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Abstract

We consider a Trotter-type-product formula for approximating the solution of a linear abstract Cauchy problem (given by a strongly continuous semigroup), where the underlying Banach space is a product of two spaces. In contrast to the classical Trotter-product formula, the approximation is given by freezing subsequently the components of each subspace. After deriving necessary stability estimates for the approximation, which immediately provide convergence in the natural strong topology, we investigate convergence in the operator norm. The main result shows that an almost optimal convergence rate can be established if the dominant operator generates a holomorphic semigroup and the off-diagonal coupling operators are bounded.

1 Introduction

The classical and rich theory of strongly continuous semigroups $\{\mathbf{T}(t)\}_{t\geq 0}$ provides a tool for solving linear abstract Cauchy problems $\dot{u}(t) = -\mathbf{C}u(t)$ for $t \geq 0$, $u(0) = x$ in a Banach space X [Paz83, Kat95, EnN00]: the linear operator $-\mathbf{C} : \mathrm{dom}(\mathbf{C}) \subset X \to X$ is a generator of a strongly continuous semigroup (denoted by $\mathrm{e}^{-t\mathbf{C}}$) if and only if for every $x \in \mathrm{dom}(\mathbf{C})$ there exists a unique solution $u(\cdot, x) \in \mathrm{dom}(\mathbf{C})$ of the abstract Cauchy problem, which is given by $\mathrm{e}^{-t\mathbf{C}}x = u(t, x)$ (we recall basics from semigroup theory in Section 2.1). Often the operator $\mathbf{C}$ is given by a sum $\mathbf{C} = \mathbf{A}+\mathbf{B}$. If $-\mathbf{A}$ is a generator, classical perturbation results (see e.g. [EnN00, Chapter III]) provide information when $-\mathbf{C}$ is again a generator. Moreover, assuming that $-\mathbf{A}$ and $-\mathbf{B}$ are generators of semigroups $\mathrm{e}^{-t\mathbf{A}}$ and $\mathrm{e}^{-t\mathbf{B}}$, respectively, then the solution operator $\mathrm{e}^{-t\mathbf{C}}$ can be approximated by the so-called Trotter-product formula

$$\lim_{n\to\infty} \left(\mathrm{e}^{-\frac{t}{n}\mathbf{A}}\mathrm{e}^{-\frac{t}{n}\mathbf{B}}\right)^n \to \mathrm{e}^{-t\mathbf{C}},$$

provided a stability condition holds for the product (see e.g. [EnN00, Corollary III.5.8]). The convergence is an immediate consequence of the more general Chernoff-product formula [Che74] and is meant in the natural topology in the theory of semigroups, the strong topology. Apart from its theoretical value, the Trotter-product formula is important in applications as it provides a way to approximate the (in general) complicated solution $\mathrm{e}^{-t\mathbf{C}}$ by subsequently applying the simpler parts $\mathrm{e}^{-t/n\mathbf{A}}$ and $\mathrm{e}^{-t/n\mathbf{B}}$ together n-times, and thus defining a numerical split-step method.

In that paper, we are interested in the case where the underlying Banach space X is given by a product $X = X_1 \times X_2$. The operator of interest consists of two parts and is given in matrix form

$$-\mathcal{C} = -\mathcal{A} + \mathcal{B} = \begin{pmatrix} -\mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{B}_2 & -\mathbf{A}_2 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \mathbf{A}_1 & \cdot \\ \cdot & \mathbf{A}_2 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \cdot & \mathbf{B}_1 \\ \mathbf{B}_2 & \cdot \end{pmatrix},$$

where $-\mathbf{A}_j : \mathrm{dom}(\mathbf{A}_j) \subset X_j \to X_j$ are generators of a semigroup, and $\mathbf{B}_1 : X_2 \to \mathbf{X}_1$, $\mathbf{B}_2 : X_1 \to X_2$ are linear operators that describe the coupling. These matrix operators occur frequently

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in physical problems: e.g. in continuum mechanics, where X_1 contains the states in a bulk and X_2 contains the states in a reservoir, the operators $\mathbf{A}_j$ describe their dynamics and $\mathbf{B}_j$ the coupling; or in quantum mechanics, where X is the state space of two combined quantum systems, each individual time-dependent system is described by $\mathbf{A}_j$, the interaction by $\mathbf{B}_1$ and $\mathbf{B}_2$.

We assume that both coupling operators $\mathbf{B}_j$ are bounded (although we discuss the unbounded case in Section 5). The operator $-\mathcal{A} : \mathrm{dom}(\mathcal{A}) = \mathrm{dom}(\mathbf{A}_1) \times \mathrm{dom}(\mathbf{A}_2) \to X$ is a generator of a semigroup on X, and its semigroup is given by the diagonal matrix of the individual semigroups $\mathrm{e}^{-t\mathbf{A}_j}$. Moreover, $\mathcal{B}$ is a bounded perturbation, and hence, $-\mathcal{C} : \mathrm{dom}(\mathcal{C}) = \mathrm{dom}(\mathcal{A}) \to X$ is a generator, too. Its semigroup $\mathrm{e}^{-t\mathcal{C}}$ defines the solution $u = u(t) = (u_1(t), u_2(t))^{\mathrm{T}}$ of the abstract Cauchy problem $\dot{u}(t) = -\mathcal{C}u(t)$, $u(0) = (x, y)^{\mathrm{T}}$ on X. As an immediate consequence one can show that (under reasonable assumptions) the solution operator of the combined system $\mathrm{e}^{-t\mathcal{C}}$ can be approximated by the Trotter-product formula $\left(\mathrm{e}^{-\frac{t}{n}\mathcal{A}}\mathrm{e}^{\frac{t}{n}\mathcal{B}}\right)^n$. However, from the practical standpoint the Trotter-product formula is often not useful because the semigroup of $\mathrm{e}^{t\mathcal{B}}$ cannot be expressed explicitly by the individual operators $\mathbf{B}_j$. Moreover, in each step an expensive evolution on the whole space $X = X_1 \times X_2$ has to be calculated.

This problem can be solved by replacing $\mathrm{e}^{-\frac{t}{n}\mathcal{A}}\mathrm{e}^{\frac{t}{n}\mathcal{B}}$ by an split-step approximation that respects the underlying product structure. The idea is to define a family of bounded operators $\mathcal{T} = \mathcal{T}(t)$ on X, consisting of two parts $\mathcal{T}(t) = \mathcal{T}_2(t)\mathcal{T}_1(t)$, where each bounded operator trajectory $\mathcal{T}_j = \mathcal{T}_j(t)$ defines the solution of the abstract Cauchy problem $\dot{u}(t) = -\mathcal{C}u(t)$, $u(0) = (x, y)^{\mathrm{T}}$, with one freezed (constant in time) component. These subsequent abstract Cauchy problems become inhomogeneous and can be solved explicitly (see Section 2.2). The split-step approximation operator is then given by

$$\mathcal{T}(t) = \begin{pmatrix} \mathrm{e}^{-t\mathbf{A}_1} & \int_0^t \mathrm{e}^{-s\mathbf{A}_1}\mathbf{B}_1 \mathrm{d}s \\ \int_0^t \mathrm{e}^{-s\mathbf{A}_2}\mathbf{B}_2 \mathrm{d}s\, \mathrm{e}^{-t\mathbf{A}_1} & \int_0^t \mathrm{e}^{-s\mathbf{A}_1}\mathbf{B}_1 \mathrm{d}s \int_0^t \mathrm{e}^{-s\mathbf{A}_2}\mathbf{B}_2 \mathrm{d}s + \mathrm{e}^{-t\mathbf{A}_2} \end{pmatrix}, \tag{AO}$$

which is the main object of investigation in the paper, see Section 3 for more details. Operator-matrix semigroups have attracted a lot of attention in the last decades: in spectral analysis [Arl02, Tre08]; in modeling and solving various types of evolution equations, see e.g. [Eng95, BaP05, LHC20, AgH21]; and in the context of split-step methods [CsN08, BC*12, BC*14]. For a recent split-step convergence analysis in the context of nonlinear gradient-flow PDEs we refer to [MRS23].

Here, under the assumption that the coupling operators $\mathbf{B}_j$ are bounded, a straight-forward computation shows that $\mathcal{T}$ is stable, i.e. for all $n \geq 0$ and $t \geq 0$, $\mathcal{T}(t/n)^n$ can be bounded. Stability provides that $\mathcal{T}(t/n)^n$ is a well-defined approximation, and, in particular, we have the strong-convergence result $\mathcal{T}(t/n)^n x \to \mathrm{e}^{-t\mathcal{C}}x$ for all $x \in X$ as $n \to \infty$, see Proposition 3.2. The main result (Theorem 4.4) is that, assuming that the semigroups of $-\mathbf{A}_1$ and $-\mathbf{A}_2$ are holomorphic, the convergence of the approximation can be improved to operator-norm convergence, and moreover, can be estimated by $O(\log(n)/n)$, which is almost the optimal convergence rate of $O(1/n)$. Operator-norm convergence for approximations of Trotter-product form have been derived first in [Rog93] for semigroups of self-adjoint operators in Hilbert spaces, and later generalized by [CaZ01] to holomorphic semigroups on Banach spaces. Here, we also assume that the semigroups are holomorphic. However, the result here shows operator-norm convergence for the approximation operators $\mathcal{T}$ where its components are not given by semigroups (although $\mathbf{B}_j$ are bounded they are not generators as they even act between different spaces). The crucial idea is to estimate $\mathcal{T}(t) - \mathrm{e}^{-t\mathcal{C}}$ for small $t \geq 0$ by evaluating their derivatives, and to show that this $O(t^2)$ behavior propagates to the whole approximation. Technical difficulties arise by the additional non-commutative feature of matrix multiplication.

The practical component of operator-norm convergence, in contrast to convergence in the strong topology, is evident: The solution can be approximated regardless the initial condition (and its generic uncertainty); moreover, a convergence-rate estimate provides a universal bound how good the approximation actually is. Moreover, the convergence proof is flexible to provide also convergence for different (but similar) approximation (see Section 4.4).

The crucial assumption for estimating the convergence rate is the holomorphicity of the dominating semigroups. In [NSZ18b], a counterexample has been constructed showing that operator-norm convergence does not hold if the semigroup of the main operator is not holomorphic. There, the construction is done by a time-dependent perturbation. Affirmative convergence result for time-dependent perturbations can be found in [NSZ17, NSZ18a, NSZ19, NSZ20]. In principle, time-dependent couplings $t \mapsto \mathbf{B}_j(t)$ can also be considered for approximations (AO). The corresponding operator-norm convergence result however is left for future work.

2 Preliminaries

In this section, $(X, \|\cdot\|)$ is a general Banach space; moreover, all operators in the paper are linear. We first recall well-known important facts from semigroup theory and (inhomogeneous) abstract Cauchy problems, see e.g. [EnN00]

2.1 Recap of semigroup theory

A family $\{\mathbf{T}(t)\}_{t\geq 0}$ of bounded linear operators on the Banach space X is called a *strongly continuous semigroup* (in the following only *semigroup*) if it satisfies the functional equation

$$\mathcal{T}(0) = \mathbf{I}, \quad \mathbf{T}(t+s) = \mathbf{T}(t)\mathbf{T}(s), \quad \text{for } t, s \geq 0,$$

and, moreover, the orbit maps $[0, \infty[\ni t \mapsto \mathbf{T}(t)x$ are continuous for all $x \in X$. In the following, the identity map is denoted by $\mathbf{I} : X \to X$. For a given semigroup its generator is a linear operator defined by the limit

$$-\mathbf{A}x := \lim_{t\to 0} \frac{1}{t} \left(\mathbf{T}(t)x - x\right)$$

on the domain $\mathrm{dom}(\mathbf{A}) = \left\{x \in X : \lim_{t\to 0} \frac{1}{t} \left(\mathbf{T}(t)x - x\right) \text{ exists}\right\}$.

It is well-known that the generator $-\mathbf{A}$ of a strongly continuous semigroup is a closed and densely defined linear operator, which uniquely determines its semigroup, which in the following will be denote by $\left\{\mathbf{T}(t) = \mathrm{e}^{-t\mathbf{A}}\right\}_{t\geq 0}$. Recall that for a semigroup $\left\{\mathbf{T}(t) = \mathrm{e}^{-t\mathbf{A}}\right\}_{t\geq 0}$, there are constants M, β such that $\|\mathbf{T}(t)\| \leq M\mathrm{e}^{\beta t}$ for all $t \geq 0$. The operator norm for a bounded operator $\mathbf{B} : X \to Y$ is defined as usual by $\|\mathbf{B}\| := \sup\{\|\mathbf{B}x\|_Y : \|x\|_X \leq 1\}$, which is a norm in the space of bounded linear operators. If $\beta \leq 0$ then the semigroup is called *bounded*; if $\|\mathbf{T}(t)\| \leq 1$, the semigroup is called a *contraction semigroup.*

The semigroup $\left\{\mathbf{T}(t) = \mathrm{e}^{-t\mathbf{A}}\right\}_{t\geq 0}$ is called a *bounded holomorphic semigroup* if its generator $-\mathbf{A}$ satisfies $\mathbf{T}(t)x \in \mathrm{dom}(\mathbf{A})$ for all $x \in X$ and $t > 0$, and if there is a constant $M_A > 0$ such that $\sup_{t>0} \|t\mathbf{A}\mathbf{T}(t)\| \leq M_A$. Recall that in this case the bounded semigroup $\{\mathbf{T}(t)\}_{t\geq 0}$ has a unique analytic continuation into the open sector $\{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \delta(\mathbf{A}) \leq \pi/2\} \subset \mathbb{C}$ of an angle $\delta(\mathbf{A}) > 0$.

In the following, we are interested in sums of operators given by a generator $-\mathbf{A}$ with semigroup $\|\mathrm{e}^{-t\mathbf{A}}\| \leq M\mathrm{e}^{\beta t}$ and a bounded operator $\mathbf{B}$. It is well-known that $-\mathbf{C} := -\mathbf{A} + \mathbf{B}$, $\mathrm{dom}(\mathbf{C}) =$

$\mathrm{dom}(\mathbf{A})$ is a generator of a semigroup $\left\{\mathrm{e}^{-t\mathbf{C}}\right\}_{t\geq 0}$ that satisfies $\|\mathrm{e}^{-t\mathbf{C}}\| \leq M\mathrm{e}^{(\beta+M\|\mathbf{B}\|)t}$ for all $t \geq 0$ (see [EnN00, Theorem III.1.3]). Moreover, if the semigroup $\left\{\mathrm{e}^{-t\mathbf{A}}\right\}_{t\geq 0}$ is a holomorphic semigroup, then so is $\left\{\mathrm{e}^{-t\mathbf{C}}\right\}_{t\geq 0}$ (see Proposition III.1.12).

We recall the following facts, which will be important for further estimates.

Lemma 2.1 (Engel-Nagel, Lemma II.1.3). *Let $-\mathbf{A}$ be a generator of a bounded semigroup $\left\{\mathrm{e}^{-t\mathbf{A}}\right\}_{t\geq 0}$. Define for $x \in X$ and $t \geq 0$ the operator*

$$\mathbf{F}_t x := \int_0^t \mathrm{e}^{-s\mathbf{A}} x \mathrm{d}s.$$

Then, we have

1 $\mathbf{F}_t : X \to X$ *is bounded with $\|\mathbf{F}_t\| \leq tM$, and for all $x \in X$ we have $\frac{1}{t}\mathbf{F}_t x \to x$ as $t \to 0$.*

2 *For all $t \geq 0$ and $x \in X$, we have $\mathbf{F}_t x \in \mathrm{dom}(\mathbf{A})$, and $\mathrm{e}^{-t\mathbf{A}}x - x = -\mathbf{A}\mathbf{F}_t x$. In particular, we have $\|\mathbf{A}\mathbf{F}_t\| \leq M + 1$.*

3 *If, in addition, $\mathbf{A} : \mathrm{dom}(\mathbf{A}) \subset X \to X$ is boundedly invertible, then $\mathbf{F}_t x = \mathbf{A}^{-1}\left(\mathbf{I} - \mathrm{e}^{-t\mathbf{A}}\right)x$.*

2.2 Inhomogeneous abstract Cauchy problems

The following split-step method is based by solving inhomogeneous abstract Cauchy problems, which are in general of the form

$$\dot{u}(t) = -\mathbf{A}u(t) + f(t)\,, \quad u(0) = u_0 \in X, \tag{iaCP}$$

where $-\mathbf{A} : \mathrm{dom}(A) \to X$ is a generator of a strongly continuous semigroup, and $f : [0, \infty[\to X$ is an inhomogeneity. The formal solution is then given by the variation of constants formula and has the form

$$u(t) = \mathrm{e}^{-t\mathbf{A}}u_0 + \int_0^t \mathrm{e}^{-(t-s)\mathbf{A}} f(s)\mathrm{d}s.$$

At least formally, one easily sees that

$$\dot{u}(t) = -\mathbf{A}\mathrm{e}^{-t\mathbf{A}}u_0 - \mathbf{A}\int_0^t \mathrm{e}^{-(t-s)\mathbf{A}} f(s)\mathrm{d}s + f(t) = -\mathbf{A}u(t) + f(t)\,.$$

There are well-known criteria on how temporal as well as spatial regularity of f determine regularity of the solution u of the inhomogeneous Abstract Cauchy problem (iaCP), see e.g. [EnN00, Chapter VI.7].

However, in our situation the formula simplifies as the inhomogeneity f will be constant in time. Indeed for $f(t) = x$, we get, with a reparametrization of the integral

$$u(t) = \mathrm{e}^{-t\mathbf{A}}u_0 + \int_0^t \mathrm{e}^{-s\mathbf{A}} x \mathrm{d}s = \mathrm{e}^{-t\mathbf{A}}u_0 + \mathbf{F}_t x. \tag{$\star$}$$

If moreover, the generator $\mathbf{A}$ is boundedly invertible then the integral can be solved using Lemma 2.1, and we have

$$u(t) = \mathrm{e}^{-t\mathbf{A}}u_0 - \mathbf{A}^{-1}\left(\mathrm{e}^{-t\mathbf{A}} - \mathbf{I}\right)x$$

The following lemma collects and summarizes these important facts, and is an immediate consequence of Lemma 2.1 and the classical well-posedness theory for linear (inhomogeneous) abstract Cauchy problems (see e.g. Engel-Nagel Corollary VI.7.8)

Lemma 2.2. *Let $x \in X$ and $u_0 \in \mathrm{dom}(\mathbf{A})$ be arbitrary. Define, for $t \in [0, \infty[$ the trajectory u by ($\star$). Then u is continuously differentiable, $u(t) \in X$ for all $t \geq 0$ and is the unique classical solution of the inhomogeneous abstract Cauchy problem $\dot{u}(t) = -\mathbf{A}u(t) + x$.*

3 Split-step method on the product space

After presenting the product-space setting, we define the split-step approximation operators. Then, we show stability (i.e. boundedness) of the time-discretized trajectories and, finally, discuss the first convergence results.

3.1 Product-space setting

For two given Banach spaces $\big(X_j, \|\cdot\|_{X_j}\big)$, we consider the Banach space $X = X_1 \times X_2$ equipped with the canonical norm, i.e.

$$\|(x, y)\|^2_{X_1 \times X_2} := \|x\|^2_{X_1} + \|y\|^2_{X_2} .$$

The identity operator on the spaces X_j is denoted by $\mathbf{I} : X_j \to X_j$, the identity operator on the whole space X is denoted by $\mathcal{I} := \begin{pmatrix} \mathbf{I} & \cdot \\ \cdot & \mathbf{I} \end{pmatrix} : X \to X$. For a bounded operator $\mathcal{B} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_2 \end{pmatrix} : X \to X$, where $\mathbf{B}_j : X_j \to X_j$ and $\mathbf{B}_{ij} : X_j \to X_i$ are bounded operators, an easy calculation shows

$$\begin{aligned} \|\mathcal{B}\begin{pmatrix} x \\ y \end{pmatrix}\|^2 &= \|\mathbf{B}_1 x + \mathbf{B}_{12} y\|^2_{X_1} + \|\mathbf{B}_{21} x + \mathbf{B}_2 y\|^2_{X_2} \leq \\ &\leq 2\left(\|\mathbf{B}_1\|^2 + \|\mathbf{B}_{21}\|^2\right)\|x\|^2_{X_1} + \left(\|\mathbf{B}_{12}\|^2 + \|\mathbf{B}_2\|^2\right)\|y\|^2_{X_1} \\ &\leq 2\max\left\{\|\mathbf{B}_1\|^2 + \|\mathbf{B}_{21}\|^2, \|\mathbf{B}_{12}\|^2 + \|\mathbf{B}_2\|^2\right\}\left(\|x\|^2_{X_1} + \|y\|^2_{X_1}\right), \end{aligned}$$

which implies the crude estimate $\|\mathcal{B}\| \leq \sqrt{2}\max\Big\{\left(\|\mathbf{B}_1\|^2 + \|\mathbf{B}_{21}\|^2\right)^{1/2}, \left(\|\mathbf{B}_{12}\|^2 + \|\mathbf{B}_2\|^2\right)^{1/2}\Big\}$.

3.2 Split-step by inhomogeneous abstract Cauchy problem

Now, we describe in detail the split-step method for approximating the solution on a product space. We consider on the space $X = X_1 \times X_2$ the operator

$$-\mathcal{C} := -\mathcal{A} + \mathcal{B}, \quad \mathcal{A} = \begin{pmatrix} \mathbf{A}_1 & \cdot \\ \cdot & \mathbf{A}_2 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \cdot & \mathbf{B}_1 \\ \mathbf{B}_2 & \cdot \end{pmatrix}, \quad -\mathcal{C}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\mathbf{A}_1 u + \mathbf{B}_1 v \\ \mathbf{B}_2 u - \mathbf{A}_2 v \end{pmatrix},$$

where each $-\mathbf{A}_j : \mathrm{dom}(\mathbf{A}_j) \subset X_j \to X_j$ are generators of a contraction semigroup and the coupling operators $\mathbf{B}_1 : X_2 \to X_1$ and $\mathbf{B}_2 : X_1 \to X_2$ are bounded. (We comment on unbounded operators in Section 5.)

Clearly, also $-\mathcal{A} : \mathrm{dom}(\mathcal{A}) = \mathrm{dom}(\mathbf{A}_1) \times \mathrm{dom}(\mathbf{A}_2) \subset X_1 \times X_2 \to X$ is a generator of a semigroup $\mathrm{e}^{-t\mathcal{A}} = \begin{pmatrix} \mathrm{e}^{-t\mathbf{A}_1} & \cdot \\ \cdot & \mathrm{e}^{-t\mathbf{A}_2} \end{pmatrix}$, which is also a contraction semigroup. Since $\mathcal{B}$ is bounded, we have that $-\mathcal{C} : \mathrm{dom}\,(\mathcal{C}) = \mathrm{dom}\,(\mathcal{A}) \subset X \to X$ is also a generator of a semigroup.

Here, we will not approximate the solution operator $\mathrm{e}^{-t\mathcal{C}}$ by the Trotter-product formula $\left(\mathrm{e}^{-t/n\mathcal{A}}\mathrm{e}^{t/n\mathcal{B}}\right)^n$. Instead, we are interested in an approximation that exploits the Block structure of the underlying state space. As for the Trotter product formula, the time-discretized iteration operator $\mathcal{T}(\tau)$ depends on the small time-step of length $\tau = t/n$ and consists of two parts, i.e $\mathcal{T} = \mathcal{T}_2\mathcal{T}_1$. Each operator $\mathcal{T}_j$ is defined by evolving only the component of X_j and leaving the other component constant. The first operator $\mathcal{T}_1$ is given by solving the inhomogeneous abstract Cauchy problem:

$$\begin{cases} \dot{u} = & -\mathbf{A}_1 u + \mathbf{B}_1 v \\ \dot{v} = & 0 \end{cases}, \text{ for } t \in [0,\infty[, \quad u(0) = u_0,\, v(0) = v_0,$$

and maps $\mathcal{T}_1(\tau) : \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \mapsto \begin{pmatrix} u(\tau) \\ v(\tau) \end{pmatrix}$. The explicit solution of the evolution equation is given by Lemma 2.2, and we have

$$\begin{pmatrix} u \\ v \end{pmatrix}(t=\tau) = \mathcal{T}_1(\tau)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\tau\mathbf{A}_1} & \int_0^\tau \mathrm{e}^{-\sigma\mathbf{A}_1}\mathrm{d}\sigma\mathbf{B}_1 \\ \cdot & \mathbf{I} \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

The second operator $\mathcal{T}_2$ is given by solving the inhomogeneous abstract Cauchy problem:

$$\begin{cases} \dot{u} = & 0 \\ \dot{v} = & \mathbf{B}_2 u - \mathbf{A}_2 v \end{cases}, \text{ for } t \in [0,\infty[, \quad u(0) = u_0,\, v(0) = v_0,$$

and maps $\mathcal{T}_2(\tau) : \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \mapsto \begin{pmatrix} u(\tau) \\ v(\tau) \end{pmatrix}$. Hence, we have

$$\begin{pmatrix} u \\ v \end{pmatrix}(t=\tau) = \mathcal{T}_2(\tau)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \cdot \\ \int_0^\tau \mathrm{e}^{-\sigma\mathbf{A}_2}\mathrm{d}\sigma\mathbf{B}_2 & \mathrm{e}^{-\tau\mathbf{A}_2} \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

Hence, we conclude that the total solution operator at time $t = \tau$ is given by

$$\begin{aligned} \mathcal{T}(\tau) = \mathcal{T}_2(\tau)\mathcal{T}_1(\tau) &= \begin{pmatrix} \mathbf{I} & \cdot \\ \int_0^\tau \mathrm{e}^{-\sigma\mathbf{A}_2}\mathrm{d}\sigma\mathbf{B}_2 & \mathrm{e}^{-\tau\mathbf{A}_2} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\tau\mathbf{A}_1} & \int_0^\tau \mathrm{e}^{-\sigma\mathbf{A}_1}\mathrm{d}\sigma\mathbf{B}_1 \\ \cdot & \mathbf{I} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{-\tau\mathbf{A}_1} & \int_0^\tau \mathrm{e}^{-\sigma\mathbf{A}_1}\mathrm{d}\sigma\mathbf{B}_1 \\ \int_0^\tau \mathrm{e}^{-\sigma\mathbf{A}_2\mathrm{d}\sigma}\mathbf{B}_2\mathrm{e}^{-\tau\mathbf{A}_1} & \int_0^\tau \mathrm{e}^{-\sigma\mathbf{A}_2}\mathrm{d}\sigma\mathbf{B}_2 \cdot \int_0^\tau \mathrm{e}^{-\sigma\mathbf{A}_1}\mathrm{d}\sigma\mathbf{B}_1 + \mathrm{e}^{-\tau\mathbf{A}_2} \end{pmatrix} \\ &=: \begin{pmatrix} \mathbf{E}_1(\tau) & \mathbf{X}_1(\tau) \\ \mathbf{X}_2(\tau)\mathbf{E}_1(\tau) & \mathbf{X}_2(\tau)\mathbf{X}_1(\tau) + \mathbf{E}_2(\tau) \end{pmatrix}, \end{aligned} \tag{AO}$$

where we have introduced the notation for the solution operator $\mathbf{E}_j$ on the diagonal and the cross terms $\mathbf{X}_j$

$$\mathbf{E}_j(\tau) = \mathrm{e}^{-\tau\mathbf{A}_j}, \quad \mathbf{X}_j(\tau) = \int_0^\tau \mathrm{e}^{-\sigma\mathbf{A}_j}\mathrm{d}\sigma\mathbf{B}_j.$$

Some easy facts regarding these operators are summarized in the next lemma, which is an trivial consequence of Lemma 2.1.

Lemma 3.1. *Let $-\mathbf{A}_j : \mathrm{dom}(\mathbf{A}_j) \subset X_j \to X_j$ be generators of contraction semigroups, and let $\mathbf{B}_1 : X_2 \to X_1$, $\mathbf{B}_2 : X_1 \to X_2$ be bounded. Define the split-step approximation operators $\{\mathcal{T}(\tau)\}_{\tau\geq 0}$ by* (AO)*. Then*

1. *For all $\tau \geq 0$, the operators $\mathcal{T}(\tau) : X \to X$ are strongly continuous bounded operators and $\mathcal{T}(0) = \mathcal{I}$,*

2 *For all $\tau \geq 0$, we have $\|\mathbf{X}_i(\tau)\| \leq \tau\|\mathbf{B}_i\|$, and there is a constant such that for all $\tau \geq 0$, we have $\|\mathbf{A}_i\mathbf{X}_i(\tau)\| \leq 2\|\mathbf{B}_i\|$.*

3 *For all $x_j \in \mathrm{dom}(\mathbf{A}_j)$, we have $\mathbf{E}_j'(\tau)x_j = -\mathbf{A}_j \mathrm{e}^{-\tau\mathbf{A}_j}x_j$; For all $x \in X_j$ we have $\mathbf{X}_j'(\tau)x = \mathbf{E}_j(\tau)\mathbf{B}_j$.*

Iterating the operators $\mathcal{T}(\tau)$ n-times, we get a trajectory till time $t \in [0,\infty[$. The main question, which is addressed in that paper, is to show convergence

$$\mathcal{T}(t/n)^n \to \mathrm{e}^{-t\mathcal{C}}.$$

3.3 Stability analysis and convergence in the strong topology

To have a useful approximation, the stability of the iterated operator $\mathcal{T}(t/n)^n$ for $t \in [0,\infty[$ has to be established.

Proposition 3.2. *Let $-\mathbf{A}_j : \mathrm{dom}(\mathbf{A}_j) \subset X_j \to X_j$ be generators of strongly continuous contraction semigroups, and let $\mathbf{B}_1 : X_2 \to X_1$, $\mathbf{B}_2 : X_1 \to X_2$ be bounded. Then for all $t \in [0,\infty[$ and $n \in \mathbb{N}$ we have*

$$\|\mathcal{T}(t/n)^n\| \leq \mathrm{e}^{t(\|\mathbf{B}_1\|+\|\mathbf{B}_2\|)}.$$

Moreover, for all $t \in [0,\infty[$, $(\mathcal{T}(t/n)^n)_{n\in\mathbb{N}}$ converges to the semigroup $\mathrm{e}^{-t\mathcal{C}}$ in the strong topology, i.e. for all $x \in X$, we have

$$\lim_{n\to\infty} \mathcal{T}(t/n)^n x = \mathrm{e}^{-t\mathcal{C}}x,$$

which is uniformly in time on bounded intervals.

Proof. Using that $-\mathbf{A}_j$ generates a contraction semigroup, we have

$$\begin{aligned}
\|\mathcal{T}_1(\tau)\begin{pmatrix}x\\y\end{pmatrix}\|^2 &= \|\mathbf{E}_1(\tau)x + \mathbf{X}_1(\tau)y\|_{X_1}^2 + \|y\|_{X_2}^2 \leq \left(\|\mathbf{E}_1(\tau)x\|_{X_1} + \|\mathbf{X}_1(\tau)y\|_{X_1}\right)^2 + \|y\|_{X_2}^2\\
&\leq \|x\|_{X_1}^2 + 2\tau\|\mathbf{B}_1\|\cdot\|x\|_{X_1}\cdot\|y\|_{X_2} + \left(1+\tau^2\|\mathbf{B}_1\|^2\right)\|y\|_{X_2}^2\\
&\leq \left(1+2\tau\|\mathbf{B}_1\|+\tau^2\|\mathbf{B}_1\|^2\right)\left(\|x\|_{X_1}^2+\|y\|_{X_2}^2\right) = (1+\tau\|\mathbf{B}_1\|)^2\left(\|x\|_{X_1}^2+\|y\|_{X_2}^2\right).
\end{aligned}$$

Hence, we get that $\|\mathcal{T}_1(\tau)\| \leq 1+\tau\|\mathbf{B}_1\|$ for all $\tau \geq 0$. Similarly, we obtain $\|\mathcal{T}_2(\tau)\| \leq 1+\tau\|\mathbf{B}_2\|$ for all $\tau \geq 0$. Hence, we get

$$\|\mathcal{T}(t/n)^n\| \leq \|\mathcal{T}_2(t/n)\|^n \cdot \|\mathcal{T}_1(t/n)\|^n \leq \left(1+\frac{t}{n}\|\mathbf{B}_2\|\right)^n\left(1+\frac{t}{n}\|\mathbf{B}_1\|\right)^n \leq \mathrm{e}^{t(\|\mathbf{B}_1\|+\|\mathbf{B}_2\|)}.$$

To prove convergence in the strong topology, we rely on the famous result of Chernoff, see e.g. [EnN00, Corollary III.5.3]. Since the stability has been already shown, it suffices to show that the derivative of $\mathcal{T}(t)$ at $t \geq 0$ is given by $-\mathcal{C}$ (which, as we already know, is a generator). For this let $(u,v)^{\mathrm{T}} \in \mathrm{dom}(\mathcal{C}) = \mathrm{dom}(\mathcal{A})$. By Lemma 3.1, we obtain

$$\begin{aligned}
\frac{1}{t}\left(\mathcal{T}(t)-\mathcal{I}\right)\begin{pmatrix}u\\v\end{pmatrix} &= \frac{1}{t}\begin{pmatrix}\mathbf{E}_1(t)-\mathbf{I} & \mathbf{X}_1(t)\\ \mathbf{X}_2(t)\mathbf{E}_1(t) & \mathbf{X}_2(t)\mathbf{X}_1(t)+\mathbf{E}_2(t)-\mathbf{I}\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}\\
&\to \begin{pmatrix}-\mathbf{A}_1u+\mathbf{B}_1v\\ \mathbf{B}_2u-\mathbf{A}_1v\end{pmatrix} = -\mathcal{C}\begin{pmatrix}u\\v\end{pmatrix},
\end{aligned}$$

as $t \to 0$. This implies $\lim_{n\to\infty}\mathcal{T}(t/n)^n x = \mathrm{e}^{-t\mathcal{C}}x$ as desired. □

The main result of the paper is to show that the convergence of the approximation $\mathcal{T}(t/n)^n \to \mathrm{e}^{-t\mathcal{C}}$ can be improved to convergence in the operator norm, and that the convergence can be estimated. To show the ideas, we shortly discuss the situation, when $-\mathbf{A}_j$ are bounded operators.

3.4 Convergence for bounded operators

Assuming that $-\mathbf{A}_j$ are bounded operators, the semigroups are given by the exponential of the generators, and we have
$$\mathrm{e}^{-\tau\mathbf{A}_j} = \mathbf{I} - \tau\mathbf{A}_j + O(\tau^2), \quad \text{as} \quad \tau \to 0\,.$$
As we will see, estimating $\mathcal{T}(t/n)^n \to \mathrm{e}^{-t\mathcal{C}}$ is similar to the unbounded case and relies on the telescopic representation of the product:

$$\mathcal{T}^n - \mathcal{S}^n = \sum_{k=0}^{n-1} \mathcal{T}^{n-1-k}(\mathcal{T}-\mathcal{S})\mathcal{S}^k.$$

The crucial idea is to get a good convergence rate for $\mathcal{T}-\mathcal{S}$, and to ensure that the remainders $\mathcal{T}^{n-1-k}$ and $\mathcal{S}^k$ can be bounded.

Indeed, in the situation of bounded operators we have, for $\tau \to 0$,

$$\begin{aligned}
\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{C}} &= \begin{pmatrix} \mathbf{E}_1(\tau) & \mathbf{X}_1(\tau) \\ \mathbf{X}_2(\tau)\mathbf{E}_1(\tau) & \mathbf{X}_2(\tau)\mathbf{X}_1(\tau) + \mathbf{E}_2(\tau) \end{pmatrix} - \{\mathcal{I} - \tau\mathcal{C}\} + O(\tau^2) \\
&= \begin{pmatrix} \mathbf{E}_1(\tau) & \mathbf{X}_1(\tau) \\ \mathbf{X}_2(\tau)\mathbf{E}_1(\tau) & \mathbf{X}_2(\tau)\mathbf{X}_1(\tau) + \mathbf{E}_2(\tau) \end{pmatrix} - \begin{pmatrix} \mathbf{I} - \tau\mathbf{A}_1 & \tau\mathbf{B}_1 \\ \tau\mathbf{B}_2 & \mathbf{I} - \tau\mathbf{A}_2 \end{pmatrix} + O(\tau^2) \\
&= O(\tau^2),
\end{aligned}$$

where we have used that

$$\mathbf{E}_j(\tau) = \mathbf{I} - \tau\mathbf{A}_j + O(\tau^2), \quad \mathbf{X}_j(\tau) = \tau\mathbf{B}_j + O(\tau^2).$$

Hence, we get

$$\begin{aligned}
\|\mathcal{T}(\tau)^n - \mathrm{e}^{-t\mathcal{C}}\| &= \|\mathcal{T}(\tau)^n - \left(\mathrm{e}^{-\tau\mathcal{C}}\right)^n\| \\
&= \|\sum_{k=0}^{n-1} \mathcal{T}(\tau)^{n-1-k}(\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{C}})\mathrm{e}^{-\tau k\mathcal{C}}\| \\
&\le \sum_{k=0}^{n-1} \|\mathcal{T}(\tau)^{n-1-k}\| \cdot \|\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{C}}\| \cdot \|\mathrm{e}^{-\tau k\mathcal{C}}\| \lesssim n\cdot\tau^2 \lesssim \frac{t^2}{n}\,.
\end{aligned}$$

Hence, the split-step method converges with order $O(n^{-1})$, which is summarized in the next proposition.

Proposition 3.3. *Let $\mathbf{A}_i : X_i \to X_i$, $\mathbf{B}_1 : X_2 \to X_1$ and $\mathbf{B}_2 : X_1 \to X_2$ be bounded. Define the split-step approximation operator family $\mathcal{T}$ by* (AO). *Then there is a constant $C = C(\|A\|, \|B\|)$ such that for all $t \ge 0$ and $n \in \mathbb{N}$ we have $\|\mathcal{T}(\tau)^n - \mathrm{e}^{-t\mathcal{C}}\| \le \frac{C}{n}t^2$.*

We note that, in general no better convergence than of order $O(n^{-1})$ can be expected. Moreover, the above calculation suggest that $\mathcal{T}_{\mathrm{B}}(\tau) := \begin{pmatrix} \mathbf{E}_1(\tau) & \tau\mathbf{B}_1 \\ \tau\mathbf{B}_2 & \mathbf{E}_2(\tau) \end{pmatrix}$, where the integrand $\mathrm{e}^{-\sigma\mathbf{A}_j}$ in the coupling term $\mathbf{X}_j(\tau)$ is replaced by the constant identity $\mathbf{I}$, can be used as another approximation. Indeed, following the lines of the proof of Proposition 3.2 the approximation based on $\mathcal{T}_\mathrm{B}$ converges on the strong topology as well. However, for proving convergence in operator norm the regularization $\mathrm{e}^{-\sigma \mathbf{A}_j}$ is needed (see also Section 4.4, where other different approximations are discussed).

4 Operator-norm convergence rate analysis

In this section, we show the convergence $\mathcal{T}(t/n)^n \to \mathrm{e}^{-t\mathcal{C}}$ in the operator norm as $n \to \infty$, and that the convergence rate can be estimated. The overall assumptions is that the operators $-\mathbf{A}_j : \mathrm{dom}\,(\mathbf{A}_j) \to X_j$ are generators of holomorphic contraction semigroups and that the linear operators $\mathbf{B}_1 : X_2 \to X_1$ and $\mathbf{B}_2 : X_1 \to X_2$ are bounded. In particular, there is a constant $M_A > 0$ such that for all $t > 0$

$$\|t\mathbf{A}_i \mathrm{e}^{-t\mathbf{A}_i}\| \le M_A\,, \tag{1}$$

which we will use frequently. Since $\mathcal{A}$ is diagonal, we easily see that $-\mathcal{A} : \mathrm{dom}(\mathcal{A}) \subset X \to X$ is a generator of a contraction semigroup. In addition, we have the same estimate $\|t\mathcal{A}\mathrm{e}^{-t\mathcal{A}}\| \le M_A$, which shows that the semigroup $\mathrm{e}^{-t\mathcal{A}}$ is holomorphic as well. Since the coupling operator $\mathcal{B}$ is bounded, also $-\mathcal{C}$ is a generator of a semigroup with $\|\mathrm{e}^{-t\mathcal{C}}\| \le \mathrm{e}^{t\|\mathcal{B}\|}$ which is also holomorphic, i.e. there is a constant $M_C > 0$ such that for all $t > 0$ we have $\|t\mathcal{C}\mathrm{e}^{-t\mathcal{C}}\| \le M_C$.

Moreover, we will use the following relations between the operator norms of $\mathbf{B}_j$ and $\mathcal{B}$:

$$\|\mathcal{B}\| = \max\{\|\mathbf{B}_1\|, \|\mathbf{B}_2\|\} \le \|\mathbf{B}_1\| + \|\mathbf{B}_2\| \le 2\|\mathcal{B}\|\,.$$

4.1 Auxiliary lemmas

We first collect important auxiliary lemmas which are used to show that main result. We frequently use the matrix multiplication rule for operators (wherever they are defined):

$$\mathcal{C}\mathcal{D} = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & \mathbf{C}_4 \end{pmatrix} \begin{pmatrix} \mathbf{D}_1 & \mathbf{D}_2 \\ \mathbf{D}_3 & \mathbf{D}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{C}_1\mathbf{D}_1 + \mathbf{C}_2\mathbf{D}_3 & \mathbf{C}_1\mathbf{D}_2 + \mathbf{C}_2\mathbf{D}_4 \\ \mathbf{C}_3\mathbf{D}_1 + \mathbf{C}_4\mathbf{D}_3 & \mathbf{C}_3\mathbf{D}_2 + \mathbf{C}_4\mathbf{D}_4 \end{pmatrix}.$$

Lemma 4.1. *Let $-\mathbf{A}_j : \mathrm{dom}\,(\mathbf{A}_j) \subset X_j \to X_j$ be generators of holomorphic contraction semigroups, and let $\mathbf{B}_1 : X_2 \to X_1$, $\mathbf{B}_2 : X_1 \to X_2$ be bounded. Then, there is a constant $C_1 = C_1(\|\mathbf{B}_1\|, \|\mathbf{B}_2\|) > 0$ such that for all $k \in \{1, \dots, n\}$ and $\tau = \frac{t}{n} > 0$ we have*

$$\|\mathcal{A}\mathcal{T}(\tau)^k\| \le C_1 \mathrm{e}^{t(\|\mathbf{B}_1\| + \|\mathbf{B}_2\|)}\,(1 + \log k) + \frac{M_A}{k\tau}.$$

Proof. Using the decomposition

$$\mathcal{A}\mathcal{T}(\tau)^k = \mathcal{A}\left(\mathcal{T}(\tau)^k - \mathrm{e}^{-\tau k\mathcal{A}}\right) + \mathcal{A}\mathrm{e}^{-\tau k\mathcal{A}},$$

we see that the second term is bounded by $\|\mathcal{A}\mathrm{e}^{-\tau k\mathcal{A}}\| \le \frac{1}{\tau k} M_A$. For the first term, we use the telescopic representation of the product $\mathcal{S}^k - \mathcal{T}^k = \sum_{j=0}^{k-1} \mathcal{S}^{k-1-j}(\mathcal{S} - \mathcal{T})\mathcal{T}^j$, and get

$$\begin{aligned} \mathcal{A}\left(\mathrm{e}^{-\tau k\mathcal{A}} - \mathcal{T}(\tau)^k\right) &= \sum_{j=0}^{k-1} \mathrm{e}^{-\tau(k-1-j)\mathcal{A}} \mathcal{A}\left(\mathrm{e}^{-\tau\mathcal{A}} - \mathcal{T}(\tau)\right)\mathcal{T}(\tau)^j = \\ &= \sum_{j=0}^{k-2} \mathrm{e}^{-\tau(k-1-j)\mathcal{A}} \mathcal{A}\left(\mathrm{e}^{-\tau\mathcal{A}} - \mathcal{T}(\tau)\right)\mathcal{T}(\tau)^j + \mathcal{A}\left(\mathrm{e}^{-\tau\mathcal{A}} - \mathcal{T}(\tau)\right)\mathcal{T}(\tau)^{k-1}\,. \end{aligned}$$

Since

$$\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{A}} = \begin{pmatrix} \cdot & \mathbf{X}_1(\tau) \\ \mathbf{X}_2\mathbf{E}_1(\tau) & \mathbf{X}_2\mathbf{X}_1(\tau) \end{pmatrix},$$

we get that there is a constant $C = C(\|\mathbf{B}_1\|, \|\mathbf{B}_2\|) > 0$ such that $\|\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{A}}\| \leq C\tau$. Moreover, we see that

$$\mathcal{A}\left(\mathrm{e}^{-\tau\mathcal{A}} - \mathcal{T}(\tau)\right) = \begin{pmatrix} \cdot & \mathbf{A}_1\mathbf{X}_1(\tau) \\ \mathbf{A}_2\mathbf{X}_2\mathbf{E}_1(\tau) & \mathbf{A}_2\mathbf{X}_2\mathbf{X}_1(\tau) \end{pmatrix},$$

which is (by Lemma 3.1) a bounded operator for $\tau > 0$ with $\|\mathcal{A}\left(\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{A}}\right)\| \leq C\tau$, $C = C(\|\mathbf{B}_1\|, \|\mathbf{B}_2\|) > 0$. Hence, we have that

$$\begin{aligned} &\|\mathcal{A}\left(\mathrm{e}^{-\tau k\mathcal{A}} - \mathcal{T}(\tau)^k\right)\| \\ &\leq \sum_{j=0}^{k-2} \|\mathrm{e}^{-\tau(k-1-j)\mathcal{A}}\mathcal{A}\| \cdot \|\mathrm{e}^{-\tau\mathcal{A}} - \mathcal{T}(\tau)\| \cdot \|\mathcal{T}(\tau)^j\| + \|\mathcal{A}\left(\mathrm{e}^{-\tau\mathcal{A}} - \mathcal{T}(\tau)\right)\| \cdot \|\mathcal{T}(\tau)^{k-1}\| \\ &\leq C\mathrm{e}^{t(\|\mathbf{B}_1\|+\|\mathbf{B}_2\|)}\left(\sum_{j=0}^{k-2} \frac{1}{(k-1-j)\,\tau}\tau + 1\right) \leq C\mathrm{e}^{t(\|\mathbf{B}_1\|+\|\mathbf{B}_2\|)}\left(\log k + 1\right), \end{aligned}$$

where we have used $\sum_{j=1}^{k-1} \frac{1}{j} \leq \log k$. This proves the claim. □

For the next two lemmas, we do not assume that the semigroups $\mathrm{e}^{-t\mathbf{A}_j}$ are holomorphic. We recall that if $\mathcal{A}$ and $\mathcal{C}$ are boundedly invertible, then the operators $\mathcal{A}^{-1}\mathcal{C}$, $\mathcal{C}\mathcal{A}^{-1}$ and their inverses are all bounded.

Lemma 4.2. *Let* $-\mathbf{A}_j : \mathrm{dom}\,(\mathbf{A}_j) \subset X_j \to X_j$ *be generators of contraction semigroups, and let* $\mathbf{B}_1 : X_2 \to X_1$, $\mathbf{B}_2 : X_1 \to X_2$ *be bounded. Moreover let* $\mathcal{A}$ *and* $\mathcal{C}$ *be boundedly invertible. Then, there is a constant* $C_2 = C_2(\|\mathcal{A}^{-1}\|, \|\mathcal{A}^{-1}\mathcal{B}\|) > 0$ *such that for all* $\tau > 0$ *we have*

$$\|\mathcal{A}^{-1}\left(\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{C}}\right)\| \leq C_2\left(1 + \mathrm{e}^{\tau\|\mathcal{B}\|}\right)\tau.$$

Proof. We have

$$\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{C}} = \mathcal{T}_2(\tau)\mathcal{T}_1(\tau) - \mathrm{e}^{-\tau\mathcal{C}} = \left(\mathcal{T}_2(\tau) - \mathcal{I}\right)\mathcal{T}_1(\tau) + \mathcal{T}_1(\tau) - \mathcal{I} + \mathcal{I} - \mathrm{e}^{-\tau\mathcal{C}},$$

and

$$\begin{aligned} \mathcal{A}^{-1}\left(\mathcal{T}_2(\tau) - \mathcal{I}\right)\mathcal{T}_1(\tau) &= \begin{pmatrix} \cdot & \cdot \\ \mathbf{A}_2^{-1}\mathbf{X}_2(\tau)\mathbf{E}_1(\tau) & \mathbf{A}_2^{-1}\mathbf{X}_2(\tau)\mathbf{X}_1(\tau) + \mathbf{A}_2^{-1}\left(\mathbf{E}_2(\tau) - \mathbf{I}\right) \end{pmatrix} \\ \mathcal{A}^{-1}\left(\mathcal{T}_1(\tau) - \mathcal{I}\right) &= \begin{pmatrix} \mathbf{A}_1^{-1}\left(\mathbf{E}_1(\tau) - \mathbf{I}\right) & \mathbf{A}_1^{-1}\mathbf{X}_1(\tau) \\ \cdot & \cdot \end{pmatrix} \\ \mathcal{A}^{-1}\left(\mathcal{I} - \mathrm{e}^{-\tau\mathcal{C}}\right) &= \mathcal{A}^{-1}\mathcal{C}\mathcal{C}^{-1}\left(\mathcal{I} - \mathrm{e}^{-\tau\mathcal{C}}\right). \end{aligned}$$

Since $\|\mathbf{A}_j^{-1}\left(\mathbf{E}_j(\tau) - \mathbf{I}\right)\| \leq \tau$ (see Lemma 2.1), we get that $\|\mathcal{A}^{-1}\left(\mathcal{T}_2(\tau) - \mathcal{I}\right)\mathcal{T}_1(\tau)\| \leq C\tau$ and $\|\mathcal{A}^{-1}\left(\mathcal{T}_1(\tau) - \mathcal{I}\right)\| \leq C\tau$, where $C = C(\|\mathcal{A}^{-1}\|)$. Moreover, we have

$$\|\mathcal{A}^{-1}\left(\mathcal{I} - \mathrm{e}^{-\tau\mathcal{C}}\right)\| \leq \|\mathcal{A}^{-1}\left(\mathcal{A} - \mathcal{B}\right)\| \cdot \mathrm{e}^{\tau\|\mathcal{B}\|}\tau \leq \left(1 + \|\mathcal{A}^{-1}\mathcal{B}\|\right) \cdot \mathrm{e}^{\tau\|\mathcal{B}\|}\tau.$$

□

Lemma 4.3. *Let $-\mathbf{A}_j : \mathrm{dom}\,(\mathbf{A}_j) \subset X_j \to X_j$ be generators of contraction semigroups, and let $\mathbf{B}_1 : X_2 \to X_1$, $\mathbf{B}_2 : X_1 \to X_2$ be bounded. Moreover let $\mathcal{A}$ and $\mathcal{C}$ be boundedly invertible. Then, there is a constant $C_3 = C_3(\|\mathbf{B}_2\|, \|\mathbf{B}_1\|, \|\mathbf{A}_1^{-1}\|, \|\mathbf{A}_2^{-1}\|) > 0$ such that for all $\tau > 0$ we have*

$$\| \left(\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{C}}\right) \mathcal{A}^{-1} \| \leq C_3 \tau^2 \mathrm{e}^{\tau\|\mathcal{B}\|} .$$

Proof. For better readability we neglect the τ-dependance for a moment. We have the decomposition

$$\mathcal{T}_2\mathcal{T}_1 - \mathrm{e}^{-\tau\mathcal{C}} = (\mathcal{I} - \mathcal{T}_2)(\mathcal{I} - \mathcal{T}_1) + \mathcal{T}_1 + \mathcal{T}_2 - \mathcal{I} - \mathrm{e}^{-\tau\mathcal{C}}.$$

The first term has the form

$$(\mathcal{I} - \mathcal{T}_2)(\mathcal{I} - \mathcal{T}_1) = \begin{pmatrix} \cdot & \cdot \\ \mathbf{X}_2 & \mathbf{I} - \mathbf{E}_2 \end{pmatrix} \begin{pmatrix} \mathbf{I} - \mathbf{E}_1 & \mathbf{X}_1 \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \mathbf{X}_2(\mathbf{I} - \mathbf{E}_1) & \mathbf{X}_2\mathbf{X}_1 \end{pmatrix},$$

which is already of order $O(\tau)$. Moreover, we have

$$(\mathcal{I} - \mathcal{T}_2)(\mathcal{I} - \mathcal{T}_1)\mathcal{A}^{-1} = \begin{pmatrix} \cdot & \cdot \\ \mathbf{X}_2(\mathbf{I} - \mathbf{E}_1)\mathbf{A}_1^{-1} & \mathbf{X}_2\mathbf{X}_1\mathbf{A}_2^{-1} \end{pmatrix},$$

which shows that there is a constant $C = C(\|\mathbf{B}_2\|, \|\mathbf{B}_1\|, \|\mathbf{A}_2^{-1}\|)$ such that the estimate

$$\| (\mathcal{I} - \mathcal{T}_2(\tau))(\mathcal{I} - \mathcal{T}_1(\tau))\mathcal{A}^{-1} \| \leq C\tau^2$$

holds.

It suffices to estimate the remaining part. We have

$$\mathcal{T}_1(\tau) + \mathcal{T}_2(\tau) - \mathcal{I} = \begin{pmatrix} \mathbf{E}_1(\tau) & \mathbf{X}_1(\tau) \\ \mathbf{X}_2(\tau) & \mathbf{E}_2(\tau) \end{pmatrix} =: \widehat{\mathcal{T}}(\tau),$$

where we have introduced the symmetric version $\widehat{\mathcal{T}}$ of $\mathcal{T}_2\mathcal{T}_1$. To estimate the difference $\widehat{\mathcal{T}}(\tau) - \mathrm{e}^{-\tau\mathcal{C}}$, we rely on the following form

$$\begin{aligned}
\left(\widehat{\mathcal{T}}(\tau) - \mathrm{e}^{-\tau\mathcal{C}}\right) &= \int_0^\tau \frac{\mathrm{d}}{\mathrm{d}\sigma}\left\{\widehat{\mathcal{T}}(\sigma)\mathrm{e}^{-(\tau-\sigma)\mathcal{C}}\right\}\mathrm{d}\sigma = \\
&= \int_0^\tau \left\{\widehat{\mathcal{T}}'(\sigma)\mathrm{e}^{-(\tau-\sigma)\mathcal{C}} + \widehat{\mathcal{T}}(\sigma)\mathcal{C}\mathrm{e}^{-(\tau-\sigma)\mathcal{C}}\right\}\mathrm{d}\sigma = \\
&= \int_0^\tau \left\{\widehat{\mathcal{T}}'(\sigma) + \widehat{\mathcal{T}}(\sigma)\mathcal{C}\right\}\mathrm{e}^{-(\tau-\sigma)\mathcal{C}}\mathrm{d}\sigma,
\end{aligned}$$

which holds on $\mathrm{dom}(\mathcal{A})$.

We compute (see Lemma 3.1)

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\widehat{\mathcal{T}}(\sigma) = \frac{\mathrm{d}}{\mathrm{d}\sigma}\begin{pmatrix} \mathbf{E}_1(\sigma) & \mathbf{X}_1(\sigma) \\ \mathbf{X}_2(\sigma) & \mathbf{E}_2(\sigma) \end{pmatrix} = \begin{pmatrix} -\mathbf{A}_1\mathbf{E}_1(\sigma) & \mathbf{E}_1(\sigma)\mathbf{B}_1 \\ \mathbf{E}_2(\sigma)\mathbf{B}_2 & -\mathbf{A}_2\mathbf{E}_2(\sigma) \end{pmatrix},$$

which provides the explicit simple form

$$\begin{aligned}
\widehat{\mathcal{T}}'(\sigma) + \widehat{\mathcal{T}}(\sigma)\mathcal{C} &= \begin{pmatrix} -\mathbf{A}_1\mathbf{E}_1(\sigma) & \mathbf{E}_1(\sigma)\mathbf{B}_1 \\ \mathbf{E}_2(\sigma)\mathbf{B}_2 & -\mathbf{A}_2\mathbf{E}_2(\sigma) \end{pmatrix} + \begin{pmatrix} \mathbf{E}_1(\sigma) & \mathbf{X}_1(\sigma) \\ \mathbf{X}_2(\sigma) & \mathbf{E}_2(\sigma) \end{pmatrix}\begin{pmatrix} \mathbf{A}_1 & -\mathbf{B}_1 \\ -\mathbf{B}_2 & \mathbf{A}_2 \end{pmatrix} \\
&= \begin{pmatrix} -\mathbf{X}_1(\sigma)\mathbf{B}_2 & \mathbf{X}_1(\sigma)\mathbf{A}_2 \\ \mathbf{X}_2(\sigma)\mathbf{A}_1 & -\mathbf{X}_2(\sigma)\mathbf{B}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1(\sigma) & \cdot \\ \cdot & \mathbf{X}_2(\sigma) \end{pmatrix}\begin{pmatrix} -\mathbf{B}_2 & \mathbf{A}_2 \\ \mathbf{A}_1 & -\mathbf{B}_1 \end{pmatrix}.
\end{aligned}$$

In particular, we have

$$\left(\widehat{\mathcal{T}}'(\sigma)+\widehat{\mathcal{T}}(\sigma)\mathcal{C}\right)\mathcal{A}^{-1}=\begin{pmatrix}\mathbf{X}_1(\sigma) & \cdot \\ \cdot & \mathbf{X}_2(\sigma)\end{pmatrix}\begin{pmatrix}-\mathbf{B}_2\mathcal{A}_1^{-1} & \mathbf{I} \\ \mathbf{I} & -\mathbf{B}_1\mathcal{A}_2^{-1}\end{pmatrix}.$$

Hence, we get that there is a constant $C=C(\|\mathbf{B}_1\|,\|\mathbf{B}_2\|,\|\mathcal{A}^{-1}\|)$ such that

$$\begin{aligned}\|\left(\widehat{\mathcal{T}}(\tau)-\mathrm{e}^{-\tau\mathcal{C}}\right)\mathcal{A}^{-1}\| &= \|\int_0^\tau\left\{\widehat{\mathcal{T}}'(\sigma)+\widehat{\mathcal{T}}(\sigma)\mathcal{C}\right\}\mathrm{e}^{-(\tau-\sigma)\mathcal{C}}\mathcal{A}^{-1}\mathrm{d}\sigma\| \\ &= \|\int_0^\tau\left\{\widehat{\mathcal{T}}'(\sigma)+\widehat{\mathcal{T}}(\sigma)\mathcal{C}\right\}\mathcal{A}^{-1}\mathcal{A}\mathcal{C}^{-1}\mathrm{e}^{-(\tau-\sigma)\mathcal{C}}\mathcal{C}\mathcal{A}^{-1}\mathrm{d}\sigma\| \\ &\le \int_0^\tau\|\left\{\widehat{\mathcal{T}}'(\sigma)+\widehat{\mathcal{T}}(\sigma)\mathcal{C}\right\}\mathcal{A}^{-1}\|\cdot\|\mathcal{A}\mathcal{C}^{-1}\|\cdot\|\mathrm{e}^{-(\tau-\sigma)\mathcal{C}}\|\cdot\|\mathcal{C}\mathcal{A}^{-1}\|\mathrm{d}\sigma \\ &\le C\int_0^\tau\sigma\mathrm{e}^{(\tau-\sigma)\|\mathcal{B}\|}\mathrm{d}\sigma\le\frac{C}{2}\tau^2\mathrm{e}^{\tau\|\mathcal{B}\|}, \end{aligned} \tag{2}$$

which proves the claim. □

4.2 Convergence result for holomorphic semigroups

We are now able to state and prove the main theorem, which show convergence in operator norm with convergence rate estimate of $O(\frac{\log n}{n})$.

Theorem 4.4. *Let $-\mathbf{A}_j:\mathrm{dom}\,(\mathbf{A}_j)\subset X_j\to X_j$ be generators of holomorphic contraction semigroups, and let $\mathbf{B}_1:X_2\to X_1$, $\mathbf{B}_2:X_1\to X_2$ be bounded. Then, there are constants $C,\eta>0$ such that for all $t\ge 0$ and $n\in\mathbb{N}$, we have*

$$\|\mathcal{T}(t/n)^n-\mathrm{e}^{-t\mathcal{C}}\|\le\frac{C}{n}\mathrm{e}^{t\eta}\mathrm{e}^{4t(\|\mathbf{B}_1\|+\|\mathbf{B}_2\|)}\left(\log n+t^2\right).$$

Proof. It is clear that nothing has to be shown for $t=0$ or $n=1$. So let $t>0$ and $n\ge 2$, and let us introduce $\tau=t/n$. If $-\mathcal{A}$ and $-\mathcal{C}$ are not boundedly invertible then, by introducing a shift $\eta>0$ they can be made invertible and the previous estimates remain unchanged. Indeed, defining $\widetilde{\mathbf{A}}_j=\mathbf{A}_j+\eta$ and $\widetilde{\mathcal{C}}=\mathcal{C}+\eta$ for $\eta>0$ such that $\widetilde{\mathbf{A}}_j,\widetilde{\mathcal{C}}$ are invertible, we observe that $\widetilde{\mathbf{E}}_j(\tau):=\mathrm{e}^{-\tau\widetilde{\mathbf{A}}_j}=\mathrm{e}^{-\tau\eta}\mathbf{E}_j(\tau)$. Clearly, $-\widetilde{\mathcal{C}}$ is a generator of a holomorphic semigroup and we have

$$\|t\widetilde{\mathcal{C}}\mathrm{e}^{-t\widetilde{\mathcal{C}}}\|=\|t\left(\mathcal{C}+\eta\right)\mathrm{e}^{-t\mathcal{C}}\mathrm{e}^{-t\eta}\|\le M_{\mathcal{C}}+t\eta\mathrm{e}^{-t\eta}\|\mathrm{e}^{-t\mathcal{C}}\|\le M_{\mathcal{C}}+\frac{1}{2}\mathrm{e}^{t(\|\mathbf{B}_1\|+\|\mathbf{B}_2\|)}, \tag{3}$$

where we have used that $x\mathrm{e}^{-x}\le\frac{1}{2}$ for all $x\ge 0$.

Moreover, we define $\widetilde{\mathbf{X}}_1(\tau):=\mathrm{e}^{-\tau\eta}\mathbf{X}_1(\tau)$. Then, we have for all $\tau\ge 0$ that

$$\mathrm{e}^{\tau\eta}\widetilde{\mathcal{T}}(\tau):=\mathrm{e}^{\tau\eta}\begin{pmatrix}\widetilde{\mathbf{E}}_1(\tau) & \widetilde{\mathbf{X}}_1(\tau) \\ \mathbf{X}_2(\tau)\widetilde{\mathbf{E}}_1(\tau) & \mathbf{X}_2(\tau)\widetilde{\mathbf{X}}_1(\tau)+\widetilde{\mathbf{E}}_2(\tau)\end{pmatrix}=\mathcal{T}(\tau).$$

Then, for all $\tau\ge 0$ we have that $\|\widetilde{\mathbf{X}}_1(\tau)\|\le\tau\|\mathbf{B}_1\|$ and

$$\|\widetilde{\mathbf{A}}_1\widetilde{\mathbf{X}}_1(\tau)\|\le\|\mathbf{A}_1\mathbf{X}_1(\tau)\|+\eta\|\mathbf{X}_1(\tau)\|,\quad\|\widetilde{\mathbf{A}}_2\mathbf{X}_2(\tau)\|\le\|\mathbf{A}_2\mathbf{X}_2(\tau)\|+\eta\|\mathbf{X}_2(\tau)\|.$$

In particular, Lemma 4.1 and Lemma 4.2 can now be adapted to the shifted situation. Moreover, we have $\widetilde{\mathbf{X}}_1'(\tau) - \mathbf{X}_1'(\tau) = -\eta \mathrm{e}^{-\tau\eta}\mathbf{X}_1(\tau)$, which shows that the estimate (2) holds, so Lemma 4.3 can also be adapted to the shifted situation. We have

$$\begin{aligned}
\mathrm{e}^{-t\widetilde{\mathcal{C}}} - \widetilde{\mathcal{T}}(t/n)^n &= \left(\mathrm{e}^{-\tau\widetilde{\mathcal{C}}}\right)^n - \widetilde{\mathcal{T}}(\tau)^n \\
&= \sum_{k=0}^{n-1} \mathrm{e}^{-\tau(n-1-k)\widetilde{\mathcal{C}}} \left(\mathrm{e}^{-\tau\widetilde{\mathcal{C}}} - \widetilde{\mathcal{T}}(\tau)\right) \widetilde{\mathcal{T}}(\tau)^k \\
&= \mathrm{e}^{-\tau(n-1)\widetilde{\mathcal{C}}}\widetilde{\mathcal{C}}\widetilde{\mathcal{C}}^{-1}\widetilde{\mathcal{A}}\widetilde{\mathcal{A}}^{-1}\left(\mathrm{e}^{-\tau\widetilde{\mathcal{C}}} - \widetilde{\mathcal{T}}(\tau)\right) + \left(\mathrm{e}^{-\tau\widetilde{\mathcal{C}}} - \widetilde{\mathcal{T}}(\tau)\right)\widetilde{\mathcal{A}}^{-1}\widetilde{\mathcal{A}}\widetilde{\mathcal{T}}(\tau)^{n-1} \\
&\qquad + \sum_{k=1}^{n-2} \mathrm{e}^{-\tau(n-1-k)\widetilde{\mathcal{C}}} \left(\mathrm{e}^{-\tau\widetilde{\mathcal{C}}} - \widetilde{\mathcal{T}}(\tau)\right)\widetilde{\mathcal{A}}^{-1}\widetilde{\mathcal{A}}\widetilde{\mathcal{T}}(\tau)^k,
\end{aligned}$$

where we have used the product $\mathcal{S}^k - \mathcal{T}^k = \sum_{j=0}^{k-1} \mathcal{S}^{k-1-j}(\mathcal{S} - \mathcal{T})\mathcal{T}^j$. Then, by Lemma 4.1, 4.2 and 4.3 , we get

$$\begin{aligned}
&\|\mathrm{e}^{-t\widetilde{\mathcal{C}}} - \widetilde{\mathcal{T}}(t/n)^n\| \\
&\quad \leq \|\mathrm{e}^{-\tau(n-1)\widetilde{\mathcal{C}}}\widetilde{\mathcal{C}}\| \cdot \|\widetilde{\mathcal{C}}^{-1}\widetilde{\mathcal{A}}\| \cdot \|\widetilde{\mathcal{A}}^{-1}\left(\mathrm{e}^{-\tau\widetilde{\mathcal{C}}} - \widetilde{\mathcal{T}}(\tau)\right)\| + \|\left(\mathrm{e}^{-\tau\widetilde{\mathcal{C}}} - \widetilde{\mathcal{T}}(\tau)\right)\widetilde{\mathcal{A}}^{-1}\| \cdot \|\widetilde{\mathcal{A}}\widetilde{\mathcal{T}}(\tau)^{n-1}\| \\
&\qquad + \sum_{k=1}^{n-2} \|\mathrm{e}^{-\tau(n-1-k)\widetilde{\mathcal{C}}}\| \cdot \|\left(\mathrm{e}^{-\tau\widetilde{\mathcal{C}}} - \widetilde{\mathcal{T}}(\tau)\right)\widetilde{\mathcal{A}}^{-1}\| \cdot \|\widetilde{\mathcal{A}}\widetilde{\mathcal{T}}(\tau)^k\| \\
&\quad \leq \frac{1}{\tau(n-1)}\left(M_{\mathcal{C}} + \frac{1}{2}\mathrm{e}^{\tau(n-1)\|\mathcal{B}\|}\right)\|\widetilde{\mathcal{C}}^{-1}\widetilde{\mathcal{A}}\| \cdot C_2\left(1 + \mathrm{e}^{\tau\|\mathcal{B}\|}\right)\tau + \\
&\qquad + C_3\tau^2\left\{C_1\mathrm{e}^{t\|\mathcal{B}\|}(1 + \log(n-1)) + \frac{M_A}{(n-1)\tau}\right\} + \\
&\qquad + \sum_{k=1}^{n-2} \mathrm{e}^{\tau(n-1-k)\|\mathcal{B}\|}\mathrm{e}^{-t\eta}C_3\tau^2\left\{C_1\mathrm{e}^{t\|\mathcal{B}\|}(1 + \log k) + \frac{M_A}{k\tau}\right\} \\
&\quad \leq \frac{1}{(n-1)}\left(M_1 + M_2\mathrm{e}^{t(\|\mathbf{B}_1\| + \|\mathbf{B}_2\|)}\right) + C_3\frac{t}{n}\left\{C_1 t\mathrm{e}^{t\|\mathcal{B}\|} + \frac{M_A}{(n-1)}\right\} + \\
&\qquad + \mathrm{e}^{t\|\mathcal{B}\|}\mathrm{e}^{-t\eta}C_3\frac{t}{n}\left\{C_1\mathrm{e}^{t\|\mathcal{B}\|}n(1 + \log n)\frac{t}{n} + M_A\log n\right\} \\
&\quad \leq \frac{C}{n}\mathrm{e}^{2t\|\mathcal{B}\|}\left(\log n + t^2\right),
\end{aligned}$$

for a constant $C > 0$, where we have used that

$$(1 + \log(n-1))\frac{1}{n} \leq 1, \quad \sum_{k=1}^{n-2}\frac{1}{k} \leq \log n, \quad \sum_{k=1}^{n-2}\log k \leq n\log n.$$

For the operators without the shift this means

$$\begin{aligned}
\|\mathcal{T}(t/n)^n - \mathrm{e}^{-t\mathcal{C}}\| = \|\left(\mathrm{e}^{t/n\eta}\widetilde{\mathcal{T}}(\frac{t}{n})\right)^n - \mathrm{e}^{t\eta}\mathrm{e}^{-t\widetilde{\mathcal{C}}}\| &= \mathrm{e}^{t\eta}\|\mathrm{e}^{-t\widetilde{\mathcal{C}}} - \widetilde{\mathcal{T}}(t/n)^n\| \\
&\leq \frac{C}{n}\mathrm{e}^{t\eta}\mathrm{e}^{2t\|\mathcal{B}\|}\left(\log n + t^2\right),
\end{aligned}$$

which shows the claimed estimate. □

4.3 Operator-norm convergence in weaker norm

Interestingly, we get on the subspace $\mathrm{dom}\,(\mathcal{A}) \subset X$ a similar convergence result as Theorem4.4 without assuming that the semigroups $\mathrm{e}^{-t\mathbf{A}_j}$ are holomorphic. For this we assume that $\mathbf{A}_j$ are boundedly invertible (as we have seen in the proof of Theorem 4.4 we could otherwise introduce a shift). We define a new operator norm for bounded operators $\mathcal{B} : X \to X$,

$$\|\mathcal{B}\|_{\mathcal{A}} := \sup_{f\in\mathrm{dom}(\mathcal{A}):\|\mathcal{A}f\|\le 1} \|\mathcal{B}f\| = \sup_{g\in X:\|g\|\le 1} \|\mathcal{B}\mathcal{A}^{-1}g\| = \|\mathcal{B}\mathcal{A}^{-1}\|.$$

If$-\mathbf{A}_j$ is unbounded, a bound on $\|\mathcal{B}\|_{\mathcal{A}}$ does not provide a bound on $\|\mathcal{B}\|$ in general. The crucial observation is Lemma 4.3, which provides an bound $\|\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{C}}\|_{\mathcal{A}} = O(\tau^2)$. Note that Lemma 4.3 here is a better estimate than the analogous results in [NSZ17, NSZ18a, NSZ18b, NSZ19, NSZ20] because the spatial regularization is only needed once to obtain an estimate of order $O(\tau)$. We refer also to [JaL00, HaO09] for comparable results related to the Trotter-product formula.

Theorem 4.5. *Let $-\mathbf{A}_j : \mathrm{dom}\,(\mathbf{A}_j) \subset X_j \to X_j$ be generators of contraction semigroups, and let $\mathbf{B}_1 : X_2 \to X_1$, $\mathbf{B}_2 : X_1 \to X_2$ be bounded. Moreover let $\mathcal{A}$ and $\mathcal{C}$ be boundedly invertible. Then, there is a constant $C = C(\|\mathbf{B}_2\|, \|\mathbf{B}_1\|, \|\mathbf{A}_1^{-1}\|, \|\mathbf{A}_2^{-1}\|) > 0$ such that for all $t > 0$ and $n \ge 1$,we have*

$$\|\mathcal{T}(t/n)^n - \mathrm{e}^{-t\mathcal{C}}\|_{\mathcal{A}} \le \frac{C}{n} t^2 \mathrm{e}^{2t(\|\mathbf{B}_1\|+\|\mathbf{B}_2\|)}\,.$$

Proof. There is nothing to show for $t = 0$ and $n = 1$. So let $t > 0$ and $n \ge 2$. Introducing, $\tau = \frac{t}{n}$ and using the product $\mathcal{S}^k - \mathcal{T}^k = \sum_{j=0}^{k-1} \mathcal{S}^{k-1-j}(\mathcal{S} - \mathcal{T})\mathcal{T}^j$, we have

$$\begin{aligned}
\left(\mathcal{T}(\tau)^n - \mathrm{e}^{-t\mathcal{C}}\right)\mathcal{A}^{-1} &= \left(\mathcal{T}(\tau)^n - \left(\mathrm{e}^{-\tau\mathcal{C}}\right)^n\right)\mathcal{A}^{-1}\\
&= \sum_{k=0}^{n-1} \mathcal{T}(\tau)^{n-k-1}\left(\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{C}}\right)\mathrm{e}^{-\tau k\mathcal{C}}\mathcal{A}^{-1}\\
&= \mathcal{T}(\tau)^{n-1}\left(\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{C}}\right)\mathcal{A}^{-1} + \left(\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{C}}\right)\mathcal{A}^{-1}\mathcal{A}\mathcal{C}^{-1}\mathrm{e}^{-\tau(n-1)\mathcal{C}}\mathcal{C}\mathcal{A}^{-1}\\
&\quad + \sum_{k=1}^{n-2} \mathcal{T}(\tau)^{n-k-1}\left(\mathcal{T}(\tau) - \mathrm{e}^{-\tau\mathcal{C}}\right)\mathcal{A}^{-1}\mathcal{A}\mathcal{C}^{-1}\mathrm{e}^{-\tau k\mathcal{C}}\mathcal{C}\mathcal{A}^{-1}.
\end{aligned}$$

By Lemma 4.3, the first and the second term can be estimated by $C\mathrm{e}^{t(\|\mathbf{B}_1\|+\|\mathbf{B}_2\|)}\tau^2$. For the sum in the last term, we have the bound $Cn\mathrm{e}^{t(\|\mathbf{B}_1\|+\|\mathbf{B}_2\|)}\tau^2$. Hence, we conclude

$$\|\left(\mathcal{T}(\tau)^n - \mathrm{e}^{-t\mathcal{C}}\right)\mathcal{A}^{-1}\| \le \frac{C}{n} t^2 \mathrm{e}^{t(\|\mathbf{B}_1\|+\|\mathbf{B}_2\|)}\,.$$

□

4.4 Other similar approximations

Analogue convergence results as 4.4 and 4.5 can also be shown for other approximations.

4.4.1 Transposed approximation

Instead on applying first $\mathcal{T}_1$ and then $\mathcal{T}_2$, one could also consider the transposed approximation $\mathcal{T}_{\mathrm{T}}(\tau) := \mathcal{T}_1(\tau)\mathcal{T}_2(\tau)$. Since, we have

$$\mathcal{T}_\mathrm{T} - \mathcal{T} = \mathcal{T}_1\mathcal{T}_2 - \mathcal{T}_2\mathcal{T}_1 = \begin{pmatrix} \mathbf{X}_1\mathbf{X}_2 & \mathbf{X}_1(\mathbf{E}_2 - \mathbf{I}) \\ \mathbf{X}_2(\mathbf{I} - \mathbf{E}_1) & \mathbf{X}_2\mathbf{X}_1 \end{pmatrix},$$

we immediately obtain that $(\mathcal{T}_\mathrm{T}(\tau) - \mathcal{T}(\tau))\,\mathcal{A}^{-1} = O(\tau^2)$, which provides an analogue of Lemma 4.2. Hence, we get for $\mathcal{T}_\mathrm{T}(t/n)^n - \mathrm{e}^{-t\mathcal{C}}$ the same operator-norm convergence result as in Theorem 4.4.

4.4.2 Symmetrized approximation

In the proof of Lemma 4.2, we had already used the symmetric approximation $\widehat{\mathcal{T}}$, which has the simple form

$$\widehat{\mathcal{T}}(t) = \begin{pmatrix} \mathbf{E}_1(t) & \mathbf{X}_1(t) \\ \mathbf{X}_2(t) & \mathbf{E}_2(t) \end{pmatrix} = \mathrm{e}^{-t\mathcal{A}} + \begin{pmatrix} \cdot & \int_0^t \mathrm{e}^{-s\mathbf{A}_1}\mathrm{d}s \\ \int_0^t \mathrm{e}^{-s\mathbf{A}_2}\mathrm{d}s & \cdot \end{pmatrix} \circ \mathcal{B}\,.$$

In Lemma 4.2 it is shown that we have an analogue result of the form $\left(\widehat{\mathcal{T}}(\tau) - \mathrm{e}^{-t\mathcal{C}}\right)\mathcal{A}^{-1} = O(\tau^2)$ holds. Hence, we get for $\widehat{\mathcal{T}}(t/n)^n - \mathrm{e}^{-t\mathcal{C}}$ the same operator-norm convergence result as in Theorem 4.4.

4.4.3 Naive solution of the integral

In the convergence result in the strong topology Proposition (3.2), we have already discussed the approximation, where the integral in the coupling term is naively solved and $\mathbf{X}_j(\tau)$ is replaced by $\tau\mathbf{B}_j$, which leads to $\mathcal{T}_\mathrm{B} = \begin{pmatrix} \mathbf{E}_1 & \tau\mathbf{B}_1 \\ \tau\mathbf{B}_2 & \mathbf{E}_2 \end{pmatrix}$. However, an analogue convergence result for the term $\left(\mathcal{T}_\mathrm{B}(\tau) - \mathrm{e}^{-\tau\mathcal{C}}\right)\mathcal{A}^{-1}$ is not clear.

5 Remarks on unbounded coupling $\mathcal{B}$

In this section, we briefly comment on the situation where the coupling between the spaces X_1 and X_2 is given by an unbounded linear operator $\mathcal{B} = \begin{pmatrix} \cdot & \mathbf{B}_1 \\ \mathbf{B}_2 & \cdot \end{pmatrix}$, $\mathbf{B}_1 : \mathrm{dom}(\mathbf{B}_1) \subset X_2 \to X_1$ and $\mathbf{B}_2 : \mathrm{dom}(\mathbf{B}_2) \subset X_1 \to X_2$. Throughout the section, we assume that $-\mathbf{A}_j$ are boundedly invertible generators of holomorphic contraction semigroups.

5.1 Existence of solution operators for the perturbed system and the inhomogeneous abstract Cauchy problem

For generators of holomorphic semigroups $-\mathbf{A}_j$ it is possible to define fractional powers $\mathbf{A}_j^\alpha$, $\alpha \in [0,1]$ interpolating between $\mathbf{A}_j$ and $\mathbf{I}$. In previous similar works (see e.g. [CaZ01, NSZ20]), it is assumed that there is an $\alpha \in [0,1[$ such that $\mathrm{dom}(\mathcal{A}^\alpha) \subset \mathrm{dom}(\mathcal{B})$ and that $\mathcal{B}\mathcal{A}^{-\alpha} : X \to X$ is bounded, or equivalently that $\mathbf{B}_1\mathbf{A}_2^{-\alpha} : X_2 \to X_1$ and that $\mathbf{B}_2\mathbf{A}_1^{-\alpha} : X_1 \to X_2$ are bounded. Then, $\mathcal{B}$ is $\mathcal{A}$-bounded with relative bound zero, and hence, the sum $-\mathcal{A}+\mathcal{B}$ is a generator of a holomorphic semigroup, by classical perturbation results [EnN00, Theorem III.2.10].

Moreover, we are going to assume that there is a $\beta \in [0,1[$ such that $\mathbf{A}_i^{-\beta}\mathbf{B}_i$ is bounded, or equivalently, that $\mathcal{A}^{-\beta}\mathcal{B}$ is bounded. Then for all $\tau \geq 0$ the split-step approximation operators $\mathcal{T}_i(\tau) : X \to X$ defined by (AO) are bounded. Indeed, we have for all $y \in X_2$ that

$$\begin{aligned}\|\int_0^\tau \mathrm{d}\sigma \mathrm{e}^{-\sigma\mathbf{A}_1}\mathbf{B}_1 y\| = \|\int_0^\tau \mathrm{d}\sigma \mathrm{e}^{-\sigma\mathbf{A}_1}\mathbf{A}_1^{\beta}\mathbf{A}_1^{-\beta}\mathbf{B}_1 y\| \leq \int_0^\tau \|\mathrm{e}^{-\sigma\mathbf{A}_1}\mathbf{A}_1^{\beta}\|\mathrm{d}\sigma \cdot \|\mathbf{A}_1^{-\beta}\mathbf{B}_1 y\| \\ \leq C_\beta \int_0^\tau \sigma^{-\beta}\mathrm{d}\sigma \cdot \|\mathbf{A}_1^{-\beta}\mathbf{B}_1\| \cdot \|y\| \leq C_\beta \tau^{1-\beta}\|\mathbf{A}_1^{-\beta}\mathbf{B}_1\| \cdot \|y\|,\end{aligned}$$

where we have used, that for generators $-\mathbf{A}$ of bounded holomorphic semigroups, there is a constant $C_\beta > 0$ such that for all $t > 0$ we have the estimate

$$\|\mathbf{A}^\beta \mathrm{e}^{-t\mathbf{A}}\| \leq \frac{C_\beta}{t^\beta}\,.$$

Hence, we get that $\mathcal{T}_1(\tau) : X \to X$ is bounded. Similarly we get that also $\mathcal{T}_2(\tau) : X \to X$ is bounded, and thus defining a bounded time-discretization $\mathcal{T}(\tau) = \mathcal{T}_2(\tau)\mathcal{T}_1(\tau)$, which satisfies $\mathcal{T}(0) = \mathcal{I}$.

5.2 Stability of the approximation family

To ensure that $\{\mathcal{T}(\tau)\}_{\tau\geq 0}$ is a reasonable approximation family, we have to show that $\mathcal{T}$ is stable, i.e. the family $\mathcal{T}(t/n)^n$ is uniformly bounded. We shortly discussed why stability is delicate and in general cannot expected under the assumptions here.

We have seen that the coupling terms in $\mathcal{T}_j$ are bounded by $O(\tau^{1-\beta})$. So we get (neglecting bounded operators)

$$\begin{aligned}\mathcal{T}(\tau) = \mathcal{T}_2(\tau)\mathcal{T}_1(\tau) &= \begin{pmatrix} \mathbf{I} & \cdot \\ \int_0^\tau \mathrm{d}\sigma \mathrm{e}^{-\sigma\mathbf{A}_2}\mathbf{B}_2 & \mathrm{e}^{-\tau\mathbf{A}_2}\end{pmatrix}\begin{pmatrix}\mathrm{e}^{-\tau\mathbf{A}_1} & \int_0^\tau \mathrm{d}\sigma \mathrm{e}^{-\sigma\mathbf{A}_1}\mathbf{B}_1 \\ \cdot & \mathbf{I}\end{pmatrix} \\ &\approx \begin{pmatrix}\mathbf{I} & \cdot \\ \tau^{1-\beta}\mathbf{I} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{I} & \tau^{1-\beta}\mathbf{I} \\ \cdot & \mathbf{I}\end{pmatrix} = \begin{pmatrix}\mathbf{I} & \tau^{1-\beta} \\ \tau^{1-\beta} & \left(\tau^{2-2\beta}+1\right)\mathbf{I}\end{pmatrix}.\end{aligned}$$

If $\beta = 0$, then we have $\begin{pmatrix}1 & \tau \\ \tau & 1+\tau^2\end{pmatrix} = \begin{pmatrix}1 & \cdot \\ \cdot & 1\end{pmatrix} + \tau\begin{pmatrix}\cdot & 1 \\ 1 & \tau\end{pmatrix}$,which has bounded powers expressed by the matrix exponential.

However, we have that on $\mathbb{R}^2$ that the matrix powers of $\begin{pmatrix}1 & \tau^{1-\beta} \\ \tau^{1-\beta} & 1+\tau^{2-2\beta}\end{pmatrix} =: \mathbf{P}(\tau^{1-\beta})$ for $\beta \in]0,1]$ are unbounded. Indeed, fixing $x = \tau^{1-\beta}$ we have that $v(x) = \left(\frac{1}{2}(-x+\sqrt{4+x^2}),1\right)^{\mathrm{T}}$ is an eigenvector of $\mathbf{P}(x)$, with $\mathbf{P}(x)v(x) = \left(1+\frac{x}{2}\left(x+\sqrt{4+x^2}\right)\right)v(x)$, and $v(x)$ is uniformly bounded for $x = \tau^{1-\beta}$ as $\tau \to 0$, with $v(x) \to (1,1)^{\mathrm{T}}$. Hence,

$$\begin{aligned}\|\mathcal{T}(t/n)^n\| &\geq \frac{1}{\|v(x)\|}\|\begin{pmatrix}1 & x \\ x & 1+x^2\end{pmatrix}^n v(x)\| = \frac{1}{\|v(x)\|}\|\left(1+\frac{x}{2}\left(x+\sqrt{4+x^2}\right)\right)^n v(x)\|. \\ &\geq \left(1+\frac{x}{2}\left(x+\sqrt{4+x^2}\right)\right)^n \geq 1+n\frac{x}{2}\left(x+\sqrt{4+x^2}\right) = nx = n^\beta t^{1-\beta},\end{aligned}$$

which tends to infinity as $n \to \infty$. This means, stability of $\mathcal{T}(\tau)$ is in general not clear, and, hence, convergence of $\mathcal{T}(\tau)^n$ to $\mathrm{e}^{-t\mathcal{C}}$ (even in the strong topology) cannot be expected.

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