On a two-scale phasefield model for topology optimization

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Abstract

In this article, we consider a gradient flow stemming from a problem in two-scale topology optimization. We use the phase-field method, where a Ginzburg–Landau term with obstacle potential is added to the cost functional, which contains the usual compliance but also an additional contribution including a local volume constraint in a penalty term. The minimization of such an energy by its gradient-flow is analyzed in this paper. We use an regularization and discretization of the associated state-variable to show the existence of weak solutions to the considered system.

1 Introduction

Topology optimization seeks to find an optimal shape for a given physical setting subject to a prescribed physical objective. In the present paper, we want to find a structure that bends as little as possible under the influence of a given load. To this end we minimize the compliance for a loaded domain subject to an elastic material response. A phase field is introduced to distinguish between void and solid parts and a Ginzburg-Landau term is added to the cost functional to avoid the creation of perforated domains. To avoid trivial solutions we add a *global volume constraint* thereby restricting the total amount of material used.

The invention of additive manufacturing increased the interest in topology optimization, since the oftentimes seemingly nature-inspired designs could now be produced easily. An important trend in additive manufacturing is the creation of porous infill structures. Compared to fully filled designs, depending on topology, size and density, these cellular structures can achieve a wide range of properties for different purposes (see [19] and the references therein). A high surface to volume ratio improves heat transfer efficiency, large numbers of internal pores are beneficial for acoustic or thermal insulators. Cell structures deform at relatively low stress levels and are thus useful for energy absorption and vibration damping. Moreover, they show a better design robustness with respect to load variation and local material deficiencies [27] as well as a significantly increased stability with respect to buckling [8], see also [11].

A well established two-stage procedure to create components with mesostructures is to begin with a topology optimization of the design space subject to a global volume constraint in order to obtain an optimal macroscopic material distribution. Then, in a second step, the solid material is replaced with an infill structure, which can be homogeneous, graded or heterogeneous, made up of regular cells or of pores with varying density as in [23]. See also [26] for an overview of strut-node mesostructures. In [21], the interior material distribution is determined via Voronoi diagrams leading to irregular honeycomb-like cell structures, prioritizing the strength-to-weight ratio. Theses two-stage approaches will necessarily provide at most sub-optimal configurations.

Recently, the authors of the present paper came up with a novel two-scale phasefield approach to create an optimal macroscopic structure while at the same time developing an optimal porous mesostructure [11]. The coupling of meso- and macroscale structures is achieved via a *local volume constraint*.

A major inspiration for this approach was the work [27] by Wu et al. on porous bone-like infill structures. But while in their case the local volumes stayed fixed throughout the runtime of the algorithm, in [11] novel stress-adaptive local volumes have been considered. This is advantageous since it allows to place more material in areas with high local stresses while in less critical areas the porosity is higher, saving weight as well as costs.

In [11], the optimal control problem has been analysed. Numerical results were derived based on a pseudo time-stepping approach for the associated gradient flow. The main novelty of the present paper is a rigorous analysis of the resulting Allen-Cahn type gradient flow system coupled to elasticity. Similar systems also arise in other problems in topology optimization [3]. Moreover, such systems have been studied intensively in the realm of crack propagation in elastic solids, see for instance [13, 24].

A different approach to approximating mesostructures has been taken in [7], where a phase-field-based topology optimization approach is used to create optimized topologies with graded density structures by introducing an additional mesoscopic density variable.

The paper is organizes as follows: In Section 2 the main ideas of phase field based topology optimization are explained, the system equations are derived and the main result is formulated. In Section 3, we provide auxiliary results for analyzing the mechanical subsystems in Hellinger–Reissner saddle-point formulation and the associated solutions operators. This enables us to show existence of solution to the coupled system in Section 4. In Section 4.1, we introduce a suitable regularization and a discretization of the system. The local-in-time existence for the approximate system is provided by a Schauder-argument in Section 4.2, which can be extended due to global *a priori* estimates. Passing to the limit in the discretization in Section 4.3 imply the existence of a solution to the system with regularized obstacle potential. Finally, passing to the limit in this regularization provides the proof of the existence of a solution according to Definition 2.1 in Section 4.4

2 Phasefield based topology optimization

2.1 Derivation of model equations

Our goal is to find the mass distribution with the stiffest material response for a design domain $\Omega \subset \mathbb{R}^d, d = 2, 3$ subject to prescribed surface and body loads f, g, and linear elastic material behaviour. Instead of maximizing the stiffness, we minimize the compliance

$$\mathcal{G}^c(\sigma, \varphi) := \int_\Omega C^{-1}(\varphi)\sigma : \sigma \mathrm{d}x$$

subject to mechanical equilibrium

$$-\operatorname{div} \sigma = \varphi g \quad \text{in } \Omega \times (0, T), \tag{1a}$$

$$\sigma - C(\varphi)\,\mathcal{E}(u) = 0 \quad \text{in } \Omega \times (0, T), \tag{1b}$$

$$u = 0 \quad \text{on } \Gamma_D \times (0, T), \tag{1c}$$

$$\sigma\mathbf{n} = f \quad \text{on } \Gamma_f \times (0, T), \tag{1d}$$

$$\sigma\mathbf{n} = 0 \quad \text{on } \Gamma \setminus \left(\overline{\Gamma}_D \cup \overline{\Gamma}_f\right) \times (0, T). \tag{1e}$$

Here, u is the displacement, σ the stress tensor, and $C(\varphi)$ the fourth-order stiffness tensor, depending on the phasefield variable $\varphi : \Omega \times (0, T) \to [0, 1]$. It acts as the control variable and describes whether locally there is material ($\varphi = 1$) or there is void ($\varphi = 0$). To avoid homogenized structures when minimizing compliance, we penalize large perimeters by adding a Ginzburg-Landau term

$$\mathcal{G}^{gl}_\beta(\varphi) = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\right) \mathrm{d}x$$

with a double obstacle potential ψ given by

$$\psi(\varphi) := \frac{1}{2}\left(\varphi - \varphi^2\right) + \psi^c(\varphi), \qquad \psi^c(\varphi) := \begin{cases} 0 & \text{if } 0 \le \varphi \le 1 \\ \infty & \text{otherwise.} \end{cases} \tag{2}$$

The parameter ε is used to control the interface width.

To avoid the trivial optimal solution of covering the whole domain with material. we introduce the volume fraction $m \in (0, 1)$ and impose the *global volume constraint*

$$\int_\Omega \varphi \mathrm{d}x = m|\Omega|, \tag{3}$$

where $|\Omega|$ denotes the Lebesque measure of the domain Ω.

As compared to earlier phasefield topology optimization approaches the main novelty is that we strive to create an optimal mesoscopic pore structure together with the optimal macroscopic material distribution. To this end, we introduce the radius r, which defines the typical length scale of the desired meso-structure and the local volume fraction μ, the fraction of material present in a local cell.

The local volume constraint only demands that at most a fraction $\mu \in (0, 1)$ of material is used in local meso-cells thereby allowing for macroscopic voids in the component. This can be described as a pointwise inequality constraint, i.e.

$$\int_{B_r(x)} \chi^c_\Omega(q)\left(\varphi(q) - \mu\right) \mathrm{d}q \le 0 \quad \text{for } x \in \Omega.$$

To assure that the integrand is evaluated only for $q \in \Omega$, we have introduced the smoothed characteristic function $\chi_\Omega^\varsigma \in C_0^2\left(\mathbb{R}^d\right)$ of the domain Ω, such that for any $\varsigma > 0$ we have $\chi_\Omega^\varsigma(x) = 0$ if $x \notin \Omega$ and $\chi_\Omega^\varsigma(x) = 1$ if $\mathrm{dist}\,(x, \Gamma) \geq \varsigma$ with a smooth transition in between. The gradient of χ_Ω^ς exists and is globally bounded by $C_\varsigma > 0$, i.e.

$$\left|\nabla\chi_\Omega^\varsigma(x)\right| \leq C_\varsigma \quad \forall x \in \mathbb{R}^d.. \tag{4}$$

Next, we will rewrite the inequality constraint as a penalty function. Using the positive part function $[x]_+ = \max\{x, 0\}$ and a proper scaling, we introduce the local volume constraint penalty term as

$$V\left(r, \varphi\right) := \int_\Omega \left[\frac{1}{r^d} \int_{B_r(x)} \chi_\Omega^\varsigma(q)\left(\varphi\left(q\right) - \mu\right)\,\mathrm{d}q\right]_+^2 \mathrm{d}x. \tag{5}$$

If the local volume fraction is restricted by μ in the whole domain, one cannot expect a larger value for the global volume fraction, m. Thus, it is sensible to choose $\mu \geq m$. In case of equality, the whole domain will be filled with mesoscale structures and holes. Introducing a porous mesostructure to a macroscopically optimized structure deteriorates its compliance. As a remedy, we allow for inhomogeneous mesostructures by introducing a stress dependency of the radius. This leads to bone-like structures (see, e.g., [22]) and an improved compliance.

All in all, the two-scale topology optimization problem can be formulated as an optimal control problem, where we want to minimize the cost functional

$$\mathcal{G}(\sigma, \varphi) = \mathcal{G}^c(\varphi, \sigma(\varphi)) + \frac{\alpha}{2}V(r(\sigma), \varphi) + \gamma\mathcal{G}_\beta^{gl}(\varphi), \tag{6}$$

subject to the state system (1) and the global volume constrained (3).This problem has been analysed in [11]. Numerical results for this optimal control problem have been derived with a pseudo-time-stepping scheme based on a gradient flow subject to the reduced cost functional $j(\varphi) = \mathcal{G}(\sigma(\varphi), \varphi)$. The resulting multifield model for twoscale topology optimization subject to a double-obstacle potential comprises a semilinear Allen-Cahn inclusion

$$\partial_t\varphi - \varepsilon\Delta\varphi + \frac{1}{\varepsilon}\partial\psi^c(\varphi) \ni F(\varphi) \quad \text{in } \Omega \times (0, T)\,, \tag{7a}$$

$$n\nabla\varphi = 0 \quad \text{on } \partial\Omega \times (0, T)\,, \tag{7b}$$

$$\varphi(0) = \varphi_0 \qquad\qquad\qquad \text{in } \Omega\,. \tag{7c}$$

Since the obstacle potential is a nonsmooth-convex functions, we work with the subdifferential formulation in (7a). Remember that the subdifferential is a set, the exact meaning of the formulation is given in Definition 2.2, below.

System (7) is coupled to the state equation (1) and the adjoint system

$$
\begin{aligned}
-\,\mathrm{div}\,\tau &= 0 & \text{in } \Omega \times (0, T)\\
\tau &= C\left(\varphi\right)\mathcal{E}\left(p\right) - \alpha\mathcal{C}\left(\varphi, \sigma\right)C\left(\varphi\right)\nabla_\sigma r\left(\sigma\right) & \text{in } \Omega \times (0, T)\\
p &= 0 & \text{on } \Gamma_D \times (0, T)\\
\tau n &= f & \text{on } \Gamma_f \times (0, T),\\
\tau n &= 0 & \text{on } \partial\Omega \setminus \left(\overline{\Gamma}_D \cup \overline{\Gamma}_f\right) \times (0, T),
\end{aligned}
\tag{8}$$

where

$$F(\varphi) := -\frac{\gamma}{\epsilon}\left(\varphi - \frac{1}{2}\right) - \alpha\int_\Omega\left[\int\!\!\!\int_{B_{r(\sigma)}(q)}\chi_\Omega^\varsigma(\zeta)\left(\varphi\left(\zeta\right) - \mu\right)\,\mathsf{d}\zeta\right]_+ \chi_{B_{r(\sigma)}(q)}\left(\cdot\right)\chi_\Omega^\varsigma(\cdot)\,\mathsf{d}q$$
$$-(C^{-1})'\left(\varphi\right)\sigma : \tau - \lambda\,, \tag{9}$$

with

$$\lambda = -\frac{1}{|\Omega|}\left(\alpha\int_\Omega\int_\Omega\left[\int\!\!\!\int_{B_{r(\sigma)}(q)}\chi_\Omega^\varsigma(\zeta)\left(\varphi\left(\zeta\right) - \mu\right)\,\mathsf{d}\zeta\right]_+ \chi_{B_{r(\sigma)}(q)}\left(x\right)\chi_\Omega^\varsigma(x)\,\mathrm{d}q\mathrm{d}x\right.$$
$$\left. + \frac{\gamma}{\epsilon}\int_\Omega\left(\varphi - \frac{1}{2}\right)\mathrm{d}x + \int_\Omega(C^{-1})'\left(\varphi\right)\sigma : \tau\mathrm{d}x\right), \tag{10}$$

and

$$\mathcal{C}\left(\varphi, \sigma\right)$$
$$= \left(\frac{1}{r(\sigma)^d}\int_{B_{r(\sigma)}(x)}\varphi\left(q\right) - \mu\,\mathrm{d}q\right)_+ \left(\frac{1}{r(\sigma)^{d+1}}\int_{B_{r(\sigma)}(x)}\left[\chi_\Omega^\varsigma(q)\nabla\varphi\left(q\right) + \nabla\chi_\Omega^\varsigma(q)\varphi\left(q\right)\right]\cdot\left(q - x\right)\mathrm{d}q\right).$$
$$\tag{11}$$

2.2 Notations and assumptions

Assumption A1. Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ be a bounded Lipschitz domain and denote its boundary by Γ.

In case of a Dirichlet boundary $\Gamma_D \subset \Gamma$ the notation

$$H_D^1\left(\Omega, \mathbb{R}^d\right) := \left\{v \in H^1\left(\Omega, \mathbb{R}^d\right) \mid v = 0 \text{ on } \Gamma_D\right\}$$

is used.

The L^2 scalar product is denoted by $(\cdot, \cdot)_{L^2}$. Duality pairings for a normed space $\mathcal{V}$ and its dual $\mathcal{V}^*$ are written via $\langle \cdot, \cdot \rangle_{\mathcal{V}^*, \mathcal{V}}$, where the subscript will be dropped if it is clear which spaces are meant.

We denote the set of all symmetric $d \times d$ matrices by $\mathbb{S}^d$. The Frobenius inner product for second order tensors $\mathcal{M}, \mathcal{N} \in \mathbb{R}^{d \times d}$ is defined by the pairwise sum of element-products

$$\mathcal{M} : \mathcal{N} := \sum_{i,j=1}^d \mathcal{M}_{ij} \mathcal{N}_{ij}$$

and their norm via

$$|\mathcal{M}| := (\mathcal{M} : \mathcal{M})^{\frac{1}{2}}.$$

We compute the product of a fourth-order tensor $C = \left[C_{ijkl}\right]_{i,j,k,l=1}^d \in \mathbb{R}^{d \times d \times d \times d}$ and a second order tensor $\mathcal{M} = \left[\mathcal{M}_{kl}\right]_{k,l=1}^d \in \mathbb{R}^{d \times d}$, namely $C\mathcal{M} \in \mathbb{R}^{d \times d}$, by

$$[C\mathcal{M}]_{ij} := \sum_{k=1}^d \sum_{l=1}^d C_{ijkl} \mathcal{M}_{kl} \quad \text{for} \quad i, j = 1, \ldots, d$$

and the norm $|C|$ via

$$|C| := \sqrt{\sum_{i,j,k,l=1}^d C_{ijkl}^2}.$$

Assumption A2. We assume that the fourth-order stiffness tensor satisfies the symmetry conditions

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk} \quad \text{for} \quad i, j, k, l = 1, \ldots, d.$$

Moreover, there exist positive constants $\Theta, \bar{\Theta}, \tilde{\Theta}$ such that for all symmetric matrices $\mathcal{M}, \mathcal{N} \in \mathbb{S}^d$ and all $a \in \mathbb{R}$ the following relationships hold:

(i) $\quad \Theta |\mathcal{M}|^2 \leq C^{-1}(a) \mathcal{M} : \mathcal{M} \leq \bar{\Theta} |\mathcal{M}|^2,$

(ii) $\quad \left|(C^{-1})'(a) \omega \mathcal{M} : \mathcal{N}\right| \leq \tilde{\Theta} |\omega| |\mathcal{M}| |\mathcal{N}|.$

The tensors C^{-1} and $(C^{-1})'$ are Lipschitz continuous with Lipschitz constants $L_{C^{-1}}$ and $L_{(C^{-1})'} > 0$ such that:

$$|C^{-1}(a) - C^{-1}(b)| \leq L_{C^{-1}} |a - b| \quad \forall a, b \in \mathbb{R}$$

and

$$\left|(C^{-1})'(a) - (C^{-1})'(b)\right| \leq L_{(C^{-1})'} |a - b| \quad \forall a, b \in \mathbb{R}.$$

For the construction of a stiffness tensor fulfilling these assumptions we refer to [2, Chapt. 2.2] and [11].

Assumption A3. The radius $r : \mathbb{S}^d \to \mathbb{R}_{>0}, \; \sigma \mapsto r(\sigma)$ is a smooth function, globally bounded in C^1 with $0 < r_{min} \leq r(\sigma) \leq r_{max} < \infty \quad \forall \sigma \in \mathbb{S}^d$ and

$$\left|D_\sigma r(\sigma)\right| \leq C \quad \forall \sigma \in \mathbb{S}^d.$$

2.3 Problem formulation and main result

Definition 2.1 (Solution of the Allen–Cahn system). The tuple $(\varphi, \sigma, u, \tau, p)$ is a *solution of the Allen–Cahn system in topology optimization with local volume constraint* (1), (7) ,(8) with the initial value $\varphi_0 \in L^\infty(\Omega, [0,1])$ with $\int_\Omega \varphi_0 \mathrm{d}x = m$, if the following conditions hold true:

i) The function $\varphi \in H^1(0,T;L^2) \cap L^2(0,T;H^1)$ is such that $\varphi \in [0,1]$ almost everywhere. There exists a $\xi \in L^q(\Omega \times [0,T])$ with $\int_\Omega \xi \mathrm{d}x = 0$ with $q \in (1,6/5)$ such that the function φ solves the PDE

$$
\begin{aligned}
\partial_t \varphi - \gamma\epsilon\Delta\varphi + \frac{\gamma}{\epsilon}\xi &= F(\varphi) &&\text{a.e. in } \Omega \times (0,T), \\
\nabla\varphi \cdot \mathbf{n} &= 0 &&\text{a.e. on } \partial\Omega \times (0,T), \\
\varphi(0) &= \varphi_0 &&\text{a.e. in } \Omega,
\end{aligned}
\tag{12}
$$

with $\mathbf{n}$ being the outer normal and F is given in (9).

ii) The function $\xi \in L^q(\Omega \times [0,T])$ can be interpreted as an element of the subdifferential of the obstacle potential ψ^c, i.e., it holds that

$$
\begin{aligned}
\xi(x,t) &\leq 0 \text{ for a.e. } (x,t) \in \Omega \times (0,T) \text{ where } \varphi(x,t) = 0\,, \\
\xi(x,t) &\geq 0 \text{ for a.e. } (x,t) \in \Omega \times (0,T) \text{ where } \varphi(x,t) = 1\,, \\
\xi(x,t) &= 0 \text{ for a.e. } (x,t) \in \Omega \times (0,T) \text{ where } \varphi(x,t) \in (0,1)\,.
\end{aligned}
$$

iii) The pair $(\sigma, u) \in L^{2q/(2-q)}(\Omega \times (0,T)) \times L^2(0,T;H^1(\Omega))$ solves the elasticity system (1) in the weak sense and the pair $(\tau, p) \in L^2(\Omega \times (0,T)) \times L^2(0,T;H^1(\Omega))$ solves the adjoint system (8) in the weak sense.

The main result of this work is as follows:

Theorem 2.2. *Under Assumptions A1, A2, A3, the Allen–Cahn system with obstacle potential has a solution φ in the sense of Definition 2.1 for any initial value $\varphi_0 \in L^\infty(\Omega, [0,1]) \cap H^1(\Omega)$ with $\int_\Omega \varphi_0 \mathrm{d}x = m$.*

Figure 1 depicts the temporal evolution of the Allen–Cahn system based on a pseudo time-stepping scheme. For details and further numerical results we refer to [11].

We prove the existence of solutions to the Allen–Cahn equations in the above sense for the nonlocal function F given in (9) and stemming from topology optimization. The proof is rather involved and we comment on the strategy below. But beforehand some remarks are in order to put the above definition into context.

Remark (Energy inequality). As the underlying equation is a gradient flow, also an energy equality is fulfilled formally. The solution constructed in the proof of Theorem 2.2 fulfills an energy inequality, *i.e.,* the solution not only fulfills Definition 2.1 but also the energy inequality

$$
\left[\frac{\gamma\epsilon}{2}\|\nabla\varphi(s)\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi(\varphi(s))\mathrm{d}x + \mathcal{G}^c(\varphi(s),\sigma(s)) + \frac{\alpha}{2}V(r(\sigma(s),\varphi(s)))\right]\Big|_0^t + \int_0^t \|\partial_t\varphi(s)\|_{L^2}^2 \,\mathrm{d}s \leq 0 \tag{13}
$$

for almost every $t \in (0,T)$. This usual estimate can be strengthened such that there exists a monotonously non-increasing function $E : [0,\infty) \to \mathbb{R}$ such that

$$
\left[\frac{\gamma\epsilon}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi^c(\varphi)\mathrm{d}x + \mathcal{G}^c(\varphi,\sigma) + \frac{\alpha}{2}V(r(\sigma,\varphi))\right] \leq E \text{ a.e. in (0,T) and } E\Big|_s^t + \int_s^t \|\partial_t\varphi(s)\|_{L^2}^2 \,\mathrm{d}\tau \leq 0
\tag{14}
$$

for a.e. $s \in [0,\infty)$ and all $t \in (s,T)$.

From the monotony of E, we infer that $\lim_{t\to\infty} E(t) = E^\infty$ such that

$$
0 = \lim_{t\to\infty}\left[E(t) - E(t+1)\right] \geq \lim_{t\to\infty}\int_t^{t+1} \|\partial_t\varphi(s)\|_{L^2}^2 \,\mathrm{d}\tau \geq 0\,.
$$

This together with (12) implies that the Euler–Lagrange equations for the associated cost functional are fulfilled in the limit, at least in some weak sense. This is exactly what we try to achieve with the gradient descent algorithm, which gives rise to the considered problem.

Thus the energy $\mathcal{G}$ given in (6) can be seen as a Lyapunov functional, which provides stability. The energy inequality is often a crucial ingredient for the derivation of a relative energy inequality [18], which is a common tool in the PDE community to

(a) initial material distribution

(b) iteration number: 1

(c) iteration number: 10

(d) iteration number: 20

(e) iteration number: 30

(f) iteration number: 40

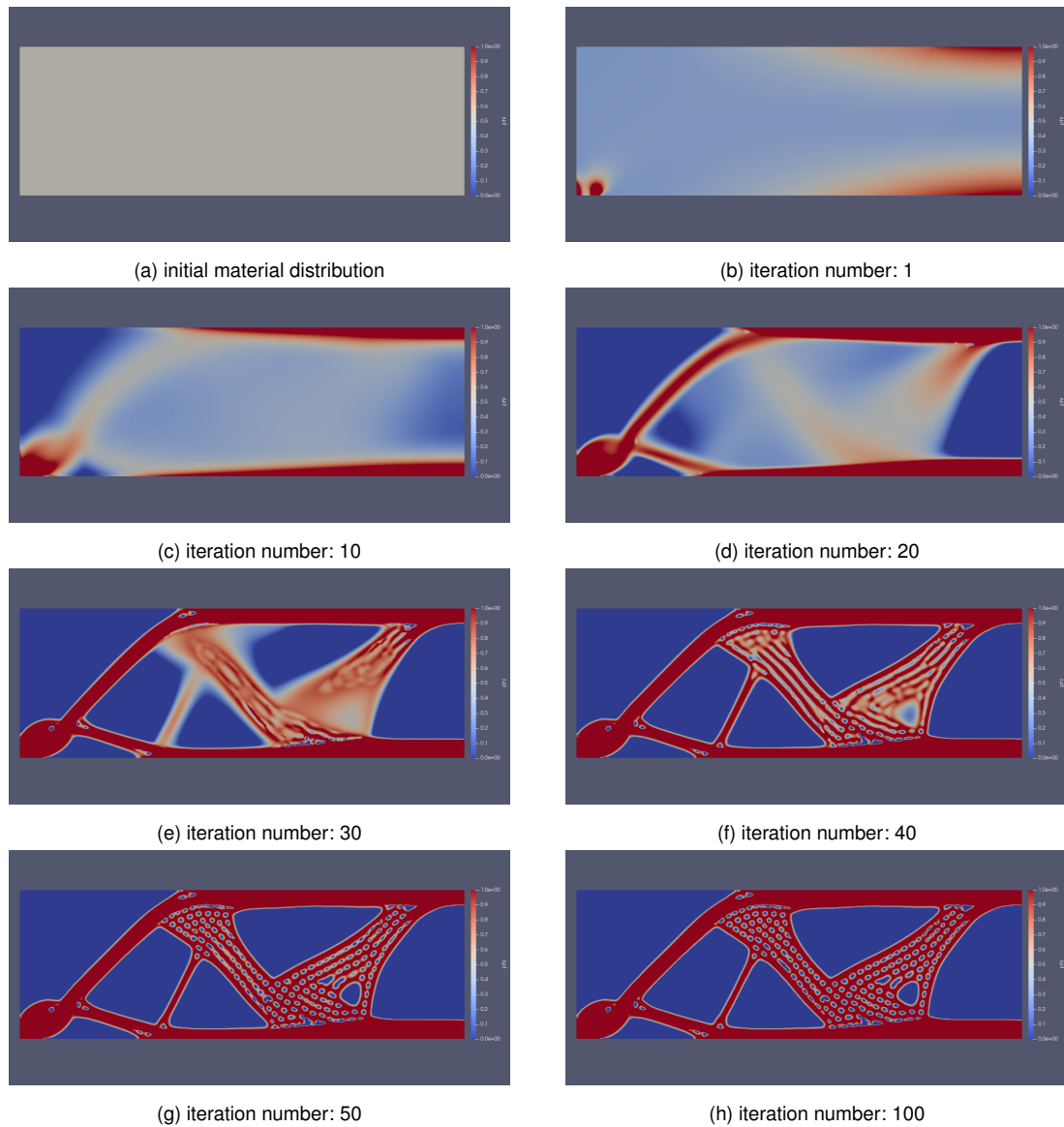(g) iteration number: 50

(h) iteration number: 100

Figure 1: Result of a pseudo-time stepping scheme for two-scale topology optimization for an MBB beam

prove weak-strong uniqueness and stability of solutions [18], consider singular limits [12] or long-time behaviour [16], or convergence of numerical schemes[1]. Thus, this technique could be of interest for future investigations.

By the energy inequality, we could not only infer the stability of solutions, but we also observe the convergence of the numerical scheme for our numerical algorithm after sufficiently many time steps. The evolution of the topology can be observed in Figure 1. For numerical details we refer to [11]

Remark (Boundary and initial conditions). The traces are well-defined almost everywhere on $\partial\Omega \times (0,T)$ and in Ω, respectively. From ii) we get $\varphi \in H^1\left(0,T;L^2(\Omega)\right) \cap L^\infty\left(0,T;H^1(\Omega)\right)$ and it follows that $\varphi \in C_w\left(0,T;H^1(\Omega)\right)$, see [20, Chapter 3, Lemma 8.1]. Additionally, comparing the terms in (12) we get $\Delta\varphi \in L^1(\Omega \times [0,T])$ and therefore $\varphi \in L^1(0,T;W^{2,1}(\Omega))$, stated differently $\varphi(t) \in W^{2,1}(\Omega)$ for almost every $t \in (0,T)$. The normal-trace operator is well-defined as a function from $W^{2,1}(\Omega)$ to $L^1(\partial\Omega)$, see for example, [9, Proposition 3.80].

Strategy for the proof: We will shortly explain the main steps in proving the above existence result. The obstacle potential ψ will be regularized in a first step by a C^2 regularization ψ_β, see Figure 2.

We will introduce a Galerkin discretization for the mechanical part of our system. For the smoothed potential, we want to show the existence of a solution of the initial value problem (26). Let us first think about the problem where the material distribution φ_β on the right-hand side is replaced by a $\bar\varphi \in H^1(0,T;L^2(\Omega,\mathbb{R})) \cap L^2(0,T;H^1(\Omega))$, which leads to the
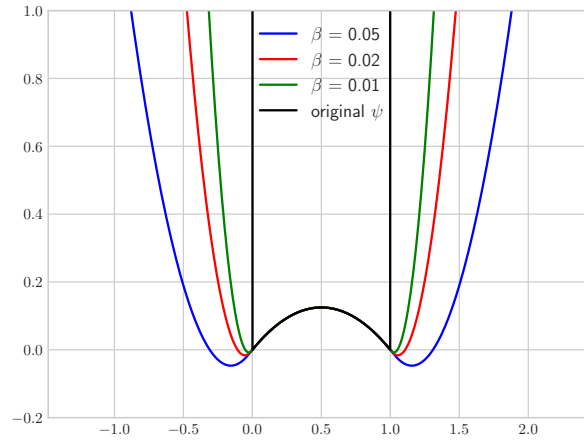
Figure 2: Approximation ψ_β of the double obstacle potential ψ for three different values of β

linear parabolic equation:

$$\begin{aligned}
\partial_t \varphi_\beta - \gamma\epsilon\Delta\varphi_\beta &= F_\beta(\bar{\varphi}) && \text{a.e. in } \Omega \times (0, T), \\
\nabla\varphi_\beta \cdot \mathbf{n} &= 0 && \text{a.e. on } \partial\Omega \times (0, T), \\
\varphi_\beta(0) &= \varphi_0 && \text{in } \Omega.
\end{aligned} \tag{15}$$

For a right-hand side $F_\beta(\overline{\varphi}) \in L^{1+\varepsilon}(\Omega \times (0, T))$ with $\varepsilon > 0$, we infer by the maximal L^p-regularity of the heat equation [10, Thm. 8.2] that the above problem (15) is uniquely solvable. Solving this equation defines an operator $\mathcal{T}, \bar{\varphi} \mapsto \varphi_\beta$. If φ_β is a fixed point of $\mathcal{T}$ then it would be a solution of problem (26). The idea is to use the Schauder fixed point theorem to show the existence of such a fixed point and therefore the existence of a local solution.

The global existence is deduced from global *a priori* estimates. In the limit of vanishing discretization, we infer a weak solution of the regularized system (15). With some nonstandard inequalities, we infer *a priori* estimates in some reflexive L^p-spaces, which allow to pass to the limit in the regularization in order to infer a solution according to Definition (2.1).

3 Auxiliary results for the mechanical subsystems

3.1 Mechanical equilibrium - the state equation

The approximate system we consider incorporates also a spatial discretization of the mechanical system. Therefore, we also need several results for the continuous and the discrete spaces. In order to avoid having two sections with very similar results, we group them together by defining $\mathcal{H}$ and $\mathcal{N}$ to include both cases. Therefore, let $\mathcal{H} \subset H_D^1(\Omega, \mathbb{R}^d)$ and $\mathcal{N} := \left\{ \eta \in L^2(\Omega, \mathbb{S}^d) \mid \exists v \in \mathcal{H} \text{ with } \mathcal{E}(v) = \eta \right\}$ be possibly finite-dimensional Hilbert spaces equipped with the $\|\cdot\|_{H_D^1}$-norm and the $\|\cdot\|_{L^2}$-norm respectively. We note that some of the results in this section only hold for the discretized spaces, since we prove them using the norm equivalence in the spatial component. Those results will be pointed out explicitly. Apart from technicalities, the proofs presented here are quite similar to the ones found in [11].

The following definition is a particular formulation of Brezzi's splitting theorem.

Definition 3.1. The pair $(u, \sigma) \in L^2(0, T; \mathcal{H}) \times L^2(0, T; \mathcal{N})$ is a *weak solution of the Hellinger–Reissner linear elasticity problem*, if it satisfies the following saddle point problem with a surface load $f \in L^\infty\left(0, T; L^3\left(\Gamma_f, \mathbb{R}^d\right)\right)$

$$\begin{aligned}
\left(C^{-1}\left(\varphi(t)\right)\sigma(t), \eta\right)_{L^2} - \left(\eta, \mathcal{E}\left(u(t)\right)\right)_{L^2} &= 0 && \forall\eta \in \mathcal{N}, \\
-\left(\sigma(t), \mathcal{E}(v)\right)_{L^2} &= -\int_{\Gamma_f} f(t) \cdot v\mathrm{d}\omega && \forall v \in \mathcal{H}
\end{aligned} \tag{16}$$

for almost all $t \in (0, T)$.

Lemma 3.2 (Existence of a solution). *Let Assumptions A1, A2 hold true. For a given phase-field $\varphi \in L^2(0, T; L^2)$ and right-hand sides $\mathbb{F} \in L^\infty\left(0, T; L^2\right)$, $\mathbb{G} \in L^\infty\left(0, T; \left(H_D^1\right)^*\right)$ there exists a unique weak solution $(u, \sigma) \in$*

$L^2(0, T; \mathcal{H}) \times L^2(0, T; \mathcal{N})$ *of the saddle point problem*

$$
\begin{aligned}
\left(C^{-1} \left(\varphi(t) \right) \sigma(t), \eta \right)_{L^2} - \left(\eta, \mathcal{E} \left(u(t) \right) \right)_{L^2} &= \langle \mathbb{F}(t), \eta \rangle && \forall \eta \in \mathcal{N}, \\
- \left(\sigma(t), \mathcal{E}(v) \right)_{L^2} &= \langle \mathbb{G}(t), v \rangle && \forall v \in \mathcal{H}
\end{aligned}
\tag{17}
$$

for almost all $t \in (0, T)$. *The following a priori estimate holds for* (u, σ)

$$
\|u\|_{L^2(0,T;H_D^1)} + \|\sigma\|_{L^2(0,T;L^2)} \le C_1 \|\mathbb{F}\|_{L^2(0,T;L^2)} + C_2 \|\mathbb{G}\|_{L^2\left(0,T;(H_D^1)^*\right)},
$$

with positive constants C_1, C_2.

Remark. We note that the right-hand sides are in L^∞ in time, but the solutions are only in L^2 in time. The result could be extended to a higher time regularity of the solutions, but we will not require this in our exposition.

Proof. According to [28, Section 30.1], the saddle point problem above is equivalent to

$$
\begin{aligned}
\left(C^{-1}(\varphi) \sigma, \eta \right)_{L^2(0,T;L^2)} - \left(\eta, \mathcal{E}(u) \right)_{L^2(0,T;L^2)} &= \langle \mathbb{F}, \eta \rangle && \forall \eta \in L^2(0,T;\mathcal{N}), \\
- \left(\sigma, \mathcal{E}(v) \right)_{L^2(0,T;L^2)} &= \langle \mathbb{G}, v \rangle && \forall v \in L^2(0,T;\mathcal{H}).
\end{aligned}
\tag{18}
$$

The assertion now follows from assumption A2, Korn's inequality and Brezzi's splitting theorem, see [5, p. 132] very similar to the proof of [11, Lem. 3.1]. □

Lemma 3.3 (The Hellinger–Reissner problem is well-posed). *Let Assumptions A1, A2 hold true. For a given phase-field* $\varphi \in L^2(0, T; L^2)$ *there exists a unique weak solution* $(u, \sigma) \in L^2(0, T; \mathcal{H}) \times L^2(0, T; \mathcal{N})$ *of the Hellinger–Reissner linear elasticity system such that Definition 3.1 is fulfilled.*

Proof. Set $\langle \mathbb{G}(t), v \rangle := - \int_{\Gamma_f} f(t) \cdot v \, \mathrm{d}\omega$. Using Hölder's inequality and the trace theorem we get

$$
\begin{aligned}
|\langle \mathbb{G}(t), v \rangle| = \left| \int_{\Gamma_f} f(t) \cdot v \, \mathrm{d}\omega \right| &\le \|f(t)\|_{L^2(\Gamma_f)} \|v\|_{L^2(\Gamma_f)} \\
&\le c_{tr} \|f(t)\|_{L^2(\Gamma_f)} \|v\|_{H_D^1(\Omega)}
\end{aligned}
$$

for almost all $t \in (0, T)$. Thus, we have $\mathbb{G}(t) \in \left(H_D^1 \right)^*$ and

$$
\|\mathbb{G}\|_{L^\infty\left(0,T;(H_D^1)^*\right)} = \operatorname*{ess\,sup}_{t \in [0,T]} \|\mathbb{G}(t)\|_{(H_D^1)^*} \le c_{tr} \operatorname*{ess\,sup}_{t \in [0,T]} \|f(t)\|_{L^2(\Gamma_f)} = c_{tr} \|f\|_{L^\infty\left(0,T;L^2(\Gamma_f)\right)}
$$

such that $\mathbb{G} \in L^\infty\left(0, T; \left(H_D^1 \right)^* \right)$. The result follows from Lemma 3.2. □

Definition 3.4 (Time-dependent control-to-state operator). Lemma 3.3 defines a function, known as the *control-to-state operator*, which maps the phase-field φ to the unique weak solution (u, σ) of the elasticity problem

$$
S : L^2(0, T; L^2) \to L^2(0, T; \mathcal{H}) \times L^2(0, T; \mathcal{N}), \quad \varphi \mapsto (u, \sigma).
$$

In the case of $\mathcal{H} = H_D^1(\Omega, \mathbb{R}^d)$ and $\mathcal{N} = L^2(\Omega, \mathbb{S}^d)$ the second component of S, i.e. the function mapping the phase-field φ to $\sigma \in L^2(0, T; L^2(\Omega, \mathbb{S}^d))$ is denoted by S_2 with

$$
S_2 : L^2(0, T; L^2) \to L^2(0, T; L^2), \quad \varphi \mapsto \sigma.
$$

In the case of $\mathcal{H} = W^k$ and $\mathcal{N} = V^k$ being k-dimensional subspaces of $H_D^1(\Omega, \mathbb{R}^d)$ and $L^2(\Omega, \mathbb{S}^d)$, respectively, the second component of S, i.e. the function mapping the phase-field φ to $\sigma^k \in L^2(0, T; V^k)$ is denoted by S_2^k with

$$
S_2^k : L^2(0, T; L^2) \to L^2(0, T; V^k), \quad \varphi \mapsto \sigma^k.
$$

Lemma 3.5. *Let Assumptions A1, A2 hold true. The control-to-state operators* S_2 *and* S_2^k *are continuous from* $L^2(0, T; L^2(\Omega, \mathbb{R}))$ *to* $L^2(0, T; \mathcal{N})$.

Proof. Show that for a sequence of controls $\{\varphi_i\}_i \subset L^2(0,T;L^2)$ converging strongly to a $\varphi \in L^2(0,T;L^2)$, it holds that the sequence of corresponding states $\{\sigma_i\}_i$ converges strongly to σ in $L^2(0,T;\mathcal{N})$. First, we show weak convergence. The *a priori* bounds of Lemma 3.2 do not depend on φ and thus i. These bonds allow to infer via usual weak-compactness results that there exists a subsequence and elements $\bar{\sigma} \in L^2(0,T;L^2)$ and $\bar{u} \in L^2(0,T;L^2)$ such that

$$\sigma_i \rightharpoonup \bar{\sigma} \quad \text{in } L^2(0,T;L^2) \quad \text{and } u_i \rightharpoonup \bar{u} \quad \text{in } L^2(0,T;H_D^1(\Omega,\mathbb{R}^d)). \tag{19}$$

We subtract the state equations (16) for φ_i, φ and write the result in the equivalent form

$$\left(C^{-1}(\varphi_i)(\sigma_i - \sigma), \eta\right)_{L^2(0,T;L^2)} - \left(\eta, \mathcal{E}(u_i - u)\right)_{L^2(0,T;L^2)}$$
$$= -\left(\left(C^{-1}(\varphi_i) - C^{-1}(\varphi)\right)\sigma, \eta\right)_{L^2(0,T;L^2)}, \tag{20}$$
$$-\left(\sigma_i - \sigma, \mathcal{E}(v)\right)_{L^2(0,T;L^2)} = 0$$

for all $\eta \in L^2(0,T;\mathcal{N})$ and $v \in L^2(0,T;\mathcal{H})$. Since $\{\varphi_i\}_i \subset L^2(0,T;L^2)$ converges strongly to a $\varphi \in L^2(0,T;L^2)$ and C^{-1} is uniformly bounded, we get pointwise convergence of $\{C^{-1}(\varphi_i)\}_i$ and via Lebesgue's dominated convergence theorem strong convergence of $\{C^{-1}(\varphi_i)\eta\}_i$ to $C^{-1}(\varphi)\eta$ in $L^2(0,T,L^2)$. Thus, the right-hand side converges to zero and therefore the left-hand side also converges to zero. Since $\{C^{-1}(\varphi_i)\mathcal{E}(v)\}_i$ converges strongly to $C^{-1}(\varphi)\mathcal{E}(v)$ in $L^2(0,T,\mathcal{N})$, we know from the uniqueness of weak solutions, see Lemma 3.2, that $\bar{\sigma} = \sigma$ and $\bar{u} = u$, where (σ, u) are the solutions of (16) associated to φ.

Notice that $\mathcal{N}$ and $\mathcal{H}$ are possibly finite-dimensional with changing cardinality k, as is the case when going to the limit in the Galerkin approximation in Proposition 4.7. To underline this we will write $\mathcal{N}^k$ and $\mathcal{H}^k$ in the finite-dimensional case. However, $u(t)$ and $\sigma(t)$ for $t \in [0,T]$ are elements of infinite-dimensional spaces. To ensure that the differences $u_i - u$ and $\sigma_i - \sigma$ are in $L^2(0,T;\mathcal{N}^k)$ and $L^2(0,T;\mathcal{H}^k)$, respectively, we need to project onto these spaces. In the case of infinite-dimensional spaces $\mathcal{N}$ and $\mathcal{H}$, this projection is just the identity function. We will use the H^1-projection in space onto $L^2(0,T;\mathcal{H}^k)$ and the L^2-projection in space onto $L^2(0,T;\mathcal{N}^k)$ and denote them by $P_{\mathcal{H}}^k$ and $P_{\mathcal{N}}^k$, respectively. We set $v = u_i - P_{\mathcal{H}}^k(u)$ and $\eta = \sigma_i - P_{\mathcal{N}}^k(\sigma)$ and get

$$\left(C^{-1}(\varphi_i)(\sigma_i - \sigma), \sigma_i - P_{\mathcal{N}}^k(\sigma)\right)_{L^2(0,T;L^2)} - \left(\sigma_i - P_{\mathcal{N}}^k(\sigma), \mathcal{E}(u_i - u)\right)_{L^2(0,T;L^2)}$$
$$= -\left(\left(C^{-1}(\varphi_i) - C^{-1}(\varphi)\right)\sigma, \sigma_i - P_{\mathcal{N}}^k(\sigma)\right)_{L^2(0,T;L^2)},$$
$$-\left(\sigma_i - \sigma, \mathcal{E}\left(u_i - P_{\mathcal{H}}^k(u)\right)\right)_{L^2(0,T;L^2)} = 0.$$

Adding productive zeros, we see that

$$\left(C^{-1}(\varphi_i)(\sigma_i - \sigma), \sigma_i - \sigma\right)_{L^2(0,T;L^2)} + \left(C^{-1}(\varphi_i)(\sigma_i - \sigma), \sigma - P_{\mathcal{N}}^k(\sigma)\right)_{L^2(0,T;L^2)}$$
$$- \left(\sigma_i - \sigma, \mathcal{E}(u_i - u)\right)_{L^2(0,T;L^2)} - \left(\sigma - P_{\mathcal{N}}^k(\sigma), \mathcal{E}(u_i - u)\right)_{L^2(0,T;L^2)}$$
$$= -\left(\left(C^{-1}(\varphi_i) - C^{-1}(\varphi)\right)\sigma, \sigma_i - P_{\mathcal{N}}^k(\sigma)\right)_{L^2(0,T;L^2)},$$
$$- \left(\sigma_i - \sigma, \mathcal{E}(u_i - u)\right)_{L^2(0,T;L^2)} - \left(\sigma_i - \sigma, \mathcal{E}\left(u - P_{\mathcal{H}}^k(u)\right)\right)_{L^2(0,T;L^2)} = 0.$$

Noticing that $\left(\sigma_i - \sigma, \mathcal{E}(u_i - u)\right)_{L^2(0,T;L^2)}$ appears twice, we reduce the set of equations to a single equation, which

helps to estimate the difference of σ and σ_i via Assumption A2,

$$
\begin{aligned}
\Theta\|\sigma_i - \sigma\|^2_{L^2(0,T;L^2)} \leq &-\left(\left(C^{-1}(\varphi_i) - C^{-1}(\varphi)\right)\sigma, \sigma_i - P^k_{\mathcal{N}}(\sigma)\right)_{L^2(0,T;L^2)} \\
&-\left(C^{-1}(\varphi_i)(\sigma_i - \sigma), \sigma - P^k_{\mathcal{N}}(\sigma)\right)_{L^2(0,T;L^2)} \\
&-\left(\sigma_i - \sigma, \mathcal{E}\left(u - P^k_{\mathcal{H}}(u)\right)\right)_{L^2(0,T;L^2)} \\
&+\left(\sigma - P^k_{\mathcal{N}}(\sigma), \mathcal{E}(u_i - u)\right)_{L^2(0,T;L^2)}.
\end{aligned}
$$

We saw above that $\left\{C^{-1}(\varphi_i)\eta\right\}_i$ converges strongly to $C^{-1}(\varphi)\eta$ in $L^2(0,T,\mathcal{H})$ and that the states converge weakly. Additionally, the projection $P^k_{\mathcal{N}}(\sigma)$ converges strongly to σ in $L^2(0,T,\mathcal{N})$ and the projection $P^k_{\mathcal{H}}(u)$ converges strongly to u in $L^2(0,T,\mathcal{H})$. Thus, as $k \to \infty$, the right-hand side tends to zero, which gives us strong convergence of $\{\sigma_i\}_i$ in $L^2(0,T;\mathcal{N})$. $\hfill\square$

We will use the following result only in the case of discretized spaces. Again, the norm equivalence in the finite-dimensional spatial component will be applied.

Lemma 3.6. *Under Assumptions A1, A2, the control-to-discretized-state operator is Fréchet-differentiable. Its derivative at* $\varphi \in L^2(0,T;L^2)$ *in direction* $\omega \in L^2(0,T;L^2)$ *is given by*

$$
(S^k)'(\varphi)\omega = (u^k_*, \sigma^k_*),
$$

where $(u^k_*(t), \sigma^k_*(t)) \in W^k \times V^k$ *is the unique weak solution of the linearized system*

$$
\begin{aligned}
\left(C^{-1}(\varphi(t))\sigma^k_*(t), \eta\right)_{L^2} - \left(\eta, \mathcal{E}\left(u^k_*(t)\right)\right)_{L^2} &= -\left((C^{-1})'(\varphi(t))\omega(t)\sigma(t), \eta\right)_{L^2} &&\forall \eta \in V^k, \\
- \left(\sigma^k_*(t), \mathcal{E}(v)\right)_{L^2} &= 0 &&\forall v \in W^k
\end{aligned}
$$

for almost all $t \in (0,t)$ *and* (u, σ) *is the unique weak solution of the Hellinger Reissner system, see Definition* (3.1). *The restriction to the second component of* $(S^k)'$ *is understood as in Definition 3.4, i.e.* $(S^k_2)'(\varphi)\omega = \sigma^k_*$.

Proof. During the proof, we frequently will use that all norms on finite dimensional spaces are equivalent. We calculate the linearized system by computing $\frac{\partial}{\partial \varphi}(\cdot)\omega(t)$ derivatives of (16), which yields

$$
\begin{aligned}
\left(C^{-1}(\varphi(t))\sigma^k_*(t), \eta\right)_{L^2} - \left(\eta, \mathcal{E}\left(u^k_*(t)\right)\right)_{L^2} &= -\left((C^{-1})'(\varphi(t))\omega(t)\sigma(t), \eta\right)_{L^2} =: \langle \mathbb{F}(t), \eta\rangle, \\
- \left(\sigma^k_*(t), \mathcal{E}(v)\right)_{L^2} &= 0.
\end{aligned}
\tag{21}
$$

We have $\mathbb{F}(t) \in L^2$ since

$$
\begin{aligned}
\left|\langle \mathbb{F}(t), \eta\rangle\right| &\leq \tilde{\Theta}\int_\Omega |\omega(t)||\sigma(t)||\eta|\,\mathrm{d}x &&\text{via Assumption A2 with } \tilde{\Theta} > 0 \\
&\leq \tilde{\Theta}\|\omega(t)\|_{L^2}\|\sigma(t)\eta\|_{L^2} &&\text{via Hölder's inequality} \\
&\leq \tilde{\Theta}\|\omega(t)\|_{L^2}\|\sigma(t)\|_{L^4}\|\eta\|_{L^4} &&\text{again via Hölder's inequality} \\
&\leq c\tilde{\Theta}\|\omega(t)\|_{L^2}\|\sigma(t)\|_2\|\eta\|_2 &&\text{via norm equivalence in finite dimensions}
\end{aligned}
$$

with $c > 0$. Lemma 3.2 then shows existence of a unique solution

$$
(u^k_*, \sigma^k_*) \in L^2(0,T;W^k) \times L^2(0,T;V^k).
$$

Now we define

$$
u_r := u^\omega - u^k - u^k_* \in L^2(0,T;W^k) \quad \text{and} \quad \sigma_r := \sigma^\omega - \sigma^k - \sigma^k_* \in L^2(0,T;V^k),
$$

where $(u^\omega, \sigma^\omega)$ is the finite-dimensional solution to the state system (16) corresponding to $\varphi + \omega$. Subtracting the linearized system (21) and the state system from the one corresponding to the control φ, we see that (u_r, σ_r) satisfies the saddle point problem

$$
\begin{aligned}
\left(C^{-1}(\varphi(t))\sigma_r, \eta\right)_{L^2} - \left(\eta, \mathcal{E}(u_r)\right)_{L^2} &= \langle \mathbb{F}_r(t), \eta\rangle &&\forall \eta \in V^k, \\
- \left(\sigma_r, \mathcal{E}(v)\right)_{L^2} &= 0 &&\forall v \in W^k,
\end{aligned}
$$

where

$$\langle \mathbb{F}_r(t), \eta \rangle := - \left(\left[C^{-1} \left(\varphi(t) + \omega(t) \right) - C^{-1} \left(\varphi(t) \right) - (C^{-1})' \left(\varphi(t) \right) \omega(t) \right] \sigma^\omega, \eta \right)_{L^2}$$
$$- \left((C^{-1})' \left(\varphi(t) \right) \omega(t)(\sigma^\omega - \sigma^k), \eta \right)_{L^2}.$$

The two terms of $\left| \langle \mathbb{F}_r(t), \eta \rangle \right|$ are investigated separately. Using Taylor's theorem and the Lipschitz continuity of $(C^{-1})'$ for the first term it holds that

$$\left| \left(\left[C^{-1} \left(\varphi(t) + \omega(t) \right) - C^{-1} \left(\varphi(t) \right) - \left(C^{-1} \right)' \left(\varphi(t) \right) \omega(t) \right] \sigma^\omega, \eta \right)_{L^2} \right|$$
$$\leq c \left\| C^{-1} \left(\varphi(t) + \omega(t) \right) - C^{-1} \left(\varphi(t) \right) - \left(C^{-1} \right)' \left(\varphi(t) \right) \omega(t) \right\|_{L^2} \| \sigma^\omega \|_2 \| \eta \|_2$$
$$\leq c \frac{1}{2} L_{(C^{-1})'} \| \omega(t) \|_{L^2}^2 \| \sigma^\omega \|_2 \| \eta \|_2 .$$

Applying Assumption A2 to the second term leads to

$$\left| \left((C^{-1})' \left(\varphi(t) \right) \omega(t)(\sigma^\omega - \sigma^k), \eta \right)_{L^2} \right| \leq c\tilde{\Theta} \| \omega(t) \|_{L^2} \left\| \sigma^\omega - \sigma^k \right\|_2 \| \eta \|_2 .$$

We note that the difference $\bar{\sigma} := \sigma^\omega - \sigma^k$ solves system (17) with $\bar{\mathbb{F}} = \left(\left(C^{-1}(\varphi) - C^{-1}(\varphi + \omega) \right) \sigma^\omega, \eta \right)_{L^2}$ and $\bar{\mathbb{G}} = 0$. From a similar estimate as above,

$$\left| \langle \bar{\mathbb{F}}, \eta \rangle \right| \leq \left\| C^{-1}(\varphi + \omega) - C^{-1}(\varphi) \right\|_{L^2} \left\| \sigma^\omega(t)\eta \right\|_{L^2} \leq C_d L_{C^{-1}} \| \omega \|_{L^2} \| \sigma^\omega \|_2 \| \eta \|_2$$

and Lemma 3.2, we infer that $\left\| \sigma^\omega - \sigma^k \right\|_2 \leq C \| \omega \|_2$. Thus, $\mathbb{F}_r(t) \in L^2(\Omega, \mathbb{S}^d)$ and via Lemma 3.2 it holds for (u_r, σ_r) that

$$\| u_r(t) \|_2 \leq C_1 \left\| \mathbb{F}_r(t) \right\|_{L^2} \leq c \left\| \omega(t) \right\|_2^2, \qquad \| \sigma_r(t) \|_2 \leq C_1 \left\| \mathbb{F}_r(t) \right\|_{L^2} \leq c \left\| \omega(t) \right\|_2^2,$$

with positive constants C_1 and c, which proves the Fréchet-differentiability of the control-to-discretized-state operator. □

3.2 Adjoint Problem

By calculating the $\frac{\partial \mathcal{L}}{\partial u}$ and $\frac{\partial \mathcal{L}}{\partial \sigma}$ derivatives, we get the saddle point problem of the adjoint system. The derivative of the local volume constraint (5) was calculated in [11]. It is given by 11.

Definition 3.7. The pair $(p, \tau) \in L^2(0, T; \mathcal{H}) \times L^2(0, T; \mathcal{N})$ is a *weak solution of the adjoint system*, if it satisfies the following saddle point problem:

$$\begin{aligned} \left(C^{-1} \left(\varphi(t) \right) \tau(t), \eta \right)_{L^2} - \left(\eta, \mathcal{E} \left(p(t) \right) \right)_{L^2} &= -\alpha \left(\mathcal{C} \left(\varphi(t), \sigma(t) \right) D_\sigma r \left(\sigma(t) \right), \eta \right)_{L^2}, \\ - \left(\tau(t), \mathcal{E} \left(v \right) \right)_{L^2} &= -\int_{\Gamma_f} f(t) \cdot v \, \mathrm{d}x \end{aligned} \tag{22}$$

for all $\eta \in \mathcal{N}$ and $v \in \mathcal{H}$ and for almost all $t \in (0, T)$.

Lemma 3.8 (The adjoint problem is well-posed). *Let Assumptions A1, A2, A3 hold true. For a given $\varphi \in L^2(0, T; H^1)$ and $(u, \sigma) \in L^2(0, T; \mathcal{H}) \times L^2(0, T; \mathcal{N})$ there exists a unique weak solution $(p, \tau) \in L^2(0, T; \mathcal{H}) \times L^2(0, T; \mathcal{N})$ of the adjoint problem such that Definition 3.7 is fulfilled.*

Proof. We aim to show that the conditions for Lemma 3.2 are fulfilled for the right-hand sides

$$\langle \mathbb{F}(t), \eta \rangle := -\alpha \left(\mathcal{C} \left(\varphi(t), \sigma(t) \right) D_\sigma r \left(\sigma(t) \right), \eta \right)_{L^2}$$

and

$$\langle \mathbb{G}(t), v \rangle := - \int_{\Gamma_f} f(t) \cdot v \mathrm{d}x.$$

Equivalently to the proof of [11, Thm 3.6] we receive $\mathbb{F}(t) \in L^2 \left(\Omega, \mathbb{S}^d \right)$. Additionally, $\| \mathbb{F}(t) \|_{L^2(\Omega)}$ is bounded. Therefore, $\mathbb{F} \in L^\infty(0, T; L^2)$. Additionally, as in Lemma 3.3, we get $\mathbb{G} \in L^\infty(0, T; \mathcal{H}^*)$ and the result follows from Lemma 3.2. □

Definition 3.9 (Time-dependent solution operator of the adjoint)**.** Lemma 3.8 defines the *solution operator of the adjoint*, which maps the phase-field φ and the stress σ to the unique weak solution (p, τ) of the adjoint problem

$$Q : L^2(0,T;H^1) \times L^2(0,T;\mathcal{N}) \to L^2(0,T;\mathcal{H}) \times L^2(0,T;\mathcal{N}), \quad (\varphi,\sigma) \mapsto (p,\tau).$$

In the case of $\mathcal{H} = H_D^1$ and $\mathcal{N} = L^2(\Omega, \mathbb{S}^d)$ the second component of Q, i.e. the function mapping the phase-field φ and the stress σ to τ is denoted by Q_2 with

$$Q_2 : L^2(0,T;H^1) \times L^2(0,T;L^2) \to L^2(0,T;L^2), \quad (\varphi,\sigma) \mapsto \tau.$$

In the case of finite-dimensional subspaces $\mathcal{H} = W^k$ and $\mathcal{N} = V^k$, which will be relevant in Section 4.2, the second component of Q, i.e. the function mapping the phase-field φ and the stress σ^k to τ^k is denoted by Q_2^k with

$$Q_2^k : L^2(0,T;H^1) \times L^2(0,T;V^k) \to L^2(0,T;V^k), \quad (\varphi,\sigma^k) \mapsto \tau^k.$$

Lemma 3.10. *Let Assumptions A1, A2, A3 hold true. The solution operators of the adjoints Q_2 and Q_2^k are continuous from $L^2(0,T;H^1 - weak) \times L^2(0,T;\mathcal{V})$ to $L^2(0,T;\mathcal{V} - weak)$, where the first and the last space space is equipped with the weak topology.*

Remark. Formulated in the notion of sequential continuity, this means that even though we only assume weak convergence of $\{\varphi_i\}_i$ in $L^2(0,T;H^1)$, we can prove weak convergence of the corresponding adjoint states, e.g. of $\{Q_2(\varphi_i,\sigma_i)\}_i$ in $L^2(0,T;\mathcal{V})$.

We note that it would also be possible to show strong convergence of the adjoint states as in the proof of Lemma 3.5, but we do not need this better convergence property in the proof.

Proof of Lemma 3.10. Let $\{\varphi_i\}_i \subset L^2(0,T;H^1)$ be a sequence of controls converging weakly to a $\varphi \in L^2(0,T;H^1)$ and denote the sequence of corresponding adjoints via $\{\tau_i\}_i$. According to [28, Section 30.1], the saddle point problem (22) is equivalent to

$$\begin{aligned}\left(C^{-1}(\varphi)\tau,\eta\right)_{L^2(0,T;L^2)} - \left(\eta, \mathcal{E}(p)\right)_{L^2(0,T;L^2)} &= -\alpha\left(\mathcal{C}(\varphi,\sigma)D_\sigma r(\sigma),\eta\right)_{L^2(0,T;L^2)},\\ -\left(\tau, \mathcal{E}(v)\right)_{L^2(0,T;L^2)} &= -\int_0^T \int_{\Gamma_f} f \cdot v \mathrm{d}x\mathrm{d}t\end{aligned} \tag{23}$$

for all $\eta \in L^2(0,T;\mathcal{N})$ and $v \in L^2(0,T;\mathcal{H})$. We subtract the adjoint equations (23) for φ_i, φ and obtain

$$\begin{aligned}\left(C^{-1}(\varphi_i)(\tau_i-\tau),\eta\right)_{L^2(0,T;L^2)} &- \left(\eta, \mathcal{E}(p_i-p)\right)_{L^2(0,T;L^2)}\\ &= -\alpha\left(\mathcal{C}(\varphi_i,\sigma_i)D_\sigma r(\sigma_i) - \mathcal{C}(\varphi,\sigma)D_\sigma r(\sigma),\eta\right)_{L^2(0,T;L^2)}\\ &\quad - \left(\left[C^{-1}(\varphi_i) - C^{-1}(\varphi)\right]\tau,\eta\right)_{L^2(0,T;L^2)}\\ -\left(\tau_i-\tau, \mathcal{E}(v)\right)_{L^2(0,T;L^2)} &= 0\end{aligned} \tag{24}$$

for all $\eta \in L^2(0,T;\mathcal{N})$ and $v \in L^2(0,T;\mathcal{H})$.

The strong convergence of $\left\{\left[\int_{B_{r(\sigma_n)}(x)} \chi_\Omega^\varsigma(\zeta)\left(\varphi_n(\zeta)-\mu\right)\mathrm{d}\zeta\right]_+\right\}_i$ has been shown in the proof of [11, Lem. 3.5]. Since $\{\varphi_i\}_i$ converges weakly in $L^2(0,T;H^1)$, we receive weak convergence of

$$\chi_\Omega^\varsigma(q)\nabla\varphi_i(q) + \nabla\chi_\Omega^\varsigma(q)\varphi_i(q)$$

and therefore strong convergence of the encompassing integral in $L^2(\Omega \times (0,T))$. This implies the existence of a dominating function for a point-wise converging subsequence via the reverse Lebesgue's theorem [6, Thm. 4.9]. Note that $\frac{1}{r(\sigma_i)^{d+1}}$ is bounded from above by $\frac{1}{r_{min}^{d+1}}$ and $B_{r(\sigma_i)}(x)$ is covered by $B_{r_{max}}$. Strong convergence of $\mathcal{C}$ in $L^2(\Omega \times [0,T])$ follows via Lebesgue's theorem on dominated convergence. This procedure is repeated for the $\mathcal{C}(\varphi_i,\sigma_i)D_\sigma r(\sigma_i)$ product since, by Assumption A3,

$$\left|D_\sigma r(\sigma_i)\right| \le C_r$$

is bounded with $C_r > 0$ and therefore, via reverse Lebesgue's and Lebesgue's theorem, the product converges strongly in $L^2(\Omega \times (0,T))$. First, the convergence of the right-hand side implies its boundedness in $L^2(\Omega \times (0,T))$ and from Lemma (3.2), we infer *a priori* an estimate on the sequence $\{\tau_i\}_i$ in $L^2(\Omega \times (0,T))$. Selecting possibly a subsequence,

we want to idetify the limit with τ. Therefore, we show that the right-hand side of (24) converges to zero. Convergence of the last part of the right-hand has already been shown in the proof of Lemma 3.5. The first part convergence due to the continuity shown above. In total, we get the strong convergence of the right-hand side in (24) such that we infer

$$\tau_i \rightharpoonup \tau \quad \text{in } L^2(\Omega \times (0, T)),$$
$$p_i \rightharpoonup p \quad \text{in } L^2(0, T; H_D^1(\Omega))$$

as $i \to \infty$.

$\square$

4 Proof of Theorem 2.2

4.1 Regularization

The double obstacle potential is not differentiable outside of $(0, 1)$. The idea is to regularize the potential, prove the existence of a solution for the smoothed potential together with an appropriate *a priori* estimate and go to the limit afterwards. To this end, we define the smoothed potential $\psi_\beta \in C^2(\mathbb{R})$ with $0 < \beta < \frac{1}{4}$ as

$$\psi_\beta(\varphi) := \frac{1}{2} \left(\varphi - \varphi^2 \right) + \psi_\beta^c(\varphi), \tag{25}$$

with the first part being quadratic and the second part being the convex function

$$\psi_\beta^c(\varphi) := \begin{cases} \frac{1}{8\beta} \left(\varphi - \left(1 + \frac{\beta}{2} \right) \right)^2 + \frac{\beta}{96} & \text{for} \quad \varphi \geqslant 1 + \beta \\ \frac{1}{24\beta^2} (\varphi - 1)^3 & \text{for } 1 < \varphi < 1 + \beta \\ 0 & \text{for } 0 \leqslant \varphi \leqslant 1 \\ -\frac{1}{24\beta^2} \varphi^3 & \text{for} -\beta < \varphi < 0 \\ \frac{1}{8\beta} \left(\varphi + \frac{\beta}{2} \right)^2 + \frac{\beta}{96} & \text{for } \varphi \leqslant -\beta \end{cases}.$$

This choice can be seen as a smoothed Yoshida approximation of the obstacle potential in order to get convex C^2 functions. This is essential for the regularity estimates in the proof of Theorem 2.2. Different choices with this property are possible, we followed [4] with the above choice. Notice that $0 \leq (\psi_\beta^c)''(\varphi) \leq \frac{1}{4\beta}$ and therefore $(\psi_\beta^c)'$ is Lipschitz continuous with Lipschitz constant $\frac{1}{4\beta}$.

We get the weak formulation of the regularized Allen–Cahn system $(\mathrm{AC})_\beta$ by replacing the double obstacle potential ψ by the regularized potential ψ_β, meaning that the subdifferential ξ and the derivative of the quadratic term is replaced with ψ_β' in the Allen–Cahn system:

$$\begin{aligned} \partial_t \varphi_\beta - \gamma \epsilon \Delta \varphi_\beta &= F_\beta(\varphi_\beta) && \text{in } \Omega \text{ and a.e. in } (0, T), \\ \nabla \varphi_\beta \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \text{ and a.e. in } (0, T), \\ \varphi_\beta(0) &= \varphi_0 && \text{in } \Omega \end{aligned} \tag{26}$$

with $\mathbf{n}$ being the outer normal, $\varphi_0 \in H^1(\Omega)$ and

$$\begin{aligned} F_\beta(\varphi_\beta) := &- (C^{-1})' \left(\varphi_\beta \right) \sigma : \tau - \lambda_\beta - \frac{\gamma}{\epsilon} \psi_\beta' \left(\varphi_\beta \right) \\ &- \alpha \int_\Omega \left[\int_{B_{r(\sigma)}(q)} \chi_\Omega^{\mathsf{S}}(\zeta) \left(\varphi_\beta(\zeta) - \mu \right) \, \mathsf{d}\zeta \right]_+ \chi_{B_{r(\sigma)}(q)}(x) \chi_\Omega^{\mathsf{S}}(x) \, \mathsf{d}q, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \lambda_\beta = -\frac{1}{|\Omega|} \Bigg(&\alpha \int_\Omega \int_\Omega \left[\int_{B_{r(\sigma)}(q)} \chi_\Omega^{\mathsf{S}}(\zeta) \left(\varphi_\beta(\zeta) - \mu \right) \, \mathsf{d}\zeta \right]_+ \chi_{B_{r(\sigma)}(q)}(x) \chi_\Omega^{\mathsf{S}}(x) \, \mathsf{d}q \, \mathsf{d}x \\ &+ \frac{\gamma}{\epsilon} \int_\Omega \psi_\beta' \left(\varphi_\beta \right) \, \mathsf{d}x + \int_\Omega (C^{-1})' \left(\varphi_\beta \right) \sigma : \tau \, \mathsf{d}x \Bigg). \end{aligned} \tag{28}$$

4.2   Existence - Regularized Allen–Cahn System with Discretized States

In order to improve the regularity properties of the approximate solutions to problem (15), we improve the regularity of $F_\beta(\bar\varphi)$ via a Galerkin approximation of the states, the solutions of the mechanical systems. We notice that u and p are elements of the Hilbert space $L^2(0,T;H^1_D(\Omega,\mathbb{R}^d))$ and pick an orthonormal basis $\{\phi_i\}_i \subset H^2_D(\Omega,\mathbb{R}^d)$ to write u and p as linear combinations of these basis functions with time-dependent coefficients. We define the finite-dimensional subspace

$$W^k := \operatorname{span}\{\phi_i, i=1,\dots,k\} \quad \text{with} \quad \operatorname{cl}_{H^1_D}\left(\lim_{k\to\infty} W^k\right) = H^1_D(\Omega,\mathbb{R}^d),$$

where $\operatorname{cl}_{H^1_D}$ denotes the closure with respect to the $\|\cdot\|_{H^1_D}$-norm. In order to have a fitting finite-dimensional subspace for σ and τ, we define

$$V^k := \operatorname{span}\left\{\mathcal{E}(\phi_i) \mid \phi_i \in W^k, i=1,\dots,k\right\} \quad \text{with} \quad \operatorname{cl}_{L^2}\left(\lim_{k\to\infty} V^k\right) = L^2(\Omega,\mathbb{S}^d).$$

Remark. We chose $\{\phi_i\}_i \subset H^2_D(\Omega,\mathbb{R}^d)$ to ensure that the space V^k, which is defined via the gradients of ϕ_i, can be embedded into L^4. This will be necessary when applying Hölder's inequality and Norm equivalences on finite dimensional spaces as in Lemma 3.6 or Lemma 21.

The Galerkin approximations of σ and τ in this space can be written as linear combinations of these basis functions with time-dependent coefficients $\{c_i(t)\}_i \subset L^\infty(0,T)$ and $\{d_i(t)\}_i \subset L^\infty(0,T)$

$$\sigma^k := \sum_{i=1}^k c_i(t)\mathcal{E}(\phi_i), \qquad \tau^k := \sum_{j=1}^k d_j(t)\mathcal{E}(\phi_j).$$

We arrive at the weak form of the Galerkin approximated, regularized Allen–Cahn system $(\mathrm{AC})^k_\beta$ by replacing the continuous stress σ and its adjoint τ with their respective discretizations σ^k and τ^k in the weak form of the regularized Allen–Cahn system $(\mathrm{AC})_\beta$:

$$\begin{aligned}
\partial_t \varphi^k_\beta - \gamma\epsilon \Delta\varphi^k_\beta &= F^k_\beta(\varphi^k_\beta) && \text{in } \Omega \text{ and a.e. in } (0,T), \\
\nabla\varphi^k_\beta \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \text{ and a.e. in } (0,T), \\
\varphi^k_\beta(0) &= \varphi_0 && \text{in } \Omega,
\end{aligned} \tag{29}$$

with $\mathbf{n}$ being the outer normal, $\varphi_0 \in H^1(\Omega)$ and

$$\begin{aligned}
F^k_\beta(\varphi^k_\beta) := &-(C^{-1})'\left(\varphi^k_\beta\right)\sigma^k : \tau^k - \lambda^k - \frac{\gamma}{\epsilon}\psi'_\beta\left(\varphi^k_\beta\right) \\
&-\alpha\int_\Omega \left[\int_{B_{r(\sigma^k)}(q)} \chi^\varsigma_\Omega(\zeta)\left(\varphi^k_\beta(\zeta)-\mu\right)\,\mathsf{d}\zeta\right]_+ \chi_{B_{r(\sigma^k)}(q)}(x)\chi^\varsigma_\Omega(x)\,\mathsf{d}q,
\end{aligned}$$

where

$$\begin{aligned}
\lambda^k = -\frac{1}{|\Omega|}\Bigg(&\alpha\int_\Omega\int_\Omega\left[\int_{B_{r(\sigma^k)}(q)} \chi^\varsigma_\Omega(\zeta)\left(\varphi^k_\beta(\zeta)-\mu\right)\,\mathsf{d}\zeta\right]_+ \chi_{B_{r(\sigma^k)}(q)}(x)\chi^\varsigma_\Omega(x)\,\mathsf{d}q\,\mathsf{d}x \\
&+\frac{\gamma}{\epsilon}\int_\Omega \psi'_\beta\left(\varphi^k_\beta\right)\,\mathsf{d}x + \int_\Omega (C^{-1})'\left(\varphi^k_\beta\right)\sigma^k:\tau^k\,\mathsf{d}x\Bigg).
\end{aligned} \tag{30}$$

Lemma 4.1. *Let Assumptions A1, A2 and A3 hold true. For $\bar\varphi \in L^2(0,T;L^2(\Omega,\mathbb{R}))$ the right-hand side $F^k_\beta(\bar\varphi)$ of the strong form of the Galerkin approximated, regularized Allen–Cahn system $(\mathrm{AC})^k_\beta$ is in $L^2(0,T;L^2)$. Furthermore, for $\bar\varphi \in L^4(0,T;W^{1,2})$ it holds that*

$$\left\|F^k_\beta(\bar\varphi)\right\|^2_{L^2(0,T;L^2)} \le \hat c\left(\|\bar\varphi\|^4_{L^4(0,T;W^{1,2})} + 1\right)$$

for some $\hat c > 0$.

Proof. Using the finite-dimensionality of σ^k, τ^k as in the proof of Lemma 3.6 and the definition of the elasticity tensor, we see that

$$\int_\Omega |(C^{-1})'\left(\bar{\varphi}(t)\right)\sigma^k(t):\tau^k(t)|\mathrm{d}x \leq \left\|(C^{-1})'\left(\bar{\varphi}(t)\right)\right\|_{L^\infty}\left\|\sigma^k(t)\right\|_2\left\|\tau^k(t)\right\|_2$$

$$=:\bar{C}\left\|\sigma^k(t)\right\|_2\left\|\tau^k(t)\right\|_2.$$

The potential term can be split via the definition (25) and we get

$$\int_\Omega |\psi'_\beta\left(\varphi_\beta(t)\right)|^2\mathrm{d}x = \int_\Omega\left|\frac{1}{2}-\varphi_\beta(t)+(\psi^c_\beta)'(\varphi_\beta(t))\right|^2\mathrm{d}x$$

$$\leq C_\beta(1+\|\varphi_\beta\|^2_{L^2(\Omega)}).$$

For the local volume constraint term it holds that

$$\int_\Omega\left|\int_\Omega\left[\int_{B_{r(\sigma)}(q)}\chi^\varsigma_\Omega(\zeta)\left(\varphi_\beta(t)\left(\zeta\right)-\mu\right)\mathsf{d}\zeta\right]_+\chi_{B_{r(\sigma)}(q)}(x)\,\chi^\varsigma_\Omega(x)\,\mathrm{d}q\right|^2\mathrm{d}x$$

$$\leq\int_\Omega\left|\int_\Omega\left(\|\varphi_\beta(t)\|_{L^1(\Omega,\mathbb{R})}+\mu|\Omega|\right)\mathrm{d}q\right|^2\mathrm{d}x\leq\int_\Omega\left||\Omega|\|\varphi_\beta(t)\|_{L^1(\Omega,\mathbb{R})}+\mu|\Omega|^2\right|^2\mathrm{d}x$$

$$\leq\int_\Omega|\Omega|^2\|\varphi_\beta(t)\|^2_{L^1(\Omega,\mathbb{R})}+\mu^2|\Omega|^4\mathrm{d}x=|\Omega|^3\|\varphi_\beta(t)\|^2_{L^1(\Omega,\mathbb{R})}+\mu^2|\Omega|^5<\infty.$$

Since λ_β is represented as an integral of the previously examined terms, it is bounded.

Estimating the right-hand side, we find

$$\int_\Omega\left|F^k_\beta(\bar{\varphi}(t))\right|^2\mathrm{d}x$$

$$\leq\int_\Omega\left|(C^{-1})'\left(\bar{\varphi}(t)\right)\sigma^k(t):\tau^k(t)\right|^2\mathrm{d}x+\int_\Omega\left|\lambda^k\right|^2\mathrm{d}x+\left(\frac{\gamma}{\epsilon}\right)^2\int_\Omega\left|\psi'_\beta\left(\bar{\varphi}(t)\right)\right|^2\mathrm{d}x$$

$$+\int_\Omega\left|\alpha\int_\Omega\left[\int_{B_{r\left(\sigma^k(t)\right)}(q)}\chi^\varsigma_\Omega(\zeta)\left(\bar{\varphi}(t)\left(\zeta\right)-\mu\right)\mathsf{d}\zeta\right]_+\chi_{B_{r\left(\sigma^k(t)\right)}(q)}(x)\,\chi^\varsigma_\Omega(x)\,\mathrm{d}q\right|^2\mathrm{d}x$$

$$\leq\bar{C}^2\left\|\sigma^k(t)\right\|^2_2\left\|\tau^k(t)\right\|^2_2+|\lambda^k|^2|\Omega|+\left(\frac{\gamma}{\epsilon}\right)^2C_\beta(1+\|\bar{\varphi}(t)\|^2_{L^2(\Omega)})$$

$$+\alpha^2\left(\|\bar{\varphi}(t)\|^2_{L^1}+\mu^2|\Omega|^2\right)|\Omega|^3\quad\text{for almost all }t\in(0,T).$$

Thus, for $\bar{\varphi}\in L^2(0,T;L^2(\Omega,\mathbb{R}))$ the right-hand side $F^k_\beta(\bar{\varphi})$ is in $L^2(0,T;L^2)$.

All terms except for the first one in $\left\|F^k_\beta(\bar{\varphi})\right\|^2_{L^2(0,T;L^2)}$ can easily be estimated via a constant $\bar{d}$ times the norm $\|\bar{\varphi}\|^2_{L^4(0,T;W^{1,2})}$. Towards the second part of the lemma, we will examine the term

$$\bar{C}^2\left\|\sigma^k\right\|^2_{L^2(0,T;L^2)}\left\|\tau^k\right\|^2_{L^2(0,T;L^2)}$$

more thoroughly. We know from Lemma 3.3 and Lemma 3.8 that

$$\|\sigma^k\|_{L^2(0,T;L^2)}\leq C_2\|\mathbb{G}\|_{L^2\left(0,T;(H^1_D)^*\right)}$$

and $\quad\|\tau^k\|_{L^2(0,T;L^2)}\leq\tilde{C}_1\|\mathbb{F}\|_{L^2(0,T;L^2)}+\tilde{C}_2\|\mathbb{G}\|_{L^2\left(0,T;(H^1_D)^*\right)},$

with

$$\langle\mathbb{G}(t),v\rangle:=-\int_{\Gamma_f}f(t)\cdot v\mathrm{d}x$$

and $\quad\langle\mathbb{F}(t),\eta\rangle:=-\alpha\left(\mathcal{C}\left(\varphi(t),\sigma(t)\right)D_\sigma r\left(\sigma(t)\right),\eta\right)_{L^2}.$

From the proof of Lemma 3.3 we get

$$\left\|\mathbb{G}(t)\right\|_{\left(H_D^1\right)^*} \le c_{tr}\left\|f(t)\right\|_{L^2(\Gamma_f)}.$$

Thus,

$$\left\|\mathbb{G}\right\|_{L^2\left(0,T;\left(H_D^1\right)^*\right)} = \left(\int_0^T \left\|\mathbb{G}\right\|_{\left(H_D^1\right)^*}^2 \, \mathrm{d}t\right)^{\frac{1}{2}} \le \left(\int_0^T c_{tr}^2\left\|f\right\|_{L^2(\Gamma_f)}^2 \, \mathrm{d}t\right)^{\frac{1}{2}} = c_{tr}\left\|f\right\|_{L^2(0,T;L^2(\Gamma_f))}.$$

From our previous calculations in the proof of Lemma 3.8, we know that

$$\left\|\mathbb{F}(t)\right\|_{L^2(\Omega)} \le \alpha \left| \left[\frac{1}{r(\sigma(t))^d}\int_{B_{r(\sigma(t))}(x)}\chi_\Omega^\varsigma(q)\left(\varphi(q,t)-\mu\right)\mathrm{d}q\right]_+ \right|$$

$$\left|\frac{1}{r(\sigma(t))^{d+1}}\int_{B_{r(\sigma(t))}(x)}\left[\chi_\Omega^\varsigma(q)\nabla\varphi(q,t)+\nabla\chi_\Omega^\varsigma(q)\varphi(q,t)\right](q-x)\,\mathrm{d}q\right| C_r$$

$$\le \alpha\frac{1}{r_{min}^d}\left(\left\|\varphi(t)\right\|_{L^1(\Omega,\mathbb{R})}+\mu\left|\Omega\right|\right)\frac{1}{r_{min}^{d+1}}\left(1+C_\varsigma\right)\left\|\varphi(t)\right\|_{H^1(\Omega,\mathbb{R})}r_{max}C_r$$

$$=: c\left\|\varphi(t)\right\|_{H^1(\Omega,\mathbb{R})}^2+d\left\|\varphi(t)\right\|_{H^1(\Omega,\mathbb{R})}.$$

Thus

$$\left\|\mathbb{F}\right\|_{L^2\left(0,T;L^2(\Omega)\right)}^2 \le \int_0^T\left[c\left\|\varphi(t)\right\|_{H^1(\Omega,\mathbb{R})}^2+d\left\|\varphi(t)\right\|_{H^1(\Omega,\mathbb{R})}\right]^2\mathrm{d}t$$

$$\le \underline{c}\left\|\varphi\right\|_{L^4(0,T;W^{1,2})}^4+\underline{d}.$$

with $\underline{c},\underline{d}>0$. Finally, we get

$$\bar{C}^2\left\|\sigma^k\right\|_{L^2(0,T;L^2)}^2\left\|\tau^k\right\|_{L^2(0,T;L^2)}^2$$

$$\le \bar{C}^2 C_2^2\left\|\mathbb{G}\right\|_{L^2\left(0,T;\left(H_D^1\right)^*\right)}^2\left(\tilde{C}_1\left\|\mathbb{F}\right\|_{L^2(0,T;L^2)}+\tilde{C}_2\left\|\mathbb{G}\right\|_{L^2\left(0,T;\left(H_D^1\right)^*\right)}\right)^2$$

$$\le \bar{b}\left\|\varphi\right\|_{L^4(0,T;W^{1,2})}^4+\bar{c}\left\|\varphi\right\|_{L^4(0,T;W^{1,2})}^2+\bar{e}$$

with $\bar{b},\bar{c},\bar{e}>0$. Together with the other terms it holds that

$$\left\|F_\beta^k(\bar\varphi)\right\|_{L^2(0,T;L^2)}^2 \le \bar{b}\left\|\bar\varphi\right\|_{L^4(0,T;W^{1,2})}^4+(\bar{c}+\bar{d})\left\|\bar\varphi\right\|_{L^4(0,T;W^{1,2})}^2+\bar{e}$$

$$\le \bar{b}\left\|\bar\varphi\right\|_{L^4(0,T;W^{1,2})}^4+\frac{(\bar{c}+\bar{d})^2}{2}+\frac{\left\|\bar\varphi\right\|_{L^4(0,T;W^{1,2})}^4}{2}+\bar{e}$$

$$\le \hat{c}\left(\left\|\bar\varphi\right\|_{L^4(0,T;W^{1,2})}^4+1\right),$$

where we used Young's inequality and defined $\hat{c} := \max\left\{\bar{b}+\frac{1}{2}, \frac{(\bar{c}+\bar{d})^2}{2}+\bar{e}\right\}$.   □

From Lemma 4.1 it follows that $\left|\int_\Omega F_\beta^k(\varphi_\beta^k)\partial_t\varphi_\beta^k\mathrm{d}x\right|$ is finite, since $\partial_t\varphi_\beta^k$ can be embedded into L^2. This means that it is promising to start the Schauder apparatus in this setting of discretized spaces. Set

$$X_T := L^4(0,T;W^{1,2})$$

and with $M>0$ and $\tilde{t}>0$ to be chosen later

$$K := \left\{\Phi\in X_T : \left\|\Phi\right\|_{L^4(0,\tilde{t};W^{1,2})}\le M\right\}$$

Solving (15) defines $\mathcal{T}: X_T\to\mathcal{T}(X_T)$, $\bar\varphi\mapsto\varphi$. To apply the Schauder fixed point theorem, we need show that K is a nonempty, closed, bounded, convex subset of the Banach space X_T, that $\mathcal{T}$ is a self-mapping on K and that $\mathcal{T}$ is compact on K.

Lemma 4.2 (Properties of K). *The set K is a nonempty, closed, bounded, convex subset of the Banach space X_T.*

Proof. We first note that K is nonempty, because the constant function $\Phi \equiv 0$ lies in K. Also, K is defined as an M-ball in the $\|\cdot\|_{L^4(0,\tilde{t};W^{1,2})}$ norm and therefore closed, bounded and convex. □

Lemma 4.3 (Self-mapping of $\mathcal{T}$). *Let Assumptions A1, A2, A3 hold true. The function $\mathcal{T} : X_T \to \mathcal{T}(X_T)$, $\bar{\varphi} \mapsto \varphi_\beta^k$ is a self-mapping on K for some $\tilde{t} < T$, i.e. $\mathcal{T} : K \to K$.*

Proof. We need to show that at least for a small $\tilde{t}$

$$\left\|\varphi_\beta^k\right\|_{L^4(0,\tilde{t};W^{1,2})} \leq M.$$

We are testing the PDE in (29) with φ_β^k and get via Hölder's and Young's inequalities

$$\int_\Omega \partial_t \varphi_\beta^k \varphi_\beta^k \mathrm{d}x - \gamma\epsilon \int_\Omega \Delta \varphi_\beta^k \varphi_\beta^k \mathrm{d}x = \int_\Omega F_\beta^k(\bar{\varphi}) \varphi_\beta^k \mathrm{d}x$$

$$\Leftrightarrow \frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi_\beta^k|^2 \mathrm{d}x + \gamma\epsilon \int_\Omega |\nabla \varphi_\beta^k|^2 \mathrm{d}x = \int_\Omega F_\beta^k(\bar{\varphi}) \varphi_\beta^k \mathrm{d}x$$

$$\Rightarrow \frac{d}{dt}\int_\Omega |\varphi_\beta^k|^2 \mathrm{d}x \leq \frac{1}{\gamma\epsilon}\int_\Omega F_\beta^k(\bar{\varphi})^2 \mathrm{d}x + \gamma\epsilon \int_\Omega (\varphi_\beta^k)^2 \mathrm{d}x.$$

Integrating over $(0,t)$ with $t \in (0,\tilde{t})$ gives

$$\int_\Omega |\varphi_\beta^k(t)|^2 \mathrm{d}x \leq \int_\Omega |\varphi_\beta^k(0)|^2 \mathrm{d}x + \frac{1}{\gamma\epsilon}\int_0^t \int_\Omega F_\beta^k(\bar{\varphi}(s))^2 \mathrm{d}x\mathrm{d}s + \gamma\epsilon \int_0^t \int_\Omega (\varphi_\beta^k(s))^2 \mathrm{d}x\mathrm{d}s.$$

We can apply a corollary of Gronwall's Lemma to get

$$\int_\Omega |\varphi_\beta^k(t)|^2 \mathrm{d}x \leq e^{t\gamma\epsilon}\int_\Omega |\varphi_\beta^k(0)|^2 \mathrm{d}x + e^{t\gamma\epsilon}\frac{1}{\gamma\epsilon}\int_0^t \int_\Omega F_\beta^k(\bar{\varphi}(s))^2 \mathrm{d}x\mathrm{d}s.$$

We are taking the essential supremum over $t \in \left[0,\tilde{t}\right]$ and arrive at

$$\left\|\varphi_\beta^k\right\|_{L^\infty(0,\tilde{t};L^2)}^2 = \operatorname*{ess\,sup}_{t\in\left[0,\tilde{t}\right]} \int_\Omega |\varphi_\beta^k(t)|^2 \mathrm{d}x \leq \frac{1}{\gamma\epsilon}\left\|F_\beta^k(\bar{\varphi})\right\|_{L^2(0,\tilde{t};L^2)}^2 e^{\tilde{t}\gamma\epsilon} + \left\|\varphi_\beta^k(0)\right\|_{L^2}^2 e^{\tilde{t}\gamma\epsilon}.$$

Furthermore, we are testing the PDE in (29) with $\partial_t \varphi_\beta^k$ and apply Young's inequality towards

$$\int_\Omega \partial_t \varphi_\beta^k \partial_t \varphi_\beta^k \mathrm{d}x - \gamma\epsilon \int_\Omega \Delta \varphi_\beta^k \partial_t \varphi_\beta^k \mathrm{d}x = \int_\Omega F_\beta^k(\bar{\varphi}) \partial_t \varphi_\beta^k \mathrm{d}x$$

$$\leq \left(\int_\Omega F_\beta^k(\bar{\varphi})^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega (\partial_t \varphi_\beta^k)^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\left(\int_\Omega F_\beta^k(\bar{\varphi})^2 \mathrm{d}x\right) + \frac{1}{2}\left(\int_\Omega (\partial_t \varphi_\beta^k)^2 \mathrm{d}x\right).$$

The last term on the right-hand side and the first term on the left-hand side can be combined. Noting $\nabla \varphi_\beta^k \cdot \mathbf{n} = 0$, we apply a partial integration on the second term and receive

$$\frac{1}{2}\int_\Omega |\partial_t \varphi_\beta^k|^2 \mathrm{d}x + \gamma\epsilon \int_\Omega \nabla \varphi_\beta^k \nabla \partial_t \varphi_\beta^k \mathrm{d}x \leq \frac{1}{2}\left(\int_\Omega F_\beta^k(\bar{\varphi})^2 \mathrm{d}x\right)$$

$$\Leftrightarrow \frac{1}{2}\int_\Omega |\partial_t \varphi_\beta^k|^2 \mathrm{d}x + \frac{\gamma\epsilon}{2}\frac{d}{dt}\int_\Omega |\nabla \varphi_\beta^k|^2 \mathrm{d}x \leq \frac{1}{2}\left(\int_\Omega F_\beta^k(\bar{\varphi})^2 \mathrm{d}x\right) = \frac{1}{2}\left\|F_\beta^k(\bar{\varphi})\right\|_{L^2}^2.$$

Since the first term is positive, it can be dropped

$$\frac{\gamma\epsilon}{2}\frac{d}{dt}\int_\Omega |\nabla \varphi_\beta^k|^2 \mathrm{d}x \leq \frac{1}{2}\left\|F_\beta^k(\bar{\varphi})\right\|_{L^2}^2.$$

At this point, we again integrate over $(0, t)$

$$\frac{\gamma\epsilon}{2} \int_\Omega |\nabla\varphi_\beta^k(t)|^2 \mathrm{d}x \le \frac{1}{2} \int_0^t \left\|F_\beta^k(\bar\varphi)\right\|_{L^2}^2 \mathrm{d}s + \frac{\gamma\epsilon}{2} \int_\Omega |\nabla\varphi_\beta^k(0)|^2 \mathrm{d}x$$

$$\Leftrightarrow \frac{\gamma\epsilon}{2} \left\|\nabla\varphi_\beta^k(t)\right\|_{L^2}^2 \le \frac{1}{2} \left\|F_\beta^k(\bar\varphi)\right\|_{L^2(0,t;L^2)}^2 + \frac{\gamma\epsilon}{2} \left\|\nabla\varphi_\beta^k(0)\right\|_{L^2}^2$$

and take the essential supremum over $t \in (0, \tilde t)$

$$\left\|\nabla\varphi_\beta^k\right\|_{L^\infty(0,\tilde t;L^2)}^2 = \operatorname*{ess\,sup}_{t\in[0,\tilde t]} \left\|\nabla\varphi_\beta^k(t)\right\|_{L^2}^2 \le \frac{1}{\gamma\epsilon} \left\|F_\beta^k(\bar\varphi)\right\|_{L^2(0,\tilde t;L^2)}^2 + \left\|\nabla\varphi_\beta^k(0)\right\|_{L^2}^2 .$$

Combining both of the previous estimates leads via Lemma 4.1 to

$$\begin{aligned}
\left\|\varphi_\beta^k\right\|_{L^\infty(0,\tilde t;W^{1,2})}^2 &\le \left\|\varphi_\beta^k\right\|_{L^\infty(0,\tilde t;L^2)}^2 + \left\|\nabla\varphi_\beta^k\right\|_{L^\infty(0,\tilde t;L^2)}^2 \\
&\le \frac{1}{\gamma\epsilon} \left\|F_\beta^k(\bar\varphi)\right\|_{L^2(0,\tilde t;L^2)}^2 e^{\tilde t\gamma\epsilon} + \left\|\varphi_\beta^k(0)\right\|_{L^2}^2 e^{\tilde t\gamma\epsilon} + \frac{1}{\gamma\epsilon} \left\|F_\beta^k(\bar\varphi)\right\|_{L^2(0,\tilde t;L^2)}^2 + \left\|\nabla\varphi_\beta^k(0)\right\|_{L^2}^2 \\
&\le \frac{2}{\gamma\epsilon} \left\|F_\beta^k(\bar\varphi)\right\|_{L^2(0,\tilde t;L^2)}^2 e^{\tilde t\gamma\epsilon} + \left\|\varphi_\beta^k(0)\right\|_{L^2}^2 e^{\tilde t\gamma\epsilon} + \left\|\nabla\varphi_\beta^k(0)\right\|_{L^2}^2 \\
&\le \frac{2\hat c}{\gamma\epsilon} \left(\|\bar\varphi\|_{L^4(0,\tilde t;W^{1,2})}^4 + 1\right) + \bar f\, e^{\tilde t\gamma\epsilon} + \bar g \\
&\le \frac{2\hat c}{\gamma\epsilon} \left(M^4 + 1\right) + \bar f\, e^{\tilde t\gamma\epsilon} + \bar g =: \mathcal D
\end{aligned}$$

$$(31)$$

where we used Lemma 4.1, the definition of K and defined $\bar f := \left\|\varphi_\beta^k(0)\right\|_{L^2}^2$ and $\bar g := \left\|\nabla\varphi_\beta^k(0)\right\|_{L^2}^2$. We have

$$\begin{aligned}
\left\|\varphi_\beta^k\right\|_{L^4(0,\tilde t;W^{1,2})} &= \left(\int_0^{\tilde t} 1 \cdot \left\|\varphi_\beta^k(t)\right\|_{W^{1,2}}^4 \mathrm{d}t\right)^{\frac{1}{4}} \le \operatorname*{ess\,sup}_{t\in[0,\tilde t]} \left\|\varphi_\beta^k(t)\right\|_{W^{1,2}} \left(\int_0^{\tilde t} 1 \mathrm{d}t\right)^{\frac{1}{4}} \\
&= \left\|\varphi_\beta^k\right\|_{L^\infty(0,\tilde t;W^{1,2})} \tilde t^{\frac{1}{4}} \le \sqrt{\mathcal D} \tilde t^{\frac{1}{4}},
\end{aligned}$$

which is smaller than M for a small enough $\tilde t$. $\qquad\square$

Lemma 4.4 (Compactness of $\mathcal T$)**.** *Let Assumptions A1, A2, A3 hold true. The operator $\mathcal T : K \to \mathcal T(K)$, $\bar\varphi \mapsto \varphi_\beta^k$ is compact from K to $\mathcal T(K)$.*

Proof. We calculate the stress σ via the control-to-state operator

$$\sigma^k = S_2^k(\bar\varphi)$$

and the adjoint state τ via the solution operator of the adjoint

$$\tau^k = Q_2^k(\bar\varphi, \sigma^k).$$

They are both part of the right-hand side

$$F_\beta^k(\bar\varphi) = F_\beta^k(S_2^k(\bar\varphi), Q_2^k(\bar\varphi, S_2^k(\bar\varphi)), \bar\varphi)$$

of the Allen–Cahn system $(\mathrm{AC})_\beta^k$, which is solved for φ_β^k,

$$\varphi_\beta^k = \mathcal T(F_\beta^k(\bar\varphi)).$$

We are viewing $\mathcal T$ as the concatenated solution operator

$$\mathcal T : \bar\varphi \mapsto F_\beta^k(\bar\varphi) \mapsto \varphi_\beta^k.$$

The continuity of $F_\beta^k(\bar\varphi) \mapsto \varphi_\beta^k$ is clear since the solution operator of the linear parabolic PDE

$$\partial_t \varphi_\beta^k - \gamma\epsilon\Delta\varphi_\beta^k = F_\beta^k(\bar\varphi)$$

is a continuous function according to [25, Thm. 8.35]. The control-to-state operator and solution operator of the adjoint have been shown to be continuous in Lemma 3.5 and Lemma 3.10. We will first show the continuity of $\bar\varphi \mapsto F_\beta^k(\bar\varphi)$.

Towards that we pick a sequence $\{\varphi_n\}_n \subset K \subset L^2(0,T;H^1)$, which converges strongly to $\bar\varphi$ in $L^2(0,T;H^1)$. We get the approximated solution of the elasticity equation with $\sigma_n^k = S_2^k(\varphi_n)$ and the approximated solution of the adjoint equation with $\tau_n^k = Q_2^k(\varphi_n, \sigma_n^k)$. For better readability we will write σ_n, τ_n and λ_n instead of σ_n^k, τ_n^k and λ_n^k. Our goal is to show that the sequence $\left\{F_\beta^k(\varphi_n)\right\}_n$ converges strongly in $L^2(0,T;L^2)$ to $F_\beta^k(\bar\varphi)$ with

$$
\begin{aligned}
F_\beta^k(\varphi_n) = &-(C^{-1})'(\varphi_n)\,\sigma_n : \tau_n - \lambda_n - \frac{\gamma}{\epsilon}\psi_\beta'(\varphi_n) \\
&-\alpha\int_\Omega \left[\int_{B_{r(\sigma_n)}(q)} \chi_\Omega^\varsigma(\zeta)\left(\varphi_n(\zeta) - \mu\right)d\zeta\right]_+ \chi_{B_{r(\sigma_n)}(q)}(x)\,\chi_\Omega^\varsigma(x)\,dq.
\end{aligned}
\tag{32}
$$

First, we are looking at the local volume constraint term prove pointwise convergence of $\int_{B_{r(\sigma_n)}(x)} \chi_\Omega^\varsigma(\zeta)\left(\varphi_n(\zeta) - \mu\right)d\zeta$ almost everywhere in Ω utilizing the notion of the symmetric difference of two sets A, B, i.e., $A\Delta B := (A \setminus B) \cup (B \setminus A)$:

$$
\begin{aligned}
&\left|\int_{B_{r(\sigma_n)}(x)} \chi_\Omega^\varsigma(\zeta)\left(\varphi_n(\zeta) - \mu\right)d\zeta - \int_{B_{r(\bar\sigma)}(x)} \chi_\Omega^\varsigma(\zeta)\left(\bar\varphi(\zeta) - \mu\right)d\zeta\right| \\
&= \left|\int_{B_{r(\sigma_n)}(x)} \chi_\Omega^\varsigma(\zeta)\left(\varphi_n(\zeta) - \mu\right)d\zeta - \int_{B_{r(\bar\sigma)}(x)} \chi_\Omega^\varsigma(\zeta)\left(\varphi_n(\zeta) - \mu\right)d\zeta \right.\\
&\qquad + \left.\int_{B_{r(\bar\sigma)}(x)} \chi_\Omega^\varsigma(\zeta)\left(\varphi_n(\zeta) - \mu\right)d\zeta - \int_{B_{r(\bar\sigma)}(x)} \chi_\Omega^\varsigma(\zeta)\left(\bar\varphi(\zeta) - \mu\right)d\zeta\right| \\
&\leq \int_{B_{r(\sigma_n)}(x)\Delta B_{r(\bar\sigma)}(x)} \left|\chi_\Omega^\varsigma(\zeta)\left(\varphi_n(\zeta) - \mu\right)\right|d\zeta + \int_{B_{r(\bar\sigma)}(x)} \left|\chi_\Omega^\varsigma(\zeta)(\varphi_n - \bar\varphi)(\zeta)\right|d\zeta \\
&\leq C\left(\sqrt{\frac{4\pi}{3}\left|r(\sigma_n(x))^d - r(\bar\sigma(x))^d\right|} + \|\varphi_n - \bar\varphi\|_{L^2}\right) \longrightarrow 0.
\end{aligned}
$$

The continuity of the control-to-state operator S_2^k leads to strong convergence of σ_n in $L^2(0,T;V^k)$, which implies almost everywhere pointwise convergence of a subsequence denoted in the same way via reverse Lebesgue [6, Thm. 4.9]. Since r is uniformly bounded we get almost everywhere convergence of a subsequence denoted in the same way, i.e. of $r(\sigma_n(x))$ to $r(\bar\sigma(x))$.

We note that the characteristic functions are almost everywhere convergent and that the pairwise product of pointwise convergent series is itself pointwise convergent, which proves the asserted pointwise convergence.

Additionally, we know that the characteristic functions as well as $\int_{B_{r(\sigma_n)}(x)} \chi_\Omega^\varsigma(\zeta)\left(\varphi_n(\zeta) - \mu\right)d\zeta$ are bounded, thus there exists a dominating function of the pairwise product series. Therefore, we can employ Lebesgue's dominated convergence theorem to get the strong convergence of the local volume constraint term.

The other terms of (32) converge strongly as well. Using Lebesgue's dominated convergence theorem yields strong convergence of $\left\{(C^{-1})'(\varphi_n)\eta\right\}_n$ to $(C^{-1})'(\bar\varphi)\eta$ in $L^2(0,T;V^k)$. Together with the weak convergence of σ_n in $L^2(0,T;V^k)$ we can conclude

$$\left((C^{-1})'(\varphi_n)\,\sigma_n, \eta\right)_{L^2(0,T;V^k)} \longrightarrow \left((C^{-1})'(\bar\varphi)\sigma^k, \eta\right)_{L^2(0,T;V^k)} \quad \forall\eta \in L^2(0,T;V^k).$$

Since τ_n converges weakly in $L^2(0,T;V^k)$ and in finite dimensions weak convergence equals strong convergence, the bilinear form

$$\left((C^{-1})'(\varphi_n)\,\sigma_n, \tau_n\right)_{L^2(0,T;V^k)} \longrightarrow \left((C^{-1})'(\bar\varphi)\,\sigma^k, \tau^k\right)_{L^2(0,T;V^k)}$$

converges as well.

Because $\{\varphi_n\}_n$ converges strongly to $\bar{\varphi}$ in $L^2(0, T; H^1)$, we get the existence of an almost everywhere convergent subsequence denoted in the same way, i.e.

$$\varphi_n(x, t) \longrightarrow \bar{\varphi}(x, t) \qquad \text{a.e. in } \Omega \times [0, T].$$

Via the reverse Lebesgue's theorem [6, Thm. 4.9], there exists a dominating function, namely $\hat{\varphi} \in L^2(0, T; L^2)$ with $\hat{\varphi}(x, t) \geq \varphi_n(x, t)$ almost everywhere. Via the definition of ψ'_β in (25), we find a dominating function for $\psi'_\beta(\varphi_n)$ in $L^2(0, T; L^2)$, i.e., for all $n \in \mathbb{N}$

$$\psi'_\beta(\varphi_n) \leq \frac{3}{8} + \frac{1 - 4\beta}{4\beta} \hat{\varphi} - \frac{1}{4\beta} \qquad \text{a.e. in } \Omega \times [0, T],$$

as we have seen in the proof of Lemma 4.1. Applying Lebesgue's dominated convergence theorem we get the strong convergence of $\left\{\psi'_\beta(\varphi_n)\right\}_n$ to $\psi'_\beta(\bar{\varphi})$ in $L^2(0, T; L^2)$. The Lagrange multiplier λ_n is identified with the other terms, which have already been shown to converge. Thus $\bar{\varphi} \mapsto F^k_\beta(\bar{\varphi})$ is continuous.

In order to show compactness of $\mathcal{T}$ it is left to show that $\mathcal{T}$ maps bounded sets into relatively compact sets. From maximal L^2-regularity, [10, Thm 8.2], we know that $(\mathrm{AC})^k_\beta$ has a solution

$$\varphi^k_\beta \in L^2(0, T; W^{2,2}) \cap W^{1,2}(0, T; L^2).$$

According to [15, Lemma 3.3], it holds that $L^2(0, T; W^{2,2}) \cap W^{1,2}(0, T; L^2)$ is compactly embedded in $L^2(0, T; W^{1,2})$.

Let $\{\varphi_n\}_n$ be a bounded series in $L^2(0, T; W^{2,2}) \cap W^{1,2}(0, T; L^2)$. By the compact embedding, it holds that there exists a convergent subsequence denoted in the same way

$$\{\varphi_n\}_n \subset L^2(0, T; W^{1,2}) \quad \text{with} \quad \varphi_n \to \underline{\varphi}.$$

Using the energy estimate (31), we know that $\{\varphi_n\}_n$ is also bounded in $L^\infty(0, T; W^{1,2})$. Together with the strong convergence of $\{\varphi_n\}_n$ to $\underline{\varphi}$ in $L^2(0, T; W^{1,2})$ and

$$\left\|\varphi_n - \underline{\varphi}\right\|_{L^4(0,T;W^{1,2})} \leq \left\|\varphi_n - \underline{\varphi}\right\|_{L^2(0,T;W^{1,2})}^{\frac{1}{2}} \left\|\varphi_n - \underline{\varphi}\right\|_{L^\infty(0,T;W^{1,2})}^{\frac{1}{2}}$$

we get the strong convergence of $\{\varphi_n\}_n$ to $\underline{\varphi}$ in $L^4(0, T; W^{1,2})$.

We have shown that for $\mathcal{T} : K \to \mathcal{T}(K)$ the set $\mathcal{T}(K)$ is relatively compact. For any bounded set $B \subset K$ it holds that $\overline{\mathcal{T}(B)} \subset \overline{\mathcal{T}(K)}$ is a closed subset of a compact set and therefore compact, meaning that $\mathcal{T}(B)$ is relatively compact. $\square$

The results are summarized in the following proposition:

Proposition 4.5. *Under Assumptions A1, A2 and A3, the linear parabolic equation* (15) *has a fixed point, i.e. the regularized Allen–Cahn system* $(\mathrm{AC})_\beta$ *has at least one solution.*

Proof. Since the conditions are met via Lemma 4.2, Lemma 4.3 and Lemma 4.4, we can apply the Schauder fixed point theorem [6, Ex. 6.26] $\square$

Lemma 4.6. *Under Assumptions A1, A2 and A3, there exists a global solution to the approximated and regularized Allen–Cahn initial value problem* (29).

Proof. We have shown the existence of a solution φ^k_β to (29) via Proposition 4.5 on a small time interval $\left[0, \tilde{t}\right]$. Now we need an *a priori* estimate to extend this to the global interval $[0, T]$. We define $\mathcal{H}$ via $\mathcal{H}^k(\varphi) = \mathcal{G}^c(\varphi, S^k_2(\varphi)) + \frac{\alpha}{2} V(r(S^k_2(\varphi)), \varphi)$ and note that $\mathcal{H}^k \geq 0$. The continuity of $\mathcal{H}^k$ is clear because of Lemma 3.5 and its differentiability follows from Lemma 3.6. As in the proof of Lemma 4.3, we test the system (29) with $\partial_t \varphi^k_\beta$. We can write

$$\int_\Omega \partial_t \varphi^k_\beta \partial_t \varphi^k_\beta \mathrm{d}x + \gamma\epsilon \int_\Omega \nabla \varphi^k_\beta \nabla(\partial_t \varphi^k_\beta) \mathrm{d}x + \frac{\gamma}{\epsilon} \int_\Omega \psi'_\beta(\varphi^k_\beta) \partial_t \varphi^k_\beta \mathrm{d}x + \frac{\partial \mathcal{H}^k(\varphi^k_\beta)}{\partial \varphi^k_\beta} \partial_t \varphi^k_\beta = 0$$

$$\Leftrightarrow \qquad \left\|\partial_t \varphi^k_\beta\right\|_{L^2}^2 + \frac{\gamma\epsilon}{2} \int_\Omega \frac{d}{dt} |\nabla \varphi^k_\beta(t)|^2 \mathrm{d}x + \frac{\gamma}{\epsilon} \int_\Omega \frac{d}{dt} \psi_\beta(\varphi^k_\beta(t)) \mathrm{d}x + \frac{d}{dt} \mathcal{H}^k(\varphi^k_\beta(t)) = 0.$$

We integrate over $[0, t]$ such that for all $t \in [0, T]$ it holds

$$\left[\frac{\gamma \epsilon}{2} \left\| \nabla \varphi_\beta^k(s) \right\|_{L^2}^2 + \frac{\gamma}{\epsilon} \int_\Omega \psi_\beta(\varphi_\beta^k(s)) \mathrm{d}x + \mathcal{H}^k(\varphi_\beta^k(s)) \right]_0^t + \int_0^t \left\| \partial_t \varphi_\beta^k(s) \right\|_{L^2}^2 \mathrm{d}s = 0. \tag{33}$$

The terms in the square brackets are finite since $\varphi_\beta^k(0)$ is admissible, thus $\partial_t \varphi_\beta^k \in L^2(0, t; L^2)$. Via a bootstrapping argument as in [28, Problem 30.2, p. 799], we can extend the local solution to a global solution of (29) on $[0, T]$. □

4.3 Existence - Regularized Allen–Cahn System

We now know that a global solution exists for every Galerkin approximation. The next step is to go to the Galerkin limit and show that the existence also holds in that case.

Proposition 4.7. *Let Assumptions A1, A2, A3 hold true. For each $0 < \beta < \frac{1}{4}$ the Allen–Cahn system with the smoothed potential, $(\mathrm{AC})_\beta$, has a solution φ_β in $L^2(0, T; W^{1,2}) \cap W^{1,2}(0, T; L^2)$.*

Proof. We know that the $\left\{ \varphi_\beta^k \right\}_k$ is bounded in $L^2(0, T; H^1)$ and that $\{\sigma\}_k, \{\tau\}_k$ are subsets of $L^2(0, T; V^k)$ and $\{u\}_k, \{p\}_k$ are subsets of $L^2(0, T; W^k)$. Previously, we have shown the existence of solutions $\left\{ \varphi_\beta^k \right\}_k$ for the Galerkin approximated, regularized Allen–Cahn system $(\mathrm{AC})_\beta^k$. According to Equation (33) there exists a $c > 0$ independent of k and β with

$$\left\| \varphi_\beta^k \right\|_{L^\infty(0,T;W^{1,2}) \cap W^{1,2}(0,T;L^2)} < c.$$

This leads to a weakly convergent subsequence also denoted by $\left\{ \varphi_\beta^k \right\}_k$ with

$$\varphi_\beta^k \rightharpoonup \varphi_\beta \quad \text{in } L^2(0, T; W^{1,2}) \cap W^{1,2}(0, T; L^2).$$

Furthermore, since the space $L^\infty(0, T; W^{1,2}) \cap W^{1,2}(0, T; L^2)$ is compactly embedded in $L^2(0, T; L^2)$, we get a strongly convergent subsequence denoted in the same way with $\varphi_\beta^k \to \varphi_\beta$ in $L^2(0, T; L^2)$. The goal is to show that φ_β is solving the regularized Allen–Cahn system $(\mathrm{AC})_\beta$. Towards that we prove the convergence of the different terms in the $(\mathrm{AC})_\beta^k$ system as k tends to infinity.

From the weak convergence of $\left\{ \varphi_\beta^k \right\}_k$ to φ_β in $W^{1,2}(0, T; L^2)$ it follows that $\left\{ \partial_t \varphi_\beta^k \right\}_k$ converges weakly to $\partial_t \varphi_\beta$ in $L^2(0, T; L^2)$. This already shows that

$$|(\partial_t \varphi_\beta^k, \omega) - (\partial_t \varphi_\beta, \omega)| = |(\partial_t \varphi_\beta^k - \partial_t \varphi_\beta, \omega)|$$

converges to zero. Similarly, from weak $L^2(0, T; W^{1,2})$ convergence we get that

$$|(\nabla \varphi_\beta^k, \nabla \omega) - (\nabla \varphi_\beta, \nabla \omega)| = |(\nabla \varphi_\beta^k - \nabla \varphi_\beta, \nabla \omega)|$$

converges to zero as well for all $\omega \in L^2(0, T; W^{1,2})$. To show convergence of the potential part we rewrite $\psi_\beta'(\varphi)$ as $\frac{1}{2} - \varphi - (\psi_\beta^c)'(\varphi)$, see (25), and apply the Lipschitz continuity of $(\psi_\beta^c)'$ towards

$$\left\| \psi_\beta' \left(\varphi_\beta^k \right) - \psi_\beta' \left(\varphi_\beta \right) \right\|_{L^2} \leq \left\| \varphi_\beta^k - \varphi_\beta \right\|_{L^2} + \frac{1}{4\beta} \left\| \varphi_\beta^k - \varphi_\beta \right\|_{L^2},$$

which converges to zero due to the strong convergence of $\left\{ \varphi_\beta^k \right\}_k$ to φ_β in $L^2(0, T; L^2)$.

It is left to show the convergence of the terms on the right-hand side. The sequence of stresses $\left\{ \sigma^k \right\}_k$ converges strongly to σ in $L^2(0, T; L^2)$ according to the continuity of the control-to-state operator proven in Lemma 3.5, since $\left\{ \varphi_\beta^k \right\}_k$ converges strongly in $L^2(0, T; L^2)$. We also get weak convergence of the adjoints $\left\{ \tau^k \right\}_k$ to τ in $L^2(0, T; L^2)$ via the continuity with respect to the weak topology of the solution operator of the adjoint proven in Lemma 3.10 using the weak converges of $\left\{ \varphi_\beta^k \right\}_k$ in $L^2(0, T; H^1)$. The convergence

$$\int_\Omega (C^{-1})'(\varphi_\beta^k) \omega \sigma^k : \tau^k \mathrm{d}x \longrightarrow \int_\Omega (C^{-1})'(\varphi_\beta) \omega \sigma : \tau \mathrm{d}x \tag{34}$$

holds for all $\omega \in L^\infty(\Omega \times (0,T))$, since $(C^{-1})'(\varphi_\beta^k)$ is bounded and since the sequences $\{\sigma\}k$ converge strongly in $L^2(\Omega \times (0,T))$, respectively, we infer a dominating function for $\{(C^{-1})'(\varphi_\beta^k)\sigma^k\}$ via the reverse Lebesgue's theorem [6, Thm. 4.9]. Thus, the sequence converges pointwise and Lebesgue's theorem we get the strong convergence in $L^2(\Omega \times (0,T))$. The weak convergence of the sequence $\{\tau^k\}$ allows to pass to the limit in (34).

The local volume constraint term

$$\left\{ \int_\Omega \left[\int_{B_{r(\sigma^k)}(x)} \chi_\Omega^\varsigma(\zeta) \left(\varphi^k(\zeta) - \mu \right) \mathrm{d}\zeta \right]_+ \chi_{B_{r(\sigma^k)}}(q)(x)\, \chi_\Omega^\varsigma(x)\, \mathrm{d}q \right\}_k$$

converges via Lebesgue, making use of the fact that boundedness was shown in the proof of Lemma 4.4 and the characteristic functions are bounded by 1. Note that λ^k and λ_β are identified via terms for which we have already shown their respective convergences. All the terms converge, thus φ_β does fulfill the Allen–Cahn system with the smoothed potential.

To show the *a priori* estimates for φ_β we are passing to the limit with $k \to \infty$ in

$$\left[\frac{\gamma\epsilon}{2}\left\|\nabla\varphi_\beta^k(s)\right\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi_\beta(\varphi_\beta^k(s))\mathrm{d}x + \mathcal{H}^k(\varphi_\beta^k(s)) \right]\Big|_0^t + \int_0^t \left\|\partial_t\varphi_\beta^k(s)\right\|_{L^2}^2 \mathrm{d}s \leq 0 \tag{35}$$

for almost all $t \in (0,T)$. However, we note that via the *a priori* estimates we only receive weak convergence in time and not the almost everywhere pointwise convergence that would seemingly be necessary here. We get around this issue by taking the essential supremum over $t \in (0,T)$, leading to

$$\operatorname*{ess\,sup}_{t \in [0,T]} \left[\frac{\gamma\epsilon}{2}\left\|\nabla\varphi_\beta^k(t)\right\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi_\beta(\varphi_\beta^k(t))\mathrm{d}x + \mathcal{H}^k(\varphi_\beta^k(t)) \right]$$
$$\leq \left[\frac{\gamma\epsilon}{2}\left\|\nabla\varphi_\beta^k(0)\right\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi_\beta(\varphi_\beta^k(0))\mathrm{d}x + \mathcal{H}^k(\varphi_\beta^k(0)) \right],$$

which implies

$$\operatorname*{ess\,sup}_{t \in [0,T]}\left\|\nabla\varphi_\beta^k(t)\right\|_{L^2}^2 + \operatorname*{ess\,sup}_{t \in [0,T]}\int_\Omega \psi_\beta(\varphi_\beta^k(t))\mathrm{d}x + \operatorname*{ess\,sup}_{t \in [0,T]}\mathcal{H}^k(\varphi_\beta^k(t)) < c,$$

with c independent of β, thus $\varphi_\beta^k \in L^\infty(0,T;H^1)$. Then, according to [17, Lemma 2.4], the inequality (35) is equivalent to

$$-\int_0^T \phi'(t)\left[\frac{\gamma\epsilon}{2}\left\|\nabla\varphi_\beta^k(t)\right\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi_\beta(\varphi_\beta^k(t))\mathrm{d}x + \mathcal{H}^k(\varphi_\beta^k(t)) \right]\mathrm{d}t$$
$$-\phi(0)\left[\frac{\gamma\epsilon}{2}\left\|\nabla\varphi_\beta^k(0)\right\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi_\beta(\varphi_\beta^k(0))\mathrm{d}x + \mathcal{H}^k(\varphi_\beta^k(0)) \right] + \int_0^T \phi(t)\left\|\partial_t\varphi_\beta^k(t)\right\|_{L^2}^2 \mathrm{d}t \leq 0$$

for all $\phi \in C^1([0,T])$ with $\phi(T) = 0$, $\phi \geq 0$, and $\phi' \leq 0$ on $[0,T]$. Since the first term in the square brackets is convex, we get its weak lower semi-continuity, see for example [12, Theorem 10.20]. The second and third term in the square brackets converge since ψ_β is continuous and $\left\{\varphi_\beta^k\right\}_k$ converges strongly in $L^2(0,T;L^2)$ together with Lemma 3.5. Therefore, defining $\mathcal{H}(\varphi) = \mathcal{G}^c(\varphi, S_2(\varphi)) + \frac{\alpha}{2}V(r(S_2(\varphi)), \varphi)$ it also holds that

$$-\int_0^T \phi'(t)\left[\frac{\gamma\epsilon}{2}\left\|\nabla\varphi_\beta(t)\right\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi_\beta(\varphi_\beta(t))\mathrm{d}x + \mathcal{H}(\varphi_\beta(t)) \right]\mathrm{d}t$$
$$-\phi(0)\left[\frac{\gamma\epsilon}{2}\left\|\nabla\varphi_\beta(0)\right\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi_\beta(\varphi_\beta(0))\mathrm{d}x + \mathcal{H}(\varphi_\beta(0)) \right] + \int_0^T \phi(t)\left\|\partial_t\varphi_\beta(t)\right\|_{L^2}^2 \mathrm{d}t \leq 0,$$

which is again equivalent to

$$\left[\frac{\gamma\epsilon}{2}\left\|\nabla\varphi_\beta(s)\right\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi_\beta(\varphi_\beta(s))\mathrm{d}x + \mathcal{H}(\varphi_\beta(s)) \right]\Big|_0^t + \int_0^t \left\|\partial_t\varphi_\beta(s)\right\|_{L^2}^2 \mathrm{d}s \leq 0$$

for almost all $t \in (0,T)$. $\qquad\square$

4.4 Existence - Allen–Cahn System

The goal of this section is to extend the existence result from the Allen–Cahn system with a regularized potential ψ_β to the original Allen–Cahn system with a double obstacle potential ψ. This is done by taking the β limit, which proves the main result.

Proof of Theorem 2.2. Since the Hellinger–Reissner elasticity system is analytically equivalent to the pure displacement ansatz, we get better regularity via the work [14] of Herzog et al. Specifically via [14, Prop. 1.2], we get that for a $c > 0$, which is independent of β and t, there exists a $2 < p < 3$ such that

$$\left\| \sigma(t) \right\|_{L^p(\Omega, \mathbb{S}^d)} \leq c$$

for almost all $t \in (0, T)$ and therefore

$$\|\sigma\|_{L^\infty(0,T;L^p(\Omega,\mathbb{S}^d))} < \infty.$$

We do not need to prove anything for τ. The higher regularity of $\sigma(t) \in L^p$ together with $\tau(t) \in L^2$ is enough to show that the right-hand side is regular enough: Noting $\frac{1}{p} + \frac{1}{2} = \frac{p+2}{2p}$ we get via Hölder's inequality

$$(C^{-1})'(\varphi_\beta(t))\sigma(t) : \tau(t) \in L^{\frac{2p}{p+2}}(\Omega, \mathbb{R}).$$

The next trick is a way to get this higher regularity onto the other terms, which was also done in [18]. We notice that there are two potential terms, one in the right-hand side F_β, see (27) and a second one in the identification of λ_β, c.f. (28). In order to prove higher regularity of $(\psi_\beta)' - \int_\Omega (\psi_\beta)' \mathrm{d}y$, remember that we have split the approximated obstacle potential into a quadratic and a convex part in (25). The derivative of the convex parts is brought to the left-hand side, whereas the derivative of the quadratic parts will stay in the right-hand side. Also, since $\partial_t \varphi_\beta \in L^2(0, T; L^2) \subset L^{\frac{2p}{p+2}}(0, T; L^2)$, we can move that term to the right-hand side as well. The argument t is not written explicitly for better readability, but the formulae hold for almost all $t \in (0, T)$. Together with the other terms, we get

$$
\begin{aligned}
\tilde{F}_\beta :=\ & F_\beta - \partial_t \varphi_\beta + \frac{\gamma}{\epsilon}(\psi_\beta^c)'(\varphi_\beta) - \frac{\gamma}{\epsilon|\Omega|} \int_\Omega (\psi_\beta^c)'(\varphi_\beta)\ \mathrm{d}x \\
=\ & -(C^{-1})'(\varphi_\beta)\,\sigma : \tau + \frac{1}{|\Omega|} \int_\Omega (C^{-1})'(\varphi_\beta)\,\sigma : \tau\ \mathrm{d}x - \partial_t \varphi_\beta \\
& + \frac{\alpha}{|\Omega|} \int_\Omega \int_\Omega \left[\fint_{B_{r(\sigma)}(q)} \chi_\Omega^\varsigma(\zeta)\,(\varphi_\beta(\zeta) - \mu)\ \mathsf{d}\zeta \right]_+ \chi_{B_{r(\sigma)}(q)}(x)\,\chi_\Omega^\varsigma(x)\ \mathrm{d}q\ \mathrm{d}x \\
& - \alpha \int_\Omega \left[\fint_{B_{r(\sigma)}(q)} \chi_\Omega^\varsigma(\zeta)\,(\varphi_\beta(\zeta) - \mu)\ \mathsf{d}\zeta \right]_+ \chi_{B_{r(\sigma)}(q)}(x)\,\chi_\Omega^\varsigma(x)\ \mathrm{d}q \\
& - \frac{\gamma}{\epsilon}\left(\frac{1}{2} - \varphi_\beta\right) + \frac{\gamma}{\epsilon|\Omega|} \int_\Omega \left(\frac{1}{2} - \varphi_\beta\right) \mathrm{d}x\ \in\ L^{\frac{2p}{p+2}},
\end{aligned}
$$

where the last equality follows from (25). We have

$$-\gamma\epsilon\Delta\varphi_\beta + \frac{\gamma}{\epsilon}f(\varphi_\beta) = \tilde{F}_\beta, \qquad \text{with } f(\varphi_\beta) = \left[(\psi_\beta^c)'(\varphi_\beta) - \frac{1}{|\Omega|}\int_\Omega (\psi_\beta^c)'(\varphi_\beta)\mathrm{d}y\right]$$

which we test by $\left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}} f(\varphi_\beta)$ for $p > 2$, leading to

$$-\gamma\epsilon \int_\Omega \Delta\varphi_\beta \left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}} f(\varphi_\beta)\mathrm{d}x\ \ + \frac{\gamma}{\epsilon}\int_\Omega \left|f(\varphi_\beta)\right|^{\frac{2p}{p+2}}\ \mathrm{d}x = \int_\Omega \tilde{F}_\beta\ \left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}} f(\varphi_\beta)\mathrm{d}x. \tag{36}$$

We are looking at the first term on the left-hand side. Due to $\nabla f = (\psi_\beta^c)''\nabla\psi_\beta^c$ it holds that

$$
\begin{aligned}
\nabla\left[\left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}}\right] &= \nabla\left[\left(\left|f(\varphi_\beta)\right|^2\right)^{-\frac{2}{p+2}}\right] = -\frac{2}{p+2}\left(\left|f(\varphi_\beta)\right|^2\right)^{-\frac{p+4}{p+2}} 2f(\varphi_\beta)(\psi_\beta^c)''\nabla\varphi_\beta \\
&= -\frac{4}{p+2}\left(\left|f(\varphi_\beta)\right|\right)^{-\frac{2p+8}{p+2}} f(\varphi_\beta)(\psi_\beta^c)''\nabla\varphi_\beta
\end{aligned}
$$

and via the chain rule

$$\nabla\left[\left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}} f(\varphi_\beta)\right] = -\frac{4}{p+2}\left(\left|f(\varphi_\beta)\right|\right)^{-\frac{4}{p+2}}(\psi_\beta^c)''\nabla\varphi_\beta + \left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}}(\psi_\beta^c)''\nabla\varphi_\beta$$

$$= \frac{p-2}{p+2}\left(\left|f(\varphi_\beta)\right|\right)^{-\frac{4}{p+2}}(\psi_\beta^c)''\nabla\varphi_\beta\,.$$

We find via multidimensional partial integration-by-parts that

$$-\gamma\epsilon\int_\Omega \nabla\cdot\nabla\varphi_\beta\left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}} f(\varphi_\beta)\mathrm{d}x = -\gamma\epsilon\underbrace{\int_{\partial\Omega}\nabla\varphi_\beta\cdot\mathbf{n}\left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}} f(\varphi_\beta)\mathrm{d}s}_{=0}$$

$$+\gamma\epsilon\int_\Omega \nabla\varphi_\beta\nabla\left[\left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}} f(\varphi_\beta)\right]\mathrm{d}x$$

$$= \gamma\epsilon\int_\Omega \frac{p-2}{p+2}\left(\left|f(\varphi_\beta)\right|\right)^{-\frac{4}{p+2}}(\psi_\beta^c)''|\nabla\varphi_\beta|^2\mathrm{d}x \geq 0\,.$$

The last inequality follows from $\gamma,\epsilon,\frac{p-2}{p+2} > 0$ and, by convexity, $(\psi_\beta^c)''(\varphi_\beta) \geq 0$. Therefore, if we drop the first term of Equation (36) we get the inequality

$$\frac{\gamma}{\epsilon}\int_\Omega\left|f(\varphi_\beta)\right|^{\frac{2p}{p+2}}\mathrm{d}x \leq \int_\Omega \tilde{F}_\beta\left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}} f(\varphi_\beta)\mathrm{d}x$$

On the right-hand side we use Hölder's and Young's inequalities to obtain

$$\int_\Omega \tilde{F}_\beta\left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}} f(\varphi_\beta)\mathrm{d}x \leq \left(\int_\Omega\tilde{F}_\beta^{\frac{2p}{p+2}}\mathrm{d}x\right)^{\frac{p+2}{2p}}\left(\int_\Omega\left|f(\varphi_\beta)\right|^{-\frac{4}{p+2}\frac{2p}{p-2}+\frac{2p}{p-2}}\mathrm{d}x\right)^{\frac{p-2}{2p}}$$

$$= \left(\int_\Omega\tilde{F}_\beta^{\frac{2p}{p+2}}\mathrm{d}x\right)^{\frac{p+2}{2p}}\left(\int_\Omega\left|f(\varphi_\beta)\right|^{\frac{2p}{p+2}}\mathrm{d}x\right)^{\frac{p-2}{2p}}$$

$$\leq C\left\|\tilde{F}_\beta\right\|_{\frac{2p}{p+2}}^{\frac{2p}{p+2}} + \frac{\gamma}{2\varepsilon}\int_\Omega\left|f(\varphi_\beta)\right|^{\frac{2p}{p+2}}\mathrm{d}x,$$

which shows by integrating in time

$$\left\|f(\varphi_\beta)\right\|_{L^{\frac{2p}{p+2}}(\Omega\times(0,T))} \leq C\left\|\tilde{F}_\beta\right\|_{L^{\frac{2p}{p+2}}(\Omega\times(0,T))}\,.$$

Thus, we know that $\left[(\psi_\beta^c)' - \frac{1}{|\Omega|}\int_\Omega(\psi_\beta^c)'\mathrm{d}y\right]$ is in $L^{\frac{2p}{p+2}}(\Omega\times(0,T))$, since $\tilde{F}_\beta$ is in $L^{\frac{2p}{p+2}}(\Omega\times(0,T))$.

We see that the derivatives of the convex parts of the smoothed potentials are bounded in a reflexive space and therefore there exists a weakly convergent subsequence

$$\{\xi_\beta\}_\beta = \left\{\left[(\psi_\beta^c)'(\varphi_\beta) - \frac{1}{|\Omega|}\int_\Omega(\psi_\beta^c)'(\varphi_\beta)\mathrm{d}y\right]\right\}_\beta \quad \text{with} \quad \xi_\beta \rightharpoonup \xi \quad \text{in} \quad L^{\frac{2p}{2+p}}(\Omega\times[0,T]).$$

Notice that $\int_\Omega\xi_\beta\mathrm{d}x = 0$ for all β. We defined $\tilde{F}_\beta$ just to make it easier to show the higher regularity. Now we want to bring the time derivative and keep the terms containing the derivative of the convex part of the potential on the left-hand side as well. Therefore, we define

$$\hat{F}_\beta := F_\beta + \frac{\gamma}{\epsilon}(\psi_\beta^c)'(\varphi_\beta) - \frac{\gamma}{\epsilon|\Omega|}\int_\Omega(\psi_\beta^c)'(\varphi_\beta)\,\mathrm{d}x = \tilde{F}_\beta + \partial_t\varphi_\beta.$$

In the following formulation all scalar products have to understood as dual pairings of $L^{\frac{2p}{2+p}}(\Omega\times[0,T])$ and its dual $L^{\frac{2p}{p-2}}(\Omega\times[0,T])$ given by the integral over $\Omega\times(0,T)$.

$$(\partial_t\varphi_\beta,\omega) + \gamma\epsilon(\nabla\varphi_\beta,\nabla\omega) + \frac{\gamma}{\epsilon}(\xi_\beta,\omega) = (\hat{F}_\beta(\varphi_\beta),\omega) \quad \forall\omega\in L^{\frac{2p}{p-2}}(\Omega\times[0,T]).$$

We know that $\partial_t\varphi_\beta\rightharpoonup\partial_t\varphi$, $\nabla\varphi_\beta\rightharpoonup\nabla\varphi$ and $\hat{F}_\beta(\varphi_\beta)\rightharpoonup F(\varphi)$ as $\beta\to 0$ in $L^{\frac{2p}{2+p}}(0,T;L^{\frac{2p}{2+p}})$. When taking the limit $\beta\to 0$, we get

$$(\partial_t\varphi,\omega) + \gamma\epsilon(\nabla\varphi,\nabla\omega) + \frac{\gamma}{\epsilon}(\xi,\omega) = (F(\varphi),\omega) \quad \forall\omega\in L^{\frac{2p}{p-2}}(\Omega\times[0,T]).$$

To show the *a priori* estimates for φ, we are passing to the limit with $\beta \to 0$ in

$$\left[\frac{\gamma\epsilon}{2}\big\|\nabla\varphi_\beta(s)\big\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi_\beta(\varphi_\beta(s))\mathrm{d}x + \mathcal{H}(\varphi_\beta(s))\right]\bigg|_0^t + \int_0^t \big\|\partial_t\varphi_\beta\big\|_{L^2}^2 \,\mathrm{d}s \le 0.$$

First, we will keep ψ_β fixed and just consider the limit $\varphi_\beta \longrightarrow \varphi$. From the usual embedding we get strong convergence in $L^2(0, T; L^2)$ and therefore there exists an almost everywhere convergent subsequence of $\{\varphi_\beta\}_\beta$ denoted in the same way and $\{\psi_\beta(\varphi_\beta)\}$ converges to $\psi_\beta(\varphi)$ almost everywhere. Since ψ is a dominating function for any ψ_β, we can apply Lebesgue's dominated convergence theorem to receive strong convergence of $\{\psi_\beta(\varphi_\beta)\}$ to $\psi_\beta(\varphi)$ in $L^2(0, T; L^2)$. Similar to the arguments used in the proof of Proposition 4.7, we arrive at

$$\left[\frac{\gamma\epsilon}{2}\big\|\nabla\varphi(s)\big\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi_\beta(\varphi(s))\mathrm{d}x + \mathcal{H}(\varphi(s))\right]\bigg|_0^t + \int_0^t \big\|\partial_t\varphi\big\|_{L^2}^2 \,\mathrm{d}s \le 0.$$

This shows that

$$\int_\Omega \psi_\beta(\varphi)\mathrm{d}x \le C \quad \forall\beta > 0.$$

We observe that the sequence of functions $\beta\psi_\beta^c$ converges to $\bar\psi$ in $C(\mathbb{R})$, where

$$\bar\psi(\varphi) := \begin{cases} (\varphi - 1)^2 & \text{if } \varphi > 1 \\ 0 & \text{if } 0 \leqslant \varphi \leqslant 1 \\ \varphi^2 & \text{if } \varphi < 0 \end{cases}.$$

From the estimate $\int_\Omega \beta\psi_\beta^c(\varphi_\beta)\mathrm{d}x \le C\beta$ and the strong convergence of φ_β, we infer in the limit that $\int_\Omega \bar\psi(\varphi)\mathrm{d}x = 0$, which implies that $\varphi \in [0, 1]$ a.e. in $\Omega \times [0, T]$ and $\psi_\beta^c(\varphi) = \psi^c(\varphi) = 0$. This allows us to replace ψ_β by ψ in the above inequality such that

$$\left[\frac{\gamma\epsilon}{2}\big\|\nabla\varphi(s)\big\|_{L^2}^2 + \frac{\gamma}{\epsilon}\int_\Omega \psi(\varphi(s))\mathrm{d}x + \mathcal{H}(\varphi(s))\right]\bigg|_0^t + \int_0^t \big\|\partial_t\varphi\big\|_{L^2}^2 \,\mathrm{d}s \le 0,$$

which implies additionally that $\varphi \in H^1\left(0, T; L^2\right) \cap L^2\left(0, T; H^1\right)$.

From the convexity of ψ_β^c we get the pointwise inequality

$$\psi_{\beta*}^c(a) - \psi_\beta^c(b) \le \psi_\beta^c(a) - \psi_\beta^c(b) \le (\psi_\beta^c)'(a)(a - b)$$

for $a \in \mathbb{R}$, $b \in [0, 1]$ and a fixed β^* with $\beta \le \beta^* < \frac{1}{4}$. Therefore, we have

$$\psi_{\beta*}^c(\varphi_\beta(x)) - \psi_\beta^c(\bar\varphi(x)) \le (\psi_\beta^c)'(\varphi_\beta(x))(\varphi_\beta(x) - \bar\varphi(x)) \tag{37}$$

for all $\bar\varphi(x) \in [0, 1]$. Let $\varepsilon > 0$. We pick test functions that are continuous on a closed interval inside of $[0, 1]$ and fulfill the volume constraint with mean m, i.e.

$$\bar\varphi \in C(\bar\Omega, [\varepsilon, 1 - \varepsilon]) \quad \text{with} \quad \int_\Omega \bar\varphi\mathrm{d}x = m.$$

Then we define the modified, β-dependent test function

$$\bar\varphi_\beta := \bar\varphi - g(\varphi_\beta)\int_\Omega \frac{\varphi_\beta - \bar\varphi}{g(\varphi_\beta)}\mathrm{d}x,$$

where

$$g(\varphi) := \sqrt{8\beta\psi_\beta^c(\varphi) + 1}.$$

Notice that for φ with $0 \le \varphi(x) \le 1$ almost everywhere in Ω, it holds that $g(\varphi) \equiv 1$. We claim that there exists a $C > 0$ such that

$$\frac{\big|\varphi_\beta(x)\big| + \big|\bar\varphi(x)\big|}{\big|g(\varphi_\beta(x))\big|} \le C.$$

For the second summand, boundedness is clear since $\bar{\varphi}$ is fixed and $\left|g(\varphi_\beta(x))\right|$ is bounded from below by 1. For the first summand, if φ_β becomes large, then (*cf.* (25)

$$g(\varphi_\beta) \leq \sqrt{\left(\varphi_\beta - \left(1 + \frac{\beta}{2}\right)\right)^2 + \frac{\beta}{96}} + 1,$$

which is bounded by an affine function in φ_β, proving the claim. We insert $\bar{\varphi}_\beta$ into (37), divide both sides by the continuous function $g(\varphi_\beta)$ and integrate over Ω

$$\int_\Omega \frac{\psi_{\beta*}(\varphi_\beta) - \psi_\beta(\bar{\varphi}_\beta)}{g(\varphi_\beta)} \mathrm{d}y \leq \int_\Omega (\psi_\beta^c)'(\varphi_\beta) \frac{(\varphi_\beta - \bar{\varphi}_\beta)}{g(\varphi_\beta)} \mathrm{d}y$$

$$\leq \int_\Omega \underbrace{\left((\psi_\beta^c)'(\varphi_\beta) - \int_\Omega (\psi_\beta^c)'(\varphi_\beta)\mathrm{d}x\right)}_{=\xi_\beta} \frac{(\varphi_\beta - \bar{\varphi}_\beta)}{g(\varphi_\beta)} \mathrm{d}y$$

$$+ \int_\Omega \int_\Omega (\psi_\beta^c)'(\varphi_\beta)\mathrm{d}x \frac{(\varphi_\beta - \bar{\varphi}_\beta)}{g(\varphi_\beta)} \mathrm{d}y \quad \forall \bar{\varphi},$$

where a zero was added in the last step. We will first show that the last term is zero. Entering the definition of the modified test function $\bar{\varphi}_\beta$ we find

$$\int_\Omega \int_\Omega (\psi_\beta^c)'(\varphi_\beta)\mathrm{d}x \frac{(\varphi_\beta - \bar{\varphi}_\beta)}{g(\varphi_\beta)} \mathrm{d}y = \int_\Omega \int_\Omega (\psi_\beta^c)'(\varphi_\beta)\mathrm{d}x \frac{\left(\varphi_\beta - \bar{\varphi} - g(\varphi_\beta) \int_\Omega \frac{\varphi_\beta - \bar{\varphi}}{g(\varphi_\beta)}\mathrm{d}x\right)}{g(\varphi_\beta)} \mathrm{d}y$$

$$= \int_\Omega (\psi_\beta^c)'(\varphi_\beta)\mathrm{d}x \left[\int_\Omega \frac{\varphi_\beta - \bar{\varphi}}{g(\varphi_\beta)}\mathrm{d}y - \int_\Omega \frac{\varphi_\beta - \bar{\varphi}}{g(\varphi_\beta)}\mathrm{d}x\right] = 0$$

and conclude

$$\int_\Omega \frac{\psi_{\beta*}(\varphi_\beta) - \psi_\beta(\bar{\varphi}_\beta)}{g(\varphi_\beta)} \mathrm{d}y \leq \int_\Omega \xi_\beta \frac{(\varphi_\beta - \bar{\varphi}_\beta)}{g(\varphi_\beta)} \mathrm{d}y. \tag{38}$$

We may now pass to the limit $\beta \to 0$. First, we will look at the right-hand side of (38): The aforementioned subsequence ξ_β converges weakly to ξ in $L^{\frac{2p}{2+p}}(\Omega \times [0,T])$. We know from strong convergence of $\{\varphi_\beta\}_\beta$ to φ in $L^2(0,T;L^2)$ that there exists a subsequence, denoted in the same way, which converges almost everywhere in $\Omega \times [0,T]$. Noting that g is continuous and $\bar{\varphi}_\beta$ is made up of φ_β, $g(\varphi_\beta)$ and the fixed function $\bar{\varphi}$, we see that $(\varphi_\beta(x) - \bar{\varphi}_\beta(x))\backslash g(\varphi_\beta(x))$ is pointwise convergent almost everywhere. We have

$$\left|\bar{\varphi}_\beta - \bar{\varphi}\right| = \left|g(\varphi_\beta) \int_\Omega \frac{\varphi_\beta - \bar{\varphi}}{g(\varphi_\beta)}\mathrm{d}x\right|$$

$$= \left|g(\varphi_\beta) \int_\Omega (\varphi_\beta - \bar{\varphi}) \left(\frac{1 - g(\varphi_\beta)}{g(\varphi_\beta)}\right)\mathrm{d}x\right|$$

$$\leq C \left|g(\varphi_\beta)\right| \int_\Omega \left|1 - g(\varphi_\beta)\right| \mathrm{d}x$$

$$\leq \sqrt{8}C \left|\Omega\right|^{\frac{1}{2}} \sqrt{\beta} \left|g(\varphi_\beta)\right| \left(\int_\Omega \psi_\beta^c(\varphi_\beta)\mathrm{d}x\right)^{\frac{1}{2}}$$

$$=: \tilde{C}\sqrt{\beta} \left|g(\varphi_\beta)\right| \quad \text{with} \quad \tilde{C} > 0.$$

It was used that $\int_\Omega (\varphi_\beta - \bar{\varphi}) \cdot 1 \mathrm{d}x = 0$ since φ_β and $\bar{\varphi}$ have the same mean value. We applied the definition of $g(\varphi_\beta)$ and Hölder's inequality in the second to last inequality. The term $\left(\int_\Omega \psi_\beta^c(\varphi_\beta)\mathrm{d}x\right)^{\frac{1}{2}}$ is bounded because of (35). The calculation above implies

$$\left\|\frac{\bar{\varphi}_\beta - \bar{\varphi}}{g(\varphi_\beta)}\right\|_{L^\infty([0,T]\times\Omega)} \leq \tilde{C}\sqrt{\beta}. \tag{39}$$

Additionally, we receive the boundedness

$$\left|\frac{\varphi_\beta(x) - \bar{\varphi}_\beta(x))}{g(\varphi_\beta(x))}\right| \leq \frac{\left|\varphi_\beta(x)\right| + \left|\bar{\varphi}(x)\right| + \tilde{C}\left|g(\varphi_\beta(x))\right|}{\left|g(\varphi_\beta(x))\right|}$$

$$\leq C + \tilde{C}.$$

We can apply Lebesgue's dominated convergence theorem to receive strong convergence of

$$\left\{ \frac{\varphi_\beta - \bar{\varphi}_\beta}{g(\varphi_\beta)} \right\}_\beta \quad \text{to} \quad \frac{\varphi - \bar{\varphi}}{g(\varphi)} = \varphi - \bar{\varphi} \quad \text{in} \quad L^{\frac{2p}{p-2}}(\Omega \times [0, T]),$$

where it was used for the equality that $\varphi \in [0, 1]$ a.e. in $\Omega \times [0, T]$ and $\psi_\beta^c(\varphi) = 0$. Putting these observations together, the right-hand side of (38) converges to $\int_\Omega \xi(\varphi - \bar{\varphi}) \mathrm{d}x$ almost everywhere in $(0, T)$.

For the convergence of the left-hand side of (38), we are making use of the strong convergence of $\{\varphi_\beta\}_\beta$ to φ in $L^2(0, T; L^2)$. As $g(\varphi_\beta)$ is bounded, we see from (39) that $\{\bar{\varphi}_\beta\}_\beta$ converges strongly to $\bar{\varphi}$ in $L^\infty([0, T] \times \Omega)$. We also know that $\bar{\varphi}(x, t) \in [\varepsilon, 1 - \varepsilon]$ almost everywhere. Thus, for all $\bar{\varphi}$ there exists a $\beta > 0$ such that $\bar{\varphi}_\beta(x, t) \in [0, 1]$ almost everywhere. Then, as seen above, it holds that $\psi_\beta^c(\bar{\varphi}_\beta) = \psi^c(\bar{\varphi}_\beta)$ almost everywhere.

From the point wise strong convergence of $\varphi_\beta(x, t) \to \varphi(x, t)$ and $\bar{\varphi}_\beta(x, t) \to \bar{\varphi}(x, t)$, we observe by the continuity of ψ_β^c as well as g that

$$\frac{\psi_\beta^c(\varphi_\beta) - \psi_\beta^c(\bar{\varphi}_\beta)}{g(\varphi_\beta)} \to \frac{\psi_\beta^c(\varphi) - \psi_\beta^c(\bar{\varphi})}{g(\varphi)} = \psi^c(\varphi) - \psi^c(\bar{\varphi}) \quad \text{for a.e.} \quad (x, t \in \Omega \times (0, T)).$$

Note that due to (39), for $\bar{\varphi} \in \mathcal{C}(\overline{\Omega}, [\varepsilon, 1 - \varepsilon])$ it holds that $\psi_\beta^c(\bar{\varphi}_\beta) = 0$ as soon as $\varepsilon :\geq \tilde{C}\sqrt{\beta}$. Therefore, we may apply Fatou's Lemma in order to pass to the limit on the left-hand side of (38) for all $\varepsilon > 0$, concluding that

$$\int_\Omega \xi(\varphi - \bar{\varphi}) \mathrm{d}x \geq \int_\Omega \psi^c(\varphi) - \psi^c(\bar{\varphi}) \mathrm{d}x \quad \forall \bar{\varphi} \in C(\overline{\Omega}, [\varepsilon, 1 - \varepsilon]) \quad \text{with} \quad \int_\Omega \bar{\varphi} \mathrm{d}x = m, .$$

Since $\varepsilon > 0$ was arbitrary, we infer that

$$\int_\Omega \xi(\varphi - \bar{\varphi}) \mathrm{d}x \geq \int_\Omega \psi^c(\varphi) - \psi^c(\bar{\varphi}) \mathrm{d}x \quad \forall \bar{\varphi} \in C(\overline{\Omega}, (0, 1)) \quad \text{with} \quad \int_\Omega \bar{\varphi} \mathrm{d}x = m, \tag{40}$$

which is the definition of the subdifferential on the space of functions with mean m.

Observing the definition of the obstacle potential (2) and the property $\varphi \in [0, 1]$ a.e. in $\Omega \times (0, T)$ and $\bar{\varphi} \in C(\overline{\Omega}, (0, 1))$, we find that the right-hand side of (40) always vanishes, *i.e.* $\int_\Omega \psi^c(\varphi(t)) - \psi^c(\bar{\varphi}) \mathrm{d}x = 0$. Now let A be a measurable set such that $\varphi(x) = 0$ for a.e. $x \in A$. We may define

$$\tilde{\varphi}(x) = \begin{cases} \tilde{\varphi}(x) = 1, & \text{if } x \in A \\ \tilde{\varphi}(x) = \varphi(x, t) - |A|/|\Omega|, & \text{if } x \in \Omega \backslash A. \end{cases}$$

We observe that $\tilde{\varphi} \in L^\infty(\Omega; [0, 1])$ with $\int_\Omega \tilde{\varphi} \mathrm{d}x = M$. Additionally, by the density of $C(\overline{\Omega}, (0, 1))$ in $L^\infty(\Omega; [0, 1])$ with respect to the weak* topology, we find a sequence $\{\tilde{\varphi}_n\}_{n \in \mathbb{N}} \subset C(\overline{\Omega}, (0, 1))$ such that $\tilde{\varphi}_n \overset{*}{\rightharpoonup} \tilde{\varphi}$. This implies by $\int_\Omega \xi \mathrm{d}x = 0$ that

$$0 \leq \lim_{n \to \infty} \int_\Omega \xi(\varphi - \tilde{\varphi}_n) \mathrm{d}x = \int_\Omega \xi(\varphi - \tilde{\varphi}) \mathrm{d}x = -\int_A \xi \mathrm{d}x + \frac{|A|}{|\Omega|} \int_{\Omega/A} \xi \mathrm{d}x = -\left(1 - \frac{|A|}{|\Omega|}\right) \int_A \xi \mathrm{d}x.$$

Note that the density only holds with respect to the weak* topology and not the norm-topology. But this is enough to pass to the limit on the left-hand side of (40). Since the A was arbitrary, we find the assertion

$$\xi(x, t) \leq 0 \text{ for a.e. } (x, t) \in \Omega \times (0, T) \text{ where } \varphi(x, t) = 0.$$

Simmilarly, we find

$$\xi(x, t) \geq 0 \text{ for a.e. } (x, t) \in \Omega \times (0, T) \text{ where } \varphi(x, t) = 1,$$
$$\xi(x, t) = 0 \text{ for a.e. } (x, t) \in \Omega \times (0, T) \text{ where } \varphi(x, t) \in (0, 1).$$

This implies that the subdifferential according to (40) coincides with the point-wise subdifferential of the obstacle potential we would expect. All conditions of Definition 2.1 are fulfilled, proving that $\varphi \in H^1(0, T; L^2) \cap L^2(0, T; H^1)$ is indeed a solution of the Allen–Cahn system with obstacle potential. $\qquad \square$

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