)$ almost surely. We know that $X^{t,x,\mathbf{I}}$ is adapted to $\mathcal{F} = \mathcal{F}^W \vee \mathcal{F}^B = \mathcal{F}^{\mathbf{I}} \vee \mathcal{F}^B$. Therefore, there exists a Borel measurable function

$$F : \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R}) \times C^0([0,T],\mathbb{R}) \longrightarrow \mathbb{R},$$

such that $\phi(X_T^{t,x,\mathbf{I}}) = F(\mathbf{I},B)$, and we can consider $\mathbf{I}$, resp. B, as random variables with values in $\mathscr{C}^{\alpha,1+}([0,T],\mathbb{R})$, resp. $C^0([0,T],\mathbb{R})$. By independence of B and W, the law of B on C^0, denoted by μ_B, defines a regular conditional distribution[7] for B given $\mathcal{F}_T^W$. Applying the disintegration formula for conditional expectations, see for instance [21, Theorem 8.5], we have

$$\begin{aligned} E\left[F(\mathbf{I},B)|\mathcal{F}_T^W\right](\omega) &= \int_{C^0([0,T],\mathbb{R})} F(\mathbf{I}(\omega),b)\mu_B(db) \\ &= \int_{C^0([0,T],\mathbb{R})} F(\mathbf{Y},b)\mu_B(db)\bigg|_{\mathbf{Y}=\mathbf{I}(\omega)} \\ &= E[\phi(X_T^{t,x,\mathbf{Y}})]\Big|_{\mathbf{Y}=\mathbf{I}(\omega)} = u^{\mathbf{I}(\omega)}(t,x), \end{aligned}$$

where the last equality follows from Theorem 3.4. □

As a direct corollary of Theorem 3.7, we can now present a pricing formula for European claims, using our pathwise solution to the RPDE (14). Let $t=0$ and $x_0 \in \mathbb{R}$, and denote by $X := X^{0,x_0}$ the unique solution to the SDE (9), starting from x_0 at time $t=0$.

Corollary 3.8 *Under Assumption 1, for $f,g \in C_b^3([0,T]\times\mathbb{R},\mathbb{R})$ and $\phi \in C_b^0(\mathbb{R},\mathbb{R}_+)$, we have*

$$y_0 := E[\phi(X_T)] = E[u^{\mathbf{I}}(0,x_0)].$$

[7] See [21, Chapter 8 p.167] for the definition of regular conditional distributions.

This suggests that to compute the price of the payoff $\phi(X_T)$ at time $t = 0$, we can take the expected value of the evaluations at $(0, x_0)$ of the pathwise solutions $u^{\mathbf{I}}$ to the random RPDE (14) with $\mathbf{Y} = \mathbf{I}$.

Another natural question is whether we can utilize the pathwise solutions $u^{\mathbf{I}}$ to the RPDE to characterize the price at any time $t \in [0, T]$. Specifically, we aim to compute the conditional expectation $E[\phi(X_T)|\mathcal{F}_t]$, where $\mathcal{F}$ represents the full filtration, defined as $\mathcal{F}_t := \mathcal{F}_t^B \vee \mathcal{F}_t^W$. The following lemma establishes the relationship between this conditional expectation and our RPDE solution $u^{\mathbf{I}}$.

Proposition 3.9 *Under Assumption 1, for $f, g \in C_b^3([0,T] \times \mathbb{R}, \mathbb{R})$ and $\phi \in C_b^0(\mathbb{R}, \mathbb{R}_+)$ we have*

$$E[\phi(X_T)|\mathcal{F}_t] = E[u^{\mathbf{I}}(t,x)|\mathcal{F}_t^W]\big|_{x=X_t} \quad \textit{a.s. for all } t \in [0,T].$$

Proof First, we can notice that $X_T := X_T^{0,x_0} = X_T^{t,X_t^{0,x_0}}$ almost surely. Indeed, using once again [26, Chapter 3 Theorem 7], we know that $X^{t,x}$ is the unique solution to the integral equation

$$X_s^{t,x} = x + \int_t^s f(u, X_u^{t,x}) v_u dW_u + \int_t^s g(u, X_u^{t,x}) v_u dB_u, \quad t \le s \le T.$$

Therefore, the process $\tilde{X} := X^{t,X_t}$ is the unique solution to the integral equation

$$\tilde{X}_s = X_t + \int_t^s f(u, \tilde{X}_u) v_u dW_u + \int_t^s g(u, \tilde{X}_u) v_u dB_u, \quad t \le s \le T.$$

But on the other hand, for $s \in [t, T]$, we have

$$\begin{aligned} X_s &= x_0 + \int_0^s f(u, X_u) v_u dW_u + \int_0^s g(u, X_u) v_u dB_u \\ &= X_t + \int_t^s f(u, X_u) v_u dW_u + \int_t^s g(u, X_u) v_u dB_u. \end{aligned}$$

By uniqueness, it follows that $X = \tilde{X}$ on $[t, T]$ and in particular $X_T = X_T^{t,X_t}$.
Next, we recall from Theorem 3.2 that

$$\mathrm{Law}\left(X^{t,x}\middle|\, \mathcal{F}_T^W \vee \mathcal{F}_t^B\right) = \mathrm{Law}\left(X^{t,x,\mathbf{I}}\middle|\, \mathcal{F}_T^W \vee \mathcal{F}_t^B\right).$$

By continuity of the RDE solution map $x \mapsto X^{t,x,\mathbf{I}}$ we can find a well-defined version of $x \mapsto E[\phi(X_T^{t,x,\mathbf{I}})|\mathcal{F}_T^W \vee \mathcal{F}_t^B]$. Applying the tower property and the Markov property for $X^{t,x,\mathbf{I}}$ yields

$$E[\phi(X_T)|\mathcal{F}_t] = E[\phi(X_T^{t,X_t})|\mathcal{F}_t](\omega) = E[E[\phi(X_T^{t,X_t,\mathbf{I}})|\mathcal{F}_T^W \vee \mathcal{F}_t^B]|\mathcal{F}_t] = E[E[\phi(X_T^{t,x,\mathbf{I}})|\mathcal{F}_T^W]|\mathcal{F}_t^W]|_{x=X_t}.$$

From Theorem 3.7 we know that $E[\phi(X_T^{t,x,\mathbf{I}})|\mathcal{F}_T^W](\omega) = u^{\mathbf{I}(\omega)}(t,x)$, and thus we can conclude

$$E[\phi(X_T)|\mathcal{F}_t] = E[u^{\mathbf{I}}(t,x)|\mathcal{F}_t^W]\big|_{x=X_t} \quad \text{almost surely.}$$

□

3.4 Special case: stochastic volatility models

In this section, we discuss two choices of f and g, where we can solve the RPDE in Theorem 3.7 explicitly. These examples correspond to stochastic volatility models in two distinct forms.

Example 1: Stochastic volatility models in Bachelier-form

First, let $f(t,x) \equiv \rho$ and $g(t,x) \equiv \sqrt{1-\rho^2}$ for some $\rho \in [-1,1]$. In this case we have a stochastic volatility model of the form

$$X_t^{t,x} = x, \quad dX_s^{t,x} = v_s\left(\rho dW_s + \sqrt{1-\rho^2}dB_s\right), \quad 0 \le t < s \le T,$$

where B and W are two independent standard Brownian motions, and v satisfies Assmuption 1. The goal now is to show that the pricing RPDE (14) reduces to a classical heat-equation, and we can give an explicit solution. Let $\phi : \mathbb{R} \to \mathbb{R}_+$ be a bounded and continuous function. Since f is constant, we have $f_0 \equiv 0$, and the RPDE (14) reduces to

$$\begin{cases} -d_t u^{\mathbf{Y}} &= \frac{1}{2}(1-\rho^2)\partial_{xx}^2 u_t^{\mathbf{Y}} \mathbf{V}_t^{\mathbf{Y}} dt + \rho \partial_x u_t^{\mathbf{Y}} d\mathbf{Y}_t^g \\ u^{\mathbf{Y}}(T,x) &= \phi(x). \end{cases} \tag{20}$$

Choosing the pair $\mathbf{Y}^{g,\epsilon}$, resp. $\mathbf{Y}^\epsilon$, as described in the proof of Theorem 3.4, we know the solution to the RPDE is defined as the unique limit of the sequence of solutions u^ϵ to the PDEs

$$\begin{cases} -\partial_t u^\epsilon &= \frac{1}{2}(1-\rho^2)\mathbf{V}_t^{\mathbf{Y}} \partial_{xx}^2 u_t^\epsilon + \rho \dot{Y}_t^\epsilon \partial_x u_t^\epsilon \\ u^\epsilon(T,x) &= \phi(x). \end{cases} \tag{21}$$

But in this case, the PDE can actually be reduced to the heat-equation

$$\begin{cases} -\partial_t v &= \frac{1}{2}(1-\rho^2)\mathbf{V}_t^{\mathbf{Y}} \partial_{xx}^2 v_t \\ v(T,x) &= \phi(x). \end{cases} \tag{22}$$

Indeed, assume v solves (22) and define $u^\epsilon(t,x) := v(t, x+\rho Y_{t,T}^\epsilon)$. Then clearly $u^\epsilon(T,x) = v(T,x) = \phi(x)$ and by the chain-rule,

$$\begin{aligned} -\partial_t u^\epsilon &= -\partial_t v(t,y)|_{y=x+\rho Y_{t,T}^\epsilon} + \rho \dot{Y}_t^\epsilon \partial_x v(t,y)|_{y=x+\rho Y_{t,T}^\epsilon} \\ &= \frac{1}{2}(1-\rho^2)\mathbf{V}_t^{\mathbf{Y}} \partial_{xx}^2 u_t^\epsilon + \rho \dot{Y}_t^\epsilon \partial_x u_t^\epsilon. \end{aligned}$$

Therefore, u^ϵ is indeed the solution to (21).
Focusing on the PDE (22), we define the heat-kernel

$$f(t,x,y) := \frac{1}{\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{-\frac{(y-x)^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}}\right\}.$$

Then one can check that the function

$$v(t,x) = \int_{\mathbb{R}} \phi(y) f(t,x,y)dy$$

is the unique solution to the heat-equation (22). Therefore, the solution to the PDE (21) is given by

$$u^\epsilon(t,x) := v(t, x+\rho Y_{t,T}^\epsilon) = \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{-\frac{(y-x-\rho Y_{t,T}^\epsilon)^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}}\right\} dy.$$

Using dominated convergence, we finally find the solution to the RPDE is explicitely given by

$$u^{\mathbf{Y}}(t,x) = \lim_{\epsilon\to 0} u^\epsilon(t,x) = \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{-\frac{(y-x-\rho Y_{t,T})^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}}\right\} dy.$$

By a direct application of Theorem 3.7, we find the following formulas in the case of stochastic volatility models.

Theorem 3.10 *Under Assumption 1, for $\phi \in C_b^0(\mathbb{R},\mathbb{R})$, the unique solution to the RPDE* (20) *is given by*

$$u^{\mathbf{Y}}(t,x) = \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{ -\frac{(y-x-\rho Y_{t,T})^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}} \right\} dy.$$

Moreover, recalling that $u(t,x,\omega) = E\left[\phi(X_T^{t,x})|\mathcal{F}_T^W\right](\omega)$, we have

$$u(t,x,\omega) = \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{2\pi(1-\rho^2)[I]_{t,T}(\omega)}} \exp\left\{ -\frac{(y-x-\rho I_{t,T}(\omega))^2}{2(1-\rho^2)[I]_{t,T}(\omega)} \right\} dy.$$

Example 2: Stochastic volatility models in Black-Scholes form

A second example comes from choosing $f(t,x) = x\rho$ and $g(t,x) = x\sqrt{1-\rho^2}$, which leads to the more common form of stochastic volatility models

$$X_t^{t,x} = x, \quad dX_s^{t,x} = X_s^{t,x} v_s \left(\rho dW_s + \sqrt{1-\rho^2} dB_s\right), \quad 0 \le t < s \le T.$$

In this case we have $f_0(t,x) = -\frac{1}{2}\rho^2 x$, and thus the RPDE reads

$$\begin{cases} -d_t u^{\mathbf{Y}} & = \left(\frac{1}{2}x^2(1-\rho^2)\partial_{xx}^2 u_t^{\mathbf{Y}} - \frac{1}{2}\rho^2 x \partial_x u_t^{\mathbf{Y}}\right) \mathbf{V}_t^{\mathbf{Y}} dt + \rho x \partial_x u_t^{\mathbf{Y}} d\mathbf{Y}_t^g \\ u^{\mathbf{Y}}(T,x) & = \phi(x). \end{cases} \tag{23}$$

Similar as before, the classical PDE with respect to the approximating rough path $\mathbf{Y}^{g,\epsilon}$ as described in the proof of Theorem 3.4, is given by

$$\begin{cases} -\partial_t u^{\epsilon} & = \frac{1}{2}x^2(1-\rho^2)\mathbf{V}_t^{\mathbf{Y}} \partial_{xx}^2 u_t^{\epsilon} + \left(\rho x \dot{Y}_t^{\epsilon} - \frac{1}{2}\rho^2 x \mathbf{V}_t^{\mathbf{Y}}\right) \partial_x u_t^{\epsilon} \\ u^{\epsilon}(T,x) & = \phi(x). \end{cases} \tag{24}$$

We can reduce that PDE to

$$\begin{cases} -\partial_t v & = \frac{1}{2}x^2(1-\rho^2)\mathbf{V}_t^{\mathbf{Y}} \partial_{xx}^2 v_t \\ v(T,x) & = \phi(x). \end{cases} \tag{25}$$

Indeed, define $\psi^{\epsilon}(t) := \rho Y_{t,T}^{\epsilon} - \frac{1}{2}\rho^2[\mathbf{Y}]_{t,T}$ and $u^{\epsilon}(t,x) := v(t, xe^{\psi^{\epsilon}(t)})$. Clearly we have $u^{\epsilon}(T,x) = v(T,x) = \phi(x)$, and by the chain-rule

$$\begin{aligned} -\partial_t u^{\epsilon} &= -\partial_t v(t,y)|_{y=xe^{\psi^{\epsilon}(t)}} - \partial_x v(t,y)|_{y=xe^{\psi^{\epsilon}(t)}} x e^{\psi^{\epsilon}(t)} \dot{\psi}^{\epsilon}(t) \\ &= \left(\frac{1}{2}y^2(1-\rho^2)\partial_{xx}^2 v(t,y)\right)\Big|_{y=xe^{\psi^{\epsilon}(t)}} \mathbf{V}_t^{\mathbf{Y}} - \left(\partial_x v(t,y) e^{\psi^{\epsilon}(t)}\right)\big|_{y=xe^{\psi^{\epsilon}(t)}} x \dot{\psi}^{\epsilon}(t) \\ &= \frac{1}{2}x^2(1-\rho^2)\mathbf{V}_t^{\mathbf{Y}} \partial_{xx}^2 u_t^{\epsilon} + x\left(\rho \dot{Y}_t^{\epsilon} - \frac{1}{2}\rho^2 \mathbf{V}_t^{\mathbf{Y}}\right) \partial_x u_t^{\epsilon}, \end{aligned}$$

and therefore u^{ϵ} is the solution to (24).
Looking directly at the PDE (25), we can again directly find a solution, which in this case is given by

$$v(t,x) = \int_{\mathbb{R}_+} \phi(y) \frac{1}{y\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{ -\frac{(\ln(y/x) - \frac{1}{2}(1-\rho^2)[\mathbf{Y}]_{t,T})^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}} \right\} dy.$$

Using dominated convergence, and applying Theorem 3.4, we find in this case

$$\begin{aligned} u^{\mathbf{Y}}(t,x) &= \lim_{\epsilon\to 0} v(t, xe^{\psi^\epsilon(t)}) \\ &= \int_{\mathbb{R}_+} \phi(y) \frac{1}{y\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{ -\frac{(\ln(y/x) - \rho Y_{t,T} + \frac{1}{2}[\mathbf{Y}]_{t,T})^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}} \right\} dy. \end{aligned}$$

We summarize the results in the following theorem.

Theorem 3.11 *Under Assumption 1, for $\phi \in C_b(\mathbb{R}, \mathbb{R}_+)$, the unique solution to the RPDE* (23) *is given by*

$$u^{\mathbf{Y}}(t,x) = \int_{\mathbb{R}_+} \phi(y) \frac{1}{y\sqrt{2\pi(1-\rho^2)[\mathbf{Y}]_{t,T}}} \exp\left\{ -\frac{(\ln(y/x) - \rho Y_{t,T} + \frac{1}{2}[\mathbf{Y}]_{t,T})^2}{2(1-\rho^2)[\mathbf{Y}]_{t,T}} \right\} dy.$$

Moreover, recalling that $u(t,x,\omega) = E\left[\phi(X_T^{t,x})|\mathcal{F}_T^W\right](\omega)$, we have

$$u(t,x,\omega) = \int_{\mathbb{R}_+} \phi(y) \frac{1}{y\sqrt{2\pi(1-\rho^2)[I]_{t,T}(\omega)}} \exp\left\{ -\frac{(\ln(y/x) - \rho I_{t,T}(\omega) + \frac{1}{2}[I]_{t,T}(\omega))^2}{2(1-\rho^2)[I]_{t,T}(\omega)} \right\} dy.$$

Remark 3.12 *The formulas for u in Theorems 3.10 and 3.11 can also be deduced from the so-called Romano-Touzi formula [27], and are already well-known. In the case where f and g are constant, one can observe that conditional on W, we have*

$$X_T^{t,x} = x + \rho \int_t^T v_s dW_s + \sqrt{1-\rho^2} \int_t^T v_s dB_s \sim \mathcal{N}(x + \rho I_{t,T}, (1-\rho^2)[I]_{t,T}),$$

which readily gives the representation for u in Theorem 3.10. Similarly, in the second example, one can notice that conditional on W, the random variable $X_T^{t,x}$ has a log-normal distribution, which leads to the latter formula of u. The approach presented in this paper therefore generalizes the understanding of the conditional process X given W in generic local stochastic volatility models, and relates them to pathwise PDEs.

3.5 Multivariate dynamics

We end this section by describing how our approach generalizes for multidimensional dynamics of the underlying process X. Let $W = (W^0, \dots, W^k)^T$ be a $(k+1)$-dimensional Brownian motion, and assume v is an $\mathbb{R}$-valued, $\mathcal{F}^{W^0}$-progressive process with bounded sample paths. Now for any $F : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^{d\times(k+1)}$, such that $F \in C_b^3(\mathbb{R}^{d+1}, \mathbb{R}^{d\times(k+1)})$, the local stochastic volatility SDE (9) generalizes to

$$X_t^{t,x} = x \in \mathbb{R}^d, \quad dX_s^{t,x} = F(s, X_s^{t,x}) v_s dW_s, \quad t < s \le T. \tag{26}$$

In particular, we have $X^{t,x} \in \mathbb{R}^d$, and for any $i = 1, \dots, d$, we have

$$\begin{aligned} (X_s^{t,x})^i &= x^i + \sum_{n=0}^k \int_t^s F^{i,n}(u, X_u^{t,x}) v_u dW_u^n \\ &= x^i + \int_t^s F^{i,0}(u, X_u^{t,x}) v_u dW_u^0 + \sum_{n=1}^k \int_t^s F^{i,n}(u, X_u^{t,x}) v_u dW_u^n \\ &= x^i + \int_t^s F^{i,0}(u, X_u^{t,x}) dI_u + \sum_{n=1}^k \int_t^s F^{i,n}(u, X_u^{t,x}) v_u dW_u^n, \end{aligned}$$

where I is the local martingale given by $I_t := \int_0^t v_s dW_s^0$. Now define the Brownian motion $B := (W^1, \ldots, W^k)$, and the vector field $G : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^{m\times k}$, with $G^{i,j}(t,x) := F^{i,j}(t,x)$ for $i = 1, \ldots, d$ and $j = 1, \ldots, k$. Then we can write the SDE (26) in the generalized form of the one-dimensional local stochastic volatility model (9), that is

$$X_t^{t,x} = x, \quad dX_s^{t,x} = F^0(s, X_s^{t,x})dI_s + G(s, X_s^{t,x})v_s dB_s, \quad t < s \leq T, \tag{27}$$

where F^0 denotes the first column of F. Existence and uniqueness of a strong unique solution to (27) can again be deduced from [26, Chapter 3 Theorem 7].

From here we can proceed exactly the same way as we did in the one-dimensional case. The main result from Section 3.1, which is Theorem 3.2, is already formulated in the multivariate framework in Appendix A.3, see Theorem A.4. Using similar arguments as in Section 3.2, for any bounded and continuous function $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$, similar as in Theorem 3.4 we can find a unique Feynman-Kac type of solution to the RPDE

$$\begin{cases} -d_t u & = L_t[u]\mathbf{V}_t^{\mathbf{Y}} dt + \Gamma_t[u] d\mathbf{Y}_t^g \quad t \in [0,T[\times \mathbb{R}^d, \\ u(T,x,\cdot) & = \phi(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{28}$$

where

$$\begin{aligned} L_t[u](x) &:= \frac{1}{2}\mathrm{Tr}\left[G(t,x)G(t,x)^T \nabla_{xx}^2 u(t,x)\right] + \langle H(t,x), \nabla_x u(t,x)\rangle \\ \Gamma_t[u](x) &:= \langle F^0(t,x), \nabla_x u(t,x)\rangle, \end{aligned}$$

where we define $H(t,x) := -\frac{1}{2}\nabla_x F^0(t,x)F^0(t,x)$, and $\nabla_x F^0(t,x)$ is the Jacobi-matrix of F^0. Moreover we can also generalize Theorem 3.7 to see that the unique solution to the RPDE above, denoted by $u = u^{\mathbf{Y}}(t,x)$, fulfills

$$u^{\mathbf{I}(\omega)}(t,x) = E\left[\phi(X_T^{t,x})|\mathcal{F}_T^{W^0}\right](\omega),$$

almost surely.

Example 3.13 *In this example we generalize the first special case of a stochastic volatility model from Section 3.4, by choosing $F(t,x) \equiv \rho = [\rho_0, \ldots, \rho_k] \in \mathbb{R}^{d\times(k+1)}$, such that for all $i = 1, \ldots, d$, we have $\sum_{j=0}^k \rho_{ij}^2 \leq 1$. In view of* (27)*, we can then write the dynamics as*

$$X_s^{t,x} = x + \rho_0 \int_t^s v_u dW_u^0 + \hat{\rho}\int_t^s v_u dB_u,$$

where ρ_0 denotes the first column of ρ, and $\hat{\rho} := [\rho_1, \ldots, \rho_k]$. By the Romano-Touzi formula [27] one can notice that conditional on W^0, the random vector $X_T^{t,x}$ is multivariate Gaussian, that is $X_T^{t,x}\big|_{\mathcal{F}_T^{W^0}} \sim \mathcal{N}(\mu, \mathbf{\Sigma})$ with

$$\mu = x + \rho_0 I_{t,T} \in \mathbb{R}^d \quad \textit{and} \quad \mathbf{\Sigma} := \hat{\rho}\hat{\rho}^T [I]_{t,T} \in \mathbb{R}^{d\times d}.$$

The same results can be achieved by explicitly solving the RPDE (28)*, which again reduces to a heat-equation as in Theorem 3.10, but in $\mathbb{R}^d$.*

4 Numerical examples

This section is devoted to present first ideas for numerical methods to price Europeans options in general local stochastic volatility models, using the theory developed in the last sections. More precisely, recall the general price dynamics

$$X_t^{t,x} = x, \quad dX_s^{t,x} = f(s, X_s^{t,x})v_s dW_s + g(s, X_s^{t,x})v_s dB_s, \quad t < s \leq T,$$

for $(t,x) \in [0,T] \times \mathbb{R}$, where v satisfies Assumption 1, and B and W are independent Brownian motions. For some bounded payoff function $\phi \in C_b(\mathbb{R}, \mathbb{R})$, we further recall $u(t,x,\omega) := E[\phi(X_T^{t,x})|\mathcal{F}_T^W](\omega)$, and we know from Theorem 3.7 that $u(t,x,\omega) = u^{\mathbf{I}(\omega)}(t,x)$ almost surely. Here $\mathbf{I}$ denotes the Itô rough path lift of $I_t = \int_0^t v_s dW_s$, and $u^{\mathbf{Y}}$ is the solution to the RPDE (14). To construct numerical methods for the RPDE, let us first replace the domain $\mathbb{R}$ by some bounded interval $[a,b]$, for some $a < b$, and consider a (for simplicity) uniform space-grid $\mathcal{S} := \{a = x_0 < x_1 < \cdots < x_N = b\}$ with $\Delta x := (b-a)/N$, and similarly a uniform time-grid $\mathcal{T} := \{0 = t_0 < t_1 < \cdots < t_J = T\}$ with $\Delta t := T/J$. Moreover, when replacing $\mathbb{R}$ with $[a,b]$ in the RPDE, we need to add appropriate boundary conditions. We choose Dirichlet boundary conditions, i.e., $\psi_a, \psi_b : [0,T] \longrightarrow \mathbb{R}$ and set $u(t,a) = \psi_a(t)$ and $u(t,b) = \psi_b(t)$. The RPDE we want to solve numerically is then of the form

$$-d_t u^{\mathbf{Y}} = L_t[u^{\mathbf{Y}}]\mathbf{V}_t^{\mathbf{Y}} dt + \Gamma_t[u^{\mathbf{Y}}]d\mathbf{Y}_t^g, \quad (t,x) \in [0,T[\times]a,b[\tag{29a}$$

$$u^{\mathbf{Y}}(t,a) = \psi_a(t), \quad t \in [0,T[\tag{29b}$$

$$u^{\mathbf{Y}}(t,b) = \psi_b(t), \quad t \in [0,T[\tag{29c}$$

$$u^{\mathbf{Y}}(T,x) = \phi(x), \quad x \in [a,b]. \tag{29d}$$

First-order finite-difference scheme

Recall from Section 3.2 Theorem 3.4, that the unique solution $u^{\mathbf{Y}}$ to the RPDE (14) is the unique limit of the solution u^ϵ to the classical PDE (15). Our first approach is to numerically approximate u^ϵ, leaving a detailed error-analysis for $|u^\epsilon - u^{\mathbf{Y}}|$ aside here. Then we proceed by using a finite-difference scheme to solve the classical PDE (15). In particular, we set $u_j^n := u^\epsilon(t_j, x_n)$, and recall the notation $\frac{d[\mathbf{Y}]_t}{dt} := \mathbf{V}_t^{\mathbf{Y}}$. Integrating over the time interval $[t_j, t_{j+1}]$ leads to

$$u_j^n = u_{j+1}^n + \int_{t_j}^{t_{j+1}} L_s[u^\epsilon](x_n)d[\mathbf{Y}]_s + \int_{t_j}^{t_{j+1}} \Gamma_s[u^\epsilon](x_n)dY_s^\epsilon. \tag{30}$$

We Replace the space derivatives in the differential operators L and Γ by finite difference quotients

$$L_{t_j}[u^\epsilon](x_n) \approx L_j^n := \frac{1}{2}g^2(t_j,x_n)\frac{u_j^{n+1} + u_j^{n-1} - 2u_j^n}{(\Delta x)^2} + f_0(t_j,x_n)\frac{u_j^{n+1} - u_j^n}{\Delta x}$$

$$\Gamma_{t_j}[u^\epsilon](x_n) \approx \Gamma_j^n := f(t_j,x_n)\frac{u_j^{n+1} - u_j^n}{\Delta x}.$$

Note that the usual stability analysis for finite difference schemes would lead to *path-dependent* conditions for Δt and Δx in the fully explicit case. Indeed, the corresponding explicit scheme was observed to be unstable unless very small time-steps were chosen. To improve stability, we instead use a implicit-explicit (IMEX) scheme. More precisely, we apply a left-point approximation for the middle integral, and a right-point approximation for the last integral in (30), which leads to the mixed implicit-explicit, backward finite-difference scheme

$$u_j^n = u_{j+1}^n + L_j^n[\mathbf{Y}]_{t_j,t_{j+1}} + \Gamma_{j+1}^n Y_{t_j,t_{j+1}}^{\epsilon},$$

for $0 \le j \le J-1, 1 \le n \le N-1$, with boundary conditions

$$u_j^0 = \psi_a(t_j), \quad u_j^N = \psi_b(t_j), \quad u_J^n = \phi(x_n). \tag{31}$$

Second-order finite-difference scheme

The first order approach appears to be very natural considering our definition of solutions to the RPDE. Indeed, we gave the RPDE only a formal meaning by defining solutions as limits of classical PDE solutions. A more direct interpretation of RPDEs in a slightly different setting is given in [9]. Extending the notion of *regular solutions* in [9, Definition 7] to our setting, we say a function $u = u^{\mathbf{Y}}(t,x) \in C^{0,2}$ is a regular solution to the RPDE (14), if

$$u^{\mathbf{Y}}(t,x) = \phi(x) + \int_t^T L_s[u^{\mathbf{Y}}]\mathbf{V}_s^{\mathbf{Y}} ds + \int_t^T \Gamma_s[u^{\mathbf{Y}}] d\mathbf{Y}_s^g \tag{32}$$

and $(\Gamma[u^{\mathbf{Y}}], \Gamma'[u^{\mathbf{Y}}]) := (\Gamma[u^{\mathbf{Y}}], -\Gamma\left[\Gamma[u^{\mathbf{Y}}]\right]) \in \mathscr{D}_{\mathbf{Y}^g}^{2\alpha}([0,T],\mathbb{R})$, where $\mathscr{D}_{\mathbf{Y}^g}^{2\alpha}$ is the space of *controlled rough paths*. The latter is now a well-defined, rough integral given by

$$\int_t^T \Gamma_s[u^{\mathbf{Y}}] d\mathbf{Y}_s^g = \lim_{|\pi|\to 0} \sum_{[v,w]\subset\pi} \left(\Gamma_v[u^{\mathbf{Y}}] Y_{v,w} - \Gamma_v\left[\Gamma[u^{\mathbf{Y}}]\right] \mathbb{Y}_{v,w}^g\right).$$

Remark 4.1 *While we firmly believe that our RPDEs can be treated in a similar manner as done in [9], addressing this goes beyond the scope of this paper. Therefore, we intentionally keep this task, as well as a full numerical analysis for both finite-difference schemes, open for future research.*

Recalling that $\mathbb{Y}_{s,t}^g = \frac{1}{2}(Y_{s,t})^2$ in the one-dimensional setting, the rough integral can be formulated in a backward representation, see [13, Chapter 5.4], given by

$$\int_t^T \Gamma_s[u^{\mathbf{Y}}] d\mathbf{Y}_s^g = \lim_{|\pi|\to 0} \sum_{[v,w]\subset\pi} \left(\Gamma_w[u^{\mathbf{Y}}] Y_{v,w} + \frac{1}{2}\Gamma_w\left[\Gamma[u^{\mathbf{Y}}]\right] Y_{v,w}^2\right).$$

By definition of Γ, we have

$$\begin{aligned}\Gamma_t\left[\Gamma[u^{\mathbf{Y}}]\right](x) &= f(t,x)\partial_x f(t,x)\partial_x u^{\mathbf{Y}}(t,x) + f^2(t,x)\partial_{xx}u^{\mathbf{Y}}(t,x)\\ &= -2f_0(t,x)\partial_x u^{\mathbf{Y}}(t,x) + f^2(t,x)\partial_{xx}u^{\mathbf{Y}}(t,x).\end{aligned}$$

Now with exactly the same idea as before, by integrating over $[t_j, t_{j+1}]$ we have

$$u^{\mathbf{Y}}(t_j,x_n) = u^{\mathbf{Y}}(t_{j+1},x_n) + \int_{t_j}^{t_{j+1}} L_s[u^{\mathbf{Y}}](x_n)\mathbf{V}_s^{\mathbf{Y}} ds + \int_{t_j}^{t_{j+1}} \Gamma_s[u^{\mathbf{Y}}](x_n) d\mathbf{Y}_s^g.$$

Denoting $u_j^n := u^{\mathbf{Y}}(t_j,x_n)$ for $0 \le j \le J$ and $0 \le n \le N$, we use the same approximations for L and Γ given in the last section, and define

$$\Gamma'_{t_j}[u](x_n) \approx (\Gamma')_j^n := -2f_0(t_j,x_n)\frac{u_j^{n+1}-u_j^n}{\Delta x} + f^2(t_j,x_n)\frac{u_j^{n+1}+u_j^{n-1}-2u_j^n}{(\Delta x)^2}.$$

Similar as in the first-order scheme, we apply a left-point approximation for the middle integral, and a right-point approximation for the rough integral in (32), which leads to the mixed implicit-explicit, second order finite-difference scheme

$$u_j^n = u_{j+1}^n + L_j^n[\mathbf{Y}]_{t_j,t_{j+1}} + \Gamma_{j+1}^n Y_{t_j,t_{j+1}} + \frac{1}{2}(\Gamma')_{j+1}^n Y_{t_j,t_{j+1}}^2,$$

for $0 \le j \le J-1, 1 \le n \le N-1$, with the same boundary conditions as in (31).

We test the two finite-difference schemes in the following two examples.

Example 4.2 (Bachelier stochastic volatility model) *Recall from Section 3.4, that for the choice $f \equiv \rho$ and $g \equiv \sqrt{1-\rho^2}$, we find the stochastic volatility model of the form*

$$X_t^{t,x} = x, \quad dX_s^{t,x} = v_s\left(\rho dW_s + \sqrt{1-\rho^2}dB_s\right), \quad t < s \le T. \tag{33}$$

As already mentioned in Remark 3.12, conditional on W, the price has a normal distribution. Applying similar techniques as in the derivation of the Black-Scholes formula, one can find an explicit expression of $u(t,x,\omega)$.

Example 4.3 (SABR local stochastic volatility model) *In a second example we consider a model leaving the classical stochastic volatility framework, namely we have the SABR dynamics $f(t,x) = \rho x^\beta$ and $g(t,x) = \sqrt{1-\rho^2}x^\beta$ for some $\beta \in (1/2, 1]$, see [19]. More precisely, we have*

$$X_t^{t,x} = x, \quad dX_s^{t,x} = (X_s^{t,x})^\beta v_s\left(\rho dW_s + \sqrt{1-\rho^2}dB_s\right), \quad t < s \le T.$$

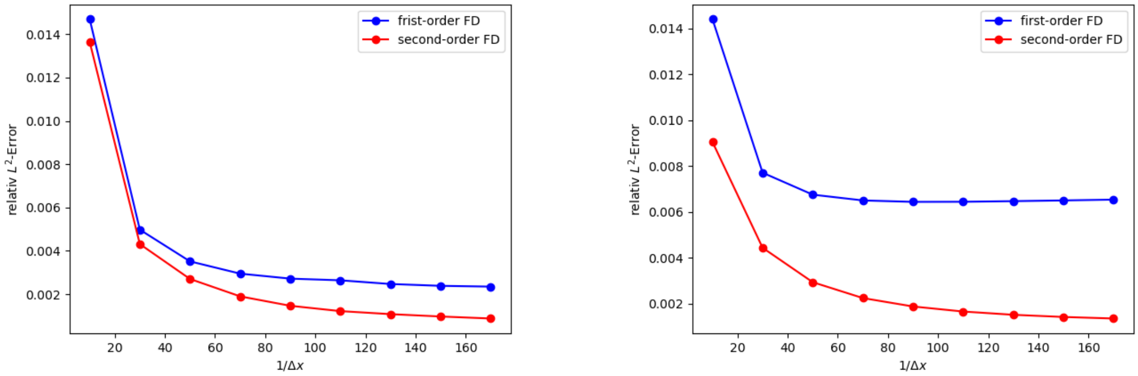
Notice that this in particular includes the case of classical stochastic volatility models in Black-Scholes form for $\beta = 1$, as described in Section 3.4. For $\beta \neq 1$, we no longer have an exact reference solution for $u(t,x,\omega)$, and we use a full Monte-Carlo simulation for comparison.

We are interested in a European put option written on the asset X, that is we consider the payoff function $\phi(x) := \max(K-x, 0)$. For all the examples in this section, we choose the rough volatility process as given in the rough Bergomi model [2], that is

$$v_t = \xi_0 \mathcal{E}\left(\eta \int_0^t (t-s)^{H-\frac{1}{2}} dW_s\right),$$

where $\mathcal{E}$ denotes the stochastic exponential. We choose $T = 1, K = 5, \xi_0 = 0.235^2, \eta = 1.9$ and $H = 0.07$, where we recall that the Hurst parameter H determines the roughness of the volatility, that is the Hölder regularity of v. Then we simulate $M = 10\,000$ paths of the pair $(I, [I]) = (\int v_t dW_t, \int V_t dt)$, with each $N = 10\,000$ uniform time-steps in $[0,1]$, we refer to [2, Section 4] for details about simulation in the rough Bergomi model.

Now having the samples $(I^{(m)}, [I]^{(m)})$ in hand, we can in particular construct samples of the Itô rough path lift $\mathbf{I}^{(m)}$ for $m = 1, \dots, M$. Along every such sample path $\mathbf{I}^{(m)}$, we solve the RPDE (14) with our finite-difference schemes, and denote by $u^{\mathbf{I}^{(m)}}$ the exact solution along the m-th sample. Recalling the identity $u^{\mathbf{I}^{(m)}}(t,x) = E[\phi(X_T^{t,x,\mathbf{I}^{(m)}})]$, see Theorem 3.4, it is tempting to think of $u^{\mathbf{I}^{(m)}}$ as the put option price written on the $(\mathcal{F}_t^B)$-price dynamics $X^{t,x,\mathbf{I}^{(m)}}$. It is however important to



(a) (b)

Figure 1: Strong relativ errors for fixed time step-size $\Delta t = 1/30$ and increasing number of space-steps, for both finite-difference scheme. In (a): Bachelier SV-model Example 4.2, and in (b): SABR SV-model Example 4.3 with $\beta = 0.6$.

note that the latter process is typically not an $(\mathcal{F}_t^B)$-martingale, and thus $u^{\mathbf{I}^{(m)}}$ could lead to arbitrage opportunities in this model. On the other hand, we can consider the Monte-Carlo approximation

$$y_0(x) = E[u^{\mathbf{I}}(0,x)] \approx \frac{1}{M}\sum_{m=1}^{M} u^{\mathbf{I}^{(m)}}(0,x) =: y_0^M(x).$$

The right-hand side then clearly converges to the (arbitrage-free) price y_0 in the corresponding LSV model, as M goes to infinity.

In both Examples 4.2 and 4.3, we apply the finite-difference schemes for the RPDEs along all samples $m = 1, \dots, M$, for simplicity we denote both by $u_{\text{FD}}^{\mathbf{I}^{(m)}}$. In Figure 1 we plot the following strong relative error with respect to the 2-norm $\|h\|_2 := \sqrt{\sum_{i=0}^{N} h(x_i)^2}$ on the space-grid, that is

$$\epsilon^M := \frac{1}{M}\sum_{m=1}^{M} \frac{\|u^{\mathbf{I}^{(m)}}(0,\cdot) - u_{\text{FD}}^{\mathbf{I}^{(m)}}(0,\cdot)\|_2}{\|u^{\mathbf{I}^{(m)}}(0,\cdot)\|_2},$$

for both examples and finite-difference schemes. The reference solution in Example 4.2 corresponds to the exact Romano-Touzi solution, and in Example 4.3 we consider a Monte-Carlo solution, obtained along $10\,000$ samples of (W, B). Moreover, we use the reference solutions to specify the boundary conditions in (29a)-(29d). It should not come as a surprise that the first-order scheme does not converge to the exact solution u, as we recall that this scheme approximates u^ϵ for some fixed ϵ, that is the solution to the classical PDE (15). Therefore, even if the finite-difference scheme were exact, we would still encounter a fixed error $|u - u^\epsilon|$, which is related to accuracy of the piecewise linear approximation I^ϵ. In Figure 2, along several samples of $(I, [I])$, we consider the Monte-Carlo solution $u(0, x, \omega)$, obtained via $M = 10\,000$ samples with each $N = 10\,000$ time-steps, together with both finite-difference solutions along each samples. Moreover, we plot the mean y_0^M, which represents an approximation for the price y_0.

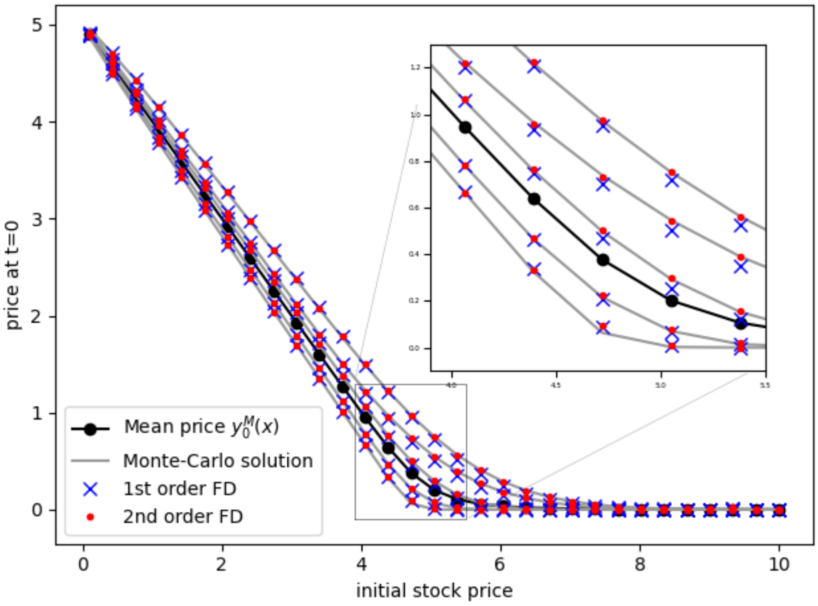


Figure 2: European put option in SABR local stochastic volatility model: Monte-Carlo values of $u^{\mathbf{I}^{(m)}}(0,x)$ vs. finite-difference solutions to the RPDE with first-order (×) and second-order (•) schemes, along several sample paths of $(I,[I])$. The black line corresponds to the mean along all $M = 10'000$ samples. We choose $\Delta t = 1/120$, $\Delta x = (b-a)/90$, $\rho = -0.4$ and $\beta = 0.6$.

A Rough path Appendix

In this section we discuss some technical details from rough path theory that we used in the main results of this paper. For any path $Y : [0,T] \to V$, recall the increment notations

$$(\delta Y)_{s,t} = Y_{s,t} := Y_t - Y_s, \quad s \le t.$$

A.1 Proofs of Section 2

Proof of Lemma 2.8 Consider the map

$$\mathcal{T} : \mathscr{C}_g^\alpha([0,T],V) \oplus C_0^{2\alpha}([0,T],\mathrm{Sym}(V\otimes V)) \longrightarrow \mathscr{C}^\alpha([0,T],V),$$

such that $\mathcal{T}((Y,\mathbb{Y}^g),\mathbb{H}) := (Y, \mathbb{Y}^g - \frac{1}{2}\delta\mathbb{H})$. It is straightforward to check that

$$\|\mathbb{Y}^g - \frac{1}{2}\delta\mathbb{H}\|_{2\alpha} = \sup_{0\le s<t\le T} \frac{|\mathbb{Y}^g_{s,t} - \frac{1}{2}\delta(\mathbb{H})_{s,t}|}{|t-s|^{2\alpha}} < \infty.$$

Moreover, for any $s \le u \le t$, we have

$$\begin{aligned}\mathbb{Y}_{s,t} := \mathbb{Y}^g_{s,t} - \frac{1}{2}(\delta\mathbb{H})_{s,t} &= \mathbb{Y}^g_{s,u} + \mathbb{Y}^g_{u,t} + Y_{s,u}\otimes Y_{u,t} - \frac{1}{2}\delta\mathbb{H}_{s,u} - \frac{1}{2}\delta\mathbb{H}_{u,t} \\ &= \mathbb{Y}_{s,u} + \mathbb{Y}_{u,t} + Y_{s,u}\otimes Y_{u,t},\end{aligned}$$

where we used the additivity of $\delta\mathbb{H}$ and Chen's relation (5). Thus it follows that $\mathcal{T}((Y,\mathbb{Y}^g),\mathbb{H}) \in \mathscr{C}^\alpha$. Now to see that $\mathcal{T}$ is injective, consider two elements $((Y,\mathbb{Y}^g),\mathbb{H})\,,((Y',\mathbb{Y}'^g),\mathbb{H}')$ and assume

$$\mathcal{T}\left((Y,\mathbb{Y}^g),\mathbb{H}\right) = T\left((Y',\mathbb{Y}'^g),\mathbb{H}'\right).$$

By definition it follows that $Y = Y'$, and using the geometricity (6), it follows that two rough path lifts of Y, $\mathbb{Y}^g$ and $\mathbb{Y}'^g$, can only differ by some $\delta F : \Delta_{[0,T]} \to V$, where δF has trivial symmetric part. Therefore, write $\mathbb{Y}'^g = \mathbb{Y}^g + \delta F$, we find

$$-\frac{1}{2}\delta\mathbb{H} = -\frac{1}{2}\delta(\mathbb{H}' - 2F)$$

taking the antisymmetric part, it follows that F is constant, hence $\mathbb{Y}^g = \mathbb{Y}'^g$. Similarly, from $\delta\mathbb{H} = \delta\mathbb{H}'$, it follows that $\mathbb{H} = \mathbb{H}'$.
On the other hand, let $\mathbf{Y} \in \mathscr{C}^\alpha$, we want to show that there exists a pair $(\mathbf{Y}^g, \mathbb{H}) \in \mathscr{C}_g^\alpha([0,T], V) \oplus C_0^{2\alpha}([0,T], \mathrm{Sym}(V \otimes V))$, such that $T(\mathbf{Y}^g, \mathbb{H}) = \mathbf{Y}$. But we know that

$$[\mathbf{Y}]_t := Y_{0,t} \otimes Y_{0,t} - 2\mathrm{Sym}(\mathbb{Y}_{0,t}) \in C_0^{2\alpha}([0,T], \mathrm{Sym}(V \otimes V)).$$

Define $\mathbb{Y}^g := \mathbb{Y} + \frac{1}{2}\delta[\mathbf{Y}]$, by the same argument as above $(Y, \mathbb{Y}^g)$ defines a rough path, and since

$$\mathrm{Sym}(\mathbb{Y}^g_{s,t}) = \mathrm{Sym}(\mathbb{Y}_{s,t}) + \frac{1}{2}Y_{s,t} \otimes Y_{s,t} - \mathrm{Sym}(\mathbb{Y}_{s,t}) = \frac{1}{2}(Y_{s,t} \otimes Y_{s,t}),$$

it is weakly geometric, and therefore the claim follows. □

Proof of Proposition 2.11 From the additivity of the Itô-integral, it is straightforward to check *Chen's relation*, see (5), for both $\boldsymbol{M}^{\text{Itô}}$ and $\boldsymbol{M}^{\text{Strat}}$. Moreover, applying Itô's formula to $M \otimes M$, one can notice that $\mathrm{Sym}\left(\mathbb{M}^{\text{Strat}}_{s,t}\right) = \frac{1}{2}M_{s,t} \otimes M_{s,t}$.
Next we consider a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$, given by

$$\tau_n := \inf\left\{t \geq 0 : \|\sigma\|_{\infty;[0,t]} \geq n\right\}.$$

Since M is continuous, and $\sigma \in L^\infty$ almost surely, we have $\tau_n \to \infty$ as $n \to \infty$. Define the localization $M^n_\cdot := M_{\tau_n \wedge \cdot}$ and similarly $\mathbb{M}^{\text{Itô},n}$ and $\mathbb{M}^{\text{Strat},n}$. Applying the Burkholder-Davis-Gundy (BDG) inequalities, see for instance [21, Theorem 18.7], for all $q \geq 2$ we have

$$E\left[|M^n_{s,t}|^q\right] \lesssim E\left[\left(\int_{s\wedge\tau_n}^{t\wedge\tau_n} |\sigma_u|^2 du\right)^{q/2}\right] \lesssim n^q |t-s|^{q/2}. \tag{34}$$

From a classical Kolmogorov continuity criterion, see [21, Theorem 4.23], it readily follows that M^n has a Hölder continuous modification for every exponent $\alpha < 1/2$, which for all $q \geq 2$ satisfies

$$E\left[\left(\sup_{0\leq s\leq t\leq T} \frac{|M^n_t - M^n_s|}{|t-s|^\alpha}\right)^q\right] < \infty. \tag{35}$$

Next, applying again BDG, we have

$$E\left[|\mathbb{M}^{\text{Itô},n}_{s,t}|^{q/2}\right] \lesssim E\left[\left(\int_{s\wedge\tau_n}^{t\wedge\tau_n} (M_{s,u})^2 |\sigma_u|^2 du\right)^{q/4}\right] \leq n^{q/2} E\left[\left(\int_{s\wedge\tau_n}^{t\wedge\tau_n} (M_{s,u})^2 du\right)^{q/4}\right].$$

But using (35), for all $\alpha < 1/2$ we have

$$E\left[\left(\int_{s\wedge\tau_n}^{t\wedge\tau_n} (M_{s,u})^2 du\right)^{q/4}\right] \leq \frac{1}{2\alpha+1}|t-s|^{q/4(2\alpha+1)} E\left[\left(\sup_{0\leq s\leq t\leq T} \frac{|M^n_t - M^n_s|}{|t-s|^\alpha}\right)^{q/2}\right].$$

Since $\alpha < 1/2$, we can conclude that

$$E\left[|\mathbb{M}^{\text{Itô},n}_{s,t}|^{q/2}\right] \leq C_2 |t-s|^{q\alpha}. \tag{36}$$

Finally, using the Itô-Stratonovich correction, we find

$$E\left[|\mathbb{M}^{\text{Strat},n}_{s,t}|^{q/2}\right] \lesssim C_2|t-s|^{q\alpha} + E\left[\left(\int_{s\wedge\tau_n}^{t\wedge\tau_n} |\sigma_u|^2 du\right)^{q/2}\right] \leq C_3|t-s|^{q\alpha}. \tag{37}$$

Combining (34),(36) and (37), we have for all $\alpha < 1/2$

$$\|M^n_{s,t}\|_{L^q} \leq C|t-s|^{\alpha}, \quad \|\mathbb{M}^{\text{Strat},n}_{s,t}\|_{L^{q/2}} \leq C|t-s|^{2\alpha}, \quad \|\mathbb{M}^{\text{Itô},n}_{s,t}\|_{L^{q/2}} \leq C|t-s|^{2\alpha},$$

for all $q \geq 2$ and some constant C. Applying the Kolmogorov criterion for rough paths, see [13, Theorem 3.1], it follows that for all $\alpha \in (1/3, 1/2)$, there exist random variables $K^n_\alpha \in L^q$, $\mathbb{K}^{\text{Itô},n}_\alpha \in L^{q/2}$ and $\mathbb{K}^{\text{Strat},n}_\alpha \in L^{q/2}$ for all $q \geq 2$, such that for all $s,t \in [0,T]$ we have

$$\frac{|M^n_t - M^n_s|}{|t-s|^{\alpha}} \leq K^n_\alpha, \quad \frac{|\mathbb{M}^{\text{Itô},n}_{s,t}|}{|t-s|^{2\alpha}} \leq \mathbb{K}^{\text{Itô},n}_\alpha, \quad \frac{|\mathbb{M}^{\text{Strat},n}_{s,t}|}{|t-s|^{2\alpha}} \leq \mathbb{K}^{\text{Strat},n}_\alpha.$$

Finally, for any $\epsilon > 0$, we can use the localization to conclude that for all $\alpha \in (1/3, 1/2)$, and R, n large enough, we have

$$P\left[\sup_{0\leq s\leq t\leq T} \frac{|M_t - M_s|}{|t-s|^{\alpha}} > R\right] \leq \frac{1}{R} E\left[K^n_\alpha\right] + P[\tau_n > T] \leq \epsilon.$$

In the same way we can show that

$$P\left[\sup_{0\leq s\leq t\leq T} \frac{|\mathbb{M}^{\text{Itô}}_{s,t}|}{|t-s|^{2\alpha}} > R\right] \leq \epsilon \text{ and } P\left[\sup_{0\leq s\leq t\leq T} \frac{|\mathbb{M}^{\text{Strat}}_{s,t}|}{|t-s|^{2\alpha}} > R\right] \leq \epsilon.$$

Therefore, by definition of the spaces $\mathscr{C}^{\alpha}$ and $\mathscr{C}^{\alpha}_g$, the claim follows. □

A.2 Rough paths with Lipschitz brackets

In this section we discuss some details about rough paths with Lipschitz brackets, which were introduced in Section 2, Definition 2.9. For $V = \mathbb{R}^d$, the correspondence from Lemma 2.8 for the space $\mathscr{C}^{\alpha,1+}$ reads

$$\mathscr{C}^{\alpha,1+}([0,T],\mathbb{R}^d) \longleftrightarrow \mathscr{C}^{\alpha}_g([0,T],\mathbb{R}^d) \oplus \mathrm{Lip}_0([0,T],\mathbb{S}^d_+),$$

where Lip_0 denotes the space of Lipschitz continuous paths starting from 0. The right hand-side can be seen as a closed subspace of $C^{\alpha} \oplus C^{2\alpha}_2 \oplus \mathrm{Lip}_0$, and is thus complete with respect to the norm

$$\|\mathbf{Y}\|_{\alpha,1+} := \|Y\|_{\alpha} + \|\mathbb{Y}^g\|_{2\alpha} + \|[\mathbf{Y}]\|_{\mathrm{Lip}}.$$

Moreover, we may identify $\mathrm{Lip}_0([0,T],\mathbb{S}^d_+)$ with $L^{\infty}_0([0,T],\mathbb{S}^d_+)$, with the Banach isometry $\|[\mathbf{Y}]\|_{\mathrm{Lip}} = \|\mathbf{V}^{\mathbf{Y}}\|_{\infty}$. The following lemma sums up the bijective relations and equivalent norms on the space of rough paths with non-decreasing Lipschitz-brackets.

Lemma A.1 *Let $\alpha \in (1/3, 1/2)$ and $\mathbf{Y} \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R}^d)$. Then we have bijections $\mathscr{C}^{\alpha,1+} \longleftrightarrow \mathscr{C}^{\alpha}_g \oplus \mathrm{Lip}_0 \longleftrightarrow \mathscr{C}^{\alpha}_g \oplus L^{\infty}_0$, with*

$$\mathbf{Y} = (Y,\mathbb{Y}) = (Y, \overbrace{\mathrm{Anti}(\mathbb{Y}) + (\delta Y)^{\otimes 2}/2}^{=\mathbb{Y}^g} - \frac{1}{2}(\overbrace{(\delta Y)^{\otimes 2} + 2\mathrm{Sym}(\mathbb{Y})}^{=\delta[\mathbf{Y}]})) = (Y, \mathbb{Y}^g - \frac{1}{2}\delta\int \mathbf{V}^{\mathbf{Y}} dt).$$

Moreover, we have

$$\|\mathbf{Y}\|_{\alpha,1+} := \|Y\|_{\alpha} + \|\mathbb{Y}^g\|_{2\alpha} + \|\delta[\mathbf{Y}]\|_{\mathrm{Lip}} = \|Y\|_{\alpha} + \|\mathbb{Y}^g\|_{2\alpha} + \|\delta\mathbf{V}^{\mathbf{Y}}\|_{\infty},$$

and the embedding $(\mathscr{C}^{\alpha,1+}, \|\cdot\|_{\alpha,1+}) \hookrightarrow (\mathscr{C}^{\alpha}, \|\cdot\|_{\alpha})$ is continuous.

Proof The bijection $\mathscr{C}^{\alpha,1+} \longleftrightarrow \mathscr{C}_g^\alpha \oplus \mathrm{Lip}_0$ follows from Lemma 2.8, where we replace $C_0^{2\alpha}$ by Lip_0, which follows directly by definition of the space $\mathscr{C}^{\alpha,1+}$. The second bijection, $\mathscr{C}_g^\alpha \oplus \mathrm{Lip}_0 \longleftrightarrow \mathscr{C}_g^\alpha \oplus L_0^\infty$ follows from the fact that $L^\infty \cong \mathrm{Lip}$. This and the fact that $\|[\mathbf{Y}]\|_{\mathrm{Lip}} = \|\mathbf{V}^{\mathbf{Y}}\|_\infty$ can for instance be found in [15, Proposition 1.37]. Finally, for all $\mathbf{Y} \in \mathscr{C}^{\alpha,1+}$, we have

$$\begin{aligned}\|\mathbf{Y}\|_\alpha := \|Y\|_\alpha + \|\mathbb{Y}\|_{2\alpha} &= \|Y\|_\alpha + \|\mathbb{Y}^g - \frac{1}{2}\delta[\mathbf{Y}]\|_{2\alpha} \\ &\leq \|Y\|_\alpha + \|\mathbb{Y}^g\|_{2\alpha} + C\|\delta[\mathbf{Y}]\|_{\mathrm{Lip}} \\ &\leq C\|\mathbf{Y}\|_{\alpha,1+},\end{aligned}$$

where the constant C changed from the second to the last line. □

As discussed in Remark 2.12, the motivation to study the space $\mathscr{C}^{\alpha,1+}$ comes from the fact that the Itô rough path lifts of local martingales, see Proposition 2.11, constitute examples of (random) rough paths with non-decreasing Lipschitz brackets. This can also be deduced from the following general result.

Lemma A.2 *Let $\alpha \in (1/3, 1/2)$ and $\mathbf{Y}^g = (Y, \mathbb{Y}^g) \in \mathscr{C}_g^\alpha([0,T], \mathbb{R}^d)$. For any $\mathbf{v} \in L^\infty([0,T], \mathbb{R}^{d\times k})$, set $\mathbf{V} := \mathbf{v}\mathbf{v}^T$ and*

$$\mathbf{Y}_t := \left(Y_t, \mathbb{Y}_{0,t}^g - \frac{1}{2}\int_0^t \mathbf{V}_s ds\right).$$

Then $\mathbf{Y} \in \mathscr{C}^{\alpha,1+}([0,T], \mathbb{R}^d)$ and $\mathbf{V}^{\mathbf{Y}} = \mathbf{V}$.

Proof Defining $\mathbb{H}_{s,t} := -\frac{1}{2}\int_s^t \mathbf{V}_u du$, it follows from Lemma 2.8 that $\mathbf{Y} \in \mathscr{C}^\alpha$. Now by Definition 2.7, we have

$$\begin{aligned}[\mathbf{Y}]_t := Y_{0,t} \otimes Y_{0,t} - 2\mathrm{Sym}(\mathbb{Y}_{0,t}) &= Y_{0,t} \otimes Y_{0,t} - 2\mathrm{Sym}(\mathbb{Y}_{0,t}^g) + \int_0^t \mathbf{V}_s ds \\ &= \int_0^t \mathbf{V}_s ds,\end{aligned}$$

where we used that $\mathbb{H}_{0,t}$ is symmetric, and $\mathbf{V} := \mathbf{v}\mathbf{v}^T \in \mathbb{S}_+^d$, and the fact that $\mathbf{Y}^g$ is a geometric rough path. But since $\mathbf{V} \in L^\infty([0,T], \mathbb{S}_+^d)$, it readily follows that $\int_0^t \mathbf{V}_s ds$ is Lipschitz with

$$\mathbf{V}_t^{\mathbf{Y}} := \frac{d[\mathbf{Y}]_t}{dt} = \mathbf{V}_t \in \mathbb{S}_+^d,$$

which finishes the proof. □

Remark A.3 *In the case of scalar rough path spaces, that is $d = 1$, it is not hard to see that every α-Hölder continuous path $Y : [0,T] \to \mathbb{R}$ has a trivial geometric lift given by $\mathbb{Y}_{s,t} := \frac{1}{2}(Y_{s,t})^2$. Thus, one can simply identify $\mathscr{C}_g^\alpha([0,T], \mathbb{R})$ with $C^\alpha([0,T], \mathbb{R})$, and in view of the one-to-one correspondences above, we have $\mathscr{C}^{\alpha,1+} \longleftrightarrow C^\alpha([0,T], \mathbb{R}) \oplus \mathrm{Lip}_0([0,T], \mathbb{R}_+) \longleftrightarrow C^\alpha([0,T], \mathbb{R}) \oplus L_0^\infty([0,T], \mathbb{R}_+)$, and the norms can be reduced to*

$$\|\mathbf{Y}\|_{\alpha,1+} := \|Y\|_\alpha + \|[\mathbf{Y}]\|_{\mathrm{Lip}} = \|Y\|_\alpha + \|\mathbf{V}^{\mathbf{Y}}\|_\infty.$$

A.3 Proof of Theorem 3.2

In this section we prove a multivariate generalization of Theorem 3.2. Let $\mathbf{Y} \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R})$, and recall the notation in Definition 2.9

$$\left([\mathbf{Y}]_t, \frac{d[\mathbf{Y}]_t}{dt}, \sqrt{\frac{d[\mathbf{Y}]_t}{dt}}\right) =: \left(\int_0^t \mathbf{V}_s^{\mathbf{Y}} ds, \mathbf{V}_t^{\mathbf{Y}}, \mathbf{v}_t^{\mathbf{Y}}\right).$$

Let $f : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $g : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times k}$ in C_b^3, and let B be a k-dimensional Brownian motion. We consider the following d-dimensional version of (11)

$$X_t^{t,x,\boldsymbol{Y}} = x, \quad dX_s^{t,x,\boldsymbol{Y}} = f(s, X_s^{t,x,\boldsymbol{Y}})d\mathbf{Y}_s + g(s, X_s^{t,x,\boldsymbol{Y}})\mathbf{v}_s^{\mathbf{Y}} dB_s, \quad t < s \leq T. \tag{38}$$

Now consider the k-dimensional local martingale

$$M_t^{\mathbf{Y}} = \int_0^t \mathbf{v}_s^{\mathbf{Y}} dB_s. \tag{39}$$

The general version of the joint-lift of $(M^{\mathbf{Y}}(\omega), Y)$ from Definition 3.1 is given by

$$Z^{\mathbf{Y}} := (M^{\mathbf{Y}}(\omega), Y), \quad \mathbb{Z}_{s,t}^{\mathbf{Y}} := \begin{pmatrix} \int_s^t M_{s,r}^{\mathbf{Y}} \otimes dM_r^{\mathbf{Y}} & \int_s^t M_{s,r}^{\mathbf{Y}} \otimes dY_r \\ \int_s^t Y_{s,r} \otimes dM_r^{\mathbf{Y}} & \mathbb{Y}_{s,t} \end{pmatrix}, \quad 0 \leq s \leq t \leq T, \tag{40}$$

where the first entry is the (canonical) Itô rough path lift of the local martingale $M^{\mathbf{Y}}$, see Proposition 2.11, $\int_s^t Y_{s,r} \otimes dM_r^{\mathbf{Y}}$ is a well-defined Itô integral, and set $\int_s^t M_{s,r}^{\mathbf{Y}} \otimes dY_r := M_{s,t}^{\mathbf{Y}} \otimes Y_{s,t} - \int_s^t Y_{s,r} \otimes dM_r^{\mathbf{Y}}$, imposing integration by parts. A general version of Theorem 3.2 can be stated as follows.

Theorem A.4 *Let $\alpha \in (1/3, 1/2)$ and $\boldsymbol{Y} \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R})$, and $M^{\mathbf{Y}}$ given in (39). Then $\boldsymbol{Z}^{\mathbf{Y}}(\omega) = (Z^{\mathbf{Y}}(\omega), \mathbb{Z}^{\mathbf{Y}}(\omega))$ defines an α'-Hölder rough path for any $\alpha' \in (1/3, \alpha)$, and for $f, g \in C_b^3$, there exists a unique solution to the rough differential equation*

$$X_t^{t,x,\mathbf{Y}} = x, \quad dX_s^{t,x,\boldsymbol{Y}}(\omega) = (g,f)(s, X_s^{t,x,\boldsymbol{Y}}(\omega))d\boldsymbol{Z}_s^{\mathbf{Y}}(\omega), \quad t < s \leq T, \tag{41}$$

for almost every ω, and $X^{t,x,\mathbf{Y}}$ defines a time-inhomogeneous Markov process. Moreover, let W be a one-dimensional Brownian motion independent of B, and assume v is $(\mathcal{F}_t^W)$-adapted such that Assumption 1 holds true. For $I_t = \int_0^t v_s dW_s$, if we choose $\boldsymbol{Y} = \boldsymbol{I}(\omega)$ the Itô rough path lift of I, then it holds that

$$\mathrm{Law}\left(X^{t,x} \middle| \mathcal{F}_T^W \vee \mathcal{F}_t^B\right) = \mathrm{Law}\left(X^{t,x,\mathbf{I}} \middle| \mathcal{F}_T^W \vee \mathcal{F}_t^B\right),$$

where $X^{t,x}$ is the unique strong solution to

$$X_t^{t,x} = x, \quad dX_s^{t,x} = f(s, X_s^{t,x})v_s dW_s + g(s, X_s^{t,x})v_s dB_s, \quad t < s \leq T. \tag{42}$$

Finally, if $v \geq 0$, then we even have indistinguishability $X^{t,x}(\omega) = X^{t,x,\mathbf{I}(\omega)}$ for a.e. $\omega \in \Omega$.

We split the proof of Theorem A.4 into the following three lemmas.

Lemma A.5 *Consider a k-dimensional local martingale of the form*

$$M_t := \int_0^t \sigma_s dB_s,$$

where σ is progressively measurable and in $L^\infty([0,T])$ almost surely. Then

1 *The lift $\boldsymbol{Z}^{\mathbf{Y}}(\omega)$, similarly constructed as in* (40) *with M, defines an α'-Hölder rough path for any $\alpha' \in (1/3, \alpha)$.*

2 *Consider a pathwise controlled rough path $(A(\omega), A'(\omega)) \in \mathscr{D}^{2\alpha'}_{\mathbf{Z}^{\mathbf{Y}}(w)}$, see [13, Chapter 4] for the definition of the space of controlled rough paths $\mathscr{D}$. Then, the rough integral*

$$\mathcal{J}(T, \omega, \boldsymbol{Y}) := \int_0^T A_s(\omega) d\mathbf{Z}^{\mathbf{Y}}_s(\omega) = \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} A_u(\omega) Z_{u,v}(\omega) + A'_u(\omega)\mathbb{Z}_{u,v}(\omega)$$

exists on a set $\Omega_0 \subseteq \Omega$ with full measure.

Proof First we show that $\boldsymbol{Z}^{\mathbf{Y}}(\omega)$ indeed defines a (random) α'-Hölder rough path for any $\alpha' \in (1/3, \alpha)$. Notice that the proof uses similar techniques as in Proposition 2.11. Define the stopping time

$$\tau_n := \inf \left\{ t \geq 0 : \|\sigma\|_{\infty;[0,t]} \geq n \right\}.$$

For any $q \geq 2$, we can apply the Burkholder-Davis-Gundy inequality, see [21, Theorem 18.7], to find

$$E\left[\left|\int_s^t \sigma^{\tau_n}_u dB_u\right|^q\right]^{1/q} \leq C_q n(t-s)^{1/2} \text{ and } |Y_{s,t}| \leq C|t-s|^{\alpha},$$

for some constants C_q and C. Moreover, we clearly have $\|\mathbb{Y}_{s,t}\|_{L^{q/2}} \leq C_1|t-s|^{2\alpha}$ for all $q \geq 2$. From Proposition 2.11, we already know that

$$\left\|\int_s^t M^{\tau_n}_{s,r} \otimes dM^{\tau_n}_r\right\|_{L^{q/2}} \leq C_2|t-s|^{2\alpha},$$

Similarly, we find

$$\left\|\int_s^t Y_{s,u} \otimes dM^{\tau_n}_u\right\|_{L^{q/2}} \lesssim \|Y\|_{\alpha}(t-s)^{\alpha+1/2}$$

and

$$\left\|\int_s^t M^{\tau_n}_{s,u} \otimes dY_u\right\|_{L^{q/2}} \leq \|M^{\tau_n}_{s,t} \otimes Y_{s,t}\|_{L^{q/2}} + \left\|\int_s^t Y_{s,u} \otimes dM^{\tau_n}_u\right\|_{L^{q/2}} \lesssim (t-s)^{2\alpha}.$$

Applying Kolomogorov for rough paths, that is [13, Theorem 3.1], it follows that $\boldsymbol{Z}^{\mathbf{Y}}_{\tau_n \wedge \cdot}(\omega)$ indeed defines a α'-Hölder rough path for any $\alpha' \in (1/3, \alpha)$. Using exactly the same localization argument as in Proposition 2.11, we can conlude that $\mathbf{Z}^{\mathbf{Y}}$ defines a α'-Hölder rough path. By [13, Theorem 4.10], the rough integral $\mathcal{J}(T, \omega, \boldsymbol{Y})$ is therefore well-defined on a set with full measure. □

Lemma A.6 *Let M be the k-dimensional local martingale from Lemma A.5, and let N be another one-dimensional local martingale, such that $[M, N] = 0$. Consider two paths $A : [0, T] \to \mathbb{R}^{d\times(k+1)}$ and $A' : [0, T] \to \mathbb{R}^{d\times(k+1)} \otimes \mathbb{R}^{k+1}$. If (A, A') is an adapted and continuous controlled rough path $(A, A') \in \mathscr{D}^{\alpha'}_{\mathbf{Z}^{\mathbf{Y}}(\omega)}$ for any $\alpha' \in (1/3, \alpha)$, then almost surely*

$$\mathcal{J}(T, \omega, \boldsymbol{Y})|_{\boldsymbol{Y}=\boldsymbol{N}(\omega)} = \left(\int_0^T A_s dZ^{\mathbf{N}}_s\right)(\omega) = \left(\int_0^T A^{k+1}_s dN_s\right)(\omega) + \left(\int_0^T A^{1:k}_s dM_s\right)(\omega) \in \mathbb{R}^d, \tag{43}$$

where A^j denotes the j-th row of A, and $A^{1:k} = [A^1, \ldots, A^k]$, and $\mathcal{J}$ is defined in Lemma A.5. The integrals on the right-hand side are Itô integrals, and $\boldsymbol{N}$ is the Itô rough path lift of N.

Remark A.7 *Let us quickly describe how to understand the two different notions of integration in* (43) *in the sense of dimension. First, since $Z := Z^{\mathbf{N}} = (M, N)$ is a $(k+1)$-dimensional local martingale, the stochastic integration in* (43) *can be understood for each component $j \in \{1, \dots, d\}$ as*

$$\left(\int_0^T A_s dZ_s\right)^j = \sum_{i=1}^{k+1} \int_0^T A_s^{j,i} dZ_s^i = \int_0^T A_s^{j,k+1} dN_s + \int_0^T A_s^{j,1:k} dM_s.$$

On the other hand, by the definition of the rough integral $\mathcal{J}$ in (ii) of Lemma A.5, we encounter the second order term $A'\mathbb{Z}$, where A' is the Gubinelli derivative[8]*. Here A' takes values in $\mathbb{R}^{d\times(k+1)} \otimes \mathbb{R}^{k+1} \cong \mathbb{R}^{d\times(k+1)^2}$, and since $\mathbb{Z} := \mathbb{Z}^{\mathbf{N}} \in \mathbb{R}^{(k+1)\times(k+1)} \cong \mathbb{R}^{(k+1)^2}$, the product $A'\mathbb{Z}$ lies again in $\mathbb{R}^d$, such that the left hand side of* (43) *is also $\mathbb{R}^d$-valued. In the simple case $d = k = 1$, which is of main interested in this paper, we simply have $A : [0, T] \to \mathbb{R}^2$ and $A' : [0, T] \to \mathbb{R}^{2\times 2}$, and the meaning of aboves terms should be clear.*

Proof Set $Z = Z^{\mathbf{N}}$ and $\mathbb{Z} = \mathbb{Z}^{\mathbf{N}}$. Since A is continuous and adapted, the stochastic integral on the right-hand side of (43) is well-defined and it is given as limit in probability

$$\int_0^T A_s dZ_s = (p) \lim_{|\pi|\to 0} \sum_{[u,v]\subset\pi} A_u Z_{u,v}.$$

Moreover, there exists a subsequence of subdivisions π^n, such that the convergence holds almost surely. By definition of the rough integral, we find

$$\mathcal{J}(T, \omega, \boldsymbol{Y})|_{\boldsymbol{Y}=\boldsymbol{N}(\omega)} = \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} A_u(\omega) Z_{u,v}(\omega) + A'_u(\omega)\mathbb{Z}_{u,v}(\omega).$$

Along the subsequence π^n we find

$$\int_0^T A_s d\mathbf{Z}_s^{\mathbf{N}} - \int_0^T A_s dZ_s = \lim_{n\to\infty} \sum_{[u,v]\subset\pi^n} A'_u \mathbb{Z}_{u,v}.$$

Assume for the moment that the processes A', N and M are bounded by some K. Changing the notation to $\pi^n = \{0 = t_0 < t_1 < \dots < t_n = T\}$ and for any $r \in \{1, \dots, d\}$, denote by $A'^{(r)}$ the r-th row of the Gubinelli derivative $A' \in \mathbb{R}^{d\times(k+1)^2}$, see also Remark A.7. Then

$$E\left[\left(\sum_{[u,v]\subset\pi^n} A'^{(r)}_u \cdot \mathbb{Z}_{u,v}\right)^2\right] = \sum_{j=0}^{n-1} E[(A'^{(r)}_{t_j}\mathbb{Z}_{t_j,t_{j+1}})^2] + 2\sum_{i<j} E[(A'^{(r)}_{t_j}\mathbb{Z}_{t_j,t_{j+1}})(A'^{(r)}_{t_i}\mathbb{Z}_{t_i,t_{i+1}})].$$

Now by definition of $\mathbb{Z}$, see (40), and the assumption $[M, N] = 0$, we have

$$\mathbb{Z} = \begin{pmatrix} \int_s^t M_{s,r} \otimes dM_r & \int_s^t M_{s,r} \otimes dN_r \\ \int_s^t N_{s,r} \otimes dM_r & \int_s^t N_{s,r} dN_r \end{pmatrix}.$$

Since A' is adapted, we can apply the tower-poperty to see that for any $t_i < t_{i+1} \le t_j < t_{j+1}$ we have

$$E[(A'^{(r)}_{t_j}\mathbb{Z}_{t_j,t_{j+1}})(A'^{(r)}_{t_i}\mathbb{Z}_{t_i,t_{i+1}})] = E[(A'^{(r)}_{t_i}\mathbb{Z}_{t_i,t_{i+1}})A'^{(r)}_{t_j} E[\mathbb{Z}_{t_j,t_{j+1}}|\mathcal{F}_{t_j}]] = 0,$$

[8] c.f. [13, Definition 4.6].

where we use the martingale property for all the stochastic integral entries of $\mathbb{Z}$, to see that $E[\mathbb{Z}_{t_j,t_{j+1}}|\mathcal{F}_{t_j}] = 0$. Thus we have

$$E\left[\left(\sum_{[u,v]\subset\pi^n} A'^{(r)}_u \mathbb{Z}_{u,v}.\right)^2\right] \le K^2 \sum_{l=1}^{(k+1)^2}\left(\sum_{j=0}^{n-1} E\left[(\mathbb{Z}^{(l)}_{t_j,t_{j+1}})^2\right]\right).$$

Using standard stochastic integral properties, we have for all $i_1 \in \{1,\dots,k\}$

$$\begin{aligned}\sum_{j=0}^{n-1}\left(\int_{t_j}^{t_{j+1}} N_{t_j,u} dM^{i_1}_u\right)^2 &\le \left(\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} N_u dM^{i_1}_u - N_{t_j}(M_{t_{j+1}} - M_{t_j})\right)^2 \\ &\le \left(\int_0^T N_u dM^{i_1}_u - \sum_{j=0}^{n-1} N_{t_j}(M_{t_{j+1}} - M_{t_j})\right)^2 \\ &\xrightarrow{n\to\infty} 0 \quad \text{in } L^1,\end{aligned}$$

since both M and N are bounded. The same holds true when we exchange the positions of M and N, and also if $M = N$. Therefore, under the boundedness assumption, we have

$$E\left[\left(\sum_{[u,v]\subset\pi^n} A'^{(r)}_u \mathbb{Z}_{u,v}.\right)^2\right] \xrightarrow{n\to\infty} 0,$$

and in this case, the stochastic and rough integral coincide almost surely.
Define now $\tau^K = \inf\{t \ge 0 : \|M_t\| \ge K \text{ or } ||A'_t\| \ge K \text{ or } \|N_t\| \ge K\}$. Since all processes are continuous, we clearly have $\tau^K \to \infty$ as $K \to \infty$. Therefore, we find

$$\begin{aligned}\mathbb{P}\left[\left|\sum_j A'^{(r)}_{t_j}\mathbb{Z}_{t_j,t_{j+1}}\right| > \epsilon\right] &\le \mathbb{P}\left[\left|\sum_{j;t_{j+1}<\tau^K} A'^{(r)}_{t_j}\mathbb{Z}_{t_j,t_{j+1}}\right| > \epsilon\right] + \mathbb{P}[\tau^K \le T] \\ &\le \frac{1}{\epsilon^2}\mathbb{E}\left[\left|\sum_{j;t_{j+1}<\tau^K} A'^{(r)}_{t_j}\mathbb{Z}_{t_j,t_{j+1}}\right|^2\right] + \mathbb{P}[\tau^K \le T] \\ &\xrightarrow{n\to\infty} \Delta(K),\end{aligned}$$

where $\Delta(K) \to 0$ as $K \to \infty$. This shows that $\sum_{[u,v]\subset\pi^n} A'_u \mathbb{Z}_{u,v} \to 0$ in probability, hence we can find another subsequence π^{n_k}, such that the convergence holds almost surely. It follows that almost surely

$$\int_0^T A_s d\boldsymbol{Z}^{\mathbf{N}}_s = \int_0^T A_s dZ_s.$$

□

To establish a Markovian structure for the process $X^{t,x,\mathbf{Y}}$, we need the following lemma, which tells us that $X^{t,x,\mathbf{Y}}$ is the limit of time-inhomogeneous Markov processes. This is the key ingredient to establish a Feynman-Kac type of result for the dynamics $X^{t,x,\mathbf{Y}}$ in Section 3.

Lemma A.8 *Let $\alpha \in (1/3, 1/2)$ and $\mathbf{Y} \in \mathscr{C}^{\alpha,1+}([0,T],\mathbb{R})$, and consider $M^{\mathbf{Y}}$ as defined in* (39). *Let $\mathbf{Y}^\epsilon$ be the rough path lift of a piecewise linear approximation*[9] *Y^ϵ of Y, such that $\mathbf{Y}^\epsilon \rightharpoonup \mathbf{Y}$, see*

[9] See [15, Chapter 5.2] for instance.

Proposition 2.5. For $(t,x) \in [0,T] \times \mathbb{R}$ assume that $X^{t,x,\mathbf{Y}}$, resp. $X^{t,x,\mathbf{Y}^\epsilon}$ is the unique solution to (41) *with respect to $\mathbf{Z}^{\mathbf{Y}}$, resp. $\mathbf{Z}^{\mathbf{Y}^\epsilon}$. Then we have*

$$X^{t,x,\mathbf{Y}^\epsilon} \xrightarrow{\epsilon \to 0} X^{t,x,\mathbf{Y}} \text{ ucp.}$$

Moreover, we have $X^{t,x,\mathbf{Y}^\epsilon} = X^{t,x,\epsilon}$ almost surely, where $X^{t,x,\epsilon}$ denotes the unique strong solution to the SDE

$$X_t^{t,x} = x, \quad dX_s^{t,x,\epsilon} = g(s, X_s^{t,x,\epsilon})\mathbf{v}_s^{\mathbf{Y}} dB_s + \left(f_0(s, X_s^{t,x,\epsilon})\mathbf{V}_s^{\mathbf{Y}} + f\left(s, X_s^{t,x,\epsilon}\right) \dot{Y}_s^\epsilon \right) ds.$$

Proof For any $\alpha' < \alpha$, assume for the moment that the rough path norms of $\mathbf{Z}^{\mathbf{Y}}$ and $\mathbf{Z}^{\mathbf{Y}^\epsilon}$ are bounded by some constant K. Then we can apply standard RDE estimates, e.g. [13, Theorem 8.5], to see that

$$\sup_{t \le s \le T} \left| X_s^{t,x,\mathbf{Y}^\epsilon} - X_s^{t,x,\mathbf{Y}} \right| \le C \varrho_{\alpha',2\alpha'}(\mathbf{Z}^{\mathbf{Y}^\epsilon}, \mathbf{Z}^{\mathbf{Y}}), \tag{44}$$

where $C = C(K, \alpha', \alpha, f, g)$ is constant. Now since $M^{\mathbf{Y}^\epsilon} = M^{\mathbf{Y}}$, for any $q \ge 1$ and $t \le s \le s' \le T$ we have

$$E[|Z_{s,s'}^{\mathbf{Y}^\epsilon} - Z_{s,s'}^{\mathbf{Y}}|^q]^{1/q} = |Y_{s,s'}^\epsilon - Y_{s,s'}| \le (s'-s)^{\alpha'} \|\mathbf{Y}^\epsilon - \mathbf{Y}\|_{\alpha',1+}.$$

In order to estimate $E[|\mathbb{Z}_{s,s'}^{\mathbf{Y}^\epsilon} - \mathbb{Z}_{s,s'}^{\mathbf{Y}}|^{q/2}]$, we use similar techniques as in the proof of Lemma A.5, by applying BDG-inequalities

$$E\left[\left| \int_s^{s'} (Y_{s,r}^\epsilon - Y_{s,r}) \otimes dM_r^{\mathbf{Y}} \right|^{q/2} \right]^{2/q} \le C_{q/2}(s'-s)^{2\alpha'} \|V^{\mathbf{Y}}\|_{\infty;[0,T]} \|\mathbf{Y}^\epsilon - \mathbf{Y}\|_{\alpha',1+}.$$

Applying similar estimates for $\left| \int_s^{s'} M_{s,r}^{\mathbf{Y}} dY_r^\epsilon - \int_s^{s'} M_{s,r}^{\mathbf{Y}} dY_r \right|^{q/2}$, we find

$$E\left[\left| \mathbb{Z}_{s,s'}^{\mathbf{Y}^\epsilon} - \mathbb{Z}_{s,s'}^{\mathbf{Y}} \right|^{q/2} \right]^{2/q} \le \tilde{C}_{q/2}(s'-s)^{2\alpha'} \|\mathbf{Y}^\epsilon - \mathbf{Y}\|_{\alpha',1+}(3\|V^{\mathbf{Y}}\|_{\infty;[0,T]} + 1)$$

Applying a Kolmogorov criterion for rough path distances, see [13, Theorem 3.3], it follows that

$$E\left[\sup_{t \le s \le T} \left| X_s^{t,x,\mathbf{Y}^\epsilon} - X_s^{t,x,\mathbf{Y}} \right|^q \right]^{1/q} \lesssim E[\varrho_{\alpha',2\alpha'}(\boldsymbol{Z}^{\mathbf{Y}^\epsilon}, \boldsymbol{Z}^{\mathbf{Y}})^q]^{1/q} \lesssim \|\mathbf{Y}^\epsilon - \mathbf{Y}\|_{\alpha',1+} \xrightarrow{\epsilon \to 0} 0, \quad \forall q \ge 1.$$

Now for the general case, that is where the rough path norms of $\mathbf{Z}^{\mathbf{Y}}$ and $\mathbf{Z}^{\mathbf{Y}^\epsilon}$ might be unbounded, we apply once again a standard localization argument, to conclude that the convergence holds in probability.
Finally, the claim $X^{t,x,\mathbf{Y}^\epsilon} = X^{t,x,\epsilon}$ follows with a standard consistency argument, similar as in the proof of Lemma A.5 but with Riemann-Stieltjes integration instead of stochastic integration. ☐

Proof of Theorem A.4 The existence and uniqueness of a solution to (41) follows from the main well-posedness result for rough differential equations, see [13, Theorem 8.4], where we only need to define the augmented path $\hat{X}_s = (s, X_s)$ and the augmented lift $\hat{\mathbf{Z}}^Y$ by including the missing cross-integrals as Riemann-Stieltjes integrals. The Markov property for the unique solution $X^{t,x,\mathbf{Y}}$ follows directly from the fact that $X^{t,x,\mathbf{Y}}$ is the ucp limit of the time-inhomogeneous Markov process $X^{t,x,\epsilon}$, see Lemma A.8. Moreover, by continuity of the Itô-Lyons map, it follows that $X_s^{t,x,\boldsymbol{I}}$ is the continuous image of the joint rough path $\boldsymbol{Z}^{\boldsymbol{I}}$, and hence measurable with respect to $\sigma(\{\boldsymbol{Z}_u^{\mathbf{I}} : u \le s\}) = \mathcal{F}_s$, and thus adapted. In Remark 2.12 we saw that $[\mathbf{I}] = [I] = \int V_t dt$ almost surely, and therefore we have $\mathbf{v}_t^{\mathbf{I}} := \sqrt{\frac{d[\mathbf{I}]_t}{dt}} = |v_t|$ and

$$Z_t = (M_t^{\mathbf{I}}, I_t) = \left(\int_0^t |v_s| dB_s, \int_0^t v_s dW_s \right).$$

Define the controlled rough path $(A, A') \in \mathscr{D}_Z^{2\alpha}$ as

$$A = (g, f)(s, X_s^{t,x,\boldsymbol{I}}), \quad A' = (D(g, f)(s, X_s^{t,x,\boldsymbol{I}})) \cdot (g, f)(s, X_s^{t,x,\boldsymbol{I}}).$$

It follows that (A, A') is adapted and continuous, and therefore we have almost surely

$$X_s^{t,x,\boldsymbol{I}(\omega)} = x + \int_t^s A_u(\omega) d\boldsymbol{Z}_u^{\mathbf{Y}}(\omega) \Big|_{\boldsymbol{Y}=\boldsymbol{I}(\omega)} = x + \left(\int_t^s A_u dZ_u \right)(\omega),$$

where we used Lemma A.5 and Lemma A.6, and the fact that $[M^{\mathbf{I}}, I] = 0$ for the last equality. This shows that $X^{t,x,\mathbf{I}}$ is indistinguishable from the unique solution to the SDE

$$d\hat{X}_t^{t,x} = x, \quad d\hat{X}_s^{t,x} = f(s, \hat{X}_s^{t,x}) v_s dW_s + g(s, \hat{X}_s^{t,x}) |v_s| dB_s, \quad t < s \leq T.$$

But by independence of W and B, one can notice that the conditional distributions of $\int_t^\cdot v_s dB_s$ and $\int_t^\cdot |v_s| dB_s$, given $\mathcal{F}_T^W \vee \mathcal{F}_t^B$, are Gaussian with covariance $\int_t^\cdot v_s^2 ds$, and thus coincide. In particular, it holds that

$$\text{Law}\left(\int_t^\cdot v_s dW_s, \int_t^\cdot |v_s| dB_s \Big| \mathcal{F}_T^W \vee \mathcal{F}_t^B \right) = \text{Law}\left(\int_t^\cdot v_s dW_s, \int_t^\cdot v_s dB_s \Big| \mathcal{F}_T^W \vee \mathcal{F}_t^B \right).$$

By strong uniqueness of the SDE (42), see [26, Chapter 3 Theorem 7], we can conclude that

$$\text{Law}(X^{t,x,\mathbf{I}} | \mathcal{F}_T^W \vee \mathcal{F}_t^B) = \text{Law}(\hat{X}^{t,x} | \mathcal{F}_T^W \vee \mathcal{F}_t^B) = \text{Law}(X^{t,x} | \mathcal{F}_T^W \vee \mathcal{F}_t^B).$$

Moreover, if we additionally assume that $v \geq 0$, then clearly $X^{t,x,\mathbf{I}} = \hat{X}^{t,x} = X^{t,x}$ almost surely. □

Remark A.9 *Finally, notice that for the geometric rough path $\mathbf{Y}^g$, we have $\mathbb{Y}^g = \mathbb{Y} + \frac{1}{2}\delta[\mathbf{Y}]$, and hence*

$$\mathbb{Z}_{s,t} := \begin{pmatrix} \int_s^t M_{s,r} dM_r & \int_s^t M_{s,r} dY_r \\ \int_s^t Y_{s,r} dM_r & \mathbb{Y}_{s,t} \end{pmatrix} = \begin{pmatrix} \int_s^t M_{s,r} dM_r & \int_s^t M_{s,r} dY_r \\ \int_s^t Y_{s,r} dM_r & \mathbb{Y}_{s,t}^g - \frac{1}{2}\delta[\mathbf{Y}]_{s,t} \end{pmatrix}$$
$$= \mathbb{Z}_{s,t}^g + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}\delta[\mathbf{Y}]_{s,t} \end{pmatrix}.$$

Now by standard properties of rough integration, see [13, Chapter 4 Example 4.14], it follows that

$$dX_s^{t,x,\mathbf{Y}} = (g, f)(s, X_s^{t,x,\mathbf{Y}}) d\mathbf{Z}_s^{\mathbf{Y}} = (g, f)(s, X_s^{t,x,\mathbf{Y}}) d\mathbf{Z}_s^{\mathbf{Y}^g} + f_0(s, X_s^{t,x,\mathbf{Y}}) d[\mathbf{Y}]_s,$$

where $f_0(s, x) := -\frac{1}{2}\nabla_x f(s, x)^T f(s, x)$.

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