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Generalized heteroclinic cycles in spherically invariant systems and their perturbations

Pascal Chossat[1] , Frédéric Guyard[1], Reiner Lauterbach[2]

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[1] Institute Non Linéaire de Nice
(CNRS et Université de Nice)
1361 Route des Lucioles
Sophia–Antipolis–06560 Valbonne
France

[2] Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstraße 39
D – 10117 Berlin
Germany

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Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
e–mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
e–mail (Internet): preprint@wias-berlin.de

Abstract

In this paper we want to investigate the effects of forced symmetry breaking perturbations, see LAUTERBACH & ROBERTS [29], as well as [28, 31], on the heteroclinic cycle which was found in the $\ell = 1$, $\ell = 2$ mode interaction by ARMBRUSTER & CHOSSAT [1, 12] and generalized by CHOSSAT and GUYARD [25, 14]. We show that this cycle is embedded in a larger class of cycles, which we call a *generalized heteroclinic cycle* (GHC). After describing the structure of this set we discuss its stability. The main problem is to find a selection principle, that is to give a mechanism which enables the physical system to select one particular heteroclinic cycle on this generalized heteroclinic cycle. After that the persistence under symmetry breaking perturbations is investigated. We will discuss also the application to the spherical Bénard problem, which was the initial motivation for this work.

Introduction

The idea that symmetry can induce intermittent-like behavior in dynamical systems, and more specifically in hydrodynamical systems, has been popularized in the late 80's by the papers of GUCKENHEIMER & HOLMES [24] in a problem related to thermal convection in a rotating domain and ARMBRUSTER, GUCKENHEIMER and HOLMES [2] in a problem of fluid flow near a wall. In each of these cases, the symmetries were simple enough that it was relatively easy to show the existence and stability of a *heteroclinic cycle* connecting nontrivial equilibria (in fact, a *homoclinic cycle* since these equilibria belong to the same group orbit). Intermittency occurs because there are initial condition near the heteroclinic cycle such that the corresponding solution spends very long time near each of the equilibria in the cycle and "jumps" to the next equilibrium in a short time. Other similar examples have been found, in particular in systems with spherical symmetry. This was observed by numerical simulations for the amplitude equations for the onset of convection in a spherical shell, by FRIEDRICH and HAKEN [19]. They pointed out the fact that intermittency occurs between axisymmetric steady-states with reversed flow directions. It was proved by ARMBRUSTER and CHOSSAT [1] that for this differential system, under certain conditions, a robust heteroclinic cycle exists. This example is of particular interest for several reasons : first, it is related to important questions in astrophysics (convection in celestial bodies); second, it deals with a highly nontrivial group action (spherical group O(3)) and although a detailed description of possible heteroclinic cycles was made in [1], there were still various unsolved questions, like the asymptotic stability of these objects, or their perturbation when the domain is allowed to rotate around an axis. Moreover, a closer look at [19] reveals that the intermittent behavior observed by FRIEDRICH and HAKEN does not completely fit to the heteroclinic cycle of ARMBRUSTER and CHOSSAT. Is it possible to understand the discrepancy (see section 1)? In the present paper we intend to study these questions. We show that for asymptotic stability to be achieved, a list of conditions on the coefficients of the amplitude equations must be satisfied. These conditions are not generic for the bifurcation problem unless some coefficients are assumed to vary as free parameters (in other words, the problem is of high codimension). It turns out however, that for the model equations of the onset of convection (Bénard problem), the range of coefficient values is consistent with such conditions. This, we thought, is enough to justify a further analysis of this problem. In our attempt to extend the validity of existence and stability of heteroclinic cycles, we have found that in fact there exists a larger sets of connections between the equilibria. The heteroclinic cycles which were studied so far consisted of isolated connecting orbits. We found that higher dimensional sets of connections could also exist, extending therefore the domain of existence of heteroclinic cycles. However, since heteroclinic cycles usually designate cycles with isolated connecting orbits, we have given the name *generalized heteroclinic cycle* (GHC) to any such higher dimensional heteroclinic cycle. This is consistent with the recent definition given in [3] to similar objects. The stability analysis of these GHCs does not follow from the "classical" results on the subject (see [26, 27]). We have performed this study by means of orbit space reduction. This technique, which consists in projecting the equivariant differential system onto the space of group orbits (each orbit being then identified with a point), proved to be useful in our case, although it is certainly not the single possible approach. Anyway, this allowed us to find sufficient conditions for the asymptotic stability of the GHC. We did not try to go further in the stability analysis (for example in proving necessity of the condition), because that was going beyond the scope of our objectives. It is interesting to notice here that essential asymptotic stability of the heteroclinic cycle (see section 4) is strongly related to the stability condition of the GHC. The next question is the forced symmetry-breaking of the (generalized) heteroclinic cycle when a perturbation is introduced into the equations, which only commutes with rotations around a given axis. In the Bénard problem, this perturbation is due to the rotation of the domain which introduces the Coriolis force in the equation of motion. We are therefore more specifically interested in perturbations which reduce the symmetry to the group $\mathbf{SO}(2) \times \mathbb{Z}_2^c$, where $\mathbb{Z}_2^c$ stands for the antipodal symmetry. Surprisingly enough, we found that such a perturbation *does not* completely destroy the heteroclinic cycle, but replaces it with a cycle involving pure equilibria as well as rotating waves (relative equilibria). This shows that when the domain rotates "slowly", as it is the case for certain planets, an intermittent convection can exist, which shows reversals of the flow. This could also be of relevant interest in the case of the Earth's core convection, although it is generally believed that in this case, inertial effects dominate convection.

The plan of the paper is as follows: in sections 1 we introduce our motivating example of spherical Bénard convection; in section 2 we set up the differential system, we describe the group action and introduce notations; in section 3 we recall the basic results of [1] and specify the stability conditions of the heteroclinic cycle; in section

4 we show the existence of the heteroclinic set and we study its stability; in section 5 we introduce the perturbation and show the persistence of a heteroclinic cycle in this case.

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1 The motivation : thermal convection in a spherical shell

1.1 The physical model and bifurcation problem

The problem of the onset of thermal convection for a fluid in a spherical shell with spherically symmetric buoyancy forces has been described by many authors (notably by [7] and [6]). Our framework will be the "classical" Bénard problem with an incompressible fluid uniformly heated from the inner boundary (core) and/or from distributed heat sources inside the fluid (radioactive material). Moreover, we shall allow for a slow rotation of the shell around a polar axis. We do not need to recall the model equations (Navier-Stokes and heat in the Boussinesq approximation) here nor the functional frame and linear stability analysis for the basic, purely conductive solution (see [8] and [19] for details). Below we list all we need to know.

- Assume first that the system does not rotate. Then the equations are invariant under the natural action of the group $\mathbf{O}(3)$ on the state variables (velocity, pressure and temperature).
- After a suitable non-dimensionalisation of the equation, the following numbers come up into the equation : $\eta = R_i/R_o$ is the aspect ratio of the spherical shell; Ra is the Rayleigh number, proportional to the temperature difference across the domain and inverse proportional to the fluid viscosity ν; $Pr = \frac{\nu}{\chi}$ is the Prandtl number (χ is the coefficient of thermal diffusivity). If Ra exceeds a critical value Ra_{crit}, the basic state of rest is linearly unstable and convection sets in through a steady-state bifurcation. The Prandtl number is assumed to be fixed in the following.
- To determine Ra_{crit} one relies on a numerical analysis of the eigenvalue problem for the linear system, which reduces to a differential system in the radial coordinate r after an expansion in spherical harmonics $Y_{\ell m}(\theta, \varphi)$ has been performed. Here $\ell > 0$ and $-\ell \leq m \leq \ell$. It can be shown that the critical modes (eigenvectors for the critical eigenvalue 0) are associated with a value

ℓ_0 of ℓ, which tends to increase when η is taken closer to 1. This means in particular that there exist values η_{ℓ_0} at which critical modes with spherical harmonics of degrees ℓ_0 and $\ell_0 + 1$ coexist.

- Each value of ℓ defines an irreducible representation of the group $\mathbf{O}(3)$. The space V of the critical modes is a representation space of $\mathbf{O}(3)$, and it is either irreducible or the sum of two irreducible representation spaces. Indeed, if $\eta = \eta_{\ell_0}$, this representation is the sum of the irreducible representations associated with ℓ_0 and $\ell_0 + 1$. Hence $\dim(V) = 4\ell_0 + 4$. The interest to consider this situation is that it leads to much more interesting dynamics than a "pure mode" bifurcation (which is always "quasi-gradient" [17]).

- Such situations, for general ℓ_0, have been considered in [14]. The case when $\ell_0 = 1$ has been studied most, for at least two reasons : first, it is the "simplest" (apart from $\ell_0 = 0$ which is excluded in the Bénard problem), still showing a lot of interesting phenomena; second, it is of some geophysical interest. In the following we focus on this case.

- Assume now that the system rotates with constant angular speed Ω around the polar axis. This introduces a new external force in the Navier-Stokes equations, namely the Coriolis force. After non-dimensionalisation, a new adimensional number appears, the Taylor number Ta, which is proportional to Ω. If Ta is taken from a neighborhood of 0, the problem can be treated as a perturbation of the completely $\mathbf{O}(3)$ invariant one. This perturbation breaks the spherical symmetry, reducing it to $\mathbf{SO}(2) \times \mathbb{Z}_2^c$, where $\mathbf{SO}(2)$ stands for the rotations around the polar axis and $\mathbb{Z}_2^c$ is the notation for the 2-element group of the antipodal symmetry. This problem was considered by [5] and, from a symmetry-breaking bifurcation point of view in the case $\ell_0 = 2$, by [10].

1.2 The amplitude equations and their third order approximation

For the local bifurcation analysis of this problem, it is suitable to perform a center manifold reduction of the model equations for a perturbation of the basic state and parameters near their critical values. For convenience, let us set $\lambda_1 = Ra - Ra_{crit}$, $\lambda_2 = \eta - \eta_{\ell_0}$ and $\epsilon = Ta$. We also denote by X the projection of the state variable (velocity, pressure, temperature) onto the space V. The center manifold theorem (see [15] for a detailed exposition of this technique), allows to reduce the local dynamics and bifurcation problem to a differential equation for X:

$$\frac{dX}{dt} = F(X, \lambda_1, \lambda_2, \epsilon) .$$

The map F admits a Taylor expansion of any order, although it is not C^∞. Taking the Taylor expansion up to a given order and expressing X in coordinates, we obtain a system of polynomial ODE's which is called the "amplitude equations". These equations have been computed by several authors and in various cases, most of the time up to order 3 only. Below we write this Taylor expansion. The relevant coefficients were computed by [19] (compare also the unpublished work [32]). In the next section we shall derive the general equivariant structure of the amplitude equations (at any order), but of course this does not provide numerical values for the coefficients. We first need some notations. In V, let x_j be the coordinate of X along the eigenvector associated with the spherical harmonic Y_{1j} ($j = -1, 0, 1$) and let y_m be the coordinate along the eigenvector associated with the spherical harmonic Y_{2m} ($m = -2, -1, 0, 1, 2$). Note that $x_{-j} = (-1)^j \bar{x}_j$ and $y_{-m} = (-1)^m \bar{y}_m$. We set $\mathbf{x} = (x_{-1}, x_0, x_1)$ and $\mathbf{y} = (y_{-2}, y_{-1}, y_0, y_1, y_2)$. We also set $\pi_1 = x_0^2 + 2x_1\bar{x}_1$ and $\pi_2 = y_0^2 + 2y_1\bar{y}_1 + 2y_2\bar{y}_2$. At order three, the amplitude equations have the form

$$\left.\begin{aligned} \dot{x}_j &= x_j(\alpha_1\ \lambda_a + \alpha_2\ \lambda_b + \gamma\ \pi_1 + \delta\ \pi_2) + \beta\ \Sigma_j^2(\mathbf{x},\mathbf{y}) + \delta'\ \Sigma_j^3(\mathbf{x},\mathbf{y},\mathbf{y}) \\ \dot{y}_m &= y_m(a_1\ \lambda_a + a_2\ \lambda_b + d\ \pi_1 + f\ \pi_2) + b\ \Upsilon_m^2(\mathbf{x},\mathbf{x}) + c\ \Upsilon_m^3(\mathbf{y},\mathbf{y}) + \\ &\qquad f'\ \Upsilon_m^4(\mathbf{x},\mathbf{x},\mathbf{y}) \quad . \end{aligned}\right\} \tag{1}$$

The bi- and trilinear terms which enter into these equations are defined in the next section. As already mentioned, the various coefficients which appear in the equations have been computed numerically. Here we shall take the results of [19] who assumed rigid boundary conditions on the inner sphere ($r = \eta$), free-type boundary conditions on the outer sphere ($r = 1$), ideally conducting boundaries. They find a critical value $\eta_1 \approx 0.2$. The linear coefficients α_1, α_2, a_1, a_2 allow to perform a change of variables

$$\lambda_1 = \alpha_1\lambda_a + \alpha_2\lambda_b \quad , \quad \lambda_2 = a_1\lambda_a + a_2\lambda_b \ . \tag{2}$$

Taking different physical conditions, e.g. different boundary conditions, would change the values of the coefficients in the amplitude equations, but we can expect that the main dynamical features would not be strongly affected. Another important property of this system is that c must be close to 0. In fact c depends on the assumption which is made about the distribution of heat sources and of mass. The general form of the temperature field and gravitational field for the fluid at rest is

$$T(r) = \frac{t_1}{r^2} + t_2 r \quad \text{and} \quad g(r) = \frac{g_1}{r^2} + g_2 r,$$

respectively, where t_j and g_j are non-negative constants. If $T(r) \equiv g(r)$, then it can be shown that $c = 0$ (see [8]). Otherwise, calculations show that c remains close to 0 (positive or negative depending on the values of the coefficients t_j and g_j). In the table below we indicate approximate values of the coefficients for three different Prandtl numbers, extrapolated from figures 6 and 7 in [19]. In these calculations, it was assumed that $t_2 = 0$ but $g_2 \neq 0$ (hence c need not be 0). Of course c can change sign under different choices of t_j and g_j, while this will not affect significantly the value of the other coefficients.

Prandtl number	0.1	1	10
β	-0.3	-0.3	-0.25
γ	-0.2	-0.4	-0.4
δ	-4.0	-1.0	-1.0
δ'	0.75	-0.23	-0.2
b	0.23	0.23	0.2
c	-0.04	0.04	0.05
d	-1.0	-1.0	-1.0
f	-1.8	-1.2	-1.0
f'	0.1	0.1	0.1

Table 1: Coefficients in the amplitude equations (from [19])

Numerical simulations of the amplitude equations by [19] have shown an interesting phenomenon: in certain regions in parameter space intermittent dynamics is observed, with "random" switching between two kinds of axisymmetric equilibria, called $\alpha-$cells and $\beta-$cells by these authors. The work of [1] allowed to understand how such behavior could occur, by establishing the existence of robust heteroclinic cycles, but it failed to show asymptotic stability. This analysis is made in section 3. However we find that the conditions for stability are not satisfied in the context described in [19]. Moreover, a careful look at [19] reveals that they do not observe a heteroclinic cycle of the type found in [1].

2 The 1-2 mode interaction

To recall the basics of $\mathbf{O}(3)$ representation theory, we mention, that there is up to equivalence precisely one irreducible representation of $\mathbf{SO}(3)$ on each real vector space of dimension $2\ell + 1$. This is referred to as the ℓ-representation of $\mathbf{SO}(3)$. Since $\mathbf{O}(3) = \mathbf{SO}(3) \oplus \mathbb{Z}_2^c$, where $\mathbb{Z}_2^c$ is the antipodal symmetry group (reflection through the origin in $\mathbb{R}^3$), a given representation can be continued in two ways to a representation of $\mathbf{O}(3)$, either the nontrivial element in $\mathbb{Z}_2^c$ acts as identity or as minus identity. One realization of the ℓ-representation is given by the action of $\mathbf{SO}(3)$ on the spherical harmonics of degree ℓ by

$$(\gamma, f) \mapsto \gamma f, \text{ where } (\gamma f)(x) = f(\gamma^{-1}x). \tag{3}$$

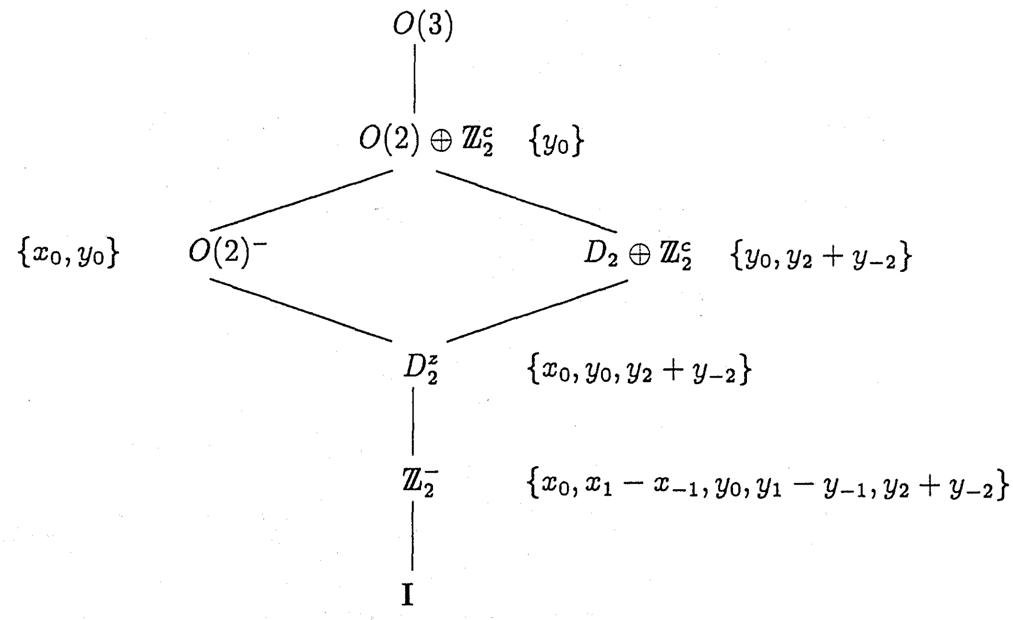

Figure 1: The poset of isotropy subgroups for the 1-2 mode interaction

The action of the antipodal symmetry on spherical harmonics is like the identity if ℓ is even and as minus identity if ℓ is odd. This is called the *natural* representation of $\mathbf{O}(3)$ on $2\ell + 1$ dimensional space. We work with this representation, which occurs naturally in the spherical Bénard problem. Let V_ℓ denote the real vector space of dimension $2\ell + 1$ with the natural $\mathbf{O}(3)$ action. We denote sums of representations by

$$V_{\ell_1,\ldots,\ell_\nu} = V_{\ell_1} \oplus \ldots \oplus V_{\ell_\nu}. \tag{4}$$

Let us now concentrate on $V_{1,2}$ an eight dimensional real vector space. We use the notation of section 1 for coordinates on this space, i.e. x_{-1}, x_0, x_1 on V_1 and $y_{-2}, y_{-1}, y_0, y_1, y_2$ on V_2, such that a general point is written as (x, y) with $x \in V_1$, $y \in V_2$. The poset of isotropy subgroups for this representation can be found in [1], for future reference it is displayed in Figure 1. The terminology for the groups is based on [23]. We recall that $\mathbb{Z}_2^c$ consists of plus/minus identity, $\mathbf{O}(2) \oplus \mathbb{Z}_2^c$ is the group of all motions of an (infinite) cylinder, $\mathbf{O}(2)^-$ is the group of the cone and $\mathbb{Z}_k$ stands for the group of rotations by $2\pi/k$, while D_k is the group of all motions of the regular $k - gon$. The group $\mathbb{Z}_2^-$ is generated by a reflection across a plane in $\mathbb{R}^3$.

2.1 Equivariant Maps

Let us begin by describing invariant functions and equivariant maps.

Lemma 2.1 *The polynomials*

$$
\begin{aligned}
\pi_1(x,y) &= x_0^2 - 2x_{-1}x_1\\
\pi_2(x,y) &= y_0^2 - 2y_{-1}y_1 + 2y_{-2}y_2\\
\pi_3(x,y) &= y_0^3 - 3y_{-1}y_0y_1 - 6y_{-2}y_0y_2 + \frac{3\sqrt{6}}{2}(y_{-1}^2y_2 + y_{-2}y_1^2)\\
\pi_4(x,y) &= -\frac{\sqrt{3}}{3}x_0^2y_0 + x_0x_{-1}y_1 + x_0x_1y_{-1} - \frac{\sqrt{3}}{3}x_{-1}x_1y_0 -\\
&\quad -\frac{\sqrt{2}}{2}x_{-1}^2y_2 - \frac{\sqrt{2}}{2}x_1^2y_{-2}\\
\pi_5(x,y) &= -2x_{-1}x_1y_{-2}y_2 - \frac{1}{2}x_0^2y_{-2}y_2\\
&\quad -\frac{\sqrt{6}}{2}y_{-2}y_0x_1^2 + \frac{3\sqrt{2}}{2}y_{-2}x_0x_1y_1 - \frac{\sqrt{6}}{2}x_{-1}^2y_0y_2\\
&\quad -y_{-1}x_0^2y_1 - \frac{\sqrt{3}}{2}y_{-1}x_0y_0x_1 + \frac{3}{4}x_0^2y_0^2 + \frac{3}{4}x_{-1}^2y_1^2 + \frac{3}{4}x_1^2y_{-1}^2\\
&\quad -\frac{\sqrt{3}}{2}x_{-1}x_0y_0y_1 + \frac{3\sqrt{2}}{2}x_{-1}x_0y_{-1}y_2 + \frac{1}{2}x_{-1}x_1y_{-1}y_1
\end{aligned}
$$

are invariant under the action of $\mathrm{O}(3)$ *and linearly independent. They generate the ring of invariant functions.*

Proof: In the appendix we show that there are five generators, provided we find five algebraically independent elements. The elements given here are clearly algebraically independent. □

Some of the generators of the module of polynomial equivariant mappings are given by [1], for the sake of convenience we recall them here and give a complete set.

Lemma 2.2 *The module of equivariant polynomial mappings over the ring of invariant polynomial is generated by the following mappings, of the form* $(0_1, \Upsilon)$, $\Upsilon^j = (\Upsilon^j_{-2}, \dots, \Upsilon^j_2)$ *or* $(\Sigma, 0_2)$, $\Sigma^j = (\Sigma^j_{-1}, \Sigma^j_0, \Sigma^j_1)$, *meaning that the equivariant has zero components in* V_1 *and the components* Υ^k_j *in* V_2 *in the first case and corresponding notation in the second case. We have for the equivariants, given in components (observe that it suffices to give the components with nonnegative upper indices only)*

$$
\Sigma^1 = \begin{cases} x_0 \\ x_1 \end{cases}
$$

$$
\Sigma^2 = \begin{cases} x_0y_0 - \frac{\sqrt{3}}{2}(x_1y_{-1} + x_{-1}y_1), \\ \frac{1}{2}\left(-x_1y_0 + \sqrt{3}x_0y_1 - \sqrt{6}x_{-1}y_2\right. \end{cases}
$$

$$\Sigma^3 = \begin{cases} \frac{3}{2}x_0y_0^2 - 2x_0y_1y_{-1} - \frac{\sqrt{3}}{2}(x_1y_0y_{-1} + x_{-1}y_0y_1) + \frac{3\sqrt{2}}{2}(x_1y_{-2}y_1 + x_{-1}y_0y_2) \\ \frac{1}{2}\left(-x_1y_1y_{-1} + \sqrt{3}(x_0y_0y_1 - x_0y_{-1}y_0) - 3x_{-1}y_1^2\right) + \sqrt{6}x_{-1}y_0y_2 + 2x_1y_{-2}y_2 \end{cases}$$

$$\Upsilon^1 = \begin{cases} y_0 \\ y_1 \\ y_2 \end{cases}$$

$$\Upsilon^2 = \begin{cases} x_0^2 + x_1x_{-1} \\ \sqrt{3}x_0x_1 \\ \sqrt{\frac{3}{2}}x_1^2 \end{cases}$$

$$\Upsilon^3 = \begin{cases} y_0^2 - y_1y_{-1} - 2y_2y_{-2} \\ y_0y_1 - \sqrt{6}y_{-1}y_2 \\ -2y_0y_2 + \frac{\sqrt{6}}{2}y_1^2 \end{cases}$$

$$\Upsilon^4 = \begin{cases} -x_0^2y_0 - 4x_1x_{-1}y_0 + \sqrt{6}(x_{-1}^2y_2 + x_1^2y_{-2}) + \sqrt{3}(x_0x_1y_{-1} + x_0x_{-1}y_1) \\ -\sqrt{3}x_0x_1y_0 - 3x_1x_{-1}y_1 + 3x_1^2y_{-1} + 3\sqrt{2}x_0x_{-1}y_2 \\ \sqrt{6}x_1^2y_0 - 3\sqrt{2}x_0x_1y_1 + 3x_0^2y_2 \end{cases}$$

$$\Upsilon^5 = \begin{cases} \sqrt{6}(x_0^2y_{-2}y_2 + \frac{3}{2}x_0^2y_{-1}y_1 + x_1^2y_{-1}^2 + \frac{5}{6}x_0^2y_2^2 + \\ \quad +x_{-1}^2y_1^2 - x_1x_{-1}y_1y_{-1} + \frac{4}{3}x_1x_{-1}y_0^2) - 5\sqrt{2}(x_{-1}x_0y_0y_1 + x_0x_1y_{-1}y_0) - \\ \quad - 2\sqrt{3}(x_0x_1y_{-2}y_1 + x_0x_{-1}y_{-1}y_2) + 2(x_{-1}^2y_0y_2 + x_1^2y_{-2}y_0) \\ \sqrt{6}(x_1^2y_{-1}y_0 + 2x_1x_{-1}y_0y_1 + \frac{3}{2}x_0^2y_0y_1) - 4\sqrt{3}x_{-1}x_0y_0y_2 - \\ \quad - 3\sqrt{2}(x_{-1}x_0y_1^2 + x_0x_1y_1y_{-1} + \frac{2}{3}x_0x_1y_0^2) + 6x_{-1}^2y_1y_2 + 3x_0^2y_{-1}y_2 \\ \sqrt{6}x_0^2y_0y_2 - 2\sqrt{3}x_0x_1y_0y_1 - 6\sqrt{2}x_{-1}x_0y_1y_2 + \\ \quad +3x_1x_{-1}y_1^2 + 6x_{-1}^2y_2^2 + \frac{3}{2}x_0^2y_1^2 + x_1^2y_0^2. \end{cases}$$

The general equivariant mapping has the form

$$\begin{aligned} \dot{x} &= A_1x + A_2\Sigma^2(x,y) + A_3\Sigma^3(x,y) \\ \dot{y} &= B_1y + B_2\Upsilon^2(x) + B_3\Upsilon^3(y) + B_4\Upsilon^4 + B_5\Upsilon^5. \end{aligned} \tag{5}$$

with A_J, B_k being functions of λ and π_1 to π_5. Third order truncation leads to equation (1).

3 The basic heteroclinic cycles in the 1-2 representation space

3.1 The classification of Armbruster & Chossat

In this section we shall essentially recall the results of Armbruster & Chossat [1] about the existence of heteroclinic cycles in the $l = 1-2$ mode interaction with $O(3)$

symmetry. We shall also fix some notations which will be used later. In addition we shall make some of the statements of [1] concerning existence and stability of heteroclinic cycles in $V_{1,2}$ more precise.

It follows from the stratification of the space $V_{1,2}$ in orbit types that there is only one symmetry axis, up to conjugacy in $\mathbf{O}(3)$, namely the axis $L = \mathrm{Fix}(\mathbf{O}(2) \times \mathbb{Z}_2^c)$. The equivariant bifurcation lemma applies to the equations restricted to L, and the bifurcation equation (after truncation at order 3) reads

$$0 = y_0(\lambda_2 + cy_0 - y_0^2)\ .$$

Of course, if $c = 0$ the bifurcation is supercritical : there are two branches of bifurcating solutions, namely $y_0 = \pm\sqrt{\lambda_2}$. If $|c|$ is considered to be a small parameter, as it turns out to be the case for the Bénard problem (see section 1), then the bifurcation is transcritical but one of the branches admits a turning point in a neighborhood of 0 and again there are two "local" bifurcating equilibria,

$$y_{\pm} = 1/2(c \pm \sqrt{c^2 + 4\lambda_2}) \tag{6}$$

In all of the following, $|c|$ will be considered such a small parameter.

Definition 3.1 *(i)* $P_1 = \mathrm{Fix}(D_2 \times \mathbb{Z}_2^c)$, $P_2 = \mathrm{Fix}(\mathbf{O}(2)^-)$ *and* $S = \mathrm{Fix}(D_2^z)$. *(ii)* α *and* β *are the equilibria in* L *with components* y_- *and* y_+ *respectively.*

Note that $L = P_1 \cap P_2$. However P_2 contains no other invariant axis, while P_1 contains in addition two conjugates of L which are obtained by rotating this axis in P_1 by angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. This is due to the fact that the subgroup D_3 of $\mathbf{SO}(3)$ acts faithfully in P_1. The rotations in P_1 correspond to elements in $\mathbf{SO}(3)$ (rotational symmetries) which we shall not need to make explicit here. Therefore within P_1 there are two copies α' and α'' of α, and two copies β' and β'' β. Also, remark that $S = P_1 \oplus P_2$.

The eigenvalues of the linearized vector field at α and β are extremely important for our analysis. The tables 2, 3 list these eigenvalues and eigenspaces at α and β respectively. We define $\tilde{\delta} = \delta + 3/2\delta'$.

Let us look at the dynamics for the system (1) restricted to the two invariant planes, and then to the 3D space S. Assume that $\lambda_2 > 0$ is fixed and c is close to 0.

3.1.1 Phase portrait in P_1

One can easily see that the assumption of small $|c|$ leads to possible secondary equilibria in P_1, bifurcating from either α or β off the invariant axis L. Their existence is subject to the condition that $c.g < 0$, where g is the coefficient of

eigenvalue	expression	eigenspace
α_0	$\lambda_1 - y_- + \tilde{\delta} y_-^2$	$\{x_0\}$
α_1 (double)	$\lambda_1 + 1/2 y_- + \delta y_-^2$	$\{x_1, x_{-1}\}$
α_2 (double)	$-3cy_-$	$\{y_2, y_{-2}\}$
α_r	$cy_- - 2y_-^2$	$\{y_0\}$
0 (double)	O(3) *invariance*	$\{y_1, y_{-1}\}$

Table 2: Eigenvalues at α

eigenvalue	expression	eigenspace
β_0	$\lambda_1 - y_+ + \tilde{\delta} y_+^2$	$\{x_0\}$
β_1 (double)	$\lambda_1 + 1/2 y_+ + \delta y_+^2$	$\{x_1, x_{-1}\}$
β_2 (double)	$-3cy_+$	$\{y_2, y_{-2}\}$
β_r	$cy_+ - 2y_+^2$	$\{y_0\}$
0 (double)	O(3) *invariance*	$\{y_1, y_{-1}\}$

Table 3: Eigenvalues at β

the fourth order term $\pi_2 \Upsilon_3$ in equations (5). The existence and stability of these solutions were first discussed by [22]. Numerical calculations by [13] and more recently by [21], have shown that $g < 0$ (always) for the Bénard problem. Therefore in this case, the secondary solutions exist only if $c > 0$.

If $c < 0$, there are no equilibria with isotropy $D_2 \times \mathbb{Z}_2^c$. In this case, the phase portrait looks like figure (9). In particular heteroclinic orbits connect β to α' and α'', and by symmetry identical connections exist between the other equilibria in P_1.

3.1.2 Phase portrait in P_2

We assume $c < 0$. The phase portrait in the "mixed-mode" plane P_2 depends on the value of λ_1 which we let vary monotonously from very negative values.

- If $\lambda < \lambda_a$, where $\lambda_a \approx 1/2(c - \sqrt{c^2 + 4\lambda_2})$, then α is a sink in P_1. Note that $\lambda_1 < 0$.
- As λ crosses λ_a, a "mixed-mode" equilibrium γ bifurcates off L (and $-\gamma$). This of course means that $\alpha_0 = 0$ at $\lambda = \lambda_a$.

- γ is a sink in P_2, until it undergoes a Hopf bifurcation at $\lambda_1 = \lambda_b \approx 1/3(c - \sqrt{c^2 + 3\lambda_2})$.
- The new born limit cycle grows until it heteroclinises on α and the equilibrium at the origin. This occurs at a value $\lambda_b < \lambda_c < 0$. As λ_1 crosses λ_c, the limit cycle disappears and a heteroclinic connection appears from α to β.
- When $\lambda = 0$, γ vanishes in the trivial equilibrium and the $\alpha \to \beta$ connection persists until a mixed mode equilibrium bifurcates from β at a value $\lambda_1 = \lambda_d$, where $\lambda_d \approx 1/2(c + \sqrt{c^2 + 4\lambda_2})$.

It follows that if $c < 0$ (close to 0) and $\lambda_c < \lambda_1 < \lambda_d$, a heteroclinic cycle is realized between type α and type β equilibria. By symmetry there is an $\mathbf{O}(3)$ group orbit of such objects. But each individual heteroclinic cycle involves three equilibria of each type (those in the plane P_1). The connections out of P_1 involve three invariant planes, which are P_2 ($\alpha \to \beta$), P_2' ($\alpha' \to \beta'$) and P_2'' ($\alpha'' \to \beta''$). To be more precise, $P_2 = \{x_0, y_0, y_{2r}\}$, $P_2' = \{x_{1r}, y_0, y_{2r}\}$ and $P_2'' = \{x_{1i}, y_0, y_{2r}\}$, with $x_1 = x_{1r} + ix_{1i}$ and $y_2 = y_{2r} + iy_{2i}$. Notice that the symmetry group of such a heteroclinic cycle is D_3 by construction.

Definition 3.2 *We denote by Type I the heteroclinic cycles connecting equilibria of type α and type β in P_1 and P_2.*

The eigenvalues of the linearized vector field at γ are given in table 4.

eigenvalue	expression	eigenspace
$\gamma_0 \in \mathbb{C}$	$Re(\gamma_0) \approx 1/2\lambda_2 + c\lambda_1 - 3/2\lambda_1^2$	$\{x_0, y_0\}$
γ_1 (double)	$\lambda_2 + \lambda_1(3/2 + c - \lambda_1)$	$\{x_1, x_{-1}, y_1, y_{-1}\}$
γ_2 (double)	$\lambda_2 - 2c\lambda_1 - \lambda_1^2$	$\{y_2, y_{-2}\}$
0 (double)	$\mathbf{O}(3)$ *invariance*	$\{x_1, x_{-1}, y_1, y_{-1}\}$

Table 4: Eigenvalues at γ

3.1.3 Phase portrait in S

As above we fix λ_2 and assume $c < 0$ and close to 0. The following succession of bifurcations was described in [1].

- At $\lambda_1 = \tilde{\lambda}_a \approx -c - \sqrt{c^2 + \lambda_2}$, the equilibrium γ undergoes a steady-state bifurcation off the plane P_2. This means that $\gamma_2 = 0$ at $\lambda_1 = \tilde{\lambda}_a$. The new born equilibria, which we note $\tilde{\gamma}$, are then sinks in S. Note also that $\tilde{\lambda}_a < \lambda_b$.

- At a value $\lambda = \tilde{\lambda}_b$, $\tilde{\gamma}$ becomes unstable through a Hopf bifurcation. As λ_1 is further increased, this limit cycle in S heteroclinises to β', α and γ, in a complicated way (occurrence of a strange attractor).
- After this "crisis", a heteroclinic connection is realized from γ directly to α' (and α'') which is a sink in S. This connection persists until a bifurcation occurs from α' off the plane P_1 in S. This happens at a positive value of λ_1.
- γ "disappears" into β' when $\lambda_1 = \tilde{\lambda}_c \approx 1/4(-c - \sqrt{c^2 + 4\lambda_2})$. The $\gamma \to \alpha'$ connection still exists at this value, and persists until a certain positive value of λ_1 is reached (value at which $\alpha_1 = 0$).

If the heteroclinic connections from α to γ and from γ to α' coexist in a certain open domain of parameter values, then again a robust heteroclinic cycle is realized, but involving now the "mixed-mode" equilibria. However, because $\lambda_b < \tilde{\lambda}_c$, this does not immediately follow from the foregoing bifurcation scenarios. It must also be noticed that the existence of the $\gamma \to \alpha'$ connection could be proved only in the limit $c = 0$. Persistence of the connection for a "large" range of values of c is numerical evidence, as well as the existence of the heteroclinic cycle. On the other hand, suppose that γ becomes unstable in P_2 due to the Hopf bifurcation. The existence of the connection from γ to α' in S leads to the existence of a connection from the limit cycle itself to α'. Then a robust heteroclinic cycle is realized between α and the limit cycle in P_2.

Definition 3.3 *We denote by Type II the heteroclinic cycles connecting equilibria of type α and type γ in P_2 and in S, and type III those connecting type α with limit cycles in P_2 and S.*

All three heteroclinic cycles have been observed numerically by [1], which indicates their asymptotic stability. The stability question turns out to be quite subtle however, and deserves some further analysis. In the next subsection we shall see under which conditions the heteroclinic cycle of type I can be stable. We shall not consider the stability of types II and III.

3.2 Stability of the heteroclinic cycles of type I

The asymptotic stability of a heteroclinic cycle, more generally the asymptotic behavior of trajectories starting in its vicinity, is an important question in view of applications. This behavior strongly depends upon the local dynamics near each equilibrium in the cycle, hence upon the *eigenvalues* at each equilibrium. KRUPA and MELBOURNE [26] have characterized the asymptotic stability in terms of these eigenvalues, under some assumptions concerning the global, geometric structure of the flow along the heteroclinic connections. Therefore the first question we can ask is whether our heteroclinic cycles belong to this class.

We denote by ξ_j, ξ_{j+1} any two equilibria in the cycle such that the unstable manifold $W^u(\xi_j)$ intersects nontrivially the stable manifold $W^s(\xi_{j+1})$. Then a basic hypothesis in [26] is the following

(H1) for each j there is a flow invariant subspace P_j such that $W^u(\xi_j) \subset P_j$ and ξ_{j+1} is a sink in P_j.

We need to introduce some more definitions. Let Γ_j be the isotropy subgroup of the equilibrium ξ_j. We consider the eigenvalues of the linearized vector field at ξ_j. We call the eigenvalues with eigendirections in $\mathrm{Fix}(\Gamma_j)$ *radial*, the eigenvalues with eigendirections tangent to $W^s(\xi_j) \cap P_{j-1}$ *contracting*, the eigenvalues with eigendirections tangent to $W^u(\xi_j) \cap P_j$ *expanding*, and the remaining eigenvalues *transverse*. By (H1), ξ_j is a sink in P_{j-1}, i.e. $W^s(\xi_j) \cap P_{j-1} = P_{j-1}$.

The stability of the type I heteroclinic cycle was already studied by [1]. It was noticed that (H1) could hardly be satisfied in a neighborhood of $(0,0)$ in the parameter plane (λ_1, λ_2). Let us be more precise. The condition $W^u(\xi_j) \subset P_j$ implies that the transverse eigenvalues have negative real part. At equilibria α and β, there is exactly one (double) transverse eigenvalue, with respective leading part

$$\begin{aligned} \alpha_1 &= \lambda_1 + 1/2y_- + \delta y_-^2 \\ \beta_1 &= \lambda_1 + 1/2y_+ + \delta y_+^2 \end{aligned}$$

and eigendirections $x_{\pm 1}$ (see table 3). Here $y_\pm$ denotes the nonvanishing component along y_0 of the equilibria α and β, i.e.

$$y_\pm = 1/2(c \pm \sqrt{c^2 + 4\lambda_2} \; .$$

Remark that if the condition is satisfied at β, i.e. with $y_0 > 0$, then it is true at α. We therefore restrict our attention to β. It was noticed in [1] that the existence of the $\alpha \to \beta$ connection in P_2 is insured, for a given positive value of λ_2, for λ_1 in an interval $(\lambda_1^-, y_+ - \tilde{\delta} y_+^2)$, where λ_1^- is strictly negative (see [1]). It is therefore sufficient to check the stability conditions at $\lambda_1 = 0$. These conditions will persist in an open interval of values of λ_1 around 0. With this particular value, we have

$$\beta_1 = 1/2y_+ + \delta y_+^2 \; .$$

The condition $\beta_1 < 0$ now reads

$$\delta < -\frac{1}{2y_+} \; , \text{ i.e.}$$

$$\delta < -(c + \sqrt{c^2 + 4\lambda_2})^{-1} \tag{7}$$

This implies $\delta < 0$ and $\delta \to -\infty$ when $\lambda_2 \to 0$, since $c < 0$ in order to insure the $\beta \to \alpha$ connection in the plane P_1. Note however that in the numerical computations of [19] for the Bénard problem, it was found that indeed δ is negative : $\delta \simeq -1$ at

$Pr \simeq 10$ and $\delta \simeq -5$ at $Pr \simeq 0.1$. It is therefore not physically irrelevant to consider situations when (H1) is fulfilled, even if $\lambda_1 > 0$.

Under (H1), a sufficient condition of asymptotic stability can easily be derived : let $-c_j$ be the real part of the least contracting eigenvalue at ξ_j, e_j be the (real part of) the largest expanding one and t_j be the maximum real part of the transverse eigenvalues (which are negative by H1). Then if

$$\Pi \min(c_j, e_j - t_j) > \Pi e_j \ , \tag{8}$$

the heteroclinic cycle is asymptotically stable. KRUPA and MELBOURNE [26] proved the necessity of this condition under some additional hypotheses, we shall come to this later. For the moment, let us check whether the above condition can be satisfied together with (H1) for the type I heteroclinic cycles.

Proposition 3.4 *Given $\lambda_2 > 0$ and $c < 0$, there exists an open interval of values of λ_1 around 0 such that if $\tilde{\delta}$ is close enough to δ, and*

$$\delta < \min\{-(c+\sqrt{c^2+4\lambda_2})^{-1},\ -3/2(c^2+4\lambda_2)^{-1}\},$$

then the type I heteroclinic cycle exists and is asymptotically stable.

Proof. In terms of the eigenvalues at α and β, the stability condition reads

$$\min\{-\alpha_2, \alpha_0 - \alpha_1\}.\min\{-\beta_0, \beta_2 - \beta_1\} > \alpha_0.\beta_2 \ .$$

It is easy to check that $-\alpha_2 < \alpha_0 - \alpha_1$ and $\beta_2 - \beta_1 < -\beta_0$, because c is close to 0. Since in addition $\alpha_2 = -3cy_-$ and $\beta_2 = -3cy_+$, the condition reduces to

$$y_-(\beta_1 - \beta_2) > y_+\alpha_0 \ .$$

Suppose $\lambda_1 = 0$, this relation becomes, after some elementary algebra,

$$0 < 3/2 + \delta y_+ - \tilde{\delta} y_-.$$

Take now $\tilde{\delta} = \delta$. Then we get the condition

$$\delta < -3/2(c^2+4\lambda_2)^{-1} \ .$$

Combined with (7), this ends the proof. □

Remark 3.5 1. *The condition in Proposition 3.4 is not satisfied in the case of intermittent dynamics as observed in [19].*

2. *In this case Figure 17 of [19] shows that the dynamics does not fit the heteroclinic cycle found by [1]. Indeed there is no trajectory following the connection in the plane P_1.*

Condition (8) can also be necessary, as was shown in [26], if some additional hypotheses are satisfied. Moreover in this case, a stability statement can still be claimed even if one of the transverse eigenvalues α_1, β_1 is positive but smaller than the corresponding expanding eigenvalue, α_0 or β_2 (see [27]). Of course (8) could not be fulfilled if both transverse eigenvalues had the foregoing property at the same time. If however this condition is again satisfied, then the heteroclinic cycle is not asymptotically stable but *essentially asymptotically stable*, meaning that trajectories converge to the cycle whenever the initial condition belongs to the complementary of a cuspoidal wedge in a neighborhood of the cycle. Therefore, for "most" initial conditions in the neighborhood, the trajectories will converge to the heteroclinic cycle. It turns out that part of the additional hypotheses are not satisfied in our case. More precisely : hypothesis (S2) of [27] reads as follows : the eigenspaces corresponding to c_j, t_j, e_{j+1} and t_{j+1} lie in the same Γ_j-isotypic component. Translated into our case, this means for example that the eigenvectors for α_2, α_1, β_2 and β_1 lie in the same $\mathbf{O}(2) \times \mathbb{Z}_2^c$-isotypic component, which is not true. Indeed, the eigenspace for α_1 and β_1 is $\{x_1, \bar{x}_1\}$ while the eigenspace for α_2 and β_2 is $\{y_2, \bar{y}_2\}$. On the former space, $\mathbf{SO}(2)$ acts by $x_1 \mapsto e^{i\varphi}x_1$, and on the latter, it acts by $y_2 \mapsto e^{2i\varphi}y_2$. So we may now ask whether (8) is still a condition of essential asymptotic stability for our heteroclinic cycle of type I. On the other hand, we may also ask whether the transverse unstable directions are still related to heteroclinic connections between the group orbits of α and β, or just escape to some other invariant object. This analysis will be the subject of the next section.

4 The heteroclinic set and its stability

4.1 Existence of an invariant sphere

Here we follow the seminal paper by FIELD [18]. We prove the existence of the invariant sphere in the case c close to zero and for $|\lambda_1 - \lambda_2|$ small. Selfadjoint refers to the spherical Bénard-problem, see CHOSSAT [8] and MOUTRANE [32] for the consequences of this assumption for the bifurcation equation. Especially it follows that $c = 0$ and b, β satisfy the relation $2\beta = -b$. So we have

Theorem 4.1 *Let $\varepsilon_k > 0$ be sufficiently small and assume $\lambda_1, \lambda_2 > 0$. Then $|\lambda_1 - \lambda_2| < \varepsilon_1(|\lambda_1| + |\lambda_2|)$, $c < \varepsilon_2$, $|b + 2\beta| < \varepsilon_2$ and the additional assumption*

$$\gamma, d < 0, \delta, \delta', f, f' < \varepsilon_3$$

imply the existence of an invariant sphere.

Proof: In the selfadjoint case with $\lambda = \lambda_1 = \lambda_2$ the hypotheses of theorem 5.1 in FIELD [18] are satisfied and for each value of $\lambda > 0$ (where we assume $\lambda = \lambda_1 = \lambda_2$)

we find an invariant sphere. Normal hyperbolicity of this sphere implies the existence of an invariant sphere as long as $\lambda_1, \lambda_2 > 0$ and $|\lambda_1 - \lambda_2|$ is sufficiently small and the deviation from the selfadjoint case is small. □

Let us make the following remarks on invariant spheres.

Remark 4.2 *1. The invariant sphere is a graph over a sphere of radius $r = \sqrt{|\lambda|}$, i.e. there exists a map $s : S(r) \to V$, such that the invariant sphere is given by $s(S(r))$.*

2. The invariant sphere is unique. This follows from Field's proof.

3. The map whose graph gives the invariant sphere is equivariant. This follows in a standard way from uniqueness.

4. The invariant sphere is normally hyperbolic.

4.2 The generalized heteroclinic cycle

In order to establish the main result on the heteroclinic set, we restrict ourselves to the space $\mathrm{Fix}(\mathbb{Z}_2^-)$. For the following we make the assumption

(H) There exists an invariant sphere.

We have seen that this hypothesis is satisfied for an open region in parameter space. Numerical computations indicate that this sphere exists for a much larger set, than Theorem 4.1 indicates.

Let us consider the dynamics in $\mathrm{Fix}(\mathbb{Z}_2^-)$ and to be more precise on the intersection of this five dimensional space with the invariant sphere, which is a topological four dimensional sphere S^4. On this space $\mathrm{Fix}(\mathbb{Z}_2^-)$ we have an action of the normalizer $N_{\mathbf{O}(3)}(\mathbb{Z}_2^-) = \mathbf{O}(2) \oplus \mathbb{Z}_2^c$ (see [17]). The invariant sphere has to be invariant under this action (if it where not for any $\gamma \in N_{\mathbf{O}(3)}(\mathbb{Z}_2^-)$ we had γS^4 is also an invariant sphere. Since all equilibria are on each invariant sphere S^4 intersects γS^4 contradicting the normal hyperbolicity). So we can restrict the flow on S^4 to the orbit space of S^4 under the action of $N_{\mathbf{O}(3)}(\mathbb{Z}_2^-)$.The action of the normalizer has $\mathbb{Z}_2^-$ as kernel, in fact we should look at the action of $\mathbf{O}(2) \oplus \mathbb{Z}_2^c/\mathbb{Z}_2^-$ which is isomorphic to $\mathbf{O}(2)$. In the following discussion groups refer to subgroups of $\mathbf{O}(2)$.

Lemma 4.3 *For this action the Hilbert map has the following form $\Theta = (\theta_0, \ldots, \theta_3)$, where the θ_j are given by*

$$\begin{aligned} \theta_0 &= y_0 \\ \theta_1 &= -x_1 x_{-1} \ (= x_1 \overline{x}_1) \end{aligned}$$

$$
\begin{aligned}
\theta_2 &= y_2 y_{-2} \ \ (= y_2 \overline{y}_2) \\
\theta_3 &= \frac{1}{2}(x_1^1 y_{-2} + x_{-1}^2 y_2.
\end{aligned}
$$

Proof: See CHOSSAT [11]. □

Theorem 4.4 *The orbit space B of the invariant sphere S^4 has the following properties*

1. *It is a three dimensional compact manifold with boundary.*
2. *The principal stratum is the interior of this manifold, which is simply connected. The corresponding orbit type is trivial.*
3. *The boundary consists of three strata: the corresponding isotropy subgroups are $\mathbb{Z}_2$, D_2 and $\mathbf{O}(2)$.*
4. *It is topologically equivalent to a manifold whose boundary is described by the equations*

$$
\begin{aligned}
\theta_1 &\geq 0 \\
\theta_2 &\geq 0 \\
\theta_3^2 - \theta_1^2\theta_2 &\leq 0 \\
\theta_0^2 + 2\theta_1 + 2\theta_2 &= 1.
\end{aligned}
$$

Remark 4.5 *We think of this orbit space as of a banana, having one ridge, for an illustration see figures 3,4.*

Proof: Clearly $\dim(B) = \dim(S^4) - 1 = 3$. The Hilbert map $\Pi = (\theta_0, \theta_1, \theta_2, \theta_3)$ sends $\mathrm{Fix}(Z_2^-)$ into a four dimensional manifold with boundary. The invariant sphere is topologically equivalent to the set $\theta_0^2 - 2\theta_1 + 2\theta_2 = 1$, so the image of the invariant sphere is topological equivalent to the image of the restriction of the Hilbert map to this set. The above statements follow easily from these considerations. □

We have the following information about the flow in this "banana".

Lemma 4.6 *If $c \neq 0$ then we have:*

1. *The two 0-dimensional strata corresponding to $\mathbf{O}(2)$-isotropy are fixed points. They correspond to equilibria of type α, β (in the terminology of [1]).*
2. *They are joined by a 1-dimensional stratum with D_2-isotropy.*
3. *On this 1-dimensional stratum we have two additional equilibria α', β' and orbits connecting these equilibria.*

4. *The two dimensional stratum is filled with solutions which are connections between α' and β'.*

Proof:

1. Zero dimensional strata obviously consist of points which are isolated in their stratum, which immediately implies that they are equilibria.
2. On a one dimensional stratum the flow consists of equilibria and connecting orbits only. In this case we have to find the solutions which (in the notation of [1]) are called α' and β' on this stratum as well as the connections shown to exist in this paper.
3. The two dimensional stratum is given by the image of the intersection of $\mathrm{Fix}(D_2^z)$ with the invariant sphere. Again [1] shows that this is filled with connections. These are precisely the connections claimed here.

□

A first main result is contained in the following theorem.

Theorem 4.7 *The flow inside the orbit space has the following features:*

1. *In the selfadjoint case, the 1-dimensional stratum on the boundary consists of equilibria only. The three dimensional stratum is filled with connections between the various equilibria on the one-dimensional stratum.*
2. *In the generic, nonselfadjoint case there are precisely two equilibria on the 1-dimensional stratum. The three dimensional stratum is filled with orbits connecting these two points.*

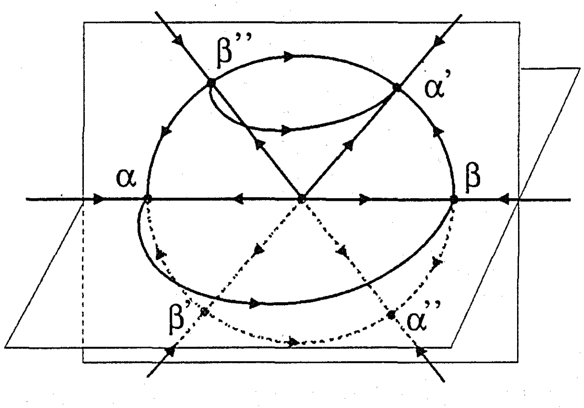

Figure 2: One of the extra connections.

Proof: The proof consists of two steps: in the first step we consider the selfadjoint case, which is somewhat degenerate. Then we use a perturbation argument to conclude the behavior in the general case. Let us begin with the selfadjoint case.

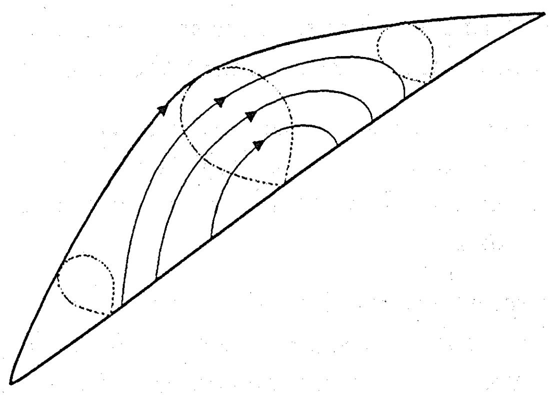

Figure 3: The flow on the banana for the cubic system when $c = 0$.

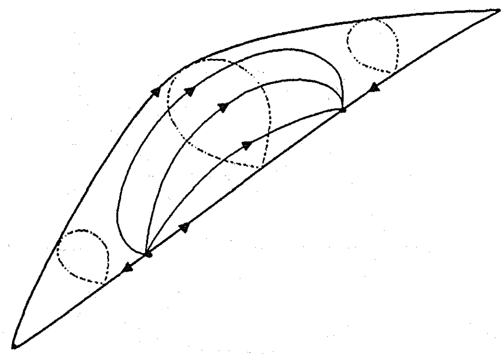

Figure 4: The flow in the case when c is small.

1. Let us begin with the selfadjoint case. Then the 1-dimensional stratum is filled with equilibria. Let $M = m(t)$, $t \in [0,1]$ be a parameterization of this stratum with $m(0)$ corresponding to α, $m(1)$ corresponding to β. One can easily check that there is a $t^* \in [0,1]$ such that for $0 \leq t < t*$ $m(t)$ has a twodimensional unstable manifold W_t^u (in the direction of M it obviously has a 0 eigenvalue) and for $t^* < t \leq 1$ it has a two dimensional stable manifold W_t^s. The intersection of the W_t^u with the twodimensional stratum consists of connections to a point $m(T(t))$, $T(t) > t^*$, these are the additional connection found by ARMBRUSTER & CHOSSAT [1]. The next step is to consider the solutions in the interior. Here we have to use some general information of dynamics in sums of representations of $\mathbf{O}(2)$, compare [1] In polar coordinates $(r_1, r_2, y_0, \phi_1, \Phi_2)$ the equation on $\mathrm{Fix}(\mathbb{Z}_2^-)$ has the form (with $\Phi = \Phi_1 - \Phi_2$)

$$
\begin{aligned}
\dot{r}_1 &= \lambda_1 r_1 - \frac{1}{2}\beta r_1\left(y_0 + \sqrt{6} r_2\cos(\phi)\right) + 2\gamma r_1^3 + \delta r_1\left(y_0^2 + 2r_2^2\right) + \\
&\quad \delta' r_1 r_2\left(2r_2 + \sqrt{6} y_0 \cos(\phi)\right) \\
\dot{r}_2 &= \lambda_2 r_2 + \frac{3}{2} b r_1^2 \cos(\phi) - 2r_2 y_0 + d r_2(y_0^2 + 2r_2^2) + 2f r_1^2 r_2 + \\
&\quad \sqrt{6} f' y_0 r_1^2 \cos(\phi) \\
\dot{y}_0 &= \lambda_2 y_0 - b r_1^2 - c(2r_2^2 - y_0^2) + d y_0(y_0^2 + 2r_2^2) + 2f y_0 r_1^2 - \\
&\quad 2f' r_1^2\left(\sqrt{6} r_2 \cos(\phi) - 2y_0\right) \\
\dot{\phi} &= -\sqrt{6}\sin(\phi)\left(\left(\frac{1}{2}b + f' y_0\right)\frac{r_1^2}{r_2} + r_2(-\beta + 2\delta')\right) \\
\frac{\dot{\phi}_1}{\dot{\phi}_2} &= -\frac{r_2^2(2\delta' y_0 - \beta)}{r_1^2(2f' y_0 + b)}.
\end{aligned}
$$

Observe that is is a general feature of such equations to allow to write an equation for the phase difference. This is not restricted to low order terms. In general these equations have singularities at $r_1 = 0$ or $r_2 = 0$. At such singularities the phase difference along solution may jump from 0 to π or vice versa, compare ARMBRUSTER, GUCKENHEIMER & HOLMES [2]. The phase difference $k\pi$ $k \in \mathbb{Z}$ are obviously invariant, as long as those solutions do not go through $r_1 = 0$ or $r_2 = 0$. Let us look at the asymptotic behavior of the phase difference along any solution. Its behavior strongly depends on the second factor, so let us study it after multiplication with r_2

$$
\left(\left(\frac{1}{2}b + f' y_0\right) r_1^2 + r_2^2(-\beta + 2\delta')\right) = b(r_1^2 + r_2^2) + f' y_0 r_1^2 + 2\delta' y_0 r_2^2.
$$

Due to the smallness assumption of f', δ' this is of one sign along solutions on the invariant sphere. Therefore

$$
\phi(t) \stackrel{t\to\infty}{\to} 0 \text{ or } \pi.
$$

The points with $\phi = 0, \pi$ correspond to the boundary of the orbit space and therefore we see that any trajectory tends to the boundary. Since we know the asymptotic behavior on the boundary we conclude that any solution has its ω-limit set on the 1-dimensional stratum.

Observe that the 1-dimensional stratum is almost normally hyperbolic. It we take out the unique point p, where the change in stability takes place we can write this stratum as a union

$$S^s \cup p \cup S^u$$

where S^s and S^u are 1-dimensional manifolds which consist of stable or unstable points respectively.

2. Here we want to use perturbation arguments based on the first case.

   The main difference between these two cases is the behavior along the 1-dimensional stratum. In the nonselfadjoint case, with c small, there are two interior solutions and the two endpoints which correspond to steady states. denote by α or β the endpoints of the 1-dimensional stratum corresponding to the equilibria α, β. the other two solutions on this stratum are labelled with α' and β' and correspond to these equilibria. They are located in S^s or S^u respectively. Especially $\alpha' \neq p \neq \beta'$. Moreover it is easy to check that there is a heteroclinic connection from β' to α' and p is on this connection. Now the result basically follows from continuous dependence on initial values, stability and instability of α' and β' respectively.

□

Theorem 4.8 *In the nonselfadjoint case, but with c close to 0, there exist heteroclinic cycles*

$$\beta' \to \alpha'' \to \beta'' \to \alpha' \to \beta'$$

such that the set of connecting orbits from $\beta' \to \alpha''$ and from $\beta' \to \alpha'$ are both three dimensional. The standard cycle

$$\beta' \to \alpha \to \beta \to \alpha' \to \beta'$$

is contained in the closure of the set of all the connections described before.

Definition 4.9 *We call the set of all the connections described in the previous theorem as the* generalized heteroclinic cycle.

Remark 4.10 *The stability analysis is more complicated than usual, cf. sect. 3. Our method is adapted to the underlying geometry.*

4.3 The stability of the generalized heteroclinic cycle

4.3.1 Reduction to the local orbit space

We study the stability of the heteroclinic set which was shown to exist in the last section. The main tool is a **local orbit space** reduction near the equilibria on the heteroclinic cycle. Of course we do not claim that there is no other way of proving the result, but we found it a convenient way to describe the relevant geometry (see [3] for another discussion of the stability of the GHC). Let us begin our discussion with the concept of a local orbit space as we shall use it.

Given a manifold of equilibria of an equivariant dynamical system, we use the slice theorem (BREDON [4]) to get a neighborhood of a given point as a product of the group orbit and the corresponding normal bundle. There is an action of the isotropy of our given point on the normal bundle. The reduction to the orbit space with respect to this action will be called the local orbit space.

If we consider a normally hyperbolic manifold $\mathcal{O}(p_0)$, then we have a fibration of a neighborhood into stable and unstable manifolds. There is a trivial flow along the group orbit and the whole dynamics will be described by the behavior in one fiber N and therefore the flow on the local orbit space gives information about the behavior in a neighborhood of $\mathcal{O}(p_0)$. This justifies the use of local orbit spaces.

As we will see, this local reduction is enough to obtain a global view of the generalized heteroclinic cycle in the orbit space $\mathrm{V}_{1,2}/\,\mathrm{O}(3)$.

All the fixed points involved in the GHC are $\mathrm{O}(2)\oplus\mathbb{Z}_c^2$ symmetric and we can restrict the calculations to the local orbit space around the point α. In a neighborhood of this point, the slice S_α to its group orbit is spanned by $x_0, x_{\pm 1}$ and $y_{\pm 2}$. We have then to determine the orbit space $S_\alpha/\,\mathrm{O}(2)\oplus\mathbb{Z}_c^2$. This action of $\mathrm{O}(2)\oplus\mathbb{Z}_c^2$ is in fact a 0-1-2 mode interaction, i.e. a sum of irreducible representations of this group associated with rotations $e^{im\theta}z_m$, $m = 0, 1, 2$ respectively.

We recall the action of $\mathrm{O}(2)\oplus\mathbb{Z}_c^2$ obtained by the restriction of the $\mathrm{O}(3)$ action on $\mathrm{V}_{1,2}$:

$$\left\{\begin{array}{lcl} \theta\, z_m & = & e^{im\theta}z_m,\ m = 1,2 \\ \kappa\, z_m & = & (-1)^m\bar{z}_m = (-1)^{\ell+m}z_{-m} \\ -\mathbb{1}\, z_m & = & (-1)^\ell z_m \end{array}\right.$$

with the additional *reality* condition ([33]) $z_{-m} = (-1)^m\bar{z}_m$. This isotropy lattice for the 0-1-2 mode interaction is given in figure (4.3.1).

Lemma 4.11 *The ring of invariant for this action of* $\mathrm{O}(2)\oplus\mathbb{Z}_2^c$ *on* S_α *is generated by the polynomials*

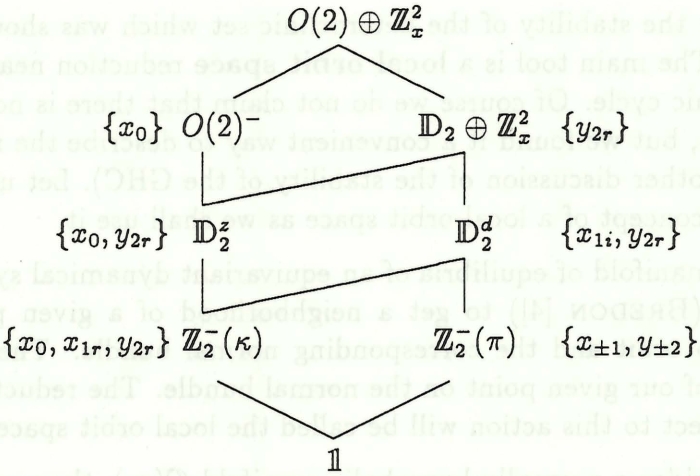

Figure 5: Isotropy lattice for the 0-1-2 mode interaction with $O(2) \oplus \mathbb{Z}_c^2$

$$\left\{ \begin{array}{lcl} \eta_0 & = & x_0^2 \\ \eta_1 & = & x_1 x_{-1} \\ \eta_2 & = & y_2 y_{-2} \\ \eta_3 & = & x_1^2 y_{-2} + x_{-1}^2 y_2 \end{array} \right. \tag{9}$$

The strata corresponding to the orbit space for this action are defined by the equations and inequalities given in table (4.3.1).

Proof. For the invariant functions note, that the representation is the same as before. Therefore we find the same invariants. The Hilbert map is then given by

$$\rho \; : \; \left\{ \begin{array}{ccc} S_\alpha & \longrightarrow & S_\alpha / O(2) \oplus \mathbb{Z}_c^2 \\ x = (x_0, x_{\pm_1}, y_{\pm 2}) & \longmapsto & (\eta_0(x), \eta_1(x), \eta_2(x), \eta_3(x)) \end{array} \right.$$

and the inequalities satisfied by the invariants are :

$$\eta_0 \geq 0, \eta_1 \geq 0, \eta_3 \geq 0 \text{ and } \eta_3^2 - \eta_1^2 \eta_2 \leq 0.$$

The equations and inequalities defining the various strata in this 4-dimensional orbit space are summed up in table (4.3.1). □

This orbit space can be seen as the "cone" defined by the equation $\eta_3^2 - \eta_1^2 \eta_2 \leq 0$ translated along the half-axis $\eta_0 \geq 0$ (see fig. (4.3.1)). However, the only cone which

stratum	defining equations
$O(2) \oplus \mathbb{Z}_2^c$	$\eta_0 = \eta_1 = \eta_2 = \eta_3 = 0$
$O(2)^-$	$\eta_0 > 0,\ \eta_1 = \eta_2 = \eta_3 = 0$
$\mathbb{D}_2 \oplus \mathbb{Z}_2^c$	$\eta_2 > 0,\ \eta_0 = \eta_1 = \eta_3 = 0$
$\mathbb{D}_2^z$	$\eta_0 > 0,\ \eta_2 > 0,\ \eta_1 = \eta_3 = 0$
$\mathbb{D}_2^d$	$\eta_0 = 0,\ \eta_1 > 0,\ \eta_2 > 0,\ \eta_3^2 = \eta_1^2\eta_2$
$\mathbb{Z}_2^-(\kappa)$	$\eta_0 > 0,\ \eta_1 > 0,\ \eta_2 > 0,\ \eta_3^2 = \eta_1^2\eta_2$
$\mathbb{Z}_2^-(\eta)$	$\eta_0 = 0,\ \eta_1 > 0,\ \eta_2 > 0,\ \eta_3^2 < \eta_1^2\eta_2$
$\mathbb{1}$	$\eta_0 > 0,\ \eta_1 > 0,\ \eta_2 > 0,\ \eta_3^2 < \eta_1^2\eta_2$

Table 5: Strata in the local orbit space $S_\alpha/\,\mathrm{O}(2) \oplus \mathbb{Z}_c^2$

possess a flow invariant surface is the cone in $\eta_0 = 0$, its surface consisting in the union of the two strata $\mathbb{D}_2 \oplus \mathbb{Z}_2^c$ and $\mathbb{D}_2^d$.

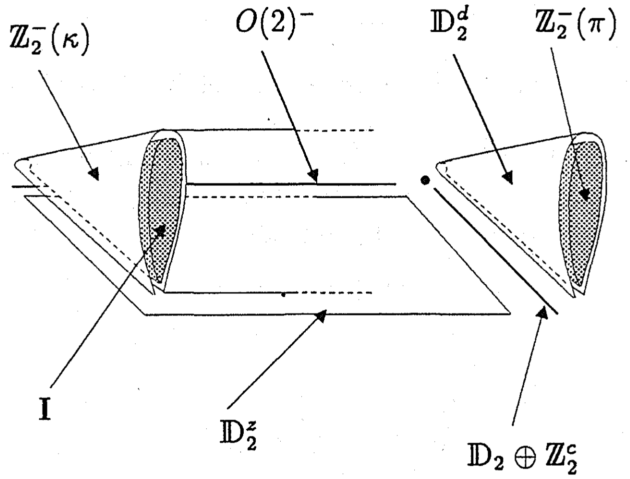

Figure 6: The stratification of the local orbit space in the point α

4.3.2 Projection of the heteroclinic set in the orbit space

The information concerning the local description of the orbit space $V_{1,2}/\,\mathrm{O}(3)$ around each fixed point of the heteroclinic set allow to give a global characterization from it. The two triplets $(\alpha, \alpha', \alpha'')$ and (β, β', β'') of conjugated points in the heteroclinic set, will be respectively projected on the points $\tilde{\alpha}$ and $\tilde{\beta}$ in the orbit space corresponding two fixed points of the projected heteroclinic set. Now the instable connections from α to β and from β to α' start respectively in the $\{y_0\}$ and $\{y_{2r}\}$ direction. The projection of the first connection is tangent to the $\mathrm{O}(2)^-$ close to the point $\tilde{\alpha}$. The projection of the second connection is included in the $D_2 \oplus \mathbb{Z}_2^c$ atrata. Using results of the previous section, it is easy to see that the $\mathbb{D}_2 \oplus \mathbb{Z}_2^c$, D_2^d and $\mathbb{Z}_2^c$ stratum near $\tilde{\beta}$ will respectively be send by the flow to the $\mathbb{D}_2 \oplus \mathbb{Z}_2^c$, $\mathbb{D}_2^d$ and $\mathbb{Z}_2^c$ stratum near $\tilde{\beta}$. The geometry of the set is rather complicated and will be studied in some detail below. A sketch of the geometry of this set is represented in figure (4.3.2).

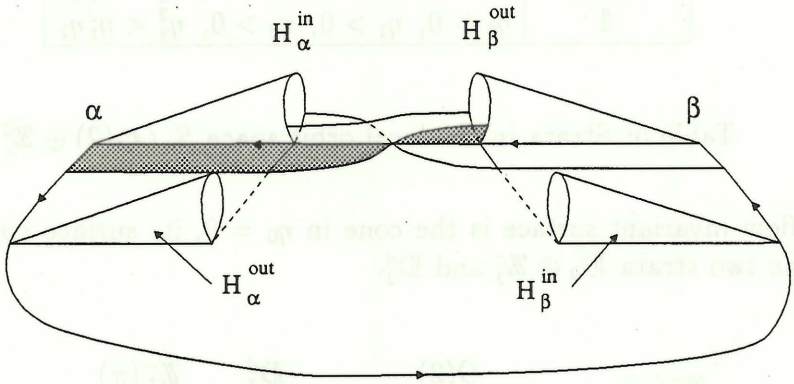

Figure 7: The projection of the heteroclinic set in the orbit space $V_{1,2}/\,\mathrm{O}(3)$

Let us first give the lowest order terms for any commuting vector field. Let x_0 denote the coordinate on the one dimensional representation, x_1, y_2 the complex coordinates on the 1 or 2 mode representation respectively.

Those equations have (in lowest order) the form

$$\begin{aligned}
\dot{x}_0 &= a_0 x_0 \\
\dot{x}_1 &= a_1 x_1 + b_1 \overline{x_1} y_2 \\
\dot{y}_2 &= a_2 y_2 + b_2 x_1^2.
\end{aligned}$$

The invariant functions supply a coordinate system on the orbit space. The equations for them read as

$$\begin{aligned}
\dot{\eta}_0 &= 2a_0\eta_0 \\
\dot{\eta}_1 &= 2a_1\eta_1 + 2b_1\eta_3 \\
\dot{\eta}_2 &= 2a_2\eta_2 + b_2\eta_3 \\
\dot{\eta}_3 &= (2a_1 + a_2)\eta_3 + 2b_1\eta_1\eta_2 + b_2\eta_1^2
\end{aligned}$$

Here it is a natural issue to ask for which terms have to be kept in order to talk about a reasonable linearization of the reduced equation. Obviously reduction and then linearization leads to an equation which differs from the one one gets by first linearizing and then projecting on the orbit space. In [16] it is shown that the correct procedure consists of first linearization and then projection.

An application of this construction gives as a linearization on the orbit space

$$\begin{aligned}
\dot{\eta}_0 &= 2a_0\eta_0 \\
\dot{\eta}_1 &= 2a_1\eta_1 \\
\dot{\eta}_2 &= 2a_2\eta_2 \\
\dot{\eta}_3 &= (2a_1 + a_2)\eta_3.
\end{aligned}$$

Let α, β denote the equilibria on the heteroclinic cycle. Let us introduce local coordinates on the relevant Poincaré sections. At each equilibrium we consider a section to the stable and unstable manifold respectively. Let us denote these by $H^{in,out}_{\alpha,\beta}$ respectively. In coordinates adapted to the geometry of these sections we have:

Definition 4.12

$$\begin{aligned}
H^{in}_{\beta} &= \{s_0 = 1 \mid s_3^2 - s_1^2 s_2 < 0\} \\
H^{out}_{\alpha} &= \{r_0 = 1 \mid r_3^2 - r_1^2 r_2 < 0\} \\
H^{out}_{\beta} &= \{p_1 + p_2 = 1 \mid p_3^2 - p_1^2 p_2 \leq 0, 0 \leq p_0 \leq \mu_\beta\} \\
H^{in}_{\alpha} &= \{q_1 + q_2 = 1 \mid q_3^2 - q_1^2 q_2 \leq 0, q_0 \leq \mu_\alpha\}.
\end{aligned}$$

The first two sets are cone like sets. The last two sets can be considered as a disc bundle over an interval. The numbers μ_β, μ_α will be chosen later. In order to study stability of the heteroclinic set we have to construct mappings between the various sections and to study their invariants. Let us denote the respective local maps by $\Phi_\alpha : H^{in}_{\alpha} \rightarrow H^{out}_{\alpha}$, $\Phi_\beta : H^{in}_{\beta} \rightarrow H^{out}_{\beta}$ and the far maps $\Psi_{\alpha,\beta} : H^{out}_{\alpha} \rightarrow H^{in}_{\beta}$ and $\Psi_{\beta,\alpha} : H^{out}_{\beta} \rightarrow H^{in}_{\alpha}$.

4.3.3 The local maps

Let us begin with the map Φ_α. We use the local description on the local orbit space due to orbit space version of the Hartman-Grobman theorem [16]. We denote the coordinates on the incoming cross section by $q_0, \ldots, q_3$ and on the outgoing cross section by $r_0, \ldots r_3$. Then the linearized flow allows to compute the dependence of the r-vector on the q-vector. For each trajectory we can compute the time necessary to travel along it from the incoming section to the outgoing section. The latter one is defined by $r_0 = 1$. This time will be called the *time of flight*. The equation

$$1 = e^{2a_0 t} q_0$$

defines the time of flight to be

$$t = -\frac{\log q_0}{2\alpha_0}.$$

This gives immediately

$$\begin{aligned} r_1(t) &= q_0^{-\frac{\alpha_1}{2\alpha_0}} q_1 \\ r_2(t) &= q_0^{-\frac{\alpha_2}{2\alpha_0}} q_2 \\ r_3(t) &= q_0^{-\frac{2\alpha_1+\alpha_2}{2\alpha_0}} q_3 \end{aligned}$$

The other simple diffeomorphism comes from the local map $\Phi_\beta : H_\beta^{in} \to H_\beta^{out}$ at β. Here we use coordinates $p_0, \dots, p_3$ on H_β^{out} and $s_0, \dots, s_3$ on H_β^{in}. The cross section H_β^{out} is defined by $p_1 + p_2 = 1$, the incoming cross section by $s_0 = 1$. Therefore we obtain the following solutions of the local differential equations:

$$\begin{aligned} p_0 &= e^{\beta_1 t} \\ p_1 &= e^{\beta_1 t} s_1 \\ p_2 &= e^{\beta_2 t} s_2 \\ p_3 &= e^{\beta_3 t} s_3, \end{aligned}$$

where we define $\beta_3 = \beta_1 + \frac{\beta_2}{2}$ and similar $\alpha_3 = \alpha_1 + \frac{\alpha_2}{2}$. The time of flight $T(s_1, s_2)$ is then given by

$$e^{\beta_1 T} s_1 + e^{\beta_2 T} s_2 = 1.$$

This equation allow to give lower and upper estimates which we shall need.

4.3.4 The far maps

Now we have to look at the far maps. Let us begin with $\Psi_{\alpha,\beta} : H_\alpha^{out} \to H_\beta^{in}$. From the geometry it is clear that (s_1, s_2, s_3) is a function of (r_1, r_2, r_3). We claim that the map is a diffeomorphism and that we can approximate it by the linearization at $(s_1, s_2, s_3) = 0$. Since $\Psi_{\alpha,\beta}(0) = 0$, we have

$$\Psi_{\alpha,\beta} = A_{\alpha,\beta} s + h.o.t,$$

where $A_{\alpha,\beta} = D_r \Psi_{\alpha,\beta}$. Since we are interested in the behavior near $r = 0$, we may neglect the higher order terms. Since the map has to preserve the strata it follows, that

$$r_3^2 - r_1^2 r_2 = 0 \Rightarrow s_3^2 - s_1^2 s_2 = 0. \tag{10}$$

Lemma 4.13 *Equation (10) has to be satisfied if we replace $\Psi_{\alpha,\beta}$ by any of its homogeneous terms. Especially Ar has to satisfy the equation. As a consequence A has the form*

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & \sqrt{A_{11}^2 A_{22}} \end{pmatrix}.$$

Proof: The first statements are clear. We have to show that A has to satisfy the given restrictions. Suppose A has the form

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Then for each triple (r_1, r_2, r_3) with $\Theta(r) = 0$ it follows $\Theta(Ar) = 0$ or

$$(A_{31}r_1 + A_{33}r_3)^2 - (A_{11}r_1 + A_{13}r_3)^2(A_{21}r_1 + A_{22}r_2 + A_{23}r_3) = 0.$$

Choosing $r_2 = r_3 = 0$ we find $A_{31}^2 = 0$ and $A_{11}^2 A_{21} = 0$. A similar conclusion can be obtained by putting $r_1 = r_3 = 0$, implying $A_{32} = 0$ and $A_{12}^2 A_{22} = 0$. Choosing $r_1 = r_3$ and $r_2 = 1$ we find (using that A has to be regular) that $A_{12} = A_{13} = 0$ and $A_{21} + A_{23} = 0$. If we set $r_1 = -r_3$, then it follows that $A_{21} = A_{23} = 0$. Altogether we find that A is diagonal and the coefficients on the diagonal satisfy

$$A_{33}^2 = A_{11}^2 A_{22}.$$

□

The next step is the to look at $\Psi_{\beta,\alpha}$. Here, we have to deal with a special complication, which we first describe on the level of the dynamics on the space V. As we have seen, we have connections from β' to α and from β' to α'. On the orbit space both connections will project onto a connection from $\pi(\beta)$ to $\pi(\alpha)$. However, those which come from the second type will pass near α and then connect to α'. On the orbit space this means that we have trajectories from $\pi(\beta)$ to $\pi(\alpha)$ which do connect immediately but on the first part they come close to $\pi(\alpha)$. However, they miss and hit on the second passage. There is nice way of identifying this behavior with group theory. The groups D_2^d and D_2^z are subgroups of the isotropy subgroup of β and the decomposition of the (local) orbit into orbit types distinguishes these two isotropy types. However as subgroups of $\mathbf{O}(3)$ these groups are conjugate and define the same orbit type. The solutions starting on the D_2^z stratum connect directly to $\pi(\alpha)$, those on the D_2^d stratum came back on a first return on the D_d^z stratum on connect on the second time. Observe that the fact, that D_2^z and D_2^d are conjugate within $\mathbf{O}(3)$ makes it possible that solutions "switch the strata". Let us collect this information in a lemma.

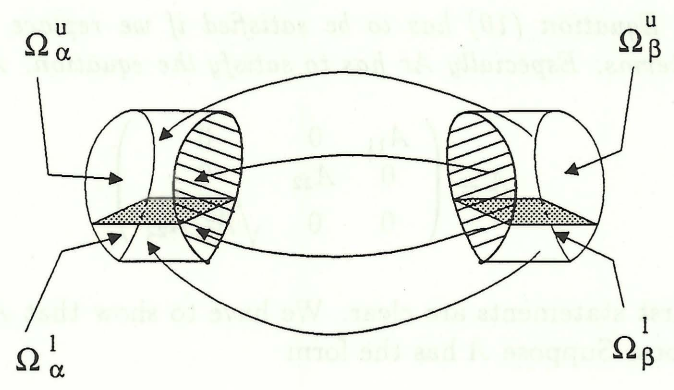

Figure 8: The disc bundles and the splitting of neighborhoods

Lemma 4.14 *The section H^{out}_β decomposes into orbit types D^d_2, D^z_2, Z^-_2 and the trivial one.*

1. *Trajectories on the orbit type D^d_2 do not hit $\pi(\alpha)$ instead they return into a neighborhood of $\pi(\beta)$ and hit the orbit type D^z_2.*
2. *Solutions on the orbit type D^z_2 connect to $\pi(\alpha)$.*

Proof: Follows from the results concerning the existence of the heteroclinic set. □

For future reference we depict the disc bundles and the way we identify the isotropy types on those bundles, see Figure 8.

We use this information to split H^{out}_β into Ω^l_β and Ω^u_β and correspondingly $\Psi_{\beta,\alpha}$ into

$$\begin{aligned} \Psi^l_{\beta,\alpha} &: \Omega^l_\beta \to H^{in}_\alpha \\ \Psi^u_{\beta,\alpha} &: \Omega^u_\beta \to H^{in}_\alpha \end{aligned}$$

The definition of these sets is as follows: choose a number μ_α sufficiently small that the local map constructed above is well defined. Consider the point in $p \in H^{out}_\beta$ which is mapped onto $(\mu_\alpha, 0, 1, 0)$. This point p defines a value p^*_1 and hence a value p^*_2 and the corresponding p_3 interval. In fact by choosing p^*_1 even smaller we can achieve that on the set $p_1 \le p^*_1$, $p^2_3 - p^2_1 p_2 < 0$ the diffeomorphism $\Psi^l_{\beta,\alpha}$ which is defined by $p \to r$, where r denotes the point where the trajectory through p hits H^{in}_α, can be approximated by the its linearization A at $p = (0, 0, 1, 0)$. In fact we will determine the form of this linearization and we will assume that A instead of $\Psi_{\beta,\alpha}$ was used to define p^*_1. This is not a loss of generality and all these quantities are well defined. Then $\Omega^l_\beta = \{(p_0, p_1, p_2, p_3) \mid p_0 \le \mu_\beta, p_1 \le p^*_1, p_2 = 1 - p_1, p^2_3 - p^2_1 p_2 \le 0\}$. $\Omega^u_\beta = H^{out}_\beta \setminus \Omega^l_\beta$. There is one more complication coming up: for the upper map $\Psi^u_{\beta,\alpha}$ we do not have a solution where we can linearize those maps and therefore it might not be justified to replace $\Psi_{\alpha,\beta}$ by its linearization. However, again the decomposition into orbit types helps. First we note that for $p \in \Omega^u_\beta$ the map $\Psi^u_{\beta,\alpha}$ will be defined as for $\Psi^l_{\beta,\alpha}$ with the difference that the trajectory does not hit H^{in}_α on the first try. This follows from the construction of p_1^*. If one chooses $p_1 > p_1^*$ we will end up with a point $(q_0, 0, 0)$ and $q_0 > \mu_\alpha$. We will denote $\Omega^l_\alpha = \Psi^l_{\beta,\alpha}(\Omega^l_\beta)$ and by Ω^u_α the complement in H^{in}_α. From the construction it follows that $\Psi^u_{\beta,\alpha}$ maps into Ω^u_α, however we cannot conclude surjectivity or other nice properties. The only thing we need is a bound in the q_0 component. The various orbit types are described by certain functions. We use linearization for such functions. The lower map $\Psi^l_{\beta,\alpha}$ can be approximated by its linear part. We find this approximation by considering how the orbit types are mapped onto each other. This gives sufficient restriction to give a clear form of the linear approximation.

Lemma 4.15 *1. $\Psi^l_{\beta,\alpha}$ can be approximated by its linear part. This part has the form:*

$$A = \begin{pmatrix} c & c & 0 \\ b & a & -a \\ b & -a & a \end{pmatrix}$$

2. For the upper part we notice that p_0 maps onto q_0 and we may assume that an estimate of the form

$$q_0 \leq M\, p_0$$

is true, where M is a positive constant.

Proof: The first part follows from linearization and identification of the strata near the fixed point, since we know that strata have to be preserved along the map. For the second part, we note that the $p_0 = 0$ maps into the set $q_0 = 0$. Since the map is differentiable, the assertion follows. □

4.3.5 The first return map

Now we can piece together this information to write the first return map.

This map is given by

$$\begin{pmatrix} q0' \\ q_1' \\ q_3' \end{pmatrix} = \Psi_{\beta,\alpha} \circ \Psi_\beta \circ \Psi_{\alpha,\beta} \circ \Psi_\alpha \begin{pmatrix} q_0 \\ q_1 \\ q_3 \end{pmatrix} .$$

We have (taking $\Psi_{\beta,\alpha}$ as $\Psi^l_{\beta,\alpha}$)

$$
\begin{aligned}
q_1' &= a_1 e^{\beta_1 T} q_0^{-\frac{\alpha_1}{\alpha_0}} q_1 - a_3 e^{\beta_3 T} q_0^{-\frac{\alpha_3}{\alpha_1}} q_3 + b e^{\beta_0 T} \\
q_0' &= c(a_1 e^{\beta_1 T} q_0^{-\frac{\alpha_1}{\alpha_0}} q_1 + a_3 e^{\beta_3 T} q_0^{-\frac{\alpha_3}{\alpha_1}} q_3) \\
q_3' &= a(a_1 e^{\beta_1 T} q_0^{-\frac{\alpha_1}{\alpha_0}} q_1 - a_3 e^{\beta_3 T} q_0^{-\frac{\alpha_3}{\alpha_1}} q_3) + b e^{\beta_0 T}
\end{aligned}
$$

Lemma 4.16 *For the limit we are interested in ($q_2 = 1$), we can take*

$$
e^{\beta_0 T} = s_2^{\frac{\beta_1}{\beta_2}}, \quad e^{\beta_3 T} = s_2^{\frac{-\beta_3}{\beta_2}} \tag{11}
$$

Proof: In the lower domain Ω^l_β, q_1 is small and q_2 is nearly 1. Therefore we look at the condition

$$
1 = e^{\beta_2 T}, \text{ or } T = -\frac{\log s_2}{\beta_2}.
$$

□

Altogether we get

Lemma 4.17 *1. If $\Psi_{\beta,\alpha} = \Psi^l_{\beta,\alpha}$ then*

$$
q_0' \le c\left(a_1(a_2 q_0^{-\frac{\alpha_2}{\alpha_0}} q_2)^{-\frac{\beta_1}{\beta_2}} q_0^{-\frac{\alpha_1}{\alpha_0}} q_1 + a_3(a_3 q_0^{-\frac{\alpha_2}{\alpha_0}} q_3)^{-\frac{\beta_0}{\alpha_2}} q_0^{-\frac{\alpha_2}{\alpha_0}} q_3\right)
$$

2. If $\Psi_{\beta,\alpha} = \Psi^u_{\beta,\alpha}$ then we find

$$
q_0' \le C\left(q_0^{\frac{\max \alpha_i}{\alpha_0}}\right)^{-\frac{\beta_0}{\beta_2}}.
$$

We recall the condition for stability of the heteroclinic cycle: if α_1, β_1¡0 and if

$$
\alpha_2\beta_1 - \alpha_1\beta_2 > \alpha_0\beta_2,
$$

then the heteroclinic cycle is stable.

To get the stability of the heteroclinic set, we have to include $\Psi^u_{\beta,\alpha}$ and we obtain:

Theorem 4.18 *The generalized heteroclinic cycle is stable if*

$$
\min\{\alpha_2\beta_0, \alpha_2\beta_1 - \alpha_1\beta_2\} > \alpha_0\beta_2. \tag{12}
$$

5 Perturbation of heteroclinic cycles by a slow rotation of the domain

Let us consider a heteroclinic cycle (simple or generalized) of the kind described in the previous sections, and suppose now that the spherical shell is allowed to rotate slowly around its "vertical" axis. This introduces some inertial terms into the equations of motion : a) a centrifugal force which we incorporate in the gradient of pressure; b) the Coriolis force, which, in the non-dimensional variables, has the form $Ta\ \mathbf{v} \times \mathbf{k}$, where $\mathbf{k}$ is the unit vector along the vertical axis Oz and Ta is the Taylor number (proportional to the angular velocity of the domain). Assuming a slow rotation comes back to assume that Ta is close to 0. For convenience, we shall use the notation ϵ instead of Ta in the following. The introduction of the Coriolis force breaks the $\mathbf{O}(3)$ invariance of the system. One easily checks that the Coriolis force commutes with, and only with, the rotations around the vertical axis (group $\mathbf{SO}(2)$) and still with the reflection through the origin (group $\mathbb{Z}_2^c$). The question is therefore to determine what happens to the heteroclinic cycles after such a perturbation. We shall see that surprisingly enough, there is persistence of a cycle as long as the perturbation is small, i.e. for ϵ close enough to 0. Since we take ϵ as a parameter in a neighborhood of 0, the center manifold reduction on the space V of critical modes for the $\mathbf{O}(3)$ invariant system is still valid (see [8] for details). We shall not proceed however to a bifurcation analysis (which would then be a codimension 4 bifurcation), but rather we shall assume the existence of a heteroclinic cycle for some fixed values of the bifurcation parameters (in particular the Rayleigh number) and $\epsilon = 0$. Then we perturb this situation with $\epsilon \to 0$.

We shall proceed in the following steps. First analyse the geometry of the action of $H = \mathbf{SO}(2) \times \mathbb{Z}_2^c$ in V. Surprisingly enough, we shall see that most of the orbit type structure of the $\mathbf{O}(3)$ action does remain. Then we can easily draw conclusions about the persistence of most of the heteroclinic connections. Finally we analyse the effect of the perturbation on the group orbits of equilibria. All this will result in the existence of a new type of heteroclinic cycle for the perturbed problem.

5.1 The action of the group H in V

We recall that $\mathbb{Z}_2^c$ acts trivially on the $\ell_0 = 2$ components (i.e. on the y_m's) and as $-\mathbb{1}$ on the $\ell_0 = 1$ components (i.e. on the x_j's). The action of $\mathbf{SO}(2)$ is as follows :

$$R_\varphi\,(x_{-1}, x_0, x_1) = (e^{-i\varphi}x_{-1}, x_0, e^{i\varphi}x_1) \tag{13}$$

$$R_\varphi\,(y_{-2}, y_{-1}, y_0, y_1, y_2) = (e^{-2i\varphi}y_{-2}, e^{-i\varphi}y_{-1}, y_0, e^{i\varphi}y_1, e^{2i\varphi}y_2) \tag{14}$$

Then it is easy to check that the isotropy subgroups of this action are, up to conjugacy in H, as follows.

1. H, with $\mathrm{Fix}(H) = \{y_0\}$

2. $\mathbf{SO}(2)$, with $\mathrm{Fix}(\mathbf{SO}(2)) = \{x_0, y_0\}$

3. $\mathbb{Z}_2 \times \mathbb{Z}_2^c$, with $\mathrm{Fix}(\mathbb{Z}_2 \times \mathbb{Z}_2^c) = \{y_0, y_{-2}, y_2\}$

4. $\mathbb{Z}_2^-$, with $\mathrm{Fix}(\mathbb{Z}_2^-) = \{x_{-1}, x_1, y_0, y_{-2}, y_2\}$

5. $\mathbb{Z}_2^c$, with $\mathrm{Fix}(\mathbb{Z}_2) = \{y_0, y_{-1}, y_1, y_{-2}, y_2\}$

6. $\mathbb{1}$, with $\mathrm{Fix}(\mathbb{1}) = V$

In the following reference to a group in this list refers to a subgroup of H (unless explicitly mentioned otherwise).

We can compare the fixed-point subspaces with those of the action of $\mathbf{O}(3)$: $\mathrm{Fix}(\mathbf{SO}(2)) = \mathrm{Fix}(\mathbf{O}(2)^-)$, $\mathrm{Fix}(\mathbb{Z}_2 \times \mathbb{Z}_2^c)$ is the span of copies of $\mathrm{Fix}(D_2 \times \mathbb{Z}_2^c)$ under the action of $\mathbf{SO}(2)$, $\mathrm{Fix}(\mathbb{Z}_2^-)$ and $\mathrm{Fix}(\mathbb{Z}_2^c)$ are unchanged (of course these isotropy subgroups are defined now up to conjugacy in H, which means that most conjugates in $\mathbf{O}(3)$ do not persist as isotropy subgroups.

5.2 Perturbation of the equilibria and their connections

We consider the equilibria α, α', α'', β, β', β'', which are involved in the heteroclinic cycles of type I when $\epsilon = 0$ and which belong to the plane $P_1 = \mathrm{Fix}_{\mathbf{O}(3)}(D_2 \times \mathbb{Z}_2^c)$. In this plane, we know that saddle-sink connections exist from β to α' and α'', from β' to α and α'' and from β'' to α and α'. The heteroclinic cycle is completed by connections from α' to β' and from α'' to β'', in the planes P_2' and P_2'' respectively (see section 3.1.2). Acting on these connections with rotations in $\mathbf{SO}(2) \subset H$, we get a one parameter family of heteroclinic cycles. The question is what happens to this object when we set $\epsilon \neq 0$.

The equilibria α and β persist since they belong to $\mathrm{Fix}(H)$. In other words, their $\mathbf{SO}(2)$ group orbit reduces to one point. This is not the case for the other equilibria. Their $\mathbf{SO}(2)$ group orbits are circles in $\mathrm{Fix}(\mathbb{Z}_2 \times \mathbb{Z}_2^c)$. When $\epsilon \neq 0$, the symmetry-breaking induces on the $\mathbf{SO}(2)$ group orbits of α' and β' a drift (see the annex). Hence these two circles of equilibria are replaced by two *rotating waves* which we denote by RW_α and RW_β respectively. It is clear that RW_α is stable in $\mathrm{Fix}(\mathbb{Z}_2 \times \mathbb{Z}_2^c)$, because when $\epsilon = 0$, α' (as well as α'') is a sink in P_1.

We now examine how the perturbation acts on the heteroclinic connections, assuming that ϵ is *close enough to 0*.

1. The $\alpha \to \beta$ connection in $\{x_0, y_0\}$ persists, because this space is still flow invariant (see above).

2. The $\beta \to \alpha'$ (and α'') connection in Fix$(D_2 \times \mathbb{Z}_2^c)$ is now replaced by a connection in Fix$(\mathbb{Z}_2 \times \mathbb{Z}_2^c)$ from β to RW_α. This is clear because the normal cross section to RW_α at the point α' (for example) is just (locally) Fix$(D_2 \times \mathbb{Z}_2^c)$.

3. For the same reason, the $\beta' \to \alpha$ connection in Fix$(D_2 \times \mathbb{Z}_2^c)$ is now replaced by a connection $RW_\beta \to \alpha$.

4. The 2 D manifold of connections which exists at $\epsilon = 0$ from RW_β to RW_α in Fix$(\mathbb{Z}_2 \times \mathbb{Z}_2^c)$, persists when $\epsilon \neq 0$.

It is important to notice that the planes P_2' and P_2'' are included in the 5 dimensional space Fix$(\mathbb{Z}_2^-)$, however the points in these planes have trivial isotropy in H. Note also that $\mathbf{SO}(2) \subset H$ acts on this space. Therefore in order to study the perturbation of the heteroclinic connections in P_2' and its copies by the action of $\mathbf{SO}(2)$, we can restrict ourselves to the space Fix$(\mathbb{Z}_2^-)$. This is what we do in the next two lemmas. For a better understanding of the geometry of connections in Fix$(\mathbb{Z}_2^-)$, we can notice that when $\epsilon = 0$, a part of the heteroclinic cycle of type I realizes a sub-heteroclinic cycles in this space (see figure 9). The $\mathbf{SO}(2)$ orbit of this sub-heteroclinic cycle is also included in Fix$(\mathbb{Z}_2^-)$.

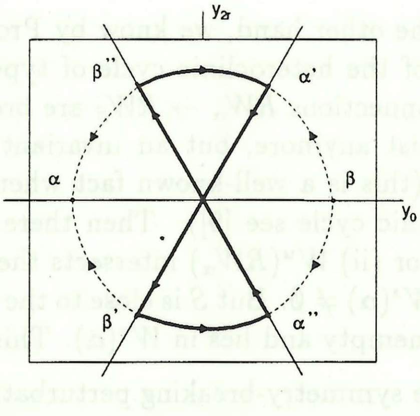

Figure 9: Projection on P_1 of the sub-heteroclinic cycle in Fix$(\mathbb{Z}_2^-)$ (solid lines).

Lemma 5.1 *The connecting orbits from RW_α to RW_β in* Fix(Z_2^-)*, do not generically persist when $\epsilon \neq 0$, close to 0.*

Proof. As noted above, the connections from RW_α to RW_β when $\epsilon = 0$ have trivial isotropy in H. The unstable manifold $W^u(RW_\alpha)$ consists precisely of these connections when $\epsilon = 0$. Let $W^s(RW_\beta)$ denote the stable manifold of RW_β. The projection of the trivial stratum on the orbit space Fix$(Z_2^-)/SO(2)$ is a 4 D manifold. The projections of RW_α and RW_β are points (equilibria for the projected vector field). The projection of $W^u(RW_\alpha)$ is a 1 D manifold and that of $W^s(RW_\beta)$ is either a 2 D or a 3 D manifold. This can be checked by a count of the unstable eigendirections at α' and of the stable eigendirections at β' in $\mathrm{Fix}(Z_2^-)$ and at $\epsilon = 0$. In any case, the sum of these dimensions is less than or equal to 4, which implies that no trajectory can belong to the intersection of the two manifolds under generic perturbations. □

Remark. This lemma does not imply that for our specific perturbed vector field, the connections $RW_\alpha \to RW_\beta$ are indeed broken. However this is observed in the numerical simulations. The splitting of the connections could also be computed in principle (like e.g. in [9]) but we do not need this : as we shall see by the next lemma, a heteroclinic cycle persists in any case when $\epsilon \neq 0$.

Lemma 5.2 *Suppose that $\epsilon \neq 0$, close to 0, and the connections $RW_\alpha \to RW_\beta$ are broken. Then one of the two following situations occurs in* $\mathrm{Fix}(Z_2^-)$ *: (i) $W^u(RW_\alpha) \subset W^s(\alpha)$; (ii) there exists a flow-invariant set S such that $W^u(RW_\alpha) \subset W^s(S)$, but in this case $W^u(S) \subset W^s(\alpha)$.*

Proof. As noted above, when $\epsilon = 0$ a heteroclinic cycle

$$RW_\alpha \to RW_\beta \to RW_\alpha$$

exists in $\mathrm{Fix}(\mathbb{Z}_2^-)$. On the other hand, we know by Proposition 3.4 that under the conditions of existence of the heteroclinic cycle of type I, α is a sink in $\mathrm{Fix}(\mathbb{Z}_2^-)$. Now, assume that the connections $RW_\alpha \to RW_\beta$ are broken. The heteroclinic cycle in $\mathrm{Fix}(Z_2^-)$ does not exist anymore, but an invariant set S (e.g. quasi-periodic orbit) can exist nearby (this is a well-known fact when forced symmetry-breaking is applied to a heteroclinic cycle see [9]). Then there are only two possibilities : (i) $W^u(RW_\alpha) \subset W^s(\alpha)$ or (ii) $W^u(RW_\alpha)$ intersects the stable manifold of S. Now notice that $W^u(RW_\beta) \cap W^s(\alpha) \neq \emptyset$. But S is close to the heteroclinic cycle if ϵ is close to 0, hence $W^u(S)$ is nonempty and lies in $W^s(\alpha)$. This proves the lemma. □

A last new feature of the symmetry-breaking perturbation is the dynamics induced on the perturbed $\mathbf{O}(3)$ group orbits of equilibria themselves. A general classification of the possible dynamics has been proposed by [29]. In our case the equilibria have isotropy $\mathbf{O}(2) \times \mathbb{Z}_2^c$, so the group orbits are 2 D manifolds. Assuming normal hyperbolicity of these orbits (which is the case here) and taking $\epsilon \neq 0$ but close enough to 0, we know that after perturbation, a 2 D flow invariant manifold persists near the former group orbit. The next lemma specifies the dynamics on this manifold.

Lemma 5.3 *Under the above assumptions, the 2 D invariant manifold which persists after perturbation of the $\mathbf{O}(3)$ group orbit of α is the union of the equilibrium α, the rotating wave RW_α, and a 2 D set of connecting trajectories from α to RW_α. A similar result holds for the perturbed $\mathbf{O}(3)$ orbit of β.*

Proof. In the appendix. □

6 Conclusion

In this paper we have been able to extend the work of [1], by the mean of recent methods about equivariant bifurcations and dynamical systems. Our main results are (i) the existence of a generalized heteroclinic cycle in the 1 - 2 mode interaction with symmetry, (ii) a sufficient condition for the stability of this GHC, (iii) the persistence of a heteroclinic cycle (or set) involving rotating waves, when the symmetry is broken by an $\mathbf{SO}(2) \times \mathbb{Z}_2^c$ equivariant perturbation (as it occurs for Bénard convection in a spherical shell which can slowly rotate around an axis). There are several questions which remain unanswered. The first one concerns the stability of the generalized heteroclinic cycle versus stability of the heteroclinic cycle (of type I). Suppose we vary one coefficient (e.g. δ) in such a way that the heteroclinic cycle of type I is asymptotically stable at the beginning, but a transverse eigenvalue (namely, β_1) becomes positive as this parameter crosses a certain critical value. Then the cycle is not asymptotically stable anymore. However, we conjecture that it is essentially asymptotically stable (see [27] for an open interval of parameter values, while it becomes completely unstable when the parameter leaves this interval. It is tempting to imagine that when this happens, the generalized heteroclinic cycle is still an attractor. However this needs not to be the case in general. We have not tried to determine analytically an interval of values of the coefficients where the condition of stability of the heteroclinic set is fulfilled. It is easy to find numerical values for this. In figure 10 we show a numerical simulation of the heteroclinic cycle for coefficient values satisfying condition (12), but at which the heteroclinic cycle of type I is clearly unstable. The Figure 10(a) shows a single trajectory which clearly follows different paths after each return. In Figure 10(b), we selected a single first return trajectory.

In order to show the impact of this work on the spherical Bénard problem we test theorem 4.18 against the values of the coefficients displayed in Table 1, for example. Choosing $\mu_1 = 0.03$ and $\mu_2 = 0.05$ the stability conditions are satisfied. This explains the observations of FRIEDRICH & HAKEN [19].

The persistence of a generalized heteroclinic cycle after a perturbation like the rotation of the spherical shell in the Bénard problem is one of the surprises of this work. We again did not intend to prove that stability is preserved under perturbations with ϵ close enough to 0. This is clear if no invariant set is created near the cycle. Numerically, the generalized heteroclinic cycle can be observed, as figure 11 shows (pictures (a) and (b)). As ϵ is increased, the trajectories pass quite far away from the rotating wave RW_β (figures 12 (a) and (b)), but the cycle is still observable.

It would be interesting to pursue this analysis when the heteroclinic cycle comes in a different $\mathbf{O}(3)$ mode interaction. Such cycles have been proved to exist in [14]. However it will then becomes very complicated, due to the high dimension of phase space (already 12 in the 2 - 3 mode interaction), and the multiplicity of heteroclinic

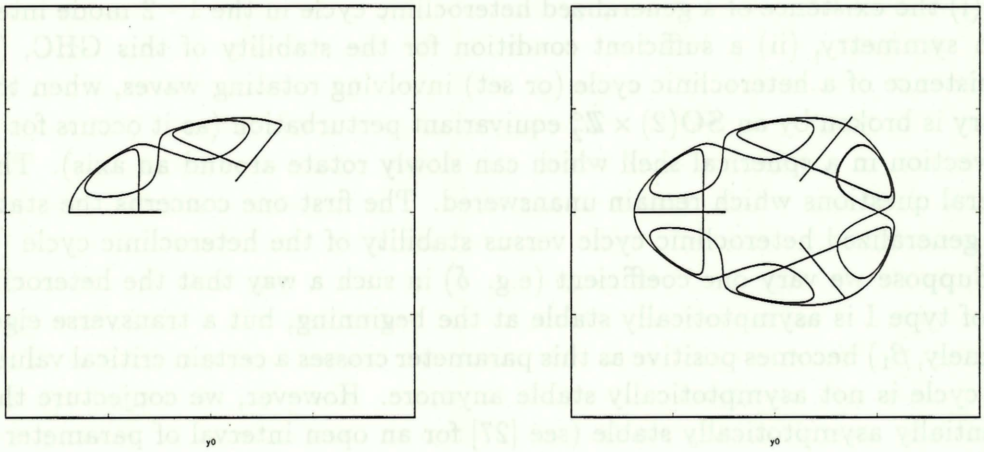

Figure 10: The heteroclinic set (a) after a certain number of returns, (b) for a single return. The trajectories are projected onto the plane $\{(y_0, y_{2r})\}$.

cycles which may coexist.

Appendix

A The $\ell = 1, \ell = 2$ mode interaction

Here, we will complete the information on the mode interaction from section 2. The Poincaré series is a well established tool in invariant theory [34]. During the last years it was also used in applications of invariant theory to equivariant bifurcation theory and equivariant dynamics [30, 31, 20]. Let $\mathcal{R}(V)$ denote the algebra of invariant polynomials over the representation V. With $\mathcal{R}_d(V)$ w denote the polynomials in $\mathcal{R}(V)$ of degree d. Similarly we define $\mathcal{M}(V)$ to be the module of equivariant polynomial mappings over $\mathcal{R}(V)$ and $\mathcal{M}_d(V)$ denotes the mappings of degree d. The

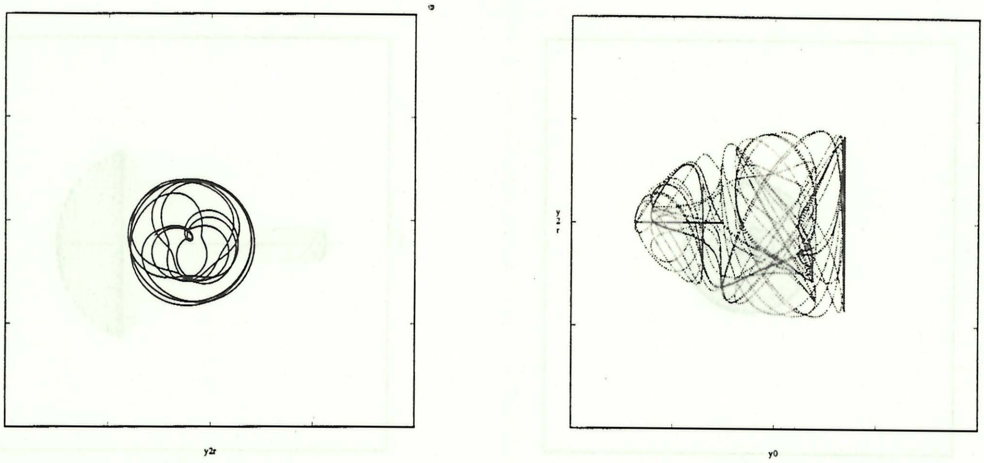

Figure 11: The perturbed heteroclinic set for $\epsilon = 0.03$. (a) projection onto the plane $\{y_2, \bar{y}_2)\}$, (b) projection onto the plane $\{y_0, y_{2r})\}$.

Poincaré–series for the ring of invariant polynomials is the formal power series

$$P^{\mathcal{R}}(s) = \sum_{d=0}^{\infty} \dim(\mathcal{R}_d(V), \tag{15}$$

and in similar fashion the Poincaré–series for the module $\mathcal{M}(V)$ is given by

$$P^{\mathcal{M}}(s) = \sum_{d=0}^{\infty} \dim(\mathcal{M}_d(V). \tag{16}$$

Both series can be determined via Molien formulas, for a reference see SATTINGER [33], SPRINGER [34]

$$P^{\mathcal{R}}(s) = \int_{\mathrm{O}(3)} \frac{1}{\det(\gamma - s1\!\!1)} d\gamma \tag{17}$$

and

$$P^{\mathcal{M}}(s) = \int_{\mathrm{O}(3)} \frac{\mathrm{tr}(\gamma)}{\det(\gamma - s1\!\!1)} d\gamma. \tag{18}$$

In order to compute the series for our representation $V_{1,2}$ we use the formulas given in [30], i.e. where these integrals were reduced to an integral over the unit circle in $\mathbb{C}$ and hence to computations of some residues. To compute these integrals we proceed

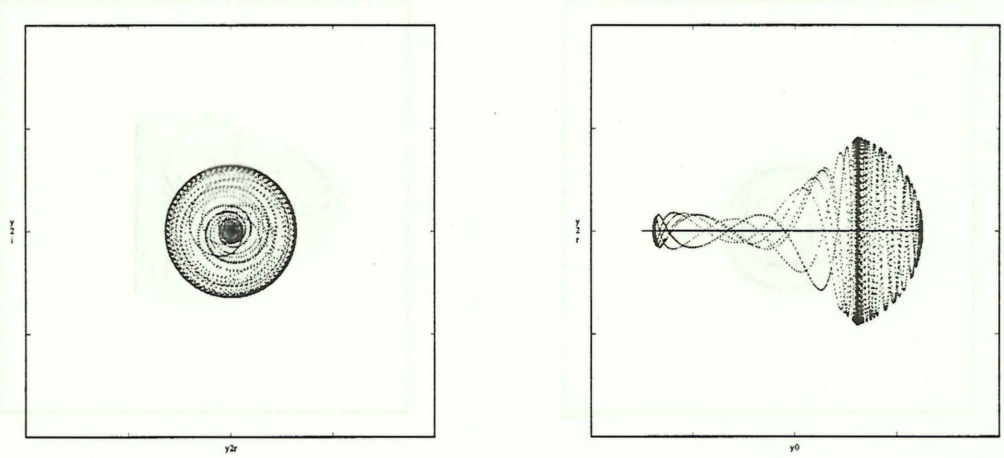

Figure 12: The perturbed heteroclinic set for $\epsilon = 0.2$. (a) projection onto the plane $\{y_2, \bar{y}_2)\}$, (b) projection onto the plane $\{y_0, y_{2r})\}$.

as in LAUTERBACH & SANDERS [30]. Some care is required for the integration over $\mathbf{O}(3) \setminus \mathbf{SO}(3)$. Using the notation as in [30] we find for the group $\mathbf{SO}(3)$ that $P^{\mathcal{R}}(s)$ for the $\ell = 1, \ell = 2$–mode interaction is given by

$$P^{\mathcal{R}}_{\mathbf{SO}(3)}(s) = \frac{1}{2}\frac{1}{2\pi i}\int\limits_{|z|=1} \frac{z^2(1-z)(z-1)}{(1-s)^2(z-s)^2(1-sz)^2(z^2-s)(1-sz^2)}dz \tag{19}$$

and similar for

$$P^{\mathcal{M}}_{\mathbf{SO}(3)}(s) = \frac{1}{2}\frac{1}{2\pi i}\int\limits_{|z|=1} \frac{z^2(1-z)(z-1)(2+2(z+z^{-1})+z^2+z^{-2})}{(1-s)^2(z-2)^2(1-sz)^2(z^2-s)(1-sz^2)}dz \tag{20}$$

Since the Weyl group for $\mathbf{O}(3)$ has four elements the respective formulas for the group $\mathbf{O}(3)$ read

$$P^{\mathcal{R}}_{\mathbf{O}(3)}(s) = \frac{1}{2}P^{\mathcal{R}}_{\mathbf{SO}(3)} + \frac{1}{4}\frac{1}{2\pi i}\int\limits_{|z|=1} \frac{z^2(1-z)(z-1)}{(1-s)(1+s)(z-s)(z+s)(1-sz)^2(z^2-s)(1-sz^2)}dz \tag{21}$$

and

$$P^{\mathcal{M}}_{\mathbf{O}(3)}(s) = \frac{1}{2}P^{\mathcal{M}}_{\mathbf{SO}(3)}(s) + \frac{1}{4}\frac{1}{2\pi i}\int_{|z|=1} \frac{z^2(1-z)(z-1)(z^2+z^{-2}}{(1-s)(1+s)(z-s)(z+s)(1-sz)^2(z^2-s)(1-sz^2)}dz \tag{22}$$

Using residue calculus one can evaluate these integrals and obtain

$$P^{\mathcal{R}}_{\mathbf{SO}(3)}(s) = \frac{1+s^6}{(1-s^2)^2(1-s^3)^2(1-s^4)}, \tag{23}$$

and

$$P^{\mathcal{M}}_{\mathbf{SO}(3)}(s) = \frac{2s+4s^2+4s^3+4s^4+2s^5}{(1-s^2)^2(1-s^3)^2(1-s^4)}. \tag{24}$$

In the case of $\mathbf{O}(3)$ we get

$$P^{\mathcal{R}}_{\mathbf{O}(3)}(s) = \frac{1}{(1-s^2)^2(1-s^3)^2(1-s^4)}, \tag{25}$$

and

$$P^{\mathcal{M}}_{\mathbf{O}(3)}(s) = \frac{2s+3s^2+2s^3+s^4}{(1-s^2)^2(1-s^3)^2(1-s^4)}. \tag{26}$$

In order to give an interpretation for the degrees of a generating set of these algebraic structures we need to find a set of algebraically independent elements of $P^{\mathcal{R}}(s)$ (for either $G = \mathbf{SO}(3)$ or $G = \mathbf{O}(3)$) with degrees indicated by the denominator of the series for the invariant polynomials, compare STURMFELS [35]. Before we continue we mention that it follows from here already that the $\mathbf{SO}(3)$ theory is significantly more difficult than the $\mathbf{O}(3)$ case. The next step consists of finding a set of algebraically independent invariant polynomials which are homogeneous of degrees 2, 2, 3, 3 and 4. These are given in Lemma 2.1. Similar considerations apply to the equivariants.

B Proof of lemma 5.3

We consider the $\mathbf{O}(3)$- group orbits of equilibria of types α and β, and we perturb them by taking ϵ close to 0 in the equations. With ϵ close enough to 0, a normally hyperbolic invariant manifold persists near each of the group orbits. The symmetry of the perturbation is $H = \mathbf{SO}(2) \times \mathbb{Z}_2^c$. The question is then to determine the kind of dynamics which is induced on this invariant manifold by the perturbation. As mentioned in section 5, the axis y_0 remains flow invariant, hence an equilibrium persists near α (as well as near β), to which we give the same name. Moreover, a rotating wave appears near the $\mathbf{SO}(2)$ orbit of α' (respectively β'), which we call

RW_α) (respectively RW_β). We shall prove that the invariant manifolds are the union of these two solutions and of a 2 D set of connecting orbits there are typically no other flow-invariant circles between these two from α (respectively β) to RW_α (respectively RW_β).

We proceed as follows : 1. Write the equations on the center manifold when $\epsilon \neq 0$, 2. show that no other relative equilibria than those listed above persist when $\epsilon \neq 0$; 3. Compute the eigenvalues along the invariant manifold at α (respectively β).

B.1 The perturbed equations on the center manifold

The equilibria α and β belong to the "pure mode" $\ell = 2$ space. In what follows we can therefore forget about the other, $\ell = 1$ components. When a center manifold reduction is performed in the case $\epsilon \approx 0$, the bifurcation equations take the following form (see [8]).

$$\dot{y}_0 = ay_0 + b(-1/2y_0^2 + 1/2y_1y_{-1} + y_2y_{-2}) + \epsilon^2 h_0 \tag{27}$$

$$\dot{y}_1 = ay_1 + b(-y_0y_1 + \sqrt{6}/2y_{-1}y_2)) + \epsilon^2 h_1 + i\epsilon k_1) \tag{28}$$

$$\dot{y}_2 = ay_2 + b(y_0y_2 - \sqrt{6}/4y_1^2) + \epsilon^2 h_2 + i\epsilon k_2 \tag{29}$$

where a and b are O(3)-invariant functions of the y_j's and h_m and k_m are even functions of ϵ and are SO(2)-invariant functions of the y_j's. The last property implies the following :

$$h_0 = h_0'y_0 + h_1''y_1\bar{y}_1 + h_1'''y_2\bar{y}_2 \tag{30}$$

$$h_1 = h_1'y_1 + h_1''\bar{y}_1y_2 \tag{31}$$

$$h_2 = h_2'y_2 + h_2''y_1^2 \tag{32}$$

Here the various functions h_0', etc... depend on ϵ and on the norms of the y_j's, and moreover $h_j'(0) = \gamma_j$ with the relations

$$\gamma_0 < \gamma_1 < \gamma_2 < 0.$$

There is a similar structure for the functions k_j and moreover we can write

$$k_j'(0) = j\beta, \quad \beta < 0.$$

In what follows we shall not need the SO(3)-invariant structure of a and b.

B.2 The perturbed relative equilibria

B.2.1 The equilibria of the unperturbed equations

We set $\epsilon = 0$. We are interested in the group orbit of equilibria on the y_0-axis. Solving equation 27 with $y_1 = y_2 = 0$, and assuming that b is in *general position*, we can write the solution :

$$\hat{y}_0 = -\frac{a}{b} = \nu.$$

Let us assume $\nu << 1$. The group orbit spanned by this solution is two dimensional (y_o is fixed by $\mathbf{SO}(2)$). Any equilibrium in this group orbit must satisfy

$$y_0^2 + 2y_1\bar{y}_1 + 2y_2\bar{y}_2 = \nu^2. \tag{33}$$

Acting by rotations around the vertical axis, we can always set $y_1 \in \mathbb{R}$, i.e. restrict to the slice on the orbit defined by that condition. A simple combination of equations (27)-(29) ($\epsilon = 0$) shows that any solution with b in general position and real y_1, must also satisfy the following relations :

$$y_2 \in \mathbb{R} \text{ and } \frac{\sqrt{6}}{2}y_0y_2 + y_2^2 - y_1^2 = 0. \tag{34}$$

Finally we can express any element of the group orbit with coordinates $y_j = e^{ij\psi}\hat{y}_j$ ($j = 0, 1, 2$), where ψ is an arbitrary phase and the $\hat{y}_j$'s are real and satisfy relations 33 and 34.

B.2.2 The relative equilibria when $\epsilon \neq 0$

Since the $\mathbf{SO}(2)$ equivariant perturbation has a vertical axis of symmetry, the relative equilibria when $\epsilon \neq 0$ must have the form

$$y_j(t) = e^{ij(\omega t+\varphi)}y_j, \quad (j = 0, 1, 2) \tag{35}$$

where ω vanishes at $\epsilon = 0$ and φ is an arbitrary phase. Replacing the amplitude variables in (27)-(29) by (35), reduces these equations to a time independent system which is just (27)-(29) with $\dot{y}_j$ replaced by $ij\omega y_j$.
As a first consequence, we can set $y_1 \in \mathbb{R}$ without restricting the generality.

Next, we look for relative equilibria which result from the perturbation of the group orbit of pure equilibria when $\epsilon = 0$. Hence we can set

$$y_j = \hat{y}_j + u_j \tag{36}$$

with u_0 and u_1 *real* (but apriori not u_2), each u_j vanishing at $\epsilon = 0$.

There are two cases where the equations for this perturbation are easily solved and lead to well-defined solutions (see [9]) :
(i) $y_1 = y_2 = 0$: the system reduces to the scalar equation (27) for u_0. This solution is $\mathbf{SO}(2)$ invariant and hence is a pure equilibrium;
(ii) $y_1 = 0$: the system reduces to three scalar equations for u_0, u_2 and ω. The solutions are now rotating waves and as ϵ tends to 0, they tend to an equilibrium on the circle defined by $y_2 = e^{i\psi}\hat{y}_2$ and $\hat{y}_2 = \sqrt{\frac{3}{2}}\hat{y}_0$ or $-\sqrt{\frac{3}{2}}\hat{y}_0$. Notice that such equilibria are just axisymmetric flows with a horizontal axis of symmetry.
We call type I and type II these two kinds of relative equilibria.

Lemma B.1 *When the norm ν is small enough, there are generically no other perturbed relative equilibria than type I and II to the axisymmetric equilibria.*

Proof. Let us assume $\hat{y}_1 \neq 0$. As noticed above we can always take $y_1 \in \mathbb{R}$ by adjusting the (arbitrary) phase. We replace $\dot{y}_j$ in equations (27)-(29) by $ij\omega y_j$. Then we can divide (28) by y_1.

The imaginary part of (28) can be readily solved for the frequency, which gives

$$\omega = -\frac{\sqrt{6}}{2}b\hat{y}_{2i} + \epsilon\beta + h.o.t.$$

(we write $y_2 = y_{2r} + iy_{2i}$).

We expand the various functional coefficients in (27)-(29). After some algebraic manipulations the remaining equations can then be replaced by the following system :

$$\begin{aligned}
0 &= b(\frac{\sqrt{6}}{2}y_0y_{2r} - \frac{1}{2}y_1^2 + y_{2r}^2 + y_{2i}^2) + (\gamma_0 - \gamma_1)\epsilon^2 y_0 + \kappa_1\epsilon y_0 y_{2i} + h.o.t. \\
0 &= b(\frac{3}{2}y_0y_{2r} - \frac{\sqrt{6}}{4}y_1^2 + \frac{\sqrt{6}}{2}y_{2i}^2) + (\gamma_2 - \gamma_1)\epsilon^2 y_{2r} + \kappa_2\epsilon y_0 y_{2i} + \delta_2\epsilon^2 y_1^2 + h.o.t. \\
0 &= b(\frac{3}{2}y_0y_{2i} + \frac{3\sqrt{6}}{2}y_{2r}y_{2i}) + (\gamma_2 - \gamma_1)\epsilon^2 y_{2i} + \kappa_3\epsilon y_1^2 + h.o.t. \\
0 &= a - by_0 - \frac{\sqrt{6}}{2}by_{2r} + \gamma_1\epsilon^2 + \kappa_4\epsilon y_{2i} + h.o.t.
\end{aligned}$$

We do not need to make the coefficients κ_j and δ_k more explicit.

We know that these equations are satisfied for $y_j = \hat{y}_j$ and $\epsilon = 0$. However one can easily check that the implicit function theorem does not apply : there is a compatibility condition. Let us set

$$y_j = \hat{y}_j + u_j, \quad j = 0, 1, 2 \ .$$

From the third equation we get

$$u_{2i} = -\frac{2\kappa_3}{3b}\frac{\hat{y}_1^2}{\hat{y}_0 + \sqrt{6}\hat{y}_{2r}}\epsilon + h.o.t. \ .$$

Replacing u_{2i} by this expression in the two first equations, combining them and identifying the terms of order ϵ^2, we obtain the following relation :

$$-3\sqrt{6}\frac{\kappa_3^2\hat{y}_1^4}{(\hat{y}_0 + \sqrt{6}\hat{y}_{2r})^2} + \frac{\sqrt{6}}{2}(\gamma_0 - \gamma_1)\hat{y}_0 - (\gamma_2 - \gamma_1)\hat{y}_{2r} - \frac{2\kappa_3}{3b}(\frac{\sqrt{6}}{2}\kappa_1 - \kappa_2)\frac{\hat{y}_1^2}{\hat{y}_0 + \sqrt{6}\hat{y}_{2r}} = 0 \ .$$

Now remember that $y_0^2 + 2y_1\bar{y}_1 + 2y_2\bar{y}_2 = \nu^2 << 1$. Hence the above relation leads to the necessary condition

$$\frac{\sqrt{6}}{2}(\gamma_0 - \gamma_1)\hat{y}_0 - (\gamma_2 - \gamma_1)\hat{y}_{2r} = 0 \ . \tag{37}$$

However numerical computations by [13] have shown that in typical cases, $\gamma_2 - \gamma_1 \approx 3(\gamma_1 - \gamma_0) > 0$.
Therefore (37) can be satisfied if there exist $\hat{y}_0$ and $\hat{y}_{2r}$ on the group orbit of basic equilibria such that

$$\frac{\sqrt{6}}{2}\hat{y}_0 + 3\hat{y}_{2r} = 0 \; .$$

By the above condition combined with (34) we finally get

$$2\hat{y}_{2r}^2 + \frac{1}{2}\hat{y}_1^2 = 0$$

which leads to a contradiction since we have assumed $\hat{y}_1 \neq 0$. □

B.3 Heteroclinic connections along perturbed group orbits of equilibria

An important consequence of lemma 1 is that there must exist connecting orbits between relative equilibria of types I and II in the invariant manifold which results from the perturbation of the group orbit of equilibria when ϵ is close enough to 0. In this section we wish to determine the direction of the flow along these connections.

Let us first make more precise the geometric structure of the perturbed group orbit when ϵ is close enough to 0. We call $\mathcal{M}_\epsilon$ this 2 dimensional, flow-invariant manifold. Hence $\mathcal{M}_0$ is the group orbit of equilibria for the unperturbed system. $\mathcal{M}_\epsilon$ contains one equilibrium y_0 and an invariant circle (rotating wave) in the subspace $y_1 \equiv 0$. Since the equilibrium is $\mathbf{SO}(2)$ invariant, the tangent plane to the manifold at y_0 is an eigenspace for a double eigenvalue. By lemma 1 it corresponds either a stable or an unstable manifold to the equilibrium, and this invariant manifold realizes a heteroclinic orbit to the rotating wave. It is therefore enough to compute this eigenvalue at y_0 in order to determine the direction of the flow along $\mathcal{M}_\epsilon$.
Notice that the tangent directions to $\mathcal{M}_0$ at y_0 are given by y_1 and $\bar{y}_1$. Indeed, if we denote by $J_\pm$ and J_3 respectively the infinitesimal generators of the action of $\mathbf{SO}(3)$ on the y_j's, then $J_3 y_0 = 0$ (rotational invariance of y_0 along the "vertical axis"), and $J_\pm y_0 = \beta_0 y_{\pm 1}$.
It follows that we only have to compute the Jacobian matrix at y_0 restricted to the coordinates y_1, $\bar{y}_1$.

Lemma B.2 *The eigenvalue along the tangent space to $\mathcal{M}_\epsilon$ at y_0 has leading real part $(\gamma_1 - \gamma_0)\epsilon^2$.*

Consequence : since $\gamma_1 - \gamma_0 > 0$ (see [13]), the connecting orbit is the unstable manifold of y_0 (therefore the stable manifold of the rotating wave).

Proof. First we solve (27) for y_0 (with $y_1 = y_2 = 0$). It gives

$$y_0 = \frac{2a}{b} + \frac{2\gamma_0}{b}\epsilon^2 + h.o.t.$$

Differentiating equation (28) with respect to y_1 and

$\bar{y}_1$ at $y_1 = y_2 = 0$, we then easily get the result. □

It is important to remark that the result does not depend on the sign of y_0. In particular when there are two kinds of equilibria on the y_0 axis, with opposite signs (which is our case of interest), the flows on the two corresponding perturbed group orbits show the same dynamical behavior.

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